

Complexity and the Character of Stock Returns: Empirical Evidence and A Model of Asset Prices Based Upon Complex Investor Learning

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ABSTRACT

Empirical evidence on the distributional characteristics of common stock returns indicates: 1) A power-law with exponent > 2 describes the positive tail behavior of the survivor function of returns ($pr(r > x) \sim x^{-\alpha}$) (Gopikrishnan et al., 1999; Plerou et al., 1999), 2) The time-series return process is characterized by autoregressive conditional heteroskedasticity (Bollerslev, Chou and Kroner, 1992; Glosten, Jagannathan and Runkle, 1993; Engle, 2004), and, 3) General nonlinear dependencies exist in the time-series of returns (Scheinkman and LeBaron, 1989; Hsieh, 1991; Brock, Hsieh and LeBaron, 1991). We propose a model of complex, self-referential learning and reasoning amongst economic agents that jointly produces security returns consistent with these general observed facts and which are supported here by empirical results presented for a benchmark sample of 50 stocks traded on the New York Stock Exchange. The market we postulate is populated by traders who reason inductively while compressing information into a few fuzzy notions which they can in turn process and analyze with fuzzy logic. We analyze the implications of such behavior for the returns on risky securities within the context of an artificial stock market model. Dynamic simulation experiments of the market are conducted from which market-clearing prices emerge, allowing us then to compute realized returns. The results indicate that the model proposed in this paper can jointly account for the presence of a power-law characterization of the positive tail of the survivor function of returns with exponent on the order of 3, for autoregressive conditional heteroskedasticity and for general nonlinear dependencies in returns. The appeal of the model is its close ties to evidence on how individuals actually reason and provides an alternative view of the influence of nontraditional learning and reasoning in complex, ill-defined capital market settings.

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1. Introduction

Empirical evidence on the distributional characteristics of common stock returns indicates: 1) A power-law tail-index close to 3 describes the behavior of the positive tail of the survivor function of returns ($pr(r > x) \sim x^{-\alpha}$) (Gopikrishnan et al., 1999; Plerou et al., 1999), 2) The time-series return process is characterized by autoregressive conditional heteroskedasticity (Bollerslev, Chou and Kroner, 1992; Glosten, Jagannathan and Runkle, 1993; Engle, 2004), and, 3) General nonlinear dependencies exist in the time-series of returns (Scheinkman and LeBaron, 1989; Hsieh, 1991; Brock, Hsieh and LeBaron, 1991). Little is known however about what behaviors on the part of investors should give rise to jointly observing these phenomena. We propose a model of complex, self-referential learning and reasoning amongst economic agents that jointly produces security returns consistent with these general observed facts. The model features investors who reason inductively through experimentation with new hypotheses while compressing information into a few fuzzy notions that they can in turn process and analyze with fuzzy logic. Our approach is motivated first by the cogent argument set forth by Arthur (1991, 1992, 1994, 1995), Arthur et al. (1997) and LeBaron et al. (1999) who conclude deductive reasoning must give way to inductive reasoning in complex, ill-defined settings and that real capital markets exhibit a high level of complexity. Second, we follow a stream of thought proposed by Smithson (1987) and Smithson and Oden (1999) amongst others who conclude human reasoning can be modeled as if the thought process is described by the application of fuzzy logic. Assuming mental behavior of this sort allows the agent to step outside the rigid confines of more traditional models. We imbed this behavior in an artificial stock market model that is utilized as a vehicle for simulating the dynamics of a market from which market-clearing security prices emerge allowing us to compute realized returns. The structure of the model extends the Santa Fe Artificial Stock Market Model studied by LeBaron, Arthur and Palmer (1999). We show that with-dividend returns computed from simulated market clearing prices for the environment we propose

exhibit positive tails in their survivor functions characterized by a power law with exponent on the order of 3, exhibit autoregressive conditional heteroskedasticity and exhibit general non-linear dependencies. We also document the presence of similar characteristics for a sample of 50 common stocks traded on the New York Stock Exchange which act here as a benchmark. The appeal of our results is twofold: First, the behavioral model we propose generates return characteristics for risky securities that are similar to what is observed for actual stock returns and second, it does so as a product of an environment in which economic agents are endowed with learning and reasoning processes that are close to what many disciplines believe is an accurate depiction of actual behavior.

This is the first successful development to our knowledge of a model of learning in a complex ill-defined setting that produces return characteristics similar to actual returns along the three characteristic dimensions just mentioned. These results are important because they bring us closer to understanding how complexity in capital markets gives rise to observed market driven data and in fact whether complexity driven by the learning and reasoning processes can produce the market outcomes observed in contrast to them being driven solely by the structure of trading.

How investors learn and interact in complex capital market environments is crucial to understanding the nature of financial security return distributions. Complexity demands an alternative approach to the analysis of markets and institutions. Leigh Tesfatsion (2002) has provided an eloquent characterization of the environment about which we speak: "Decentralized market economies are complex adaptive systems, consisting of large numbers of adaptive agents involved in parallel local interactions. These local interactions give rise to macroeconomic regularities such as shared market protocols and behavioral norms which in turn feed back into the determination of local interactions. The result is a complicated dynamic system of recurrent causal chains connecting individual behaviors, interaction networks, and social welfare outcomes." (pg. 1). The analysis of such systems has borrowed heavily from the general modeling framework employed in the interdisciplinary field of complex adaptive systems. The approach has its roots in work begun and continuing at the Santa Fe Institute. Examples of work focusing on the behavior of security prices include Arthur, Holland, LeBaron, Palmer and Taylor (1997), Brock and Hommes (1998), LeBaron, Arthur and Palmer (1999), Tay and Linn (2001). Tesfatsion (2002)

and LeBaron (2000, 2004) provide reviews of this literature.¹

Our model is motivated by the discrepancy between the idealized well-defined environment that is commonly assumed in neoclassical financial market models and the complex ill-defined markets that are observed in practice. Neoclassical financial market models are generally designed within the context of a well-defined setting so that economic agents are able to logically deduce the expected prices of securities which they in turn employ when setting their demands for those securities.² Real stock markets do not however typically conform to the severe restrictions required to guarantee such behavior. The actual market environment is usually much more ill-defined. The dilemma is that in an ill-defined environment the ability to exercise deductive reasoning breaks down, making it impossible for individuals to form precise and objective price expectations. This implies participants would need to rely on some alternative form of reasoning to guide their decision-making. We conjecture that individual reasoning in an ill-defined setting can be described as an inductive process. The application of inductive reasoning involves the formulation of tentative hypotheses to fill in the gaps left by incomplete information. These hypotheses are then continually tested in the market and revised as agents seek to improve their understanding of market behavior. Agents generate predictions by the application of an inductive reasoning process in which they rely on fuzzy decision-making rules due to limits on their ability to process and condense information.

Our study is close in spirit to a contemporaneous study by Gabaix et al. (2004) which focuses on a model of large fluctuations in stock returns and is motivated by the presence of a power-law decay in the survivor function of returns as well as trading volume. Our study however differs in several important ways from theirs. First, we present a model that jointly produces a power-law decay in the survivor function for returns, autoregressive conditional heteroskedasticity and general nonlinear dependencies. The central focus of Gabaix et al. is explaining what gives rise to the power-law decay. Second, our model focuses on the influence of learning and reasoning by agents and the influence of non-traditional aspects of these activities on the distributions of returns. Gabaix et al. present an insightful model built up from assumptions about the structure of trading and the search for trading partners. We on the other hand, choose to minimize these influences in order to highlight how agents learn and reason in an ill-defined environment. In this way both studies provide important and new insights into what factors

potentially give rise to the features of stock returns already mentioned.

The paper is organized as follows. In Sections 2-5, we begin by presenting empirical results on the existence of a power law in the behavior of the survivor function of common stock returns, on the presence of autoregressive conditional heteroskedasticity in stock returns and on the general existence of non-linear dependencies in stock returns. Our focus is on a sample of 50 common stocks traded on the New York Stock Exchange. Section 6 goes on to summarize how agents learn in the model and the market environment. Section 7 describes the dynamic simulation experiments of the model. Section 8 presents an analysis of the returns computed from the market clearing prices generated in the artificial stock market drawing comparisons with the results presented in Sections 2-5 for the benchmark sample of stock returns. Section 9 presents our conclusions

2. The Data and Descriptive Statistics

The data examined herein are comprised of a) daily with-dividend return series for 50 actively traded NYSE-listed common stocks, and b) 10 return series computed from simulations of the artificial stock market model. We defer our analysis of the data from the artificial stock market simulations until Section 10 following our discussion of the model's structure. Instead we begin by focusing on the characteristics of the stock returns for the 50 NYSE-listed stocks in order to establish a benchmark for comparison. The source of the stock return data is the CRSP Daily Return file, and the period included is 01/03/63 through 12/31/98.³

Table 2 presents summary statistics along with Jarque-Bera test statistics for normality.⁴ The sample statistic names are listed in the left-most column. Each cell of the table contains a summary measure for the sample statistic computed over all of the cases associated with either the actual security return series (Column heading: SAMPLE) or the return series computed for each of the artificial stock market experiments (Column heading: ASM). For example after computing the mean daily return for each of the 50 stock return series obtained from the CRSP files, we computed the mean of these means to be .0006. The value is reported in the most northwest cell of the table. The remaining cells report alternative summary measures. The tables that are discussed in later sections are constructed similarly.

Notable amongst the results reported in Table 2 is the typical large kurtosis in sample returns and the positive skewness. Kurtosis for a normal distribution should equal 3 while skewness should equal 0. Both measures for the sample return series deviate from these benchmarks. The Jarque-Bera test statistics soundly reject normality for each series.

3. Power-Law Tail Behavior of the Survivor Function

Panel A of Figure 1 presents a graph of the positive tail of the empirical survivor functions for the normalized returns of the 50 NYSE-listed common stocks in the sample.⁵ The normalized return is defined as the actual return less the mean computed over the sample period divided by the standard deviation of the return computed over the same calendar period. The pattern observed is consistent with what is commonly referred to as a power-law,

$$pr(r_i > x) \sim \frac{1}{x^{\alpha_i}} \quad (3.1)$$

where α_i is the exponent characterizing the power-law tail-index for security i . Note from the figure the similarity in the pattern across the 50 stocks. We compute an overall estimate of α for the panel of stocks using the methods developed in Hill (1975). Given that the tail behavior appears to be rather stable for samples containing the 50 to 100 largest values of r_i , in our computation of the Hill's estimate $\hat{\alpha}_i$ we use cutoff values x_{cutoff} that will give us between 50 to 100 data points beyond x_{cutoff} . Our estimation uses about 3% of the data for any given series. Husman et al. (2001) have found sample sizes of this size to be optimal in the estimation of the exponent. For each stock, we compute the Hill's estimate of $\hat{\alpha}_i$ for 50 samples that contain from 50 to 100 values beyond x_{cutoff} . We then compute the mean of the sample estimates. Finally we compute the mean of the means across the 50 stocks along with the cross-sectional standard deviation. We find the following result:

Positive Tail Exponent: $\alpha = 3.3615 \pm 0.55871$

where the range shown is \pm one standard deviation and 3.3615 is the grand mean. These results are in general agreement with results presented by Plerou et al. (1999) and others who conclude that a power-law exponent based on the positive tail on the order of 3 describes stock returns for a wide range of return

frequencies including daily data.

4. Nonlinear Dependence

A series exhibiting no serial correlation may nevertheless exhibit non-linear dependence (Granger and Andersen (1978)). We therefore first filter the data for linear dependencies. One source of dependence not captured however by linear filtering is the one brought on by the now well-documented autoregressive conditional heteroscedasticity (ARCH) that has been identified in many financial time series (for instance, Bollerslev, Chou and Kroner (1992), Engle (2004)).⁶

We focus our attention on the TARCH or threshold autoregressive conditional heteroskedasticity class of models (Glosten, Jagannathan and Runkle (1993)). These models account for the possibility that the market reacts in an asymmetric fashion to good and bad news and have been shown to have good explanatory power. The general structure of the ARMA(m,n)-TARCH(p,q) model for a return series $r_{i,t}$ is composed of an equation for the mean and an equation for the conditional variance:

$$r_t = a_0 + \sum_{\tau=1}^m \phi_{\tau} r_{t-\tau} + \varepsilon_t - \sum_{\omega=1}^n \theta_{\omega} \varepsilon_{t-\omega} \quad (4.1)$$

$$\sigma_t^2 = \omega + \sum_{\eta=1}^q \alpha_{\eta} \varepsilon_{t-\eta}^2 + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{\sigma=1}^p \beta_{\sigma} \sigma_{t-\sigma}^2 \quad (4.2)$$

where we suppress the i subscript, $d_t = 1$ if $\varepsilon_t < 0$ and 0 otherwise. ε_t represents a random error in principle driven by exogenous information shocks, and σ_t^2 is the conditional variance. The indicator variable d_t captures the asymmetric reaction of the market to good and bad news.

We begin by filtering each of the 50 return series for ARMA-TARCH effects. We fit ARMA-TARCH models to each of the data series using maximum likelihood methods. The AIC and BIC criteria are used to determine the appropriate number of lags.⁷ Panel A of Table 3 reports summary statistics for the autocorrelations of the consequent residual series and the corresponding Q-statistics.⁸ We infer from the summary data for the Q statistics reported in Table 3 that autocorrelation is not present in any of the series.

Panel B of Table 3 reports the corresponding serial correlation data for the squares of the residuals and the

respective Q-statistics utilizing the autocorrelations of the squared residuals. If statistically significant ARCH effects persist the Q-statistics for the squared residuals will be large (Gourieroux and Jasiak (2001, pp. 129-130)).⁹ We cannot reject the null of no autocorrelation in the squared residuals at conventional statistical levels. The results shown in Table 3 lead us to infer that the residuals computed as a result of fitting ARMA-TARCH models to the sample data are statistically indistinguishable from IID and that the series do not exhibit ARCH-type behavior.

5. The BDS Test for Dependence

A test developed by Brock, Dechert and Scheinkman (1987) (see also Brock, Dechert, LeBaron and Scheinkman (1996)) has power to reject the null hypothesis of independently and identically distributed data against several alternative hypotheses including models exhibiting nonlinear dependence.¹⁰ The data we examine are the residual series after filtering for ARMA-TARCH effects. The test then amounts to a test of the null hypothesis of an IID process against the alternative of a nonlinear deterministic or a nonlinear stochastic process. The BDS test statistic has an asymptotic normal distribution under the null hypothesis of IID data. Details on the test statistic are presented in an Appendix. A positive BDS statistic suggests some form of clustering is present on a too frequent basis, or in other words, patterns occur too frequently relative to what would be expected if the data exhibited no dependencies.

The BDS statistic tests the propensity of a series to cluster within a distance between points of ε . If the data are IID the probability of the distance between any pair of points being less than or equal to ε will be constant. In contrast, if the filtered data exhibit nonlinear dependence then this will not tend to be the case. We set the parameter ε to 1.5 and 2 times the standard deviation of the residuals. Kanzler (1999) in an extensive study of the behavior of the BDS statistic has found these settings for ε minimize the failure of the BDS statistic to approximate the normal distribution in finite samples under the null hypothesis of IID data. The BDS test statistic has an asymptotic normal distribution under the null hypothesis of IID data. Details on the test statistic are presented in an Appendix including an elaboration on the definition of the 'embedding dimension' n .¹¹ A positive BDS statistic suggests some form of clustering is present on a too frequent basis, or in other words, patterns occur too frequently relative to what would be expected if the data exhibited no dependencies. We reject the null hypothesis at the .5% level whenever

$$|BDS_n(\varepsilon, T)| > 2.57.$$

Table 4 shows the BDS test consistently rejects IID for the ARMA-TARCH residuals for the sample data series. Separately in unreported results we fit ARMA – EGARCH models (Nelson, 1991) to the series.¹² Our conclusions upon computing the BDS statistics for the residuals of the ARMA – EGARCH specifications are largely the same, the EGARCH specifications do not remove the nonlinear dependence from the data.

We conclude from these results that the sample of common stocks jointly exhibit a power-law decay in the positive tail of the survivor function of returns, exhibit autoregressive conditional heteroskedasticity and exhibit nonlinear dependencies after both linear filtering as well as filtering out the influence of conditional heteroskedasticity. We now turn to a discussion of our model of investor behavior and then to the empirical experiments and how those results compare to those found for the actual stock returns.

6. The Model

6.1 Learning and Hypothesis Development by Agents

The argument that individuals will form their expectations by induction in ill-defined environments has been suggested as an alternative to the deductive model usually invoked (for instance, Arthur (1991, 1992, 1994, 1995), Arthur et al (1997), Blume and Easley (1995), LeBaron et al (1999), Rescher (1980)). Induction is a means of finding the best available answers to questions that transcend the information at hand.

Inductive reasoning follows a two-step process: possibility-elaboration and possibility-reduction. The first step involves creating a spectrum of plausible alternatives based on our experience and the information available. In the second step, these alternatives are tested to see how well they answer “the question” being asked or how well they connect the existing incomplete premises to explain the data observed. The alternative offering the *best-fit connection* is then accepted as a viable explanation. Subsequently when new information becomes available or when the underlying premises change, the *fit* of the current *connection* may degrade. When this happens it will be replaced by a new *best-fit* alternative.

Under this scheme of rationalizing, each individual in the market continually creates a multitude of “market hypotheses” (this corresponds to the possibility-elaboration step discussed above). These hypotheses, which represent the individuals’ subjective expectations models of what moves the market price and dividend, are then simultaneously tested for their predictive ability. The hypotheses identified as reliable will be retained and acted upon in buying and selling decisions. Unreliable hypotheses will be dropped or changed (this corresponds to the possibility-reduction step) and ultimately replaced with new ones. This process is carried out repeatedly as individuals receive new information and adapt in a constantly evolving market.

Agents learn market-level data (specific market descriptors) their past actions have in turn helped generate. In this way there is a self-referential element that contributes to the complexity of the environment. They use these market-level data in conjunction with hypotheses they hold to compute predictions about future prices. Agents may hold multiple hypotheses that can be used in predicting future stock prices. Agents also learn about the predictive accuracy of the hypotheses they hold. Each period an agent selects to use which of his extant hypotheses is most accurate based on its squared-error forecasting accuracy in the immediate past.

The hypotheses held by any agent are however not static but rather evolve over time as he engages in inductive learning and experimentation. Based upon the data at-hand the agent selects to retain hypotheses, modify hypotheses through combining features of existing hypotheses and experiment with new hypotheses. The agent continually checks her extant hypotheses for the degree of their ex-post accuracy and parsimony.¹³ A hypothesis that has proven to be accurate (in a prediction sense) has a high probability of being retained in the set of hypotheses utilized by the agent in the next round. An inaccurate hypothesis has a high probability of being replaced by a new hypothesis.

Agents actively engage in the generation of new hypotheses through what we will call Combination Experiments and Individual Experiments. Combination Experiments involve combining elements of pairs of accurate hypotheses in an attempt to discover even more accurate hypothesis. Such activity is assumed to occur with fixed probability π . A new hypothesis so generated then replaces (with high probability) a low accuracy hypothesis amongst the set of hypotheses held by the agent. Individual Experiments involve the modification of already

existing hypotheses by alteration of the parameters associated with the hypothesis. Individual Experiments occur with probability $(1-\pi)$. Therefore Individual Experiments only occur when Combination Experiments do not. Agents revise their hypotheses every k periods.¹⁴ The new, untested hypotheses that are created will not in-and-of themselves, cause disruptions because they will be acted upon only if they prove to be accurate. This avoids brittleness and provides what machine-learning theorists call “gracefulness” in the learning process.

6.2 Learning, Price Predictions and Demands

The model begins with a dividend, d_t , announced publicly at time t and which follows an autoregressive process of the form

$$d_t = \bar{d} + \rho(d_{t-1} - \bar{d}) + v_t, \quad (8.1)$$

where, $v_t \sim N(0, \sigma_v^2)$.

Agents observe past prices and the dividend. The forecasting model employed by any agent k is assumed to be linear in past prices and dividends and is defined by

$$\hat{E}_t[p_{t+1} + d_{t+1}] = a_{k,j}(p_t + d_t) + b_{k,j}, \quad (8.2)$$

where $a_{k,j}$ and $b_{k,j}$ are the parameters selected by agent k based upon one of the hypotheses j in his set of hypotheses. The hypothesis selected as a basis for the choice of the parameters is the most accurate hypothesis held by the agent based upon the performance of the hypothesis in predicting the price plus dividend during the immediate past.

The process by which the agent draws conclusions about the parameters a and b involves the application of fuzzy logic.¹⁵ Each agent in the model forms personal expectations using what we will refer to as a rule base (hypothesis). Each rule base contains a set of *conditional forecast* rules that guide decision-making. Each rule base can be thought of as a subjective “market hypothesis” held by the individual. The rules themselves reflect the individual's application of fuzzy logic as a guide in coalescing information. The individual is permitted to hold up to five different hypotheses about the price process.

Each agent summarizes historic data available at date t with what we will call market descriptors, or

information ‘bits’. Five market descriptors [$p * r/d$, $p/MA(5)$, $p/MA(10)$, $p/MA(100)$, and $p/MA(500)$] are computed. The variables r , p and d are the interest rate, price, and dividend respectively. The variable $MA(n)$ denotes an n -period moving average of prices. Thus, the first information bit reflects the current price in relation to the current dividend and is a “fundamental” bit. The remaining four bits are “technical” bits indicating whether the price history exhibits a trend or similar characteristic. Agents use the information bits when formulating conclusions about a and b . We refer to the bits as [$fund$, $tech1$, $tech2$, $tech3$, $tech4$].

Figure 2 illustrates the underlying reasoning process employed by each agent. The five graphs on the left side of the figure describe the agent’s inclinations to assign weight to the conclusion that a value for a specific information bit is a member of one of four different characteristic sets. The curves themselves can be thought of as ‘Membership Functions’. Refer to the boxed graph identified by $fund$. The four curves within the box from left to right portray the Membership Functions for assigning the conclusion that the observed value for $fund$ is low, moderately low, moderately high or high, respectively. The Membership Functions for each information bit play an important role in the evaluation of each market hypothesis. A clarifying example of the connection will be discussed shortly when we consider Figure 3. For now, we continue our description of Figure 2.

Each agent maintains a set of five different hypotheses about the evolution of prices with each hypothesis presenting a different scenario for the values of a and b . Each hypothesis, which we refer to as a rule base, is composed of four rules. A rule defines a conditional relation between the extent to which the observed information bits have specific characteristics (low, moderately low, moderately high, high) and specific characteristics (low, moderately low, moderately high, high) for the forecast model parameters (a) and (b). Each Rule has the general form: If $fund$ is x_1 and $tech1$ is x_2 and $tech2$ is x_3 and $tech3$ is x_4 and $tech4$ is x_5 , then a is y_1 and b is y_2 . We define $x_1, x_2, x_3, x_4, x_5 \in \{0,1,2,3,4\}$ and $y_1, y_2 \in \{1,2,3,4\}$ where the codes 1,2,3,4 correspond to the characteristics (low, moderately low, moderately high, high) and the code 0 implies the information bit is not used in the evaluation of the rule. Therefore, as should be clear, the parameters a and b each are associated with the characteristics (low, moderately low, moderately high, high).

The potential characteristics of the parameters a and b are described by the two graphs on the far right of

Figure 2. Each of these graphs portrays four curves. The curves are the Membership Functions for a and b . Referring to the graph for the parameter b , the arrow points to the Membership Function associated with the characteristic ‘moderately low’.

Figure 3 illustrates the process by which values for the parameters a and b are determined at a point in time for a given hypothesis. The agent observes (computes) a value for each of the information bits. Each information bit is associated with one of the first five columns of graphs (from left to right). The values are listed across the top of the figure. There are four rules for the hypothesis. The specification of each rule is provided in the box at the bottom of the figure. Each of the rules is associated with a row of graphs in the figure. The consequent part of each rule associated with the parameters a and b are displayed in the last two columns of graphs. To understand the mechanics of the process consider Rule 2: “If $tech1$ is moderately low and $tech4$ is moderately high, then a is moderately low and b is moderately low”. Notice that this rule does not make use of the bits $fund$, $tech2$ or $tech3$. Hence the only graphs on row 2 of interest are those for $tech1$ and $tech4$. The graph for $tech1$ (on row 2) displays only the Membership Function for the set ‘moderately low’, since this is the condition to be tested by the rule. Further, the graph for $tech4$ (on row 2) displays only the graph for moderately high since it is the only condition to be tested. Finally the graphs for a and b (on row 2) display only the curves for ‘moderately low’ and ‘moderately low’ respectively, since these are the outcomes associated with the satisfaction of the conditions placed on the information bit values by Rule 2.

The graphs show the values for $tech1$ and $tech4$ each lie within the Membership Functions associated with the conditions laid out in Rule 2. However, the value for $tech4$ is a weak member of the “Moderately High” Membership Function. Rule 2 specifies that the two conditions must jointly occur in order for the rule to be satisfied. The weakness of the result for $tech4$ acts like an upper bound on the agent interpreting the joint condition as having been satisfied. That is the weight given by the agent to the satisfaction of the joint condition is limited by the weakness of the membership of $tech4$ in the moderately high set. This upper bound translates into a constraint on the weight the agent attaches to the outcomes specified by the rule for the parameters a and b (on row 2). The agent concludes that while Rule 2 is somewhat satisfied, the conclusions for a and b must be tempered due to the fact that

the observed value for *tech4* is only a partial member of the “Moderately High” membership function. The tempering process is reflected in the weight the agent assigns to the conclusions about *a* and *b*. These weights are shown by the shaded areas under the curves in the two graphs for *a* and *b* (on row 2). Weighted conclusions about *a* and *b* are generated for each rule (each row).

The final step in generating point values for *a* and *b* comes through first aggregating the weighted conclusions for *a* and *b* across rules. The aggregation is done by summing the shaded areas and is shown in the two graphs in the southeast corner of the figure. Smithson (1987) and Smithson and Oden (1999) present results suggesting individuals reason as if by the application of fuzzy logic. When confronted with choices similar to what are presented here, individuals tend to utilize the point estimate that is most central on a weighted basis, generally referred to as the centroid. We assume agents in the model use the centroid, which is identified with a bold vertical bar in the final two graphs located in the southeast corner of the figure.

Each agent forecasts next period’s price and dividend ($\hat{E}_{i,t}[p_{t+1} + d_{t+1}]$) using the forecast parameters from the rule base (hypothesis) in her set that has proven to be the most recently accurate. The hypothesis that has performed best, in terms of a moving-average squared forecast error is the one selected as the basis for making a prediction about next period’s price.

The share demand by agent *k* equals

$$x_{k,t} = \frac{\hat{E}_{k,t} [p_{t+1} + d_{t+1}] - p_t(1+r)}{\lambda \hat{\sigma}_{k,t,p+d}^2} \quad (8.3)$$

where p_t is the price of the risky asset at time *t*, λ is the degree of risk aversion, r is the relevant risk-free interest rate for the time horizon and the expectation (prediction) and variance estimate are conditional on hypothesis *j* the most recently accurate of the hypotheses held by the agent.¹⁷ Agents in the model know that equation (8.3) will hold in a homogeneous rational expectations equilibrium when the degree of risk aversion is constant across individuals. However, the fact that they must use induction to form and modify hypotheses and that they use fuzzy rules when forming expectations, means that they never know if the market is actually in equilibrium. Faced with this conundrum, we assume agents select to use (8.3) when setting their demands, knowing that sometimes the market

will be in equilibrium and that sometimes it will not. Each agent is endowed with one share and hence the market clearing condition is

$$\sum_{k=1}^N x_{k,t} = N. \quad (8.4)$$

The market clearing price, p_t , is found by summing equation (8.3) over all agents and then setting the sum equal to N , the number of shares available.¹⁸ Once the market clears, the price and dividend at time t are revealed and the accuracies of the rule bases are updated.

7. The Market Experiments

We examine the implications of our conjecture about the learning and reasoning processes for the underlying hidden structure of the risky security's returns. Learning frequency refers to the frequency at which hypothesis revisions occur. When the learning frequency is high, agents will revise their rule bases (hypotheses) more often. Learning frequency will play a key role in determining the structure of the rule bases and how closely agents' price expectations. When learning frequency is high, agents will revise their beliefs frequently their hypotheses are more likely to be influenced by transient behavior in the time series of market variables. These factors together will make it difficult for agents to converge on a common equilibrium price expectation. In contrast, when learning frequency is low, agents will have more time between revising their rule bases to test their hypotheses. Furthermore, their hypotheses will also tend to be based on longer horizon features in the time series of market variables. Consequently, agents are more likely to converge on a common equilibrium price expectation. The learning frequency parameter is defined to be high in the experiments we present here.¹⁹

The model's parameters that are common to all the experiments are tabulated in Table 1. The learning frequency is equal to 30. This means that the agents learn on average once every 30 periods. Combination Experiments occur with a fixed probability of .2 at each revision date. Individual Experiments occur with a fixed probability of .8 at each revision date. A complete description of the process by which experimentation by agents is implemented is provided in an Appendix.²⁰ We began with a random initial configuration of rules. We then simulated the market for 10,000 periods to allow any asymptotic behavior to emerge. Subsequently, starting with the

configuration attained at $t = 10,000$ we simulated an additional 2,500 periods to generate price data for the statistical analysis discussed in the next section. The returns examined in the next section utilize the latter 2,500 price observations and are computed as

$$r_t = \frac{p_t + d_t}{p_{t-1}} - 1. \quad (9.1)$$

8. Results

The columns labeled 'ASM' in Table 1 and all subsequent tables present summary results for the 10 trial experiments of the artificial stock market. Return first to Table 1. Recall the columns headed SAMPLE present summary results for the returns on the 50 NYSE-traded stocks. A comparison of the summary statistics for the experiments (ASM) and for the actual return series (SAMPLE) indicates a close correspondence in the numbers. Thus at this level we can conclude our choice of parameters for the experiments yields characteristics that are similar to those found in the SAMPLE data. In particular, the experimental data exhibit large positive kurtosis and positive skewness. The null hypothesis of normality is consistently rejected for the experimental data.

Panel B of Figure 1 presents the positive tails of the empirical survivor functions for the normalized returns of the 10 experimental cases. The pattern in the graphed lines for the experiments is similar to the pattern in the benchmark returns as shown in Panel A of the figure. Further the Hill estimate for the overall α tail-index assuming similar specifications of a power-law for the decay in the survivor function yields:

Positive Tail: $\alpha = 3.3017 \pm 0.41649$ (one std. dev.)

The result compares favorably with the result for the benchmark stocks. Positive Tail: $\alpha = 3.3615 \pm 0.55871$ (one std. dev.). We conclude the experimental data exhibit a power-law exponent for the positive tail of the survivor function of returns that is highly similar to what is observed for the SAMPLE data.

Table 3 presents the autocorrelations and Q-statistics for the residuals and squared residuals of the experimental series computed after filtering for ARMA-TARCH effects. The computed Q-statistics are of a similar magnitude to those shown for the actual data, and indicate the null cannot be rejected in either Panel A or Panel B.

Thus we conclude the residuals exhibit IID characteristics and that autocorrelation is not apparent in the squared residuals of the experimental data.

The columns labeled ASM in Table 4 present summary data on the BDS test statistics for the experimental series using the residuals from the ARMA-TARCH models. The tests consistently reject the null hypothesis of no dependence in the data. Hence, we conclude the tests suggest nonlinear dependencies are present in the experimental data as is also shown for the SAMPLE data.

In conclusion, the empirical results for the experimental data compare favorably with the results for the benchmark returns of the NYSE-traded stocks.

9. Conclusions

The paper began by establishing three basic results for the daily returns on a benchmark sample of 50 common stocks traded on the New York Stock Exchange. First, the positive tail of the empirical survivor functions for these stocks exhibits a power-law decay ($pr(r > x) \sim x^{-\alpha}$) with exponent between 3 and 4. Second, the series exhibit both ARMA structure and autoregressive conditional heteroskedsticity. Third, nonlinear dependencies remain present in the data even after filtering for ARMA and TARCH effects.

We go on to present an alternative model of learning and reasoning behavior in capital markets where the environment that investors operate in is complex and ill-defined. Supported by extant evidence from research in psychology and other related disciplines, we argue that investors who are limited in their abilities to process extensive amounts of data and who must make decisions in an ill-defined environment will rely on their innate abilities to analyze in fuzzy terms and reason inductively. We show that these traits can be faithfully captured by a genetic-fuzzy classifier system. We assert that models endowing agents with such learning and reasoning processes may account for some of the documented empirical characteristics found in the actual stock returns of our benchmark sample. As such, we imbed the learning and reasoning process in an artificial stock market model and conduct dynamic simulation experiments to generate market-clearing prices for a risky security. We compute implied returns using the prices generated by the experiments and go on to analyze these returns using the same methods applied to our benchmark sample. We find the characteristics of the returns from our experimental market

conform to those for the benchmark sample. Specifically, the positive tails of the empirical survivor functions of the experimental return data exhibit a power-law with exponent in the range 3-4, exhibit autoregressive conditional heteroskedasticity and ARMA effects, and after filtering out the ARMA – TARCH effects from the data, are associated with BDS test statistics suggesting nonlinear dependencies are still present in the data. These results are the same as the results found for the returns on the benchmark sample.

The results indicate that the model proposed in this paper can jointly account for the presence of a power-law decay in the positive tails of the survivor function of returns, for autoregressive conditional heteroskedasticity and for general nonlinear dependencies. The framework offers an alternative perspective on what generates the behavior of financial security returns that extends beyond the traditional paradigms. A useful extension of our work would be to marry the learning and reasoning process we propose with a model of the structure of trading similar to that proposed in Gabaix et al. (2004). We leave that endeavor for future research.

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Endnotes

¹ In Tay and Linn (2001) we examine amongst other things, the time series of prices for a related model and in particular how those prices deviate from rational expectations prices. That study does not address the jointly observed behaviors in returns mentioned at the beginning of this section and which are the focus of this study. See also Arthur, Holland, LeBaron, Palmer, and Tayler (1997) and Palmer, Arthur, Holland, LeBaron and Tayler (1994). Leigh Tesfatsion of Iowa State University maintains a comprehensive website devoted to agent-based computational economics and finance:

(<http://www.econ.iastate.edu/tesfatsi/acc.htm>). LeBaron (2000, 2004) presents a review of the foundation articles.

² An excellent example is the general equilibrium model developed in Brock (1982).

³ Returns from the period October 19-21, 1987 are excluded from each series.

⁴ The Jarque-Bera statistic is used to test the hypothesis that a given set of data is drawn from a normal distribution. The test statistic measures the difference of the skewness and kurtosis of the series with those from the normal distribution. The statistic is computed as:

$$JB = \frac{N-k}{6} \left(S^2 + \frac{1}{4}(K-3)^2 \right)$$

where S is the skewness, K is the kurtosis, and k represents the number of estimated parameters used to create the series (Jarque and Bera (1987)). Under the null hypothesis of a normal distribution, the Jarque-Bera statistic is distributed as χ^2 with 2 degrees of freedom.

⁵ Let y be a random variable with cumulative probability distribution function $F_y(x)$. The survivor function $S_y(x)$ is defined as: $S_y(x) = 1 - F_y(x) = pr(y > x)$.

⁶ See Engle (1982) for the original development of the ARCH model.

⁷ See Akaike (1973, 1987); Akaike Information Criterion, AIC; Schwarz (1978); Bayesian Information Criterion, BIC.

⁸ ρ_t refers to the autocorrelation of the relevant series at lag t . $Q(L)$ refers to the Q -statistic computed up to a lag L . The Q -statistic is computed in the following manner (Ljung and Box (1979)):

$$Q_L = T(T-2) \sum_{i=1}^L \frac{\rho_i^2}{(T-i)}$$

Under the null hypothesis of no autocorrelation the Q -statistic is distributed χ^2 with L df.

⁹ Engle (1983) develops a Lagrange multiplier test for the presence of ARCH. The test statistic is computed easily in three steps. First compute the residuals from the linear filtering process. Square the residuals and then regress the squared residual at t on the squared residual at $t-1$. The R^2 from this regression multiplied by the number of observations equals the test statistic. The statistic is asymptotically distributed chi-squared with 1 df under the null hypothesis of conditional homoskedasticity. An alternative test involves the examination of the Ljung-Box Q -statistics computed from the autocorrelations of the squared residuals. The squared residuals have expectation equal to the variance. Rejection of no autocorrelation in the squared series (ie. large Q -statistics) is consistent with rejection of conditional homoskedasticity.

¹⁰ See also, Brock, et al. (1991), Brock, et al. (1996), LeBaron (1997) and Kanzler (1999) for further discussion and analysis of the BDS test. Several alternative tests have been proposed in the literature including a test developed by Kaplan (1994) and the test of White (1989). Barnett et al. (1997) show that for the large sample experiments they study the BDS test always rejects the null when it should be rejected.

¹¹ The Appendix appears in the Supplement accompanying this paper. The BDS statistics were computed using the MATLAB routine developed by Blake LeBaron of Brandeis University.

¹² The EGARCH model proposed by Nelson (1991) has the following structure:

$$\ln(\sigma_t^2) = \omega + \sum_{\tau=1}^p \beta_\tau \ln \sigma_{t-\tau}^2 + \sum_{\eta=1}^q \alpha_\eta \left[\frac{|e_{t-\eta}|}{\sigma_{t-\eta}} - E \left\{ \frac{|e_{t-\eta}|}{\sigma_{t-\eta}} \right\} \right] + \sum_{\eta=1}^q \gamma_\eta \frac{e_{t-\eta}}{\sigma_{t-\eta}}$$

where $E \left\{ \frac{|e_{t-\eta}|}{\sigma_{t-\eta}} \right\} = \sqrt{\frac{2}{\pi}}$ under the assumption of normal errors. We estimate the model by maximum likelihood methods.

¹³ Forecast precision is measured by the inverse of

$$e_{k,j,t}^2 = (1 - \theta) e_{k,j,t-1}^2 + \theta \left[(p_t + d_t) - E_{k,j,t-1}(p_t + d_t) \right]^2$$

where k indexes the individual agent and j ($=1,2,3,4,5$) indexes an hypothesis held by individual k . We set $\theta=.02$. The variable $e_{k,j,t}$ is the deviation of the actual price plus dividend from the price plus dividend prediction for made by individual k using hypothesis j for period t . The fitness of each hypothesis is calculated as

$$f_{k,j,t} = -e_{k,j,t}^2 - \beta s.$$

where β is a constant and s is the number of information bits utilized. The fitness measure imposes higher costs on hypotheses that product larger squared forecast errors and which employ a greater amount of information. Similar specifications are employed by LeBaron et al. (1999) and Tay and Linn (2001). We set $\beta=.000005$ in all simulations.

¹⁴ The mechanism utilized in modeling learning by induction is a genetic algorithm. Comprehensive discussions on Genetic Algorithms include Holland and Reitman (1978), Holland, Holyoak, Nisbett and Thagard (1986), Goldberg (1989). The process of recombining and experimentation are referred to as crossover and mutation in the literature on genetic algorithms.

¹⁵ L. A. Zadeh (1962, 1965) developed the original exposition on fuzzy logic. We return to its application

and the definition of fuzzy rules in the context of individual decision-making in a later section. Smithson (1987) and Smithson and Oden (1999) amongst others present evidence on reasoning and the human thought process that suggests the assertion that individuals reason as if by the axioms of fuzzy logic is supported.

¹⁶ The moving average of the squared forecast error for the hypothesis selected serves as the estimate of the variance.

¹⁷ The optimal demand function is derived from the first order condition of expected utility maximization of agents with exponential utility of consumption under the condition that the random variable of interest is normally distributed. However, when the distribution of stock prices is non-Gaussian the above connection to the maximization of expected utility under an exponential utility function no longer exists, so in those cases we simply take this demand function as given.

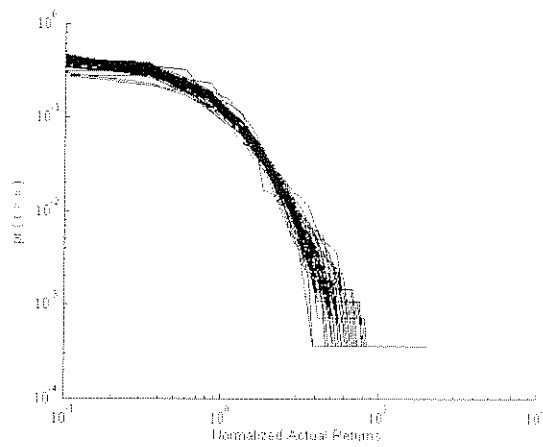
¹⁸ The following equation is used to proxy for the forecast variance ($\sigma_{k,j,t,p+d}^2$) that appears in the denominator of equation (8.3): $e_{k,j,t}^2 = (1-\theta)e_{k,j,t-1}^2 + \theta[(p_t + d_t) - E_{k,j,t-1}(p_t + d_t)]^2$. The equation represents the moving average of the squared forecast error for the hypothesis selected. A similar specification has been employed by LeBaron et al. (1999) and Tay and Linn (2001). We set the parameter $\theta = .02$ in the simulations. Results not reported suggest our overall conclusions are not sensitive to the selection of θ .

¹⁹ The model was coded and simulated using MATLAB, a product of the MathWorks, Inc.

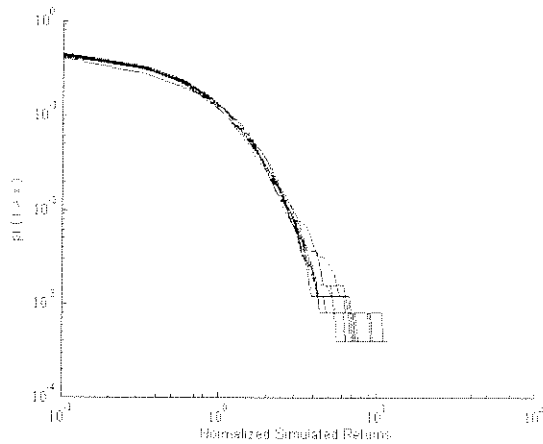
²⁰ The Appendix appears in the Supplement accompanying this paper.

Figure 1

Positive Tail of the Empirical Survivor Function, $pr(r > x)$, of Normalized Daily Stock Returns
Panel A: Daily Returns on 50 Common Stocks Traded on the New York Stock Exchange from the period 01/03/63 through 12/31/98
Panel B: Returns from 10 Simulation Experiments of the Artificial Stock Market Model Parameterized as Shown in Table 1



Panel A



Panel B

Note: Normalized daily returns are computed as the actual return minus the mean return all divided by the standard deviation of the return.

Figure 2
Structure of the Fuzzy Inference System Used by Agents

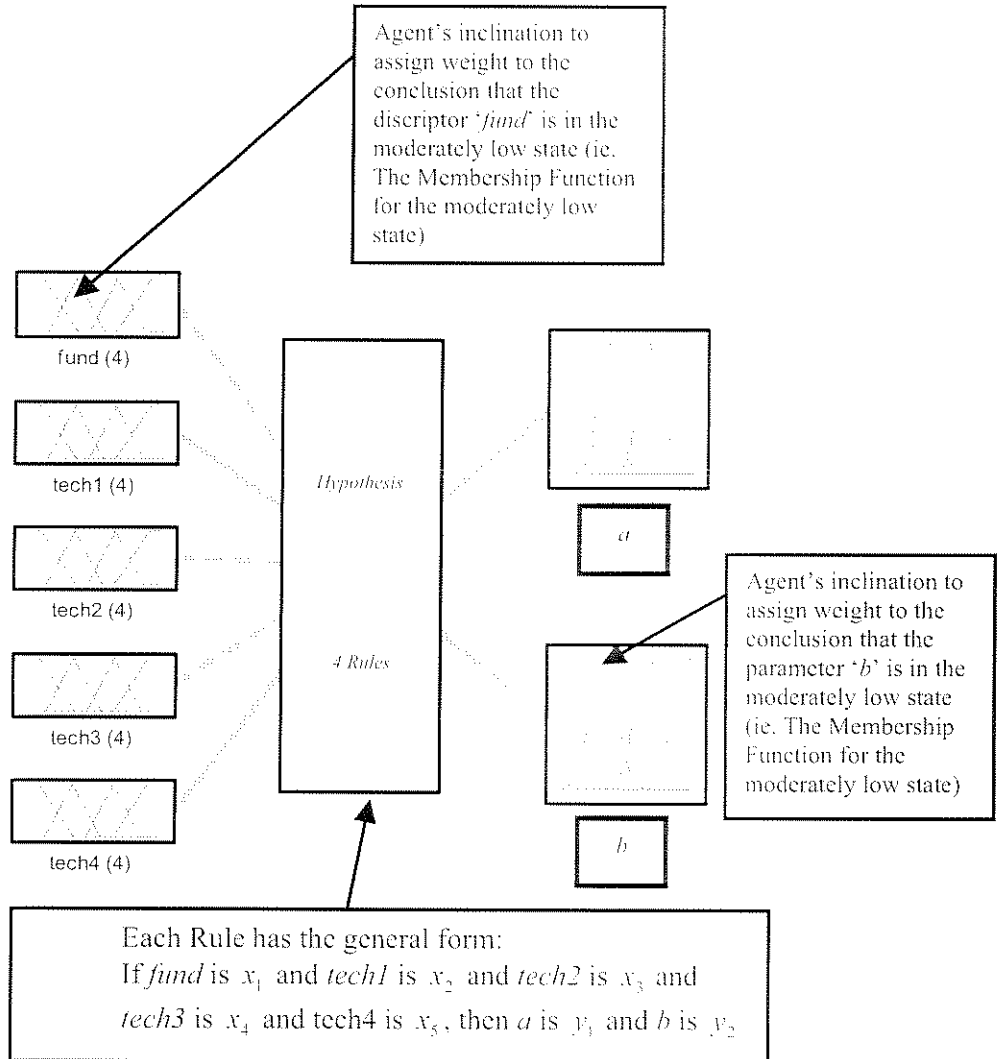
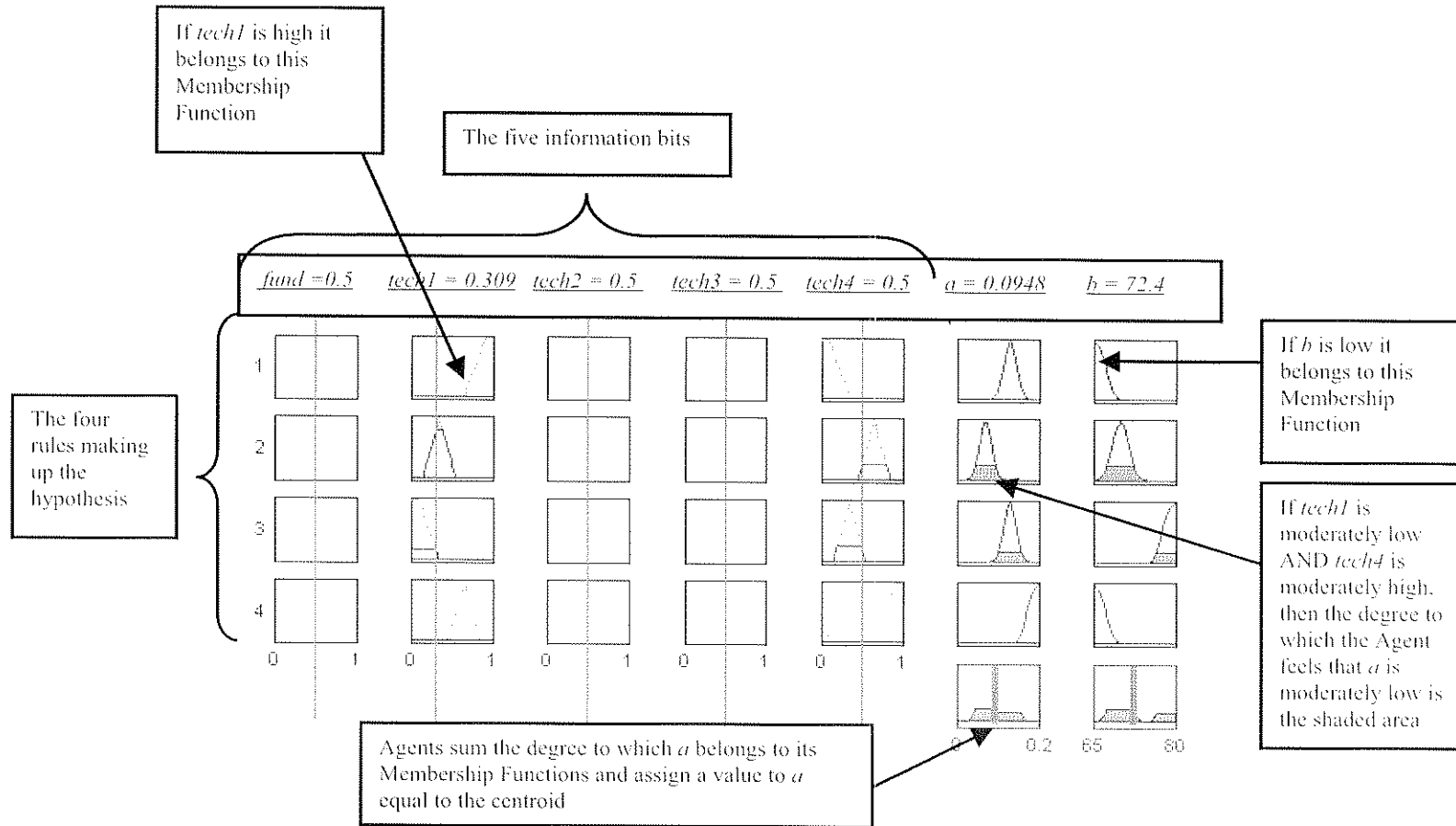


Figure 3
Illustration of the Process By Which an Agent Maps Information Into Conclusions About the Values of the Parameters a and b Using a Fuzzy Logic Reasoning System



The descriptors, $fund$, $tech2$ and $tech3$ are excluded from the hypothesis

Rule 1: If $tech1$ is high and $tech4$ is low, then a is moderately high and b is low

Rule 2: If $tech1$ is moderately low and $tech4$ is moderately high, then a is moderately low and b is moderately low

Rule 3: If $tech1$ is low and $tech4$ is moderately low, then a is moderately high and b is high

Rule 4: If $tech1$ is moderately high and $tech4$ is high, then a is high and b is low

Table 1
Parameter Values For the Artificial Stock Market Model

Description of Parameters	Values
Mean Dividend (\bar{d} in eq. (8.1))	0.0137
Autoregressive Parameter (ρ in eq. (8.1))	0.1
Variance of error term in eq. (8.1)	0.00040502
Risk-free Interest Rate (r)	0.00016783
Risk Aversion Parameter (λ in eq. (8.3))	0.5
Number of Information Bits for Market Descriptors	5
Number of Forecast Model Parameters	2
Probability of Combination Experimentation	0.2
Probability of Individual Experimentation	0.8
Number of Agents (also Number of Shares)	25
Number of Rule Bases (Hypotheses) per Agent	5
Number of Fuzzy Rules per Rule Base (Hypothesis)	4

Note: Common parameter values in the simulations of an artificial stock market in which a risky security is traded and investors are subjective expected utility of final wealth maximizers. Investors represented in the model employ induction and reason as if by fuzzy logic when forming their expectations. The process is modeled as a genetic-fuzzy classifier system. We use the daily 1-year T-bill rates in the secondary market from July 15, 1959 to September 5, 2001. In the simulation, we divide this interest rates series by 365 to obtain the approximate daily interest rates. The value given in the table is the mean daily interest rate. The Probability of Combination Experimentation is the probability that elements of two hypotheses will be split and combined. The Probability of Individual Experimentation is the probability that an agent will have one of his rule bases subjected to random change. When a particular rule base is selected for experimentation, the probability that any individual bit is changed equals 0.5.

Table 2**Summary Statistics for the Sample Returns**

Statistic	Mean		Median		Std Dev		Maximum		Minimum	
	Actual	ASM	Actual	ASM	Actual	ASM	Actual	ASM	Actual	ASM
Mean	0.0006	0.0001	0.0006	0.0001	0.0003	0.0000	0.0014	0.0002	-0.0003	0.0001
Median	0.0000	-0.0003	0.0000	-0.0003	0.0000	0.0001	0.0000	-0.0001	0.0000	-0.0005
Maximum	0.1580	0.1764	0.1400	0.1779	0.0771	0.0290	0.4000	0.2155	0.0500	0.1174
Minimum	-0.1360	-0.0805	-0.1250	-0.0706	0.0598	0.0219	-0.0500	-0.0582	-0.3300	-0.1204
St Dev	0.0194	0.0166	0.0179	0.0166	0.0084	0.0013	0.0507	0.0186	0.0084	0.0144
Skewness	0.4025	0.9226	0.4016	0.8745	0.4038	0.2883	1.5053	1.3739	-0.7138	0.4985
Kurtosis	10.8717	11.8025	9.3995	11.5272	3.8730	3.6920	20.9968	17.2438	5.9347	6.1766
Jarque-Bera	9029.435	9734.930	5055.975	8096.276	9095.018	7178.316	37611.410	21813.950	1000.435	1154.639

Table 3

Autocorrelations and Ljung-Box Q statistics for ARMA-TARCH Residuals

Panel A: ARMA-TARCH residual statistic

Statistic	Mean		Median		St Dev		Max		Min	
	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM
$\rho(1)$	0.0074	0.0040	0.0100	0.0030	0.0180	0.0157	0.0340	0.0270	-0.0450	-0.0330
$\rho(2)$	-0.0048	-0.0046	-0.0035	-0.0070	0.0195	0.0193	0.0370	0.0390	-0.0470	-0.0270
$\rho(3)$	0.0008	-0.0031	-0.0005	0.0005	0.0194	0.0132	0.0480	0.0160	-0.0340	-0.0210
$\rho(4)$	-0.0016	0.0088	0.0020	0.0060	0.0161	0.0114	0.0230	0.0290	-0.0430	-0.0080
$\rho(5)$	-0.0027	0.0044	0.0000	0.0035	0.0172	0.0204	0.0300	0.0290	-0.0500	-0.0260
$\rho(6)$	-0.0071	0.0000	-0.0070	-0.0080	0.0195	0.0215	0.0380	0.0520	-0.0450	-0.0170
$\rho(7)$	-0.0063	0.0045	-0.0075	0.0020	0.0182	0.0231	0.0350	0.0470	-0.0440	-0.0310
$\rho(8)$	-0.0046	0.0069	-0.0030	0.0060	0.0188	0.0210	0.0410	0.0350	-0.0520	-0.0240
$\rho(9)$	0.0041	0.0089	0.0035	0.0070	0.0203	0.0133	0.0430	0.0360	-0.0540	-0.0150
$\rho(10)$	0.0055	0.0116	0.0095	0.0140	0.0192	0.0201	0.0390	0.0360	-0.0400	-0.0170
$Q(5)$	4.725	3.341	4.532	3.105	2.734	1.562	10.332	6.728	0.337	1.244
$Q(10)$	10.199	8.577	10.170	8.074	3.322	3.331	16.629	14.131	4.069	3.095
$Q(20)$	20.687	19.545	21.925	20.404	4.907	5.994	30.284	27.181	10.006	10.613

Note: Under the null hypothesis of no autocorrelation the Q -statistic is distributed χ^2 with L df. Reference values for $\text{Prob}(\chi^2 < c) = .995$: $c = [16.75 \text{ (df= 5), } 25.19 \text{ (df= 10), } 40 \text{ (df= 20)}]$.

Table 3 (continued)

Autocorrelations and Ljung-Box Q statistics for ARMA-TARCH Residuals

Panel B: ARMA-TARCH squared residual statistics

Statistic	Mean		Median		St Dev		Max		Min	
	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM
$\rho(1)$	0.0049	0.0240	0.0000	0.0235	0.0153	0.0159	0.0420	0.0550	-0.0150	0.0010
$\rho(2)$	-0.0036	0.0008	-0.0060	0.0010	0.0112	0.0055	0.0260	0.0090	-0.0260	-0.0060
$\rho(3)$	-0.0025	-0.0038	-0.0040	-0.0035	0.0124	0.0104	0.0300	0.0090	-0.0230	-0.0210
$\rho(4)$	-0.0012	0.0022	-0.0040	-0.0025	0.0128	0.0140	0.0360	0.0390	-0.0220	-0.0090
$\rho(5)$	0.0005	-0.0041	-0.0025	-0.0070	0.0167	0.0093	0.0510	0.0160	-0.0290	-0.0140
$\rho(6)$	0.0007	0.0011	0.0020	-0.0070	0.0136	0.0208	0.0400	0.0590	-0.0240	-0.0100
$\rho(7)$	-0.0020	-0.0036	-0.0035	-0.0065	0.0136	0.0075	0.0320	0.0140	-0.0260	-0.0110
$\rho(8)$	-0.0109	-0.0054	-0.0100	-0.0050	0.0110	0.0068	0.0270	0.0060	-0.0340	-0.0140
$\rho(9)$	-0.0030	0.0071	-0.0085	0.0015	0.0187	0.0213	0.0360	0.0610	-0.0370	-0.0120
$\rho(10)$	0.0000	-0.0041	-0.0030	-0.0040	0.0172	0.0049	0.0430	0.0050	-0.0300	-0.0110
$Q(5)$	2.736	3.033	2.028	2.226	2.031	2.518	9.494	8.102	0.181	0.724
$Q(10)$	6.205	5.594	6.016	4.600	3.558	4.347	16.369	13.715	0.824	1.109
$Q(20)$	14.379	9.382	15.145	9.825	6.194	4.779	26.641	15.536	2.555	2.415

Note: Under the null hypothesis of no autocorrelation the Q -statistic is distributed χ^2 with L df. Reference values for $\text{Prob}(\chi^2 < c) = .995$: $c = [16.75 \text{ (df= 5), } 25.19 \text{ (df= 10), } 40 \text{ (df= 20)}]$.

Table 4**BDS Test Statistics for ARMA-TARCH Residuals**

Panel A: $\nu = 1.5x$

Dimension n	Mean		Median		St Dev		Max		Min	
	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM
2	8.736	13.625	8.487	12.792	3.064	2.777	16.209	17.447	2.522	10.325
3	10.330	15.452	9.880	14.654	3.353	2.823	19.365	20.298	4.665	12.564
4	11.262	16.629	10.535	15.768	3.568	2.695	21.684	21.651	6.113	13.690
5	12.029	17.523	11.325	16.711	3.783	2.722	23.578	22.882	6.204	14.612
6	12.720	18.398	11.857	17.726	4.031	2.745	25.208	24.117	6.610	15.507
7	13.349	19.291	12.518	18.500	4.325	2.965	26.727	25.746	7.010	16.190
8	14.002	20.281	13.094	19.436	4.646	3.186	28.060	27.482	7.413	17.030
9	14.673	21.320	13.546	20.574	5.057	3.449	29.634	29.280	7.660	17.633
10	15.365	22.487	13.783	21.832	5.547	3.776	31.246	31.359	7.980	18.326

Note: The null hypothesis of IID is rejected at the .5% level when $|BDS_n(\nu, T)| > 2.57 \cdot \sigma$; σ : standard deviation of the return series.

Table 4 (continued)

BDS Test Statistics for ARMA-TARCH Residuals

Panel B: $\varepsilon = 2s$

Dimension n	Mean		Median		St Dev		Max		Min	
	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM	SAMPLE	ASM
2	8.364	14.765	8.244	14.220	2.848	3.102	15.135	21.298	1.659	10.846
3	9.604	15.886	9.205	15.284	2.999	3.071	17.251	22.370	3.688	12.338
4	10.278	16.389	9.836	15.630	3.068	2.885	17.586	22.455	4.896	12.736
5	10.800	16.621	10.247	16.088	3.103	2.739	18.731	22.299	5.308	13.058
6	11.207	16.848	10.615	16.418	3.152	2.590	19.748	22.222	5.724	13.663
7	11.522	17.062	11.069	16.449	3.240	2.566	20.401	22.273	6.141	14.063
8	11.846	17.253	11.436	16.547	3.319	2.543	20.928	22.285	6.500	14.365
9	12.163	17.455	11.603	16.905	3.453	2.515	21.459	22.305	6.785	14.579
10	12.443	17.707	11.590	17.235	3.609	2.509	21.945	22.325	7.073	14.880

Note: The null hypothesis of IID is rejected at the .5% level when $|BDS_n(\varepsilon, T)| > 2.57$; ε : standard deviation of the return series.