Abstract

Students and teachers are facing challenges in teaching and learning fractions. The concepts related to fractions, ratios, and proportions are learned in primary school to cast a solid foundation in proportional reasoning for the understanding of more-advanced mathematics in the secondary school curriculum. In the Numeracy Development Project (NDP), a New Zealand school mathematics professional development project, more effort is called for in the areas of fractions and proportion due to the unsatisfactory performance of students and the difficulty of advancement at the higher year levels in these areas. Games are noted as being a useful tool in teaching mathematics and adopted as one type of activity in the NDP to create an enjoyable environment for effective classroom teaching. Although teachers and parents agree that games help to improve children’s knowledge and students also find learning through playing games is fun, there is still doubt on what and how students learn about mathematics through playing games. Nevertheless, researchers have agreed on the value of games for learning mathematics.

The aim of this study is to develop an interactive computer game on fractions, and investigate the learning of fractions that follows as a result of playing the game. A pilot study was conducted to test the manipulation of fraction cards to develop a theoretical framework for the computer game. The computer game focuses on comparing sizes of fractions in an environment that enables players to visualize representations of fractions, manipulate parts of fractions and apply their knowledge of fractions. The developed computer game was used in the main study and used to collect relevant data to identify the effects of the game in learning fractions. The improvement of students after playing the computer game was identified by comparing the results between pre and post tests and pre and post maths tasks. In particular, changes in students’ strategies for comparing sizes of fractions in the mathematics tasks were examined. Finally, questionnaires were used to determine if students enjoy learning through the game.

This study investigates students’ learning of fractions in three categories of mathematical ability, namely high achievers, average students and cause-for-concern students. High achievers improved on the use of numerical strategies while average and cause-for-concern students improved on the use of representations of fractions. The improvements in ordering fractions after using the game demonstrate its benefits. This is consistent with the
notion of promoting the use of games as educational tools for learning. Based on the findings of this study, a framework of students’ strategies is developed to present a developmental picture of students’ fractional thinking. Suggestions are given to teachers to move students to more sophisticated strategies and develop their mathematical concepts based on their current mathematical thinking.
Acknowledgement

I wish to express my appreciation to:

My supervisors, Professor Dr. Derek Holton, Dr. Chris Linsell and, Dr. John Shanks for their continued guidance, patience and encouragement.

The staff and other postgraduate students in the Department of Mathematics and Statistics, for their ongoing support throughout my study. In particular, Mr. Brian Niven, the Statistics Consultant, who offered advice on the statistical analyses. Naomi, George and Palani, who also studied in the field of mathematics education, and who generously shared their ideas and research activities.

Mr. Ben Anderson, who assisted in the programming of the Tower Trap computer game.

The Year 8 students, the school teachers and the school principals, for their willingness to participate in the research and for welcoming me into their classrooms.

Ms. Pauline Brook, from the Student Learning Centre, who helped me to improve my writing style and use of English.

My good friends, Shelley, Ralf, Claudine, Didem, Mami, Hiang Loon, Vivian, Jung-hi and Gina whose friendship was especially important when I was away from family.

My family, for their continuing encouragement and inspiration, which provided me with confidence and enthusiasm to pursue this research.
TABLE OF CONTENTS

ABSTRACT .............................................................................................................................................................................. ii
ACKNOWLEDGEMENT .............................................................................................................................................................. iv
TABLE OF CONTENTS ............................................................................................................................................................... v
LIST OF TABLES ........................................................................................................................................................................ xiii
LIST OF FIGURE .......................................................................................................................................................................... xvi

CHAPTER 1  INTRODUCTION
1.1 Learning fractions .............................................................................................................................................................. 1
   1.1.1 Numeracy Development Project (NDP) .......................................................................................................................... 1
   1.1.2 Framework of students’ strategies ................................................................................................................................. 4
1.2 Mathematics games ................................................................................................................................................................. 5
   1.2.1 Activities in teaching mathematics ............................................................................................................................... 5
   1.2.2 Challenges to the use of educational computer games ................................................................................................. 6
1.3 Research objectives ................................................................................................................................................................. 8
1.4 Research questions ................................................................................................................................................................. 8
1.5 Research scope ......................................................................................................................................................................... 9
1.6 Research significance ............................................................................................................................................................... 9
1.7 Outline of chapters ................................................................................................................................................................. 10

CHAPTER 2  FRACTIONS
2.1 Part-whole construct ............................................................................................................................................................. 13
   2.1.1 Fraction constructs .......................................................................................................................................................... 13
      2.1.1.1 Part-whole ................................................................................................................................................................. 14
      2.1.1.2 Ratio ........................................................................................................................................................................ 15
      2.1.1.3 Quotient ................................................................................................................................................................. 16
      2.1.1.4 Operator ................................................................................................................................................................. 17
      2.1.1.5 Measure ................................................................................................................................................................. 18
      2.1.1.6 Interrelationship among various constructs ............................................................................................................. 18
   2.1.2 Part-whole thinking in the New Zealand Number Framework (NZNF) ........................................................................... 20
   2.1.3 Represent part-whole relation using divided quantity diagrams ....................................................................................... 22
   2.1.4 The use of partitioning in learning fractions .................................................................................................................. 23
   2.1.5 Understanding of fraction sizes ......................................................................................................................................... 25
2.2 Comparing sizes of fractions ................................................................. 29
  2.2.1 Students’ reasoning and thinking ...................................................... 29
  2.2.2 Drawing divided quantity diagrams: Equal-parts and equal-wholes .... 31
  2.2.3 Using benchmarks ............................................................................ 34
  2.2.4 Finding common denominators .......................................................... 36

2.3 Towards a better understanding of fractions ........................................... 40
  2.3.1 Pirie and Kieren’s model of the growth of mathematical understanding 40
      2.3.1.1 First level - Primitive Knowing .................................................. 41
      2.3.1.2 Second level - Image-Making ...................................................... 42
      2.3.1.3 Third level - Image-Having ......................................................... 43
      2.3.1.4 Fourth level - Property Noticing .................................................. 43
      2.3.1.5 Fifth level - Formalizing .............................................................. 44
      2.3.1.6 Sixth level - Observing ................................................................. 44
      2.3.1.7 Seventh level - Structuring ............................................................ 45
      2.3.1.8 Eighth level - Inventising ............................................................. 45
  2.3.2 Students’ strategies and sophistication levels ....................................... 46
  2.3.3 Continuities between novices and masters ......................................... 53
      2.3.3.1 Strategies of novices and masters ................................................ 53
      2.3.3.2 Shared characteristics of the knowledge system of novices and masters ............................................................................... 54
      2.3.3.3 Reconceptualisation of students’ misconceptions of fractions .... 55

CHAPTER 3  COMPUTER GAMES

3.1 Definition of computer games ................................................................. 59

3.2 Computer games for learning ................................................................. 61
  3.2.1 New technology, new learning ............................................................ 62
  3.2.2 Key design of educational computer games ........................................ 63
      3.2.2.1 Rules ............................................................................................ 65
      3.2.2.2 Goals and objectives ................................................................. 65
      3.2.2.3 Outcomes and feedback .............................................................. 66
      3.2.2.4 Conflict/Competition/Challenge/Opposition ............................... 67
      3.2.2.5 Interaction ................................................................................. 68
      3.2.2.6 Representation or story ............................................................. 69

3.3 Computer games for learning mathematics .......................................... 70
  3.3.1 Mini games for instruction ................................................................. 70
  3.3.2 Constructing mathematical concepts in computer games ................. 72
CHAPTER 5  DESIGNING AND DEVELOPING *TOWER TRAP*

5.1 Structure of *Tower Trap* ................................................................. 122
  5.1.1 Learning objectives of ordering fractions from the smallest to the largest and vice versa ............................................................... 124
  5.1.2 Problem context of forming fraction brick staircases .................... 124
  5.1.3 Visualize and manipulate representations of fractions .................. 124
  5.1.4 Fraction bricks ................................................................................ 125
    5.1.4.1 Tall and long bricks ................................................................. 126
    5.1.4.2 Visible bricks ........................................................................... 128
    5.1.4.3 Broken bricks .......................................................................... 129
    5.1.4.4 Hidden bricks .......................................................................... 133
    5.1.4.5 Sequence of brick types in *Tower Trap* .................................. 134
    5.1.4.6 Difficulty levels of fractions .................................................... 135
  5.1.5 Strategies for comparing and ordering fractions ............................ 139
  5.1.6 Visible, broken to hidden bricks: Concrete to abstract ................. 139
  5.1.7 Incorporation of students’ needs into the development of *Tower Trap* ..................................................................................... 140

5.2 Computer game features .................................................................... 144
  5.2.1 Game world ................................................................................... 144
  5.2.2 Game play ...................................................................................... 145
  5.2.3 Game flows .................................................................................... 149

5.3 Pedagogy of the game ........................................................................ 153
  5.3.1 Key ideas pages ............................................................................... 154
  5.3.2 Specific instructions ....................................................................... 155
  5.3.3 Cognitively oriented feedback ......................................................... 158
  5.3.4 Tips pages ....................................................................................... 160
  5.3.5 Alternate questions ......................................................................... 161
  5.3.6 Game user data ............................................................................... 162

CHAPTER 6  STUDENTS’ EVALUATION AND GAME PLAY IN *TOWER TRAP*

6.1 Students’ evaluation of the game ....................................................... 165
  6.1.1 Playing the game ........................................................................... 169
  6.1.2 Game features ............................................................................... 169
  6.1.3 Learning fractions ......................................................................... 171
  6.1.4 Teaching aids ................................................................................. 174

6.2 Number of attempts taken at game levels .......................................... 175
  6.2.1 Introduction levels ......................................................................... 176
CHAPTER 7 STUDENTS’ STRATEGIES TOWARDS A FRAMEWORK

7.1 Students’ strategies for comparing and ordering fractions...........................181
   7.1.1 Main strategies of students with different mathematical abilities..........182
       7.1.1.1 High achievers .................................................................183
       7.1.1.2 Average students ..............................................................185
       7.1.1.3 Cause-for-concern students ............................................186
   7.1.2 Using a number of different types of strategies.................................188
   7.1.3 Relevance of the sets of fractions to be compared to the strategies adopted..189
       7.1.3.1 High achievers .................................................................190
       7.1.3.2 Average students ..............................................................191
       7.1.3.3 Cause-for-concern students ............................................192

7.2 Students’ strategies and mathematical understanding.................................193
   7.2.1 Big numbers are equal to big fractions...........................................193
   7.2.2 Big numbers are equal to small fractions...........................................194
   7.2.3 \( \frac{1}{2} \) is the biggest fraction.........................................................194
   7.2.4 Drawing divided quantity diagrams .................................................195
   7.2.5 Big denominators are equal to small fractions.................................199
   7.2.6 Numerators and denominators.......................................................201
   7.2.7 Converting fractions into equivalent form........................................201
   7.2.8 Using benchmarks ...........................................................................207
   7.2.9 Partial reasoning, no reasoning and incorrect reasoning....................211

7.3 Discussion ......................................................................................................213
   7.3.1 Matching strategies with sets of fractions ........................................213
   7.3.2 A framework of fractional strategies and thinking............................215
   7.3.3 Mathematical understanding underlying students’ strategies............217
   7.3.4 Comparison of strategies’ framework with literature.......................220

CHAPTER 8 IMPROVEMENT OF STUDENTS’ KNOWLEDGE OF FRACTIONS

8.1 Individual performance .................................................................................225
8.1.1 Sam ................................................................................................................. 226
  8.1.1.1 Pre and post maths tasks .................................................................. 226
  8.1.1.2 Pre and post maths tests ................................................................. 228
  8.1.1.3 Computer game play ....................................................................... 229
  8.1.1.4 Questionnaires ................................................................................. 231
8.1.2 Mary ............................................................................................................... 232
  8.1.2.1 Pre and post maths tasks ................................................................ 233
  8.1.2.2 Pre and post maths tests ................................................................. 235
  8.1.2.3 Computer game play ....................................................................... 235
  8.1.2.4 Questionnaires ................................................................................. 237
8.1.3 Peter ................................................................................................................ 238
  8.1.3.1 Pre and post maths tasks ................................................................ 239
  8.1.3.2 Pre and post maths tests ................................................................. 240
  8.1.3.3 Computer game play ....................................................................... 241
  8.1.3.4 Questionnaires ................................................................................. 243
8.1.4 Comparison of the responses of the three students ........................................ 245
8.2 Changes of students’ strategies between the pre and post maths tasks ............... 247
  8.2.1 Use of representations of fractions .......................................................... 248
    8.2.1.1 Big denominators are equal to small fractions ............................... 249
    8.2.1.2 Circular divided quantity diagrams ............................................. 250
    8.2.1.3 Alternative for students who are unable to provide good reasoning 252
    8.2.1.4 Big numbers are equal to small fractions ....................................... 254
    8.2.1.5 Reflect fractional thinking ............................................................ 254
    8.2.1.6 Interpreting fractions differently ................................................... 256
  8.2.2 Use of numerical strategies in the post task ............................................... 257
    8.2.2.1 Changes from using representations of fractions to converting
          fractions into a common denominator, percentages or decimals ......... 258
    8.2.2.2 Changes between different numerical strategies ............................ 259
    8.2.2.3 Improvement in using benchmarks ................................................. 262
8.3 Achievement differences between the pre and post tests .................................. 264
  8.3.1 School A and School C ............................................................................ 264
  8.3.2 Comparison between experimental group and control group .................... 267
  8.3.3 Students’ selections in the tests ............................................................... 270

CHAPTER 9 MOVING STUDENTS TOWARDS ADVANCED STRATEGIES
9.1 Identifying students’ strategies ........................................................................ 278
<table>
<thead>
<tr>
<th>9.2</th>
<th>Continuities of mathematical thinking</th>
<th>283</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2.1</td>
<td>Numerical strategies based on divided quantity thinking</td>
<td>284</td>
</tr>
<tr>
<td>9.2.2</td>
<td>Requirement of fundamental number knowledge in algorithmic procedures</td>
<td>285</td>
</tr>
<tr>
<td>9.3</td>
<td>Instructional approaches to move students to higher strategy</td>
<td>286</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Peter: Big numbers are equal to small fractions</td>
<td>287</td>
</tr>
<tr>
<td>9.3.2</td>
<td>Mary: Big denominators are equal to small fractions</td>
<td>288</td>
</tr>
<tr>
<td>9.3.3</td>
<td>Sam: Numerical strategies</td>
<td>292</td>
</tr>
</tbody>
</table>

**CHAPTER 10  CONCLUSIONS**

<table>
<thead>
<tr>
<th>10.1</th>
<th>Design of a good computer game for learning fractions</th>
<th>301</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1.1</td>
<td>Theoretical framework of the game - The structure of Tower Trap</td>
<td>301</td>
</tr>
<tr>
<td>10.1.2</td>
<td>Intrinsic integration of content - Forming fraction brick staircases</td>
<td>301</td>
</tr>
<tr>
<td>10.1.3</td>
<td>Pedagogical approach of the game - Fraction brick types</td>
<td>302</td>
</tr>
<tr>
<td>10.1.4</td>
<td>Students’ positive evaluation of the game for learning fractions</td>
<td>302</td>
</tr>
<tr>
<td>10.2</td>
<td>Students’ strategies for comparing sizes of fractions</td>
<td>303</td>
</tr>
<tr>
<td>10.2.1</td>
<td>Students’ mathematical abilities and use of strategies</td>
<td>303</td>
</tr>
<tr>
<td>10.2.2</td>
<td>Continuities of mathematical thinking</td>
<td>304</td>
</tr>
<tr>
<td>10.2.3</td>
<td>Students’ strategies toward a framework</td>
<td>305</td>
</tr>
<tr>
<td>10.2.4</td>
<td>Part-whole concept</td>
<td>307</td>
</tr>
<tr>
<td>10.3</td>
<td>Changes of strategies</td>
<td>308</td>
</tr>
<tr>
<td>10.3.1</td>
<td>High achievers: Improvement on the use of numerical strategies</td>
<td>309</td>
</tr>
<tr>
<td>10.3.2</td>
<td>Average and cause-for-concern students: Improvement on the use of representations of fractions</td>
<td>309</td>
</tr>
<tr>
<td>10.4</td>
<td>Limitation of this study</td>
<td>310</td>
</tr>
<tr>
<td>10.4.1</td>
<td>Allocated time for playing the game</td>
<td>310</td>
</tr>
<tr>
<td>10.4.2</td>
<td>Design of the study</td>
<td>311</td>
</tr>
<tr>
<td>10.5</td>
<td>Recommendations for future research</td>
<td>311</td>
</tr>
<tr>
<td>10.5.1</td>
<td>A range of difficulty categories</td>
<td>311</td>
</tr>
<tr>
<td>10.5.2</td>
<td>Improper fractions or fractions bigger than unit 1</td>
<td>312</td>
</tr>
<tr>
<td>10.5.3</td>
<td>Decimal fractions</td>
<td>314</td>
</tr>
<tr>
<td>10.5.4</td>
<td>Tall and long fraction bricks</td>
<td>314</td>
</tr>
<tr>
<td>10.5.5</td>
<td>Integrate numerical strategies with the computer game context</td>
<td>315</td>
</tr>
<tr>
<td>10.5.6</td>
<td>Instructional assessment and materials</td>
<td>315</td>
</tr>
<tr>
<td>10.5.7</td>
<td>Enhance the understanding of students who used partial strategies</td>
<td>315</td>
</tr>
</tbody>
</table>

**REFERENCES** | 317 |
## LIST OF TABLES

Table 1.1: The Number Framework (Ministry of Education, 2008a, 2010a) ......................... 3
Table 2.1: Descriptions of the strategies for comparing fractions (Ministry of Education, 2010b, para. 4) ................................................................. 48
Table 2.2: Strategies and students’ understanding of fractions ............................................... 52
Table 3.1: Previous definitions of computer games ................................................................. 59
Table 3.2: Key defining characteristics of computer games .................................................... 60
Table 3.3: Complex games ................................................................................................... 71
Table 3.4: Extrinsic, intrinsic and constructivist integrations games (Kafai et al., 1998, p. 160-163) ............................................................................................ 78
Table 3.5: Example of computer mathematics games ............................................................ 80
Table 4.1: Mixed methods research (Greene et al., 1989) ..................................................... 88
Table 4.2: Number of students from Schools A to C ............................................................. 92
Table 4.3: Best fit approach (Ministry of Education, 2010d) ................................................ 93
Table 4.4: Knowledge of the Number Framework (Ministry of Education, 2008a) ............... 94
Table 4.5: Strategies of the Number Framework (Ministry of Education, 2008a) ............... 95
Table 4.6: Categories of students (Ministry of Education, 2010d) ...................................... 97
Table 4.7: Schools and number of students involved in the study ........................................ 98
Table 4.8: Research sampling according to the characteristics of quasi-experiment .......... 99
Table 4.9: Set of fractions used in the pilot study ................................................................. 102
Table 4.10: Knowledge of fractions assessed in the tests ..................................................... 105
Table 4.11: Sets of fractions used in the pre and post maths tasks ...................................... 108
Table 4.12: School A and C participants’ populations .......................................................... 110
Table 4.13: Categories of students’ strategies .................................................................... 112
Table 4.14: Partial reasoning, incorrect reasoning and no reasoning ................................. 115
Table 5.1: Set of fractions used in the game levels ............................................................... 137
Table 5.2: Strategies for comparing fractions ..................................................................... 138
Table 5.3: Strategies for comparing fractions ..................................................................... 149
Table 5.4: Feedback on fractions ....................................................................................... 159
Table 5.5: Set of fractions used in original and alternate questions ................................... 162
Table 5.6: Scores for various attempts ............................................................................... 163
Table 6.1: Students evaluation of game, by school ......................................................... 166
Table 6.2: Evaluation of high achievers, average and cause-for-concern students from School A ................................................................. 167
Table 6.3: Evaluation of high achievers, average and cause-for-concern students from School C ...................................................................................................................... 168
Table 7.1: Data of students’ strategies ................................................................................................................................. 182
Table 7.2: Examples of students’ reasoning ............................................................................................................................ 200
Table 7.3: Levels of mathematical understanding underlying students’ strategies ................................................................. 218
Table 7.4: Different opinions on students’ strategies ............................................................................................................. 221
Table 8.1: Sam’s results in the pre and post maths tasks ........................................................................................................... 227
Table 8.2: Sam’s results in the pre and post tests for questions 3, 5 and 6 .................................................................................. 228
Table 8.3: Sam’s attempts in ordering tall visible and broken bricks ............................................................................................ 229
Table 8.4: Sam’s attempts in ordering long visible and broken bricks .......................................................................................... 230
Table 8.5: Sam’s attempts in ordering hidden tall and long bricks .............................................................................................. 230
Table 8.6: Sam’s rating in questionnaires ................................................................................................................................. 231
Table 8.7: Mary’s results in the pre and post maths tasks .......................................................................................................... 233
Table 8.8: Mary’s results in the pre and post tests for question 4 ................................................................................................. 235
Table 8.9: Mary’s attempts in ordering tall visible and broken bricks .......................................................................................... 235
Table 8.10: Mary’s attempts in ordering long visible and broken bricks ....................................................................................... 236
Table 8.11: Mary’s attempts in ordering hidden tall and long bricks ............................................................................................ 236
Table 8.12: Mary’s rating in questionnaires .............................................................................................................................. 237
Table 8.13: Peter’s results in the pre and post maths tasks ........................................................................................................... 239
Table 8.14: Peter’s results in the pre and post tests for questions 3, 5 and 6 ................................................................................ 240
Table 8.15: Peter’s attempts in ordering tall visible and broken bricks .......................................................................................... 241
Table 8.16: Peter’s attempts in ordering long visible and broken bricks ....................................................................................... 242
Table 8.17: Peter’s attempts in ordering hidden tall and long bricks ............................................................................................ 242
Table 8.18: Peter’s rating in questionnaires .............................................................................................................................. 243
Table 8.19: Diagrams drawn in the pre and post tasks of question 5 .......................................................................................... 251
Table 8.20: Improved arguments from the pre to the post tasks ................................................................................................... 253
Table 8.21: Referring to representations of fractions in the post task ........................................................................................... 254
Table 8.22: Diagrams drawn in the post task ............................................................................................................................ 255
Table 8.23: “Piece” as shaded parts ................................................................................................................................................ 255
Table 8.24: Refer to representations of fractions in the post task ............................................................................................... 256
Table 8.25: Students’ reasoning given in the post task ................................................................................................................ 259
Table 8.26: Students’ change from converting fractions into percentages in the pre task to finding a common denominator in the post task ........................................................................................................ 260
Table 8.27: Students’ use of a benchmark in the pre task and finding a common denominator in the post task ................................................................. 261
Table 8.28:  Reasoning given in the pre and post tasks of question 5 ................................... 262
Table 9.1:  Sam’s reasoning and strategies ................................................................. 279
Table 9.2:  Mary’s reasoning and strategies ............................................................... 280
Table 9.3:  Peter’s reasoning and strategies ................................................................. 281
Table 9.4:  Relate diagrams to finding a common denominator ................................... 285
Table 9.5:  Rectangular divided quantity diagrams that represent fractions .......... 288
Table 9.6:  Relate diagrams to finding a common denominator ................................... 290
Table 9.7:  Relate diagrams to converting fractions into percentages .................... 290
Table 9.8:  Relate diagrams to using the benchmark of a half .................................. 291
Table 9.9:  Problems and reasons of using particular strategies for ordering fractions .. 293
Table 9.10: Instructional approaches to move students to higher strategies .......... 295
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>A continuous quantity (Ministry of Education, 2008d, p. 5)</td>
<td>14</td>
</tr>
<tr>
<td>2.2</td>
<td>A set of discrete objects (Ministry of Education, 2008d, p. 5)</td>
<td>14</td>
</tr>
<tr>
<td>2.3</td>
<td>A ratio 3:5 for the shade of grey to white mixture (Ministry of Education,</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>2008d, p. 6)</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>A rate 15:10 for the price of pineapples (Ministry of Education, 2008d, p.</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>6)</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>A diagram of the problem (Ministry of Education, 2008c, p. 79)</td>
<td>16</td>
</tr>
<tr>
<td>2.6</td>
<td>Strip diagrams (Ministry of Education, 2008d, p. 69)</td>
<td>18</td>
</tr>
<tr>
<td>2.7</td>
<td>Conceptual scheme for instruction on rational numbers (Behr et al., 1983)</td>
<td>19</td>
</tr>
<tr>
<td>2.8</td>
<td>Circular and rectangular quantity diagrams</td>
<td>22</td>
</tr>
<tr>
<td>2.9</td>
<td>Fraction circles (Ministry of Education, 2008d)</td>
<td>23</td>
</tr>
<tr>
<td>2.10</td>
<td>Representations of $\frac{1}{2}$ and $\frac{3}{5}$</td>
<td>31</td>
</tr>
<tr>
<td>2.11</td>
<td>Varying wholes</td>
<td>32</td>
</tr>
<tr>
<td>2.12</td>
<td>Circular, rectangular and triangular divided quantity diagrams (Newstead</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>&amp; Murray, 1998)</td>
<td></td>
</tr>
<tr>
<td>2.13</td>
<td>Incorrect representation of $\frac{3}{4}$ (Steinle &amp; Price, 2008)</td>
<td>33</td>
</tr>
<tr>
<td>2.14</td>
<td>Shapes that have been divided into quarters (Baturo, 2004)</td>
<td>34</td>
</tr>
<tr>
<td>2.15</td>
<td>Rectangular divided quantity diagrams representing equivalent fractions</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{3} = \frac{4}{12}$</td>
<td></td>
</tr>
<tr>
<td>2.16</td>
<td>Rectangular divided quantity diagrams representing equivalent fractions</td>
<td>39</td>
</tr>
<tr>
<td>2.17</td>
<td>The eight understanding layers of Pirie and Kieren’s model (1994b)</td>
<td>41</td>
</tr>
<tr>
<td>2.18</td>
<td>Rectangular divided quantity diagram</td>
<td>42</td>
</tr>
<tr>
<td>2.19</td>
<td>Strategies for comparing fractions (Darr &amp; Fisher, 2006; Maguire et al.,</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>2007)</td>
<td></td>
</tr>
<tr>
<td>4.1</td>
<td>Research design</td>
<td>86</td>
</tr>
<tr>
<td>4.2</td>
<td>Mixed-method design (Johnson &amp; Onwuegbuzie, 2004)</td>
<td>89</td>
</tr>
<tr>
<td>4.3</td>
<td>End of Year 8 Curriculum Expectations (Tagg &amp; Thomas, 2007)</td>
<td>96</td>
</tr>
<tr>
<td>4.4</td>
<td>Student categories defined in this study</td>
<td>98</td>
</tr>
<tr>
<td>4.5</td>
<td>Research processes</td>
<td>100</td>
</tr>
<tr>
<td>4.6</td>
<td>A two-dimensional staircase</td>
<td>101</td>
</tr>
<tr>
<td>4.7</td>
<td>Front and back sides of a fraction card</td>
<td>101</td>
</tr>
</tbody>
</table>
Figure 5.35: Game menu shows all levels are completed .................................................... 152
Figure 5.36: Top of the tower ............................................................................................... 153
Figure 5.37: Game checkpoint ............................................................................................. 153
Figure 5.38: Key idea page ................................................................................................... 154
Figure 5.39: Instructions to move the bricks ........................................................................ 155
Figure 5.40: Instructions to climb up the staircase ............................................................... 155
Figure 5.41: Instructions to try again ................................................................................... 156
Figure 5.42: Order long bricks from the largest to the smallest ........................................... 157
Figure 5.43: Select the blue parts of the tall broken bricks .................................................. 157
Figure 5.44: Order the hidden tall bricks ............................................................................ 158
Figure 5.45: Game feedback ................................................................................................ 159
Figure 5.46: A game tips page ............................................................................................ 161
Figure 5.47: Players could stop and continue the game ....................................................... 163
Figure 6.1: Mean attempts across all types of fraction bricks ............................................ 175
Figure 6.2: Tall broken bricks $\frac{1}{5}$ and $\frac{1}{3}$ ........................................................................ 177
Figure 6.3: The possible ordering of tall broken bricks ........................................................ 177
Figure 7.1: Using two strategies to compare three fractions .............................................. 183
Figure 7.2: Strategies used by the 15 high achievers .......................................................... 184
Figure 7.3: Strategies used by the 50 average students ......................................................... 185
Figure 7.4: Strategies used by the 42 cause-for-concern students ...................................... 186
Figure 7.5: Number of types of strategies used by high achievers ..................................... 188
Figure 7.6: Number of types of strategies used by average students .................................. 188
Figure 7.7: Number of types of strategies used by cause-for-concern students ............... 188
Figure 7.8: Number of correct ordering of fractions and types of strategies .................... 189
Figure 7.9: Strategies used by the 15 high achievers for the 10 questions in the maths tasks ................................................................................................................. 190
Figure 7.10: Strategies used by the 50 average students for the 10 questions in the maths tasks ................................................................................................................. 191
Figure 7.11: Strategies used by the 42 cause-for-concern students for the 10 questions in the maths tasks ................................................................................................................. 192
Figure 7.12: Divided quantity diagrams drawn by students ............................................... 196
Figure 7.13: Divided quantity diagrams drawn in question 4 ............................................. 197
Figure 7.14: Divided quantity diagrams with similar size .................................................... 197
Figure 7.15: Divided quantity diagrams with unequal whole ............................................. 198
Figure 7.16: Divided quantity diagrams drawn in question 5 .............................................. 198
Figure 7.17: Converting fractions into percentages in question 3 ........................................ 203
Figure 7.18: Percentages estimation for fractions in the pre and post tasks of question 4... 203
Figure 7.19: Students’ incorrect ordering of fractions .......................................................... 205
Figure 7.20: Students’ incorrect ordering of fractions .......................................................... 205
Figure 7.21: Using two strategies to compare three fractions .............................................. 206
Figure 7.22: Students’ reasoning in the maths tasks ............................................................ 211
Figure 7.23: Strategies and types of fractions ...................................................................... 214
Figure 7.24: Strategies’ framework for ordering fractions ................................................... 216
Figure 8.1: The number of students who were WR in the maths tasks .............................. 247
Figure 8.2: The number of students RR and change their strategies in the maths tasks .... 248
Figure 8.3: Circular divided quantity diagrams .................................................................. 250
Figure 8.4: Circular divided quantity diagrams that are equally divided by the students .. 252
Figure 8.5: Circular divided quantity diagrams that are unequally divided by the students ............................................................................................................ 252
Figure 8.6: Circular divided quantity diagrams .................................................................. 257
Figure 8.7: Achievement differences between the pre and post tests ............................... 265
Figure 8.8: Achievement differences between the pre and post tests for high achievers... 266
Figure 8.9: Achievement differences between the pre and post tests for average students ............................................................................................................ 266
Figure 8.10: Achievement differences between the pre and post tests for cause-for-concern students ............................................................................................................ 267
Figure 8.11: Comparison between experimental group and control group ...................... 267
Figure 8.12: Finding the range of fractions ......................................................................... 270
Figure 8.13: Representing $\frac{2}{5}$ in rectangular divided quantity diagrams .................... 271
Figure 8.14: Representing $\frac{3}{5}$ in rectangular divided quantity diagrams ...................... 271
Figure 8.15: Divided quantity diagrams in Question 2 ......................................................... 271
Figure 8.16: $\frac{2}{5}$ and $\frac{2}{3}$ shaded blue on the broken bricks .............................................. 272
Figure 8.17: Representation of $\frac{2}{5}$ dragged from the broken brick ............................... 272
Figure 8.18: Comparing and ordering fractions with unlike denominators in Question 5 273
Figure 8.19: Broken bricks ................................................................................................. 274
Figure 9.1: The cause-for-concerns student’s reasoning .................................................... 282
Figure 9.2: Strategies’ framework for ordering fractions ................................................... 283
Figure 9.3: Students’ reasoning .......................................................................................... 284
Figure 9.4: Students’ reasoning .......................................................................................... 286
Figure 9.5: Fractions $\frac{1}{2} < \frac{2}{3}$ ............................................................................................ 287
Figure 9.6: Fractions $\frac{1}{5} < \frac{2}{6}$ ............................................................................................ 287
Figure 10.1: Strategies’ framework for ordering fractions ................................................... 305
Figure 10.2: Diagrams drawn by cause-for-concern students in question 3 ..................... 307
Figure 10.3: Difficulty options of easy, normal and hard .................................................... 312
Figure 10.4: Forming the improper fraction $\frac{3}{2}$ ................................................................. 313
Figure 10.5: Forming the improper fraction $\frac{5}{3}$ ................................................................. 313
Figure 10.6: Representations of decimal fractions using broken bricks .......................... 314
1.1 Learning fractions

Fractional or rational numbers that relate to the quantities of “how many” and “how much” are important in everyday-life situations (Hunting & Sharpley, 1988; Anthony & Walshaw, 2007). For example, “fair shares” determined by young children, shopping needs such as price per litre or price per kilogram, discounts presented as percentage reductions in price, and speed limits that are presented as the relationship between distance and time. Students begin to learn concepts related to fractions, ratios, and proportions in primary school. If students experience difficulties with fractions, they will have problems with other domains in mathematics such as algebra, measurement, and ratio and proportion concepts (Lamon, 2007). For example, students need to be advanced multiplicative thinkers and have a solid foundation in proportional reasoning to engage with algebraic learning effectively (Wu, 1999; Litwiller, 2002; Brown & Quinn, 2006; Lamon, 2007). The ability for abstract mathematical ideas in the context of fractions benefits students in algebraic learning (Wu, 1999). Fractions, ratios, and proportions are described as the most mathematically rich and cognitively complicated areas in primary school mathematics. However, they are difficult for teachers to teach as well as for students to learn (Yoshida & Shinmachi, 1999; Smith III, 2002; Naiser, Wright & Capraro, 2004; Ni & Zhou, 2005; Holton, 2007; Ward and Thomas, 2007; Young-Loveridge, Taylor, Hawera & Sharma, 2007). Hence, research on the learning of fractions contributes to the body of knowledge regarding the challenges faced by students and teachers alike when teaching and learning fractions.

1.1.1 Numeracy Development Project (NDP)

The 1995 Third International Mathematics and Science Study (TIMSS) identified the poor performance of New Zealand students in number (place value, fractions, and computation), measurement, and algebraic concepts. As such, a school mathematics professional development project, the Numeracy Development Project (NDP), was
implemented following a pilot in 2000, as a step of reform action in education taken by the Ministry of Education in New Zealand. The project is aimed at improving students’ performance in mathematics through enhancing the professional capability of teachers. The fundamental goal of the project is to develop teachers’ ability to better understand students’ number strategies and knowledge, as well as their stages of development from using less sophisticated to more sophisticated strategies. Ultimately, it is hoped improvement in teacher knowledge leads to an improvement in the number achievement of students (Ministry of Education, 2008a, 2010a).

Research has shown an improvement in the standard of New Zealand children’s mathematical performance over the years since the beginning of the NDP (Holton, 2006). This can be seen from students’ achievements in a variety of standard tests outside of the NDP such as Progressive Achievement Tests (PAT), Assessment Tools for Teaching and Learning (asTTle), the Trends in Mathematics and Science Study (TIMSS), and National Education Monitoring Project (NEMP) (Thomas & Tagg, 2006). However, the performance of students on proportion and ratio is lower than expected (Young-Loveridge, 2005, 2006; Tagg & Thomas, 2007) and advancement in these areas is more difficult at the higher year levels. Therefore, more effort is called for in the areas of fractions and proportions in the NDP (Holton, 2007), and research needs to focus on: i) promoting students’ multiplicative and proportional thinking, and ii) improving learning and achievement for students identified as “at risk” and “cause for concern” (Ministry of Education, 2008b). In addition, the misunderstandings that students are likely to encounter and the misconceptions that they may bring to class are being focused on in professional development to support effective teaching and learning programmes in fractions and proportional thinking (Ministry of Education, 2008b).

This study collected data from NDP classrooms in New Zealand schools. The basis of the NDP is the Number Framework, which helps teachers, parents, and students to understand the requirements of the Number Knowledge and Number Strategy sections of The New Zealand Curriculum (Ministry of Education, 2010a, para. 4). The Number Strategy section describes the mental processes students use to estimate answers and solve operational problems with numbers. The Number Knowledge section describes the key items of knowledge that students need to learn (Ministry of Education, 2010a, para. 5).
The strategy section of the framework consists of a sequence of global stages (i.e., nine stages from Stage 0 to Stage 8). Progress through the stages indicates an expansion in knowledge and in the range of strategies that students have acquired. Data on the students’ current stage on the framework were obtained from the intermediate school teachers who had assessed their students using the Numeracy Project Assessment Interview (NumPA) (Ministry of Education, 2008c) at the beginning of the school year. The framework provides teachers with valuable information about their students’ knowledge and mental strategies. Students’ progress is continually monitored by their teachers to ensure that they are in the correct ability group (Ministry of Education, 2010b). Table 1.1 shows the key features of each strategy stage of the Number Framework (Ministry of Education, 2010a). The lower five stages of the framework (Stage 0 to Stage 4) focus on counting and the four upper stages involve the use of increasingly complex part-whole strategies.

### Table 1.1: The Number Framework (Ministry of Education, 2008a, 2010a)

<table>
<thead>
<tr>
<th>Stage</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 0: Emergent</td>
<td>The student is unable to consistently count a given number of objects because they lack knowledge of counting sequences and/or one-to-one correspondence.</td>
</tr>
<tr>
<td>Stage 1: One-to-one Counting</td>
<td>The student is able to count a set of objects or form sets of objects but cannot solve problems that involve joining and separating sets.</td>
</tr>
<tr>
<td>Stage 2: Counting from One on Materials</td>
<td>The student is able to count a set of objects or form sets of objects to solve simple addition and subtraction problems. The student solves problems by counting all the objects.</td>
</tr>
<tr>
<td>Stage 3: Counting from One by Imaging</td>
<td>The student is able to visualize sets of objects to solve simple addition and subtraction problems. The student solves problems by counting all the objects.</td>
</tr>
<tr>
<td>Stage 4: Advanced Counting</td>
<td>The student uses counting on or counting back to solve simple addition or subtraction tasks.</td>
</tr>
<tr>
<td>Stage 5: Early Additive Part-Whole</td>
<td>The student uses a limited range of mental strategies to estimate answers and solve addition or subtraction problems. These strategies involve deriving the answer from known basic facts, (e.g. doubles, fives, making tens).</td>
</tr>
<tr>
<td>Stage 6: Advanced Additive/Early Multiplicative Part-Whole</td>
<td>The student can estimate answers and solve addition and subtraction tasks involving whole numbers mentally by choosing appropriately from a broad range of advanced mental strategies (e.g. place value positioning, rounding and compensating or reversibility). The student uses a combination of known facts and a limited range of mental strategies to derive answers to multiplication and division problems, (e.g. doubling, rounding or reversibility).</td>
</tr>
<tr>
<td>Stage 7: Advanced Multiplicative Part-Whole</td>
<td>The student is able to choose appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors, (e.g. place value partitioning, rounding and compensating, reversibility).</td>
</tr>
<tr>
<td>Stage 8: Advanced Proportional Part-Whole</td>
<td>The student can estimate answers and solve problems involving the multiplication and division of fractions and decimals using mental strategies. These strategies involve recognising the effect of number size on the answer and converting decimals to fractions where appropriate. These students have strongly developed number sense and algebraic thinking.</td>
</tr>
</tbody>
</table>
1.1.2 Framework of students’ strategies

In NDP classrooms, students’ strategies are of greater concern to the teacher than the difficulty of mathematical problems. It is the most sophisticated arithmetic strategy used by students but not the most difficult arithmetic problem answered by students that is to be considered to categorise students into different mathematical groups for instruction (Ministry of Education, 2008c). Students’ strategies are investigated to gain an insight into students’ thinking (Kafai, Franke & Battey, 2002). Students’ strategies which reflect their thinking and problem-solving processes are advocated as the focus of the pedagogical content knowledge of teachers (Young et al., 2007) and instructional strategies are modified to build on students’ possible conceptual schema (Olive & Vomvoridi, 2006).

In addition to the fractional thinking and concepts underlying the correct strategies of fractions, the misconceptions that cause the prevalence of various invalid strategies should be taken into consideration in the instruction (Gould, 2005; Steinle & Price, 2008). It is noted that students’ representations are relevant to their conceptual knowledge of fractions (Bulgar, 2009) because students express their ideas when constructing diagrams for mathematical problem solving (Ministry of Education, 2008d). Teachers need to focus not only on the incorrectly drawn diagrams, but also on the hidden misconceptions of fractions. It is important for teachers to understand students’ conceptions and misconceptions to tailor lessons according to students’ needs (Ward, 1999; Tirosh, 2000). Students’ incomplete thinking of fractions is often neglected in classroom teaching, hence it becomes necessary for teachers to remind students of their mistakes in the process of constructing the knowledge of fractions (Steinle & Price, 2008).

The Number Framework which set out a sequence of global stages to outline students’ numerical development provides a structure for teachers to identify students’ knowledge and strategy stage. Grouping students’ mathematical thinking into broad stages enables the framework to become the basis for assessing students’ levels of thinking in numbers and informing the next stage of learning. The problematic areas of knowledge that are common to many students are identified to form the core of the class knowledge teaching at the warm-up phase of each lesson and inform teachers of which strategy stage to assign to the students (Ministry of Education, 2008c). By understanding the development of students’ mathematical thinking, teachers can create learning opportunities that are closely aligned with potential
student development (Kafai et al., 2002). However, it is difficult for teachers to identify students’ current levels of thinking which would help them to understand students’ difficulties and support these students’ learning by building on their thinking, rather than imposing their own thinking (Steinberg, Empson & Carpenter, 2004). This study intended to provide an extension to the Number Framework by giving additional yet significant information on students’ strategies for ordering fractions. Besides assessing students’ levels of thinking, such a framework is useful for designing curriculum (Jones, Langrall, Thornton & Mogill, 1997) and planning instruction (Empson, Junk, Dominguez, & Turner, 2005).

1.2 Mathematics games

Mathematics games have been used as activities for teaching and learning mathematics. Nevertheless, using computer games in education can be challenging to teachers, parents as well as students.

1.2.1 Activities in teaching mathematics

Materials and activities have become the characteristics of new practice in the NDP. In order to recognise students’ needs in selecting the right activities for the planned instruction (Ell, 2007), teachers are encouraged to choose activities for students in different groups of ability that will support them in both strategy and knowledge development (Trinick & Stephenson, 2005). Games are one type of activity used in the NDP to create a pleasant environment for effective classroom teaching. Teachers and parents agree that games help improve children’s knowledge (Fisher & Neill, 2007). Students also find learning through playing games fun (Young-Loveridge, 2005). Learning with fun is effective (Lepper & Cordova, 1992; Bragg, 2007). “Learning develops quickly and easily through exploration and fun” (Dryden & Vos, 2005, p.309). Teachers are urged to choose interesting and fun ways to engage students in learning (Sullivan, 2010). Students’ perception of fun is associated with their performance in mathematics (Bragg, 2003). By providing enjoyable yet mathematically focused activities, games enable students to engage in meaningful dialogue about the mathematical concepts and strategies underpinning the games (Bragg, 2003, 2007).
In reforming the Mathematics Curriculum in New Zealand, emphasis was placed on increasing use of context and activities (Linsell, 2005). Among the popular activities proposed by Lovitt and Clarke (1988), games are noted as being a useful tool in teaching mathematics. Games for learning mathematics have been found to engage and motivate students (Koirala & Goodwin, 2000). The mathematics games children play (e.g., simple card and board games), involve numerous mathematical concepts (UNESCO, 2004), improve young children’s number knowledge (Peters, 1998), resolve students’ conceptual obstacles and improve their achievement in mathematics (Onslow, 1990). Further, mathematics games help mathematics learning through the advantage of providing meaningful situations, motivating students through enjoyment, developing positive attitudes towards mathematics, increasing learning through increased interaction between players, allowing learning at different levels, diagnosing and assessing learning, providing 'hands-on' interactive tasks for both school and home, encouraging independent learning and minimising language barriers (Davies, 1995; Booker, 2000; Koirala & Goodwin, 2000; Reusser, 2000). These positive aspects of games are enhanced in the availability of multimedia technology. Computer mathematics games allow for visualization of difficult mathematical concepts and engagement with the details of mathematical procedure through manipulating objects (Betz, 1995; Darragh, 1996; Klawe, 1998; Presmeg, 2006).

1.2.2 Challenges to the use of educational computer games

Educational computer games provide a new form of education called *educational entertainment* or *edutainment*, which concurrently aims to entertain as well as educate. According to Buckingham and Scanlon (2005), “edutainment relies heavily on visual material, on narrative or game-like formats, and on more informal, less didactic styles of address” (p. 46). The edutainment materials are used to satisfy parents’ expectations about what count as valid education, and qualify as pleasurable and entertaining for children to use in their leisure time. However, parents and teachers who did not grow up with the new media often distrust and are sceptical about the educational concepts and actions of the new media (Fromme, 2003). They only accept the use of computers for more serious types of computer activities such as writing texts, drawing graphs, programming and using learning software (e.g., interactive geometry software). Many teachers are doubtful about whether students learn mathematics through playing games and they want to be specific about the learning outcomes that could be obtained from games. This view is supported by Hodgson, Man and Leung.
Teachers also worry about whether students learn and achieve learning objectives when they play games by themselves (Ell, 2007).

Most mathematics software and websites were advertised and claimed as mathematics games merely because they could provide fun learning with numbers. Play is good to motivate learning (Prensky, 2001a), but many other factors are required to produce educational games. The key structural elements that make games engaging and well suited for learning are proposed by Prensky (2001a) (See Section 3.2.2). These elements are rules, goals and objectives, outcomes and feedback, conflict/competition/challenge/opposition, interaction and, representation or story. These game elements were lacking in many of the games found on the web that were mainly developed to provide interactive practice to students. These games were usually just transformed from text into a digital form either in a “fill in the blank” or “multiple choices” format and thus did not make a good use of the special properties that a digital form can offer.

With the advantages of multimedia, the experience using computers can be enhanced through the combination of text, audio, images, animation, video and interactivity. Most of the time, these media are added to games to attract users but do not by themselves guarantee a good game. It is worthwhile highlighting that players’ actual experience is more important than great graphics to determine a really fun and exciting game (Prensky, 2001a). Therefore, the various formats of technological multimedia should be better utilized to integrate mathematical concepts with game contexts and adapt educational theories to game approaches. Using games in the educational process can be beneficial if pedagogical and educational factors are also considered despite these games being designed as entertainment products (Moreno-Ger, Burgos, Sierra, & Fernández-Manjón, 2008).

Challenges to the use of new technologies always exist with ongoing educational practice. Buckingham (2005) indicates some significant gaps in current knowledge of media education research in schools and one of the most obvious gaps is the little work being done on computer games. Research on the development of children’s understanding or learning progression using computer games is particularly required. As remarked by Kafai et al. (2002), among the educational software developed on rational numbers, less focus is given to developing the equivalence of fractions than operations on fractions. The topic of comparing sizes of fractions is considered as a critical area of mathematics. Research has shown that
students face difficulty with fractional quantities due to their limited knowledge of comparing fractions. Students may operate (e.g., add and subtract) on numerators and denominators as if they were two independent numbers, ignoring the value of the fraction as a whole (Newstead & Murray, 1998; Stafylidou & Vosniadou, 2004). Comparing fractions helps students to understand the quantitative values that fractions represent. The understanding of unit size is an important idea to complement the counting and sharing parts emphasised in the part-whole fractions (Sophian & Wood, 1997). Therefore, this study intended to develop a game about sizes of fractions that included the six game elements which made the game engaging as well as educational.

1.3 Research objectives

The purpose of this study is to develop a computer game on fractions that will improve the learning of fractions when the game is played by students with different mathematical abilities.

The objectives of the research are:
1. To develop a computer game focusing on comparing sizes of fractions in an environment that enables players to visualize representations of fractions, manipulate parts of fractions and apply their knowledge of fractions.
2. To identify students’ strategies for ordering fractions and to develop a framework of students’ strategies based on the findings.
3. To identify any improvement in students after playing the computer game.

1.4 Research questions

In order to achieve the above objectives, the following research questions were identified. I will discuss these in more depth in chapter 4.

1. What needs to be considered in developing a computer game that would enable students to compare and order fractions?
2. What strategies are used by students in ordering fractions? How can these strategies be classified as in a framework?
3. What improvements, if any, are there in students’ ability as a result of playing the game?
1.5 Research scope

This study focuses on the topic of comparing sizes of fractions. This is one of the areas of fractions that use knowledge of order and equivalence. The game developed in this research project of ordering rectangular blocks that represent sizes of fractions, is called “Tower Trap”. The learning objective of this computer game is to order fractions with like and unlike denominators from the smallest to the largest and vice versa. For this study, students played the computer game in one session to enable assessment of the immediate effect of the game. However, the game can be played across several sessions, which may be useful for students needing revision or practice.

1.6 Research significance

The literature on fractions and game will show that less focus is given to developing the equivalence of fractions among educational software (Kafai et al., 2002) and students’ reasoning about order and equivalence of fractions in partitioning (Nunes, 2008). An equal and concurrent emphasis should be given to both part-whole concept and fraction size and order. This could be achieved by comparing and ordering fractional representations (i.e., rectangular divided quantity diagrams) in the computer game of Tower Trap. Therefore, this research study will examine these aspects with a view to adding to research literature.

The computer game on comparing and ordering fractions provides a mechanism for learning the interaction between numerators and denominators. This helps students overcome misconceptions including comprehending fractions as two independent numbers based on prior knowledge of whole numbers. Students’ understanding of order and sizes of fractions will affect their learning of fractional quantity later (Behr, Lesh, Post & Silver, 1983; Behr, Wachsmuth, Post, & Lesh, 1984; Behr & Post, 1992; Ministry of Education, 2008d).

The framework of students’ strategies developed will enable teachers to better understand student thinking (Young-Loveridge et al., 2007). This will help in the selection of teaching methods to move students from basic to advanced strategies for fractions (Steinberg et al., 2004).
If the computer game on comparing and ordering fractions can be shown to be beneficial to students with different mathematical abilities; teachers, parents and students can be more confident and convinced of using games as educational tools for learning fractions. The effectiveness of game design affects the parents and teachers from either encouraging or discouraging students to play the game (Hong, Cheng, Hwang, Lee & Chang, 2009).

1.7 Outline of chapters

The following is an outline of the remaining chapters of this thesis:

Chapter 2 and Chapter 3 provide a literature review on fractions and computer games. Chapter 2 Fractions highlights the part-whole construct of fractions as fundamental in acquiring rational number knowledge. Subsequently, this chapter discusses the understanding of the unit size of fractions and a range of strategies used by students for comparing sizes of fractions. A model of the growth of mathematical understanding is referred to in analysing the use of simple to more complicated strategies.

Chapter 3 Computer Games defines computer games, describes key structural elements of making a game engaging, clarifies pedagogical approach of mathematics games and discusses criteria for reviewing educational software or games about fractions.

Chapter 4 Research Methodology explains the purpose of the research, the research paradigm of mixed methods, research sample, tools and process involved in the data collection and analysis of the qualitative and quantitative data.

Chapter 5 to Chapter 8 present findings to answer research questions 1 to 3. Chapter 5 and Chapter 6 discuss the development of computer games that enables students to learn fractions (research question 1). Chapter 5 also discusses the research method. In fact, the Tower Trap computer game described in Chapter 5 is used to collect data in the main study. Chapter 7 investigates students’ strategies (research question 2) and Chapter 8 shows the improvement (if there is any) after playing the game (research question 3).

In Chapter 5 Designing and Developing Tower Trap, a theoretical framework of the structure of Tower Trap is designed to integrate instructional factors and game elements, to
assure the mathematical inquiry of the game. This chapter also discusses the findings of the pilot study on the benefits of the game and the concerns of students for the development of a computer game that caters for different learning needs. Finally, the game features and the game pedagogy are elaborated.

In Chapter 6 Students’ Evaluation and Game play in Tower Trap, students’ evaluation of the quality of the game and its effectiveness in teaching and learning fractions is investigated. In addition, the chapter presents the number of attempts taken by students to order fractions at different game levels. Using students’ number of attempts, the game’s difficulty level for students with different mathematical abilities was determined.

In Chapter 7 Students’ Strategies Towards a Framework, the strategies of high achievers, average and cause-for-concern students for comparing sizes of fractions in maths tasks are examined. Levels of mathematical understanding involved in students’ strategies are investigated. The strategies used by students with different mathematical abilities which involve different levels of mathematical thinking are framed to describe a progression of students’ fractional thinking.

The positive effects of Tower Trap in learning fractions are discussed in Chapter 8 Improvement of Students’ Knowledge of Fractions. The first section focuses on the performance of three individuals (a high achiever, an average student and a cause-for-concern student) before analysing the performance of the three mathematical ability groups overall in the pre and post maths tasks and tests. The second section examines the beneficial changes in students’ strategies in the pre and post maths tasks. The third section analyses the achievement differences between the pre and post tests particularly on the improvement of being wrong (W) in the pre test but right (R) in the post test (i.e., WR).

Following on from the discussions on the framework of students’ strategies and their improvement in the use of strategies, Chapter 9 Moving Students Towards Advanced Strategies provides suggestions to teacher to improve students’ strategies. Based on the work of the three individual students (as discussed in Chapter 8), this chapter explains how to identify students’ current level of mathematical thinking and misconceptions through their use of strategies, shows the continuities of mathematical thinking through the use of similar strategies between students with different mathematical abilities, and proposes instructional
approaches that suit particular strategies so that students moves from current level of strategic thinking to a higher one.

In Chapter 10 Conclusions, the research findings which answer the research questions on the Tower Trap and the learning of fractions are presented. This chapter concludes the findings of the research on the design of a good computer game for learning fractions and students’ strategies for comparing sizes of fractions. Finally, limitations of this study are explained and recommendations for future research are given.
CHAPTER 2

FRACTIONS

This chapter discusses the way students learn and improve in the knowledge and strategies of fractions. There are three sections in this chapter. Section 2.1 provides a background of five constructs of fractions generally and of the part-whole construct in particular. The Numeracy Development Project (NDP) emphasizes part-whole thinking for numerical development and for learning fractions. The part-whole relation is represented using divided quantity diagrams. Partitioning based on students’ whole number knowledge and the understanding of fraction size and order are important for developing fraction concepts. Section 2.2 discusses strategies for comparing sizes of fractions with particular emphasis on the divided quantity diagrams. A discussion of the knowledge of equal-partitioning and equal-whole is also included. Apart from the divided quantity diagrams, using benchmarks and finding common denominators are also presented. Section 2.3 elaborates on students’ progression towards a better understanding of fractions. The model of Pirie and Kieren (1989, 1994a, 1994b) is used to analyse the growth of students’ mathematical understanding levels which is related to the sophistication of the strategies used by the students to compare fractions.

2.1 Part-whole construct

This section firstly explains fraction constructs including the part-whole construct of fractions and discusses the importance of part-whole thinking in the New Zealand Number Framework (NZNF) and the representations of the part-whole relation using divided quantity diagrams. Subsequently this section highlights the use of partitioning for learning fractions and the understanding of fraction size and order for developing advanced fraction concepts.

2.1.1 Fraction constructs

Fractions can be interpreted in different ways using several constructs. Part-whole, ratio or rate, operator, quotient and measure are the five interrelated constructs of rational
numbers proposed in 1976 and revised later in 1988 by Kieren (Kieren, 1976; Kieren, 1988). The following are the interpretations of rational numbers based on the various constructs.

2.1.1.1 Part-whole

Part-whole describes the relation of a part of a quantity to its total amount. For example, in the fraction $\frac{3}{5}$, the number 5 shows how many parts it is divided into and the number 3 shows how many fifths (i.e., pieces) are taken. Basic to the part-whole notion is partitioning a whole, which is the process of partitioning a continuous quantity into a number of equal-sized subparts or a set of discrete objects into equal-sized subsets. The equal parts can be composed or unitised and recomposed or reunitised to form the initial whole (Behr, Lesh, Post & Silver, 1983; Baturo, 2004). An example of composing or unitising is constructing one whole when given two-thirds. The part-whole construct provides a basis on which to introduce fraction language and symbols. For example, one part of a whole that has been divided into two equal parts has a fraction of one-half or $\frac{1}{2}$. Students compare the parts to the whole in continuous quantity and discrete objects (Ministry of Education, 2008d, p. 5). As an example for a continuous quantity, a square is divided into four equal parts and the fraction $\frac{3}{4}$ comprises three parts out of four divided parts (Figure 2.1). A set of four circles is an example of discrete objects where the fraction $\frac{3}{4}$ comprises three circles out of four circles (Figure 2.2).

<table>
<thead>
<tr>
<th>Image removed for copyright reasons</th>
<th>Image removed for copyright reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Figure 2.1: A continuous quantity</strong> (Ministry of Education, 2008d, p. 5)</td>
<td><strong>Figure 2.2: A set of discrete objects</strong> (Ministry of Education, 2008d, p. 5)</td>
</tr>
</tbody>
</table>
2.1.1.2 Ratio

Ratio is the quantitative relation between two amounts showing the number of times one value contains or is contained within the other (Oxford University Press, 2011). Ratio expresses a fractional relationship between two quantities. This relationship exists in comparing parts with parts or (sometimes) parts with the whole (Ministry of Education, 2008d, p. 3). Ratio is written as $a:b$ and interpreted as “for every $a$ there are $b$” or “for every $b$ there are $a$”. As ratio conveys the notion of relative magnitude, it is considered as a comparative index rather than as a number (Behr et al., 1983). The following example shows the shade of grey to white mixture has a ratio 3:5. Both use measures of colour which are in the same units (Figure 2.3).

![Image removed for copyright reasons](image)

**Figure 2.3: A ratio 3:5 for the shade of grey to white mixture (Ministry of Education, 2008d, p. 6)**

When the units are different, the ratio involves a rate (Ministry of Education, 2008d, p. 6). The following shows a rate 15:10 for 15 pineapples that cost $10 (Figure 2.4).

![Image removed for copyright reasons](image)

**Figure 2.4: A rate 15:10 for the price of pineapples (Ministry of Education, 2008d, p. 6)**
Ratios can be written as proportions. Two ratios are in proportion to one another when they are equal. For example, 3:5 is equivalent to \( \frac{3}{8} \) or \( \frac{5}{8} \). Behr et al. (1983) noted the use of proportions is a very powerful problem-solving tool in a variety of physical situations and problem settings that require comparisons of magnitudes. For example, students can use the trigonometric ratio to make measurement for the following problem: “An aircraft has climbed to an altitude that is 600 metres higher than where it took off. The distance on the ground it has travelled is 2600 metres. What has been the average angle of climb for the aircraft?” (Ministry of Education, 2008d, p. 79). The problem can be described using the following diagram (Figure 2.5):

![Image removed for copyright reasons](image)

**Figure 2.5: A diagram of the problem (Ministry of Education, 2008c, p. 79)**

### 2.1.1.3 Quotient

Quotient is a result obtained by dividing one quantity by another (Oxford University Press, 2011). The symbol \( \frac{a}{b} \) is also used as a way of writing the quotient \( a \div b \) (Behr et al., 1983). The example \( 2 \div 3 = \frac{2}{3} \) shows the fraction as a quotient.

The following pizza sharing problem provides an example which describes quotients as the answers to sharing division problems: “Three girls share two pizzas equally. How much pizza does each girl get?” (Ministry of Education, 2008d, p. 4).

The 2 pizzas shared among 3 girls results in each girl getting \( 2 \div 3 = \) two thirds of a pizza.
The same result is obtained if six girls share four pizzas equally, where each girl gets \( 4 ÷ 6 = \frac{4}{6} \) of a pizza.

\[ 2 ÷ 3 = 4 ÷ 6 \quad \text{because} \quad \frac{2}{3} = \frac{4}{6} \, . \]

The above examples show that the quotient construct allows the same equal shares in different situations (Ministry of Education, 2008d, p. 7).

### 2.1.1.4 Operator

Operator is a symbol or function denoting an operation (Oxford University Press, 2011). The symbol \( \frac{a}{b} \) can refer to a function that operates on some number, quantity, object, or set. For example, in finding \( \frac{3}{4} \) of 100, the \( \frac{3}{4} \) operates on the 100 by multiplying by 3 and dividing by 4, which yields 75 (i.e., \( 100 \times 3 = 300 \) and \( 300 ÷ 4 = 75 \)). As such, the \( \frac{3}{4} \) is thought of as a function applied to 100. Behr et al. (1983) and Behr, Harel, Post and Lesh (1992) state the numerator causes an extension of the quantity it operates, while the denominator causes a contraction. In the above example, the numerator 3 causes an extension (i.e., increase) of 100, while the denominator 4 causes a contraction (i.e., decrease). This leads to the following paired interpretations of the numerator and denominator:

- **stretcher-shrinker**
  
  When operating on continuous objects such as length, say \( L \), which stretches \( L \) to \( a \) times its length and then shrinks it by \( a \) factor of \( b \).

  For example, a big bottle can fill up 1 litre of water and a small bottle can only fill up two-thirds as much as the big bottle. To obtain the volume of the small bottle, 1 litre is multiplied by two and then divided by three.

- **multiplier-divider**
  
  It transforms a discrete set with \( n \) elements to a set with \( na \) elements and then this number is reduced to \( \frac{na}{b} \). For example, if \( \frac{2}{3} \) is operating on 15 (\( \frac{2}{3} \) of 15 or \( \frac{2}{3} \times 15 \)), multiply 15 by two and then divide it by three.
2.1.1.5 Measure

Measure is a quantity contained in another an exact number of times (Oxford University Press, 2011). Measure reconceptualises the part-whole notion of fraction. Fractions as measurement units, refers to how much there is of a quantity relative to a specified unit of that quantity (Behr et al., 1983). Measure is the idea of finding out how many times a fraction or ratio fits into a given fraction or ratio (Ministry of Education, 2008d, p. 5). An example is one whole is divided by a non-unit fraction in the following problem context:

“Amy’s car has a full tank of petrol. Each trip takes five-eighths of a tank. How many trips can she do on a full tank?”

This can be represented using strip diagrams (Figure 2.6) and recorded symbolically as

\[
1 \div \frac{5}{8} = \frac{8}{5} = \frac{1}{1} \quad \text{trips.}
\]

As shown in the diagram, 1 full tank is equal to \(\frac{8}{8}\) and 1 trip uses \(\frac{5}{8}\) tank of petrol.

Therefore, having 1 full tank of fuel allows Amy to complete 1 trip with balance of \(\frac{3}{8}\) tank of petrol. Since 1 trip requires \(\frac{5}{8}\) tank of petrol, the balance of \(\frac{3}{8}\) tank of petrol will allow another \(\frac{3}{5}\) of trip. This gives a total of \(1\frac{3}{5}\) trips with 1 full tank of petrol.

2.1.1.6 Interrelationship among various constructs

Having many constructs, fractions are complex to teach and learn. Students need to learn each of these constructs as well as the interconnections between them to gain a complete understanding of rational numbers (Kieren, 1976) and to become generalised proportional thinkers (Ministry of Education, 2008d, p. 2).
Behr et al. (1983) developed Kieren’s idea further and highlighted the interrelationships among the various constructs (Figure 2.7).

![Image removed for copyright reasons](image)

**Figure 2.7: Conceptual scheme for instruction on rational numbers (Behr et al., 1983)**

Partitioning and part-whole are connected to other constructs including ratio, operator, quotient and measure using solid arrows. The solid arrows denote the fundamental nature of partitioning and the part-whole construct to learning other constructs of rational numbers. The dotted arrows represent the links between the constructs above and their application to equivalence, multiplication, problem solving and addition. This forms the relationship as elaborated as follows:

- The ratio construct develops the concept of equivalence.
  In renaming \( \frac{2}{3} \) as \( \frac{4}{6} \), the relationship between 2 and 3 or 2:3 is equivalent to the relationship between 4 and 6 or 4:6.
- The operator promotes an understanding of multiplication
  “Fractions can operate on other numbers in a multiplicative way” (Ministry of Education, 2008d, p. 55). For example, \( \frac{2}{3} \) of 15 means two-thirds of fifteen (i.e., \( \frac{2}{3} \times 15 \)), where fifteen is divided by three and multiplied by two.
- The measure helps addition of like fractions.
A unit can be split into smaller equal units such as \( \frac{1}{3} \), \( \frac{1}{3} \), \( \frac{1}{3} \). The same quantity emerges in adding like fractions, in which \( \frac{1}{3} + \frac{1}{3} \) is equivalent to \( \frac{2}{3} \).

- All of the constructs including quotient are useful in problem solving of fractions. For example, \( \frac{2}{3} \) means the division of 2 by 3 or \( \frac{4}{6} \) means the division of 4 by 6 or \( 4 \div 6 \). The divisions \( 2 \div 3 \) and \( 4 \div 6 \) give the same result. Quotients are the answers to sharing division problems.

Charralambous and Pitta-Pantazi (2005) provide empirical validity to Behr et al.’s (1983) model by analysing data of 646 fifth and sixth graders’ performance on fractions. The findings support the basic role of the part-whole concept in developing other constructs of fractions. They affirm the preponderance of the part-whole notion in teaching rational numbers and justify the traditional instructional approach of using the part-whole notion as the inroad to teaching fractions.

### 2.1.2 Part-whole thinking in the New Zealand Number Framework (NZNF)

The part-whole construct is used in the NDP in different ways both as a construct of fractions and a strategy in computing. This study focused mainly on the part-whole construct as fractions. This section discusses the importance of part-whole thinking in a student’s numerical development. In the NZNF stages which describe students’ numerical progressions, part-whole thinking is focused on in the upper stages (Young-Loveridge & Wright, 2002) while counting is emphasised in the lower stages. Counting involves one way of solving problems and restricts the size of numbers. For example, a student recognises that “7” represents all seven objects and counts on from there: “8, 9, 10, 11, 12” to solve \( 7 + 5 \).

According to Ministry of Education (2008a, p.3), “students understand that the end number in a counting sequence measures the whole set and can relate the addition or subtraction of objects to the forward and backward number sequences by ones, tens, etc.” With part-whole (computing) thinking, students “recognise numbers as abstract units that can be treated simultaneously as wholes or can be partitioned and recombined” (Ministry of Education, 2008a, p. 4). Students are able to derive results from related known facts. For example, \( 7 + 5 \) can be completed by partitioning 5 into 2 parts, which are 3 and 2 so that the 3 can be added.
with 7 (i.e., \(7 + 3\)) to make 10 and the remaining 2 can be added with the 10 to get 12. The strategies that students use can be represented in various ways such as the following (Ministry of Education, 2008a):

a) Compensatory knowledge

Addition problem: \(7 + 5 = 7 + (3 + 2) = (7 + 3) + 2 = 10 + 2 = 12\)

Addition problem: \(7 + 5 = (2 + 5) + 5 = 2 + (5 + 5) = 2 + 10 = 12\)

b) Standard place value partitioning

Addition problem: \(85 + 23 = (80 + 20) + (5 + 3) = 100 + 8 = 108\)

c) Standard place value with tidy numbers and compensation

Subtraction problem: \(85 - 29 = 85 - 30 + 1 = (80 - 30) + (5 + 1) = 50 + 6 = 56\)

d) Reversibility

Subtraction problem: \(85 - 29 = \square \text{ as } 29 + \square = 85\)

\(29 + (1 + 50 + 8) = 85 \text{ so } 85 - 29 = 59\)

e) Halving and halving

Division problem: \(52 \div 4 \text{ as } 52 \div 2 = 26, \ 26 \div 2 = 13 \text{ therefore } 52 \div 4 = 13\)

Dividing by four is the same as dividing by two twice

f) Using unit fractions and conversion from percentages

Word problem: What is the selling price of a shirt which normally cost $36 after a 25% discount?

25% is \(\frac{1}{4}\), so the selling price is \(\frac{3}{4}\) of $36. \(36 \div 4 = 9, \ \frac{3}{4} = 3 \times \frac{1}{4} = 3 \times 9 = 27\).

In conclusion, part-whole thinking “opens up the world of large numbers and multiple strategies” (Ministry of Education, 2010b, para. 4).

It is a major challenge for teachers to teach part-whole thinking to school children who are expected to be part-whole thinkers by the end of Year 3 (Ministry of Education, 2010b). Students encounter difficulties in transitioning into part–whole stages although many students are able to progress through the counting stages (Young-Loveridge, 2006). Teachers try their best to help their students acquire part-whole strategies, yet a large number of students still persist in using counting (Young-Loveridge, 2007). The persistent counters lack knowledge of place value, basic facts, and number sequence forwards (i.e., 1, 2, 3, …) and backwards.
These are the components of knowledge that provide a foundation for the students to become part-whole thinkers.

2.1.3 Represent part-whole relation using divided quantity diagrams

The part-whole relation of fractions is presented explicitly using the partitioned quantity method for representing fractions. Reeve and Pattison (1996) describe the area model where each unit is divided into segments of equal size and shape, and the relations between segments and the unit are used to represent the simple part-whole relations explicitly. In this study, we use the term ‘divided quantity diagram’ to refer to a drawing of the area model. Partitioning shapes on paper produces divided quantity diagrams that enable students to present the part-whole relation of fractions. Divided quantity or partitioned quantity refers to “a whole quantity that has been divided into some numbers of equal-sized parts” (Smith III, 2002, p.4). A given fractional quantity can be expressed by shading the parts (which represented numerators) from the divided parts comprising a unit (which represent denominators). Divided quantity diagrams are figures that are divided into equal-parts to compare shaded area and total area. For example, the fraction $\frac{1}{2}$ is represented by shading one of two equal-parts using the circular and rectangular quantity diagrams in Figure 2.8.

![Figure 2.8: Circular and rectangular quantity diagrams](image)

Many part-whole problems used in instruction are based on area models of circular shapes that directly build on students’ informal knowledge of equal sharing (Moseley, 2005). Children begin to learn about part-whole fractions from sharing half of a whole between two people (Lamon, 1996; Nunes & Bryant, 1996; Moseley, 2005; Darr & Fisher, 2006). Fraction circle sheets of paper are used in the NDP classroom for students to fold and cut into parts as
a way of representing fractions as an area. For example, students are asked to cut the circles into halves, label the halves using symbols and cut out the pieces (Figure 2.9).

Figure 2.9: Fraction circles (Ministry of Education, 2008d)

Such activities enable students to relate the number of parts obtained (cut) from one whole (of the circle sheet) to the number of parts chosen. In the process students come to understand “the size of a fraction involves coordinating both the numerator (top number) and the denominator (bottom number)” (Ministry of Education, 2008d, p. 20). Although it is easy to show on a circle when one whole unit is formed, other shapes such as hexagons, squares and rectangles are used for partitioning activities that involve the repeated halving strategy (e.g., halves, quarters and eighths). More complex partitions on larger number of pieces (e.g., 6ths, 10ths and 12ths) and on odd numbers of pieces (e.g., 3rds and 5ths) (Maguire, Neil & Fisher, 2007) can also be used. Extensive activity in partitioning wholes of many different sized objects in different ways contributes to students’ development of part-whole fraction thinking (Smith III, 2002).

2.1.4 The use of partitioning in learning fractions

Partitioning is a key construct in early equal-sharing strategies (Lamon, 1996; Empson, 1999). A collection or quantity can only be shared equally among a group of people for fair shares (Pothier & Sawada, 1983; Hunting & Sharpley, 1988) after being partitioned into equal-parts. The number of people sharing the items with will determine how the item is partitioned (Ministry of Education, 2008d, p. 16). For example, a cake is divided into thirds if sharing with three people. Through partitioning, students apply their informal knowledge of fractions in learning fractions. Darr and Fisher (2006) propose partitioning as an important idea for teaching fractions. This involves (i) “the ability to divide an object or objects into a
Partitioning objects as a way to solve equal sharing problems is the key mathematical idea in constructing fractional concepts (Ministry of Education, 2008d, 2010c). In sharing an object, the students realise that the denominator tells how many parts the whole has been divided into and the numerator tells how many of those parts have been chosen. For example, \( \frac{1}{5} \) shows that one (a whole) is divided into five equal parts (fifths) and that one of those parts is taken. The English language one-fifths provides the same meaning to the fractional concept as the symbol for the fraction \( \frac{1}{5} \). The special words of halves, thirds, and quarters (fourths) cause a barrier to students in generalising the meaning of fractions. “It is not until fifths (five-ths), sixths, sevenths, and so on are encountered that the “ths” suffix code becomes evident” (Ministry of Education, 2008c, p. 16). Fractional units of one can be split up (partitioned) and repackaged (re-unitised) unlike the basis of whole-number counting. By splitting up ones through forming unit fractions (e.g., \( \frac{1}{2} \)) and recombining several of these new units to form fractions (e.g., \( \frac{2}{5} \)), the students coordinate the link between the numerator and denominator in fraction symbols. (Ministry of Education, 2008d, p. 26). The part-whole concept of a unit fraction as being one of several equal parts of a given whole “arise(s) in the context of partitioning and discussing the resulting pieces and their size relative to a discrete unit (Empson, 1999, p. 289).

By partitioning fractions into parts and treating each part like a whole number, fractions are linked to counting numbers. The NDP refers to fractions in terms of the number of pieces so that students understand the numbers represented in a fraction symbol. “The bottom number (denominator) indicates how many pieces make up the whole. The top number (numerator) tells us how many pieces there are. It counts the number of pieces.” (Ministry of Education, 2008d, p. 11). According to Mack (1990, 1995), students develop meaning for fraction symbols and procedures based on their informal knowledge of partitioning. He elaborates on this as follows:
• The symbolic representations of fractions are drawn on students’ prior knowledge of whole numbers and fractions in terms of parts of a whole.

For example, a student might explain that \( \frac{3}{4} \) or three fourths means three pieces out of four equal-sized pieces.

• Students conversion of mixed numerals and improper fractions is based on knowledge of fractions equivalent to one.

For example, a student might write “\( \frac{2}{2} \frac{2}{2} \frac{1}{2} \)” and explain three and half as the mixed numeral for the improper fraction \( \frac{7}{2} \).

• In comparing fractions students develop the realisation that the more parts a unit is divided into, the smaller the parts become.

For example, a student might comment that sixths have smaller pieces than quarters and so \( \frac{1}{6} \) is smaller than \( \frac{1}{4} \).

• Students think about fractions as the numbers of pieces that can be combined or removed in adding and subtracting fractions with common denominators.

For example, you have three fifths of a pizza, and I give you one fifth more of a pizza. How much pizza do you have?

A student might explain “there are four pieces of pizza”, meaning four fifths of a pizza. Other comments such as “taking a piece a way” and “one piece left” were related to the thinking of the fraction as a specific number of parts and each part was treated as a separate unit. By representing fractions with a counting of the partitions that would complete the whole, the attempt to add and subtract both the fractional parts and the wholes becomes reasonable.

2.1.5 Understanding of fraction sizes

An understanding of unit size is required to complement the counting and sharing parts of fractions emphasised in partitioning and the part-whole concept. The idea of relative amount, or “how much of one thing there is compared with another”, is proposed by Sophian and Wood (1997) as complementary to children’s understanding of sharing in their intuitive knowledge of fractional quantities. The concept of relative amount provides a foundation for
understanding the quantitative values that fractions represent. An emphasis on “comparison of quantities” in the Number Framework is proposed by Young-Loveridge (2008) considering Catherine Sophian’s (2003a, 2008) research which challenges the “counting first” perspective in the NDP. According to Sophian (2008), comparison of quantities is more fundamental to mathematics learning than counting.

Steffe and Olive (2010) promote partitive reasoning that transcends part-whole reasoning in developing conceptions of fractions as quantities. The partitive fraction scheme allows students to conceive fractions as lengths rather than solely as parts of wholes. This scheme involves the following operations or mental actions in generating fractional knowledge (Hackenberg, 2007, 2010; Steffe & Olive, 2010).

- *partitioning* as the mental action of dividing a unit (whole) into equal-sized parts
- *disembedding* as taking a part out of a whole unit without mentally destroying the whole
- *iterating* as repeatedly instantiating a part of a unit in order to make a larger amount

The following discussion is based on a situation of getting a chocolate bar that is two-thirds of the rectangular bar. The part-whole concept refers “one-third” as how much one out of three equal parts of the bar is of the whole bar. With a partitive fractional scheme, “one-third” is conceived as a part that is one of three parts marked off on the bar, disembed the part and iterated it three times, to produce a three-part bar. To make two-thirds of the bar, the bar can be partitioned into three equal parts, take out one of those parts, and iterate it to make two parts. In this way, the part-whole conception of fractions is transcended in that a unit fractional part (i.e., one-third) can be disembedded from the partitioned whole and iterated to make another fractional part of the partitioned whole (i.e., two-thirds).

It was shown from the above discussion that partitive reasoning differs from the part-whole concept in the explicit use of fractional language. The students who rely on the part-whole concept use “one-third” to refer to the relation \( \frac{1}{3} \) and use the statement “out of” to describe \( \frac{1}{3} \) as one out of three equal parts. They involve in the explicit numerical one-to-many comparison (one-to-third) to deal with two numerals comprising a fraction, numerator 1 and denominator 3 (Steffe, 2002). On the other hand, the students who have constructed a
partitive fraction scheme can *disembed* one-third from the whole and *iterate* one-third to make another fractional part of the whole (Hackenberg, 2005, p. 56).

The splitting operation plays a critical role in moving from part-whole to partitive concepts. According to Steffe and Olive (2010), the splitting operation is the composition of partitioning and iterating, in which partitioning and iterating are conceived as inverse operations and unified. Without the ability of splitting, the meaning for two-thirds remains tied to part-whole concepts; in which it comes from it being part of a whole, not from it being a fraction that is two times one-third. Splitting a bar involves more than partitioning a bar into three parts, but also iterating any part of the partitioned bar to make the partitioned bar (i.e., the whole). Splitting involves the multiplicative relationship between the partitioned bar (i.e., one-thirds) and the given bar (i.e., the whole): two-thirds is one-third two times.

Moving from part-whole to partitive concepts is one of the developmental hurdles with regard to conceptualizing fractions and splitting plays a critical role in providing meaning to partitive conception (Norton & Hackenberg, 2010). Furthermore, the splitting operation is necessary in students’ construction of advanced fraction schemes which can contribute to students’ development toward algebraic reasoning.

A strong focus on counting but with limited understanding of the comparison of quantities in the early years may cause a difficulty for students when learning about fractional quantity later. In partitioning a whole into a predetermined number of equal parts, many students do not seem to produce immediate insights into equivalence and order of fractional quantities. For example, these students do not see it as necessary that halves from two identical wholes are equivalent (Nunes, 2008). The students neglect the value of the fraction as a whole and treat the components of a fraction as separate numbers. They refer to fractional parts as pieces that can be counted and combined arithmetically just like whole number quantities (Mack, 1995). This belief of fraction consisting of two independent numbers emerges directly from the initial theory of natural numbers (Paik & Mix, 2003; Stafylidou & Vosniadou, 2004). Inappropriate application of whole-number schemes affects students’ success in working with fraction concepts and operations (Newstead & Murray, 1998). In the examples of previous work, Paik & Mix (2003) show 6 to 8 year old students misinterpret fraction names by trying to map the numerator and denominator onto the number of shaded and unshaded parts of the fractions. Stafylidou and Vosniadou (2004) show students ranging
in age from 10 to 16 year old order the numbers from the smallest to the biggest: 1, $\frac{1}{7}$, $\frac{4}{3}$, $\frac{5}{6}$ with the reason that “the smallest goes first and the other numbers follow” (p. 510). In solving the problem of estimating $\frac{12}{13} + \frac{7}{8}$, Behr, Wachsmuth, Post and Lesh (1984) also found 13 year old students added the numerator 12 to the numerator 7 (= 19) and the denominator 13 to the denominator 8 (= 21). In adding fractions $\frac{3}{4} + \frac{7}{8}$, Young-Loveridge, Taylor, Hawera and Sharma (2007) found Year 7-8 students added the numerators 3 and 7 and the denominators 4 and 8 to obtain $\frac{10}{12}$ as the solution, although only like units can be combined or compared in the addition and subtraction of fractions. For example, by finding a common denominator of 8, $\frac{3}{4} = \frac{6}{8}$ can be added with $\frac{7}{8}$. The above suggests that students’ inability to see a fraction as a quantity causes them to perceive numerators and denominators as two independent numbers.

In investigating the traditional use of partitioning as the starting point for teaching fractions, Nunes (2008) notes currently the usual practice is to give the initial focus on representations before promoting students’ reasoning about order and equivalence of fractions. Nevertheless she suggests schools could develop students’ quantitative reasoning of fractions before or at the same time students are learning fractional representations. While partitioning, students need to anticipate that the right number of cuts produces the right number of equal parts and exhausts the whole (Nunes, 2008).

This study intended to give equal and concurrent emphasis to both part-whole concept and fraction size and order. This could be achieved by comparing and ordering fractional representations (i.e., rectangular divided quantity diagrams) in the computer game of Tower Trap. This will be explained in Chapter 5 Section 5.1.4.

This section has described the foundational importance of the part-whole construct for rational number development, the focus of the NZNF for developing students’ numerical strategies, and the part-whole relation based on divided quantity diagrams. The connection of partitioning with whole number knowledge assists students’ construction of fraction concepts.
Nevertheless, the understanding of unit size is critical to complement the counting and sharing parts of fractions stressed in partitioning. The development of fractional quantities requires partitioning, disembedding, iterating and splitting as the operations in generating fractional knowledge.

2.2 Comparing sizes of fractions

This section discusses reasoning given by the students to explain the way they order fractions. The common strategies used by students to compare and order fractions include drawing divided quantity diagrams, using benchmarks and finding common denominators. The knowledge of equal-partitioning and equal-whole required in drawing divided quantity diagrams is also elaborated.

2.2.1 Students’ reasoning and thinking

Students’ fractional thinking is reflected from their reasoning in solving problems with fractions. In the NDP classroom, students have been encouraged to explain their thinking about fractions and the methods they have used to solve problems with fractions (Darr & Fisher, 2006; Ministry of Education, 2008a, 2008d). For instance, diagrams are drawn by students to express their ideas and verbal explanations are given by students to describe their thinking. As an example, the explanation “two-thirds is more than one-half while one-quarter is less than one-half” reveals how $\frac{1}{4}$ is ordered smaller than $\frac{2}{3}$.

In an investigation of Year 7–8 New Zealand students’ strategies when adding fractions, Young-Loveridge et al. (2007) found that students’ verbal explanations and written recording provides “a rich source of data about students’ thinking and problem-solving processes” (p. 83). These findings about students’ thinking ensure their understanding of fractions reaches a deep and connected level; meanwhile, the identified students’ strategies inform teachers about the next steps to take in the students’ development. Consistently, Gould (2005) gained a much richer insight into students’ fraction concept images by examining their explanations given to the task of comparing sizes of fractions. In fact, students’ reasoning is worth investigating, not only to reveal the concepts and thinking about fractions underlying the correct strategies, but also to determine the misconceptions causing the prevalence of
various incorrect strategies for ordering fractions (Gould, 2005; Steinle & Price, 2008). Quite a lot of “incorrect” reasoning is provided to the “correct” answers given by students (Gould, 2005). For example, $\frac{12}{13}$ is bigger than $\frac{9}{10}$ because the numerator 12 is bigger than the numerator 9 while the denominator 13 is bigger than the denominator 10.

Sharp and Adams (2002) noticed some students used symbols and pictures to communicate their thinking after the fact but not all pictures drawn by these students matched their answers. Plausibly the paper-and-pencil information is used by these students to describe their thinking rather than to do their thinking. Students experience great difficulty generalising to any sort of formal procedure due to their inability to describe their earlier thinking. No consistent strategy can be inferred if insufficient data is given by the students to describe their thinking.

Students are sometimes unable to give any reasoning to their solution of mathematical problems on operating with fractions. They are considered as unable to give reasoning in the following situations:

- Students give answers only
- Students repeat the answers or provide tautology
- Students provide irrelevant reasoning

Young-Loveridge et al. (2007) show that the students who add fractions correctly without any explanation are unsure of their answers and impute their correct answers to be a lucky guess. Sharp and Adams (2002, p. 346) relate conceptual understanding to “too mentally quick to need symbol algorithm” for a few students who calculate everything mentally when solving problems involving division of fractions. These students consistently blurt out correct answers but are neither willing nor able to explain their methods. There are a number of potential reasons for this finding:

(a) an inability for students to document their thinking using a consistent procedure;
(b) an inability to describe and write their method onto paper using mathematical concepts and expressions.
2.2.2 Drawing divided quantity diagrams: Equal-parts and equal-wholes

Drawing divided quantity diagrams is one of the strategies students used to compare sizes of fractions. This is the common strategy used by the students to show their ideas and explain the comparison of fractions (Ministry of Education, 2008d, p. 21). The knowledge of equal-partitioning and equal-whole is particularly focused on in this section because of its critical role in the representation of the sizes of fractions (Lamon, 1996; Newstead & Murray, 1998; Yoshida & Shimanchi, 1999; Mack, 2001; Yoshida & Sawano, 2002; Olive & Vomvoridi, 2006). In the NDP instruction, equal shares of fractional parts and equal-sized wholes are stated as the mathematical ideas needed for learning to find halves, quarters, and other fractions of shapes (Ministry of Education, 2008d, 2010d). The equal-parts and the parts to whole relations are pre-requisite fractional concepts required to construct partitioning operations (Olive & Vomvoridi, 2006).

Equal-partitioning and equal-whole are the mathematical relations underlying the partitioned quantity of the representation of fractions. Moseley (2005) found that the students who receive a curriculum emphasising the part-whole relations of the rational number domain often focus more on the superficial surface of the part-whole representations than the underlying mathematical relation. These students extend the features of part-whole reasoning in comparing sizes of fractions. For example, the students try to match surface features to show two fractions \( \frac{1}{2} \) and \( \frac{3}{5} \) are the same using circular diagrams and ignore the relations between the quantities. Figure 2.10 shows fifths are unequally divided and three parts shaded are close to one half.

![Figure 2.10: Representations of \( \frac{1}{2} \) and \( \frac{3}{5} \)](image)
Moseley (2005) stresses that not only the surface feature of visual appearance, but the underlying operation that fits the mathematical structure is necessary to interpret representations of fractions meaningfully.

Lack of understanding of equal-whole leads students to the mistake of comparing fractions with different wholes in terms of size and shape. They misunderstand the connection between the sizes of the denominators of fractions and the relative size of the wholes. This is the typical error shown in the diagrams drawn by students where the size of the whole for each fraction represented is drawn to be directly proportional to the size of the denominator (i.e., the larger the denominator, the larger the size of whole) (Peck & Jencks, 1981; Yoshida & Kuriyama, 1995; Gould, 2005). This is most likely to appear on rectangular divided quantity diagrams drawn to represent fractions. Figure 2.11 provides an example of this error where the sizes of the whole drawn when comparing two fractions (i.e., $\frac{1}{5}$ and $\frac{1}{3}$) are not the same. Instead, the sizes of the whole are drawn proportional to the denominator (i.e., the whole of $\frac{1}{5}$ is bigger than the whole of $\frac{1}{3}$).

![Figure 2.11: Varying wholes](image)

On the other hand, the concept of equal-partitioning is the basis of partitioning a unit into parts for the representation of fractions. When asked to show $\frac{3}{4}$ in three different ways in the investigation of students’ concepts of fractions, Newstead and Murray (1998) documented circular, rectangular and triangular divided quantity diagrams drawn by students (Figure 2.12).
Figure 2.12: Circular, rectangular and triangular divided quantity diagrams (Newstead & Murray, 1998)

These diagrams are divided into four parts with three parts shaded. The divided parts are in equal sizes except for the parts in the triangular diagram. Newstead and Murray (1998) suggest that the generalisation to a triangle of the partitioning of a shape into four parts is due to the children’s limited concept of equal-partitioning. Steinle and Price (2008) also documented students representing $\frac{3}{4}$ using a circle which is divided into 4 parts with unequal areas (using 3 vertical lines) (Figure 2.13). The students recognise only familiar pictures or icons without paying attention to the essential features of equal areas of fractions.

Figure 2.13: Incorrect representation of $\frac{3}{4}$ (Steinle & Price, 2008)

Baturo (2004) glimpses into students’ part-whole fraction understanding using a variety of prototypic and nonprototypic representations of fractions as shown in Figure 2.14. Students normally represent fractions using circles and this is the prototypic representation of fractions. On the hand, the nonprototypic representations are the other shapes that are untypically used by students to represent fractions such as squares, rectangles, rhombuses, parallelogram and oval. The students were asked to select the shapes that had been divided into quarters. The students’ ability to recognise only the prototypical images was revealed when they accepted circles G and J although these circles were unequally partitioned. Students had a limited ability to recognise nonprototypical images when they simply reject other unfamiliar shapes such as a rhombus F although it was equally partitioned.
By focusing on partitioning a variety of prototypic and nonprototypic wholes, Baturo (2004) showed that the teacher was able to challenge the students’ understanding at a greater depth than before. This helped students to correct their mistakes made earlier with the representations of fractions and subsequently enhanced students’ mathematics learning outcomes.

There is doubt about using children’s informal knowledge of partitioning as a basis for fraction instruction. Simply assuming students are aware of the need of partitioning a whole into equal-parts without mentioning this explicitly in the approximate representations used by teachers may cause some students to overlook the important concept of equal partition requirement when dividing a unit into fractions (Olive & Vomvoridi, 2006). On the other hand, if teachers emphasise equal-parts and equal-whole as the underlying relations for the representations of fractions, perhaps students’ misconceptions could be avoided and resolved, or at least prevented from being exacerbated.

2.2.3 Using benchmarks

The strategy of using benchmarks can be used to compare sizes of fractions. When comparing two fractions, a third number is sometimes used because the third number may easily be seen to be bigger than one of the compared fractions and smaller than the other. This third number is called a benchmark (or a reference point). ‘0’, ‘$\frac{1}{2}$’ and ‘1’ are the ‘third numbers’ that are commonly used as benchmarks for comparing proper fractions (Behr et al., 1984; Smith III, 2002; Ministry of Education, 2010f). For example, when the benchmark of
is used to compare two fractions $\frac{8}{11}$ and $\frac{7}{15}$, which is more than $\frac{1}{2}$ is bigger than $\frac{7}{15}$ which is less than $\frac{1}{2}$.

In the NDP, benchmarks are promoted for comparing sizes of fractions at different strategy stages, which are Using Imaging and Using Number Properties. The following describes the visual display given to students in comparing $\frac{1}{2}$ and $\frac{3}{4}$ at the stage of Using Imaging (Ministry of Education, 2008d).

*Shielding:* Model comparing the size of fractions by placing some pizza pieces under paper plates. For example, place one-half under one plate and three-quarters under another. Label the plates by writing the symbols $\frac{1}{2}$ and $\frac{3}{4}$ on top. Ask the students which plate has the most pizza under it. Listen for the students to use benchmarks for comparison. Often these involve equivalent fractions, such as “Two-quarters is one-half, so three-quarters will have more” (p. 21).

An understanding of fractional parts is required at this stage to develop benchmarks for zero, half and one.

- Fractional parts are equal sized parts to make a whole or a unit. A whole or unit can be an object or a collection of things and is counted as “1”.
- Fractional parts are named by the number of equal parts that are needed to make a unit. For example, fifths require five equal parts to make a unit.
- The more fractional parts used to make a whole, the smaller the parts. For example, tenths are smaller than sixths.

(Ministry of Education, 2010f, para. 4)

Fractional parts are considered for comparisons of fractions. For example, $\frac{7}{8}$ and $\frac{5}{6}$. “$\frac{7}{8}$ is closer to one because eightths are smaller parts than sixths and $\frac{7}{8}$ is $\frac{1}{8}$ smaller than 1 and $\frac{5}{6}$ is $\frac{1}{6}$ smaller than one. As $\frac{1}{8}$ is smaller than $\frac{1}{6}$, $\frac{7}{8}$ is closer to one” (Ministry of Education, 2010f, para. 8).
When using benchmarks at the stage of Using Number Properties, students access only to their calculators rather than to anything visual. Students are expected to justify their calculation steps and encouraged to estimate and use leveraging from known benchmark fractions (Ministry of Education, 2008d). For example, $\frac{5}{11}$ is slightly less than one-half or 50% (which is equal to $\frac{5.5}{11}$).

According to Behr et al. (1984), applying this strategy requires generalised and abstract thinking and estimating skill. They observed different explanations specified by students when comparing the given fractions to a third number, in which “an amount or number of pieces was needed to complete a whole or attain a reference point” or “an amount or number of pieces in excess of a whole or beyond a reference point” (Behr et al., 1984, p. 328). For example, three quarters ($\frac{3}{4}$) needs one quarter to make a whole; two thirds ($\frac{2}{3}$) needs one thirds to make a whole. One third is bigger than one quarter, so three quarters is bigger than two thirds.

### 2.2.4 Finding common denominators

A strategy for comparing sizes of fractions included in the teaching of fractions is to convert fractions to fractions with a common denominator. A common denominator is the common multiple of the denominators of two or more fractions (Ministry of Education, 2008d, p.66). For example, 12 is the common denominator for fractions $\frac{1}{4}$ and $\frac{1}{6}$. Fractions with unlike denominators need to be converted to equivalent fractions with a common denominator. By finding the common denominator 12, the fraction $\frac{1}{4} = \frac{3}{12}$ and the fraction $\frac{1}{6} = \frac{2}{12}$ can now be compared. This method is used in the NDP instruction for the addition and subtraction of fractions with unlike denominators and finding fractions between two other fractions. Using common denominators requires students to generalise the number properties rather than rely on images of the materials (Ministry of Education, 2008d). In finding the common denominator 12 for fractions $\frac{1}{4}$ and $\frac{1}{6}$, the students needed to coordinate the numerators and
denominators of fractions: \( \frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12} ; \frac{1}{6} = \frac{1 \times 2}{6 \times 2} = \frac{2}{12} \). They compared equivalent fractions \( \frac{3}{12} \) and \( \frac{2}{12} \) in order to determine that \( \frac{1}{4} \) was bigger than \( \frac{1}{6} \) without referring to the image of fractions.

Percentages are equivalent fractions with a common denominator of 100. Students are encouraged to use the conversions of percentages to compare fractions. Converting fractions into percentages makes the comparison easier. This method is mostly used to compare fractions with unlike denominators (Ministry of Education, 2008d). Fractions \( \frac{1}{4} \) and \( \frac{1}{6} \) can be converted into decimals (\( \frac{1}{4} = 0.25 \) and \( \frac{1}{6} = 0.1667 \)) or percentages (\( \frac{1}{4} = 25\% \) and \( \frac{1}{6} = 16.67\% \)) for numerical comparisons.

Sharp and Adams (2002) propose two ways of developing the understanding of common denominators of fractions. Prior knowledge of whole numbers provides a base for children to establish meaning for the common denominator algorithm (i.e., the process to find a common denominator). The example of comparing \( \frac{1}{3} \) and \( \frac{1}{4} \) is used to show how the understanding of common denominators of fractions is developed based on whole number knowledge and diagrams. Fractions \( \frac{1}{3} \) and \( \frac{1}{4} \) can only be compared directly if they have the same denominator. Fraction \( \frac{1}{3} \) is multiplied by \( \frac{4}{4} \) to obtain the common denominator 12

\[ \left( \frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12} \right) \]. Similarly, fraction \( \frac{1}{4} \) is multiplied by \( \frac{3}{3} \) to obtain the common denominator 12

\[ \left( \frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12} \right) \]. So \( \frac{4}{12} > \frac{3}{12} = \frac{1}{4} \).

Students draw pictures to develop a common denominator algorithm (Sharp & Adams, 2002). The pictures drawn enable students to understand the mathematical situations and “literally counted on their pictures to get an answer” (Sharp & Adams, 2002, p. 345). By drawing a picture to find equivalent fractions, students are led to construct concepts rather than just memorising the formula for converting fractions into a common denominator.
Subdividing the area of a square using horizontal and vertical lines is a way to show simultaneously two superimposed fractions in creating equivalent fractions. For instance, to convert thirds into twelfths, start by modelling a third with horizontal lines and filling in the area that represents $\frac{1}{3}$. Then subdivide each fractional part into 4 parts using vertical lines (Figure 2.15). Examine the diagram to determine the number of subdivisions in the shaded region which becomes the new numerator. Write the equation that the model expresses which is $\frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12}$.

![Figure 2.15: Rectangular divided quantity diagrams representing equivalent fractions](image)

These skills are prerequisite for developing the common denominator procedure symbolically or technically. Figure 2.16 shows the conversions of $\frac{1}{3}$ and $\frac{1}{4}$ to $\frac{4}{12}$ and $\frac{3}{12}$, respectively. Four out of 12 parts are shaded in a) while 3 out of 12 parts are shaded in b). More parts are shaded in a) than b). Therefore, fraction $\frac{1}{3}$ is bigger than fraction $\frac{1}{4}$.
The dominance of procedural over conceptual aspects of student learning is a concern of Sophian and Madrid (2003b). The procedure of converting fractions to common denominators is preferred by students than the simpler but equally effective alternative strategies in working with different-denominator fractions. For example, fractions \( \frac{1}{5}, \frac{1}{4} \) and \( \frac{1}{3} \) are converted into the common denominator 60: \( \frac{1}{5} = \frac{1}{5} \times \frac{3}{3} \times \frac{4}{4} = \frac{12}{60}; \quad \frac{1}{4} = \frac{1}{4} \times \frac{3}{3} \times \frac{5}{5} = \frac{15}{60}; \quad \frac{1}{3} = \frac{1}{3} \times \frac{4}{4} \times \frac{5}{5} = \frac{20}{60} \). Hence, \( \frac{15}{60} \) is a fraction found between \( \frac{12}{60} \) and \( \frac{20}{60} \). Based on the understanding of the parts divided on the equal-size wholes, the fraction \( \frac{1}{4} \) is found between \( \frac{1}{5} \) and \( \frac{1}{3} \). Students’ conceptual understanding of fractions is important to facilitate their procedural learning of finding common denominators (Sophian & Madrid, 2003b). With limited understanding of the mental picture of fractions, students can rely on the common denominator method to compare fractions (Ward, 1999). Focusing only on common denominators causes students to neglect the possibility of drawing conclusions in other ways about magnitude relations between different-denominator fractions.

Consistent with Sophian and Madrid (2003b), Young-Loveridge et al. (2007) gave attention to the over-reliance of students on computational procedures in a task of adding two fractions. They found that students rely greatly on the complex computational procedure of finding a common denominator for the fractions without acquiring a deep conceptual understanding of fractions. Some of the students found a common denominator of 32 rather
than using the relationship of eighths to quarters when adding $\frac{3}{4}$ and $\frac{7}{8}$. Young-Loveridge et al. (2007) suggest that this may be due to the limited understanding of the students for comparing equivalent fractions using a common denominator. Young-Loveridge et al. (2007) also indicate the inappropriate use of the fraction language such as “three over four” rather than “three fourths or three quarters” reflecting students’ work is rule-based algorithms rather than meaningful fractional quantities computation.

This section has discussed students’ strategies for comparing sizes of fractions. The reasoning given by students to justify their methods of ordering fractions reflects their fractional thinking. Drawing divided quantity diagrams to compare fractions requires the basic knowledge of equal-whole and equal-partitioning whereas numerical conversions and using benchmarks involve number properties.

2.3 Towards a better understanding of fractions

A model for analysing the growth of mathematical understanding is referred to in this section to relate students’ strategies to understanding levels. Following this, the students’ strategies and sophistication levels and the continuities between novices and masters are discussed. In this context, ‘novices’ refers to students who have just learned fractions in the classroom; ‘masters’ refers to students whose reasoning of fractions is accurate, direct, flexible, and confident.

2.3.1 Pirie and Kieren’s model of the growth of mathematical understanding

Pirie and Kieren (1989, 1992, 1994a, 1994b) develop a model for observing and describing the growth of mathematical understanding. They define understanding as a whole, dynamic, levelled but non-linear, recursive process. The model contains eight embedded layers of understanding, which range outwards from Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing and Structuring to Inventising (Figure 2.17).
The first four levels are the inner modes of mathematical understanding which represent the core of context-dependent and, local know-how in mathematics. Formalising is the first formal mode of understanding which embeds these less formal understanding levels or modes. The subsequent, outer, formal modes of understanding (i.e., Observing and Structuring to Inventising) embed the formalising and thus embed the earlier, inner modes of informal understanding.

The following are the explanations and illustrations of the entire chain of layers from primitive knowing to inventising (Pirie & Kieren, 1989, 1992, 1994a, 1994b) with reference to learning sizes and order of fractions.

### 2.3.1.1 First level - Primitive Knowing

Primitive knowing is the starting place for students to grow any particular mathematical concept (Pirie & Kieren, 1989). For the growth of initial understanding of sizes of fractions, teachers assume that the students already know the language and construction of
individual fractions. These students have been introduced to the mathematical concept of a fraction as a number that can represent part of a whole: \( \frac{1}{2} \) is one part of two, \( \frac{2}{3} \) is two parts of three and so on. The part-whole concept assists students to construct meaning for fraction symbols. In the Numeracy Development Project (NDP), students identify symbols for halves, quarters, thirds and fifths before ordering fractions at Stage 5 (i.e., Early Additive) (Ministry of Education, 2008a). Students are introduced to the symbols and words related to models of fractions. For example, “the word one-half is recorded as well as the symbol \( \frac{1}{2} \)” (Ministry of Education, 2008c, p. 11). Students understand what the numbers represent in a fraction symbol in the ways that “the bottom number (denominator) indicates how many pieces make up the whole” and “the top number (numerator) tells us how many pieces there are. It counts the number of pieces.” (Ministry of Education, 2008c, p. 11). This involves an understanding of equal partitioning of sets and shapes.

### 2.3.1.2 Second level - Image-Making

The first recursion (i.e., mathematical operations) occurs at the second level of Image Making. The students form images out of the previous knowing and capability. Their actions may involve physical objects, figures, graphics or symbols. Specific images are made by the students to convey the meaning of mental image and these images are not limited to pictorial representations. As examples, divided quantity diagrams (refer to Section 2.2.2) are usually drawn by the students for the given, specific fractions in ordering fractions. Fractions \( \frac{1}{2} \) and \( \frac{2}{5} \) are represented in the following divided quantity diagrams (Figure 2.18):

![Figure 2.18: Rectangular divided quantity diagram](image-url)
Since these diagrams have an individual quality, the students’ effective action here is related to the images of specific fractions. To represent \( \frac{1}{2} \), the rectangle is divided into two equal parts and one part is shaded. To represent \( \frac{2}{5} \), the rectangle is divided into five equal parts and two parts are shaded.

### 2.3.1.3 Third level - Image-Having

The students develop more divided quantity diagrams until these action-tied specific images are generalized and replaced by a form for the images. A mental image is constructed and this involves the understanding level of Image Having. This is the first level of abstraction which can only be achieved by the students themselves by recursively building on images based in action.

When more diagrams were drawn to represent specific images at the previous level of image making, the students realise that the bigger the denominator the more equal parts are divided on a whole of circle or rectangle. At this level the students do not need to draw divided quantity diagrams to compare simple fractions. They have now replaced their actions on representing fractions physically with an image big denominators are equal to small fractions. They can order \( \frac{1}{3} \) smaller than \( \frac{1}{2} \) based on the constructed mental image.

### 2.3.1.4 Fourth level - Property Noticing

As the students talk about the mathematical concept as objects, they can examine and notice features or distinctions among the objects. This involves the fourth level of Property Noticing and the property is closely related to the mathematical images formed by the students earlier. The aspects of ones images can now be manipulated or combined to construct context specific and relevant properties.

In recognizing fractions \( \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \text{ and } \frac{7}{8} \) which are progressively closer to 1, the students examine the pictorial and mental images formed at the previous levels about these fractions. They notice the common features on the fractional parts in which each
fraction needs one part to make a whole. Fraction \( \frac{2}{3} \) needs \( \frac{1}{3} \) to make a whole, \( \frac{4}{5} \) needs \( \frac{1}{5} \) to make a whole, \( \frac{6}{7} \) needs \( \frac{1}{7} \) to make a whole, and so on. Based on these images which consider the gap between fractions and the whole, a context specific and relevant property of parts to make a whole is constructed.

2.3.1.5 Fifth level - Formalizing

In growing to the fifth level of Formalizing, the students abstract methods or commonalities from previous images to characterize the noticed properties. With a self-conscious thought, they formalize their understanding and transform the images of the concept into mathematical definitions or algorithms. Their understanding of the mathematical concept is not limited to visual representations but extended to classes of mathematical objects. They can solve the mathematical problems using only the number concepts and symbols related to the concept.

Converting fractions into a common denominator, decimals and percentages is a formal way to compare fractions. The students recognize the patterns of the singular images of fractions formed in the informal activities with particular fractions. These include drawing divided quantity diagrams and using the mental image big denominators are equal to small fractions. By characterizing the properties of these images using a formal character, they can now compare all fractions sizes using the symbols only.

2.3.1.6 Sixth level – Observing

The students move from a Formalizing to an Observing level of understanding when the formulas or the methods become pieces of a possible theory rather than techniques for computation used in the earlier level. The students reflect on and coordinate the formal activity involved in formalizing and express such coordination as conjectures for the concept. They can now look for patterns or formalisms in solving mathematical problems.

When the students want to know the truth for comparing all sizes of fractions, the numerical conversion or methods become pieces of a possible theory to them. Based on the
complexity of fractions, they can decide whether to compare fractions using a common denominator, percentages or decimals.

2.3.1.7 Seventh level - Structuring

In their attempts to think about formal observations as a theory, the students gain an understanding of Structuring. At this level the students observe their own thought structures, see the consequences of their thoughts and organise their thoughts consistently. They are aware of the inter-relationships of a collection of conjectures. They make statements for justification or verification through logical or meta-mathematical argument independent of physical or even algorithmic actions.

The students need to logically situate the consequences observed at the previous level. They can explain that finding a common denominator is simple for fractions that contain denominators as the factors of another denominator. For example, common denominator 12 is found for fractions $\frac{1}{3}$ and $\frac{5}{12}$ from multiplying denominator 3 by 4. On the other hand, finding a common denominator for fractions with big numbers involves complicated multiplication. Converting standard simple fractions into percentages is simple as their percentage equivalents can be recalled instantly. For example, $\frac{1}{2} = 50\%$ and $\frac{1}{4} = 25\%$.

However, long division is required to convert certain fractions into percentages. For example, $\frac{5}{12} = 41.66\%$. Considering the consequences of each strategy, the strategy that suits the specific numerical situation can be selected.

2.3.1.8 Eighth level - Inventising

Inventising is the highest level of understanding where the students create a new way of thinking about the concept. With a full structured understanding of the concept, they are able to initiate a sequence or structure of thought for the growth of a totally new concept.

In 1984, Behr et al. suggested the strategy of using benchmarks was invented by the students to determine the order and the equivalence of fractions. The students produced a new idea for comparing sizes of fractions by developing a third number as the benchmark. For
example, a half (i.e., \( \frac{1}{2} \)) was referred to when comparing three-ninths (i.e., \( \frac{3}{9} \)) and three-sixths (i.e., \( \frac{3}{6} \)). The fractions were compared with the third number (i.e., \( \frac{1}{2} \)) based on the way of comparison they had developed previously. They found three-ninths did not cover one half or less than one half while three-sixths was one half. Therefore, they decided three-ninths was less than three-sixths. Nevertheless, the benchmark strategy is not considered to be inventising in this thesis because this strategy has been introduced in the NDP classroom.

Walter and Gibbons (2010) suggest the behaviours of questioning ideas, reorganizing ideas, and validating ideas are only demonstrated by the students at the understanding layer for inventising. At this outermost layer, the students culminate their understanding. On the other hand, the behaviours that the students demonstrate most frequently at other layers are offering, building and explaining ideas.

Pirie and Kieren’s model of the growth of mathematical understanding is one of the new, more recent theoretical frameworks of mathematical understanding. Researchers have adopted the model for the analysis of students’ mathematical understanding by investigating students’ problem-solving behaviours (Walter & Gibbons, 2010) and the understanding pathway in making sense of fraction multiplication (Kyriakides, 2010).

### 2.3.2 Students’ strategies and sophistication levels

The above showed some of the strategies used by students for ordering fractions at different understanding levels. This section provides frameworks of students’ strategies that support the above discussion. According to Darr and Fisher (2006), Maguire et al., (2007) and Ministry of Education (2010d), students’ strategies can become increasingly sophisticated with practice (Figure 2.19). These strategies include attempting to use whole number knowledge, drawing pictures, identifying fractions with the same denominator or numerator, benchmarking fractions to well known fractions and using equivalent fractions (Table 2.1). In teaching fractions, emphasis can be given to a range of strategies used by the students for ordering and comparing fractions. The ability to use different strategies demonstrates the development of students’ understanding of fractions (Smith III, 2002; Darr & Fisher, 2006; Maguire et al., 2007; Ministry of Education, 2010d).
Figure 2.19: Strategies for comparing fractions (Darr & Fisher, 2006; Maguire et al., 2007)
### Table 2.1: Descriptions of the strategies for comparing fractions (Ministry of Education, 2010b, para. 4)

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misconceptions - Attempting to use whole number knowledge</td>
<td>Students will sometimes attempt to apply their knowledge of whole numbers to compare fractions. For instance, they will say 7/10 is greater than 4/5 as 7 is greater than 4. These students need to be challenged about the meaning of fractions. Using concrete materials can help them check their thinking.</td>
</tr>
<tr>
<td>Drawing pictures</td>
<td>Students will often draw pictures to compare the size of fractions. These pictures can be useful to support or explain their understanding. However, they can sometimes be misleading, especially when the fractions are close together in size.</td>
</tr>
</tbody>
</table>
| Identifying fractions with the same denominator or numerator | Students will often use their part-whole understandings of fractions to reason about fraction size. For instance, when a student understands that the size of the parts that make up a fraction depend on the size of the denominator, they can quickly see that fractions with the same denominator or numerator can be easily compared.  
  **Unit fractions**  
  Students can compare unit fractions by using their understanding of the meaning of the denominator. They can be aware that "the larger the denominator the smaller the fraction", because the whole is broken into more parts, therefore each must be smaller.  
  **Fractions with the same numerator**  
  Student comparing fractions with the same numerator is a derivation of comparing unit fractions. For example, if one unit fraction is larger than another then any common count of the parts will be larger, e.g., 3/4 is larger than 3/7 because 1/4 is larger than 1/7, and there is the same number of parts being compared.  
  **Fractions with the same denominator**  
  Students can compare fractions with the same denominator by comparing only the numerator. This is almost no different from comparing whole numbers as the numerator is simply the count of units. For instance, 5/7 is greater than 4/7, just because you have more of the same size part. |
| Benchmarking fractions to well known fractions | Students will sometimes use a familiar fraction as a benchmark or reference point when comparing fractions. For instance, they will note that 9/10 is greater than 7/8, as 9/10 is only 1/10 away from a whole, while 7/8 is 1/8 away.  
  A half can be a useful benchmark. Students often develop the understanding that a fraction is the same as 1/2 when the denominator is exactly twice the numerator. They can use this relationship to place a fraction in relation to a half and then make comparisons with other fractions. For instance, 2/3 is greater than 1/2, while 5/11 is less than a 1/2. This means 2/3 must be greater than 5/11. Some students who are able to identify when a fraction is greater or smaller than a half will not be able to tell you how close the fraction is to 1/2. Asking how far away a fraction is from 1/2 can be a useful follow up question to encourage further understanding. |
When comparing fractions, students will sometimes convert one or more of the fractions to an equivalent fraction with the same (common) denominator as the other fraction or fractions. For example, Which fraction is larger 3/4 or 1/2? Students may recognise that they can easily compare fractions if they both have the same denominator, and they may know that 2/4 is the same as (equivalent to) 1/2. Using this knowledge of equivalent fractions they can compare 3/4 and 2/4.

Example, Which fraction is larger 2/3 or 3/5? Students may look for the lowest common multiple, in this case 15, and rename the fractions equivalent fractions, so the question becomes: Which fraction is larger 10/15 or 9/15?

One common way to find equivalent fractions is work upwards from both fractions until a common denominator is found (trial and error). Another way is to recognise the relationship between both denominators and have a sense of what the denominator needs to be. Students then either extrapolate or cross multiply to work out what the numerator should be.

Note: cross multiply is to multiply the numerator of each (or one) side by the denominator of the other side. Mathematically, to cross multiply is to go from \( \frac{a}{b} \cdot \frac{c}{d} \) to \( ad = bc \).

Some students will also convert fractions to equivalent decimals and percentages to make comparisons. Students use many methods to convert fractions to equivalent fractions or decimals and percentages including cross-multiplying. These methods can often rely on learned processes that mask the students’ true understanding of fraction size.

The above strategies are learned in school and included in the NDP instruction to move students from Stage 4, Advanced Counting to Stage 5, Early Additive Part-whole. Students’ strategies are developed in the NDP teaching model for number through the phases of Using Materials, Using Imaging, and Using Number Properties. Students’ strategies have been investigated in research focusing on fourth graders (Behr et al., 1984), Year 6 students (Gould, 2005) and upper elementary, middle, and high school students (Smith III, 2002). Their findings about students’ strategies are consistent with the strategies developed in the NDP.
In the activity of ordering fractions from the smallest to the largest, students use equipment such as paper circles or commercial fraction kits in the phase of Using Materials. The students who achieve success proceed to the next phase of Using Imaging. They think about fraction sizes and image the way they have compared them using manipulatives at the previous phase to help them anticipate the ordering of fractions. The students often draw a picture to show their ideas or explain the comparison using benchmarks which often involve equivalent fractions. For example, “Two-quarters is one-half, so three-quarters will be more than two-quarters” (Ministry of Education, 2008d, p. 21). The use of pictures and benchmarks as one of the students’ strategies is also documented by Behr et al. (1984), Smith III (2002) and Gould (2005). The students use manipulative aids including pictures, area model or divided quantity diagrams. They use benchmarks such as a half. As an example, two fifths (\(\frac{2}{5}\)) is less than two thirds (\(\frac{2}{3}\)) because two fifths is less than one half while two thirds is more than one half. When comparing fractions with unlike denominators, students refer to denominator only (Behr et al., 1984) and consider bigger denominator is smaller fraction (Gould, 2005). For example, one fifth (\(\frac{1}{5}\)) is less than one thirds (\(\frac{1}{3}\)) because the bigger the denominator is, the smaller the pieces get.

The strategy of “numerator and denominator” was noticed by Behr et al. (1984), Smith III (2002) and Gould (2005) where the students referred to both the numerators and the denominators of the fractions. When comparing fractions with the same numerators, “the child referred to both the numerators and the denominators, indicating that the same number of parts was present (the numerators) but that the fraction with the larger (or largest) denominator had the smaller (or smallest) sized parts” (Behr et al., 1984, p. 327). When comparing fractions with the same denominators, “the child referred to both the numerators and the denominators, indicating that the size of the parts was the same (the denominators) but that there were more parts (the numerators)” (Behr et al., 1984, p. 329). An example is two fifths (\(\frac{2}{5}\)) is less than two thirds (\(\frac{2}{3}\)) because two pieces are taken in each, but the pieces in fifths are smaller than the pieces in thirds. The students who use the numerator and denominator strategy based their thinking on a mental image of their experience with manipulative aids, which can refer to the inverse compensating relation between the number of equal parts of a whole and their size (Behr et al., 1984).
In the next phase of *Using Number Properties*, the students have a good understanding of coordinating the numerator and denominator of fractions without using materials or images to make comparisons. Their justifications relate to relationships within the symbols (Ministry of Education, 2008c). In finding a common denominator, 
\[
\frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12}, \quad \frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12}.
\]
Therefore, \(\frac{1}{4} < \frac{1}{3}\). In converting fractions into percentages, \(\frac{1}{3} = 33.33\%\) and \(\frac{1}{4} = 25\%\). In converting fractions into decimals, \(\frac{1}{3} = 0.3333\) and \(\frac{1}{4} = 0.25\). Smith III (2002) and Gould (2005) found only students from Year 6 and above used conversion to common denominator, decimals or percentages.

Apart from Darr and Fisher (2006), Maguire et al. (2007) and Ministry of Education (2010d), the investigation of Stafylidou and Vosniadou (2004) is also related to students’ understanding levels of fractions. Stafylidou and Vosniadou (2004) asked students to compare and order fractions and justify their responses. They revealed three main constructs from students’ responses: fractions as two independent numbers; fractions as parts of a whole; and fractions as relations between two numbers.

The first construct (i.e., two independent numbers) has emerged from students’ prior knowledge of whole numbers and the value of fractions increases according to the increase of the value of the whole numbers involved in the fractions. This is directly related to the presupposition the unit is the smallest of all fractions, which is consistent with the base-theory of natural numbers. Stafylidou and Vosniadou (2004) noticed another contradictory belief that emerged from whole number knowledge is the value of the fraction increases as the value of the numerator (or denominator) decreases. This is directly related to the presupposition the unit is bigger than all fractions, which is created as a transitional phase in the process of understanding fractions in terms of a unit divided into parts. It could be argued that the more parts the less value a fraction has.

In the second construct (i.e., part of a whole), the numerator and the denominator of a fraction are seen as two numbers connected with a relationship of part-whole of the unit and the unit is the biggest fraction. Only the third explanatory framework (i.e., relations between numerator and denominator) involves the understanding of the relation between numerator
and denominator and the infinite value of fractions which can be smaller, equal or bigger than the unit.

A consistency is found between the strategies identified by Darr et al. (etc) and students’ understanding of fractions documented by Stafylidou and Vosniadou (2004) (Table 2.2). This includes:

- The misconception of attempting to use whole number knowledge involves the understanding of fractions as two independent numbers.
- The strategies of drawing diagrams and identifying factors with the same denominator or numerator require the part-whole understanding of fractions.
- Benchmarking fractions to well known fractions and using equivalent fractions imply the understanding of fractions as the relation between numerators and denominators.

|---|---|
| **Misconceptions - Attempting to use whole number knowledge** | Two independent numbers
The value of fractions increases when the value of the numerator (or denominator) increases or when the numbers that comprise it (either the numerator or the denominator) increase.
The value of the fraction increases as the value of the numerator (or denominator) decreases or when the numbers that comprise it (either the numerator or the denominator) decrease. |
| **Drawing pictures**<br>**Identifying factors with the same denominator or numerator** | Part of a whole
A fraction always represents a quantity smaller than the unit.
A fraction with a numerator equal to the denominator is equivalent to the unit. |
| **Benchmarking fractions to well known fractions**<br>**Using equivalent fractions** | Relations between numerator/denominator
A fraction is bigger than the unit.
Fractions are not seen as two separate numbers, but rather a quantity obtained from the quotient relation between two numbers. |
2.3.3 Continuities between novices and masters

This section provides empirical data of students’ strategies that are framed in terms of increasing sophistication by showing the strategies of novices and masters. The shared characteristics of the knowledge system of novices and masters, and reconceptualisation of students’ misconceptions are also discussed.

2.3.3.1 Strategies of novices and masters

Smith III, diSessa and Roschelle (1993) investigated the knowledge and reasoning of novices and masters on tasks involving order and equivalence relations between fractions. Ten upper elementary students were taken as novices, and three middle school and eight senior high students represented masters. A diversity of strategy application was noticed from the reasoning of both masters and novices (averaging of 7 different strategies for both). Masters use specific tools that are well suited to particular numerical situations, while novices are restricted to the manipulation of a mental model of divided quantity. Neither group’s reasoning heavily relied on the strategies of conversion to common denominator or conversion to decimals. The numerical conversion methods involve application of whole-number or decimal knowledge which is sufficient to solve all kinds of fraction problems and hence heavily emphasized in textbooks and instruction. In contrast, students tend to use strategies that are not explicitly taught but tailored for solving specific classes of problems.

A similar finding can be seen in Behr, Wachsmuth and Post (1985) when investigating students’ cognitive strategies in constructing a two-addend rational number sum close to 1. High performers use a flexible and spontaneous application of fraction order and equivalence concepts and benchmarks, whereas low performers tend to apply concepts in a constrained or inaccurate manner. High performers carry out algorithmic procedures including common denominators and one of them was able to compute the standard algorithm mentally to determine the sum of the generated fractions. This requires the ability to store a long sequence of memory units and manipulate symbols mentally. An incorrect algorithm was detected from the work of a low performer, who actually preferred to work on an algorithm or rule that could be used to obtain answers than to work with the manipulative aids provided.
2.3.3.2 Shared characteristics of the knowledge system of novices and masters

Smith III et al. (1993) discuss the shared characteristics of the knowledge system of novices and masters. When comparing $\frac{12}{24}$ and $\frac{8}{16}$, masters and novices both use the half of relationship within the components of fractions to affirm their equivalence. Both the numerators 12 and 8 are half of the denominators, 24 and 16, respectively. The masters and novices generalized this strategy to establish the equivalence of other fractions:

- Fractions $\frac{1}{4}$ and $\frac{6}{24}$ are equivalent because the numerators 1 and 6 are fourth of denominators 4 and 24, respectively.
- Fractions $\frac{4}{6}$ and $\frac{6}{9}$ are equivalent because the numerators 4 and 6 are two thirds of denominators 6 and 9, respectively.

The differences between the two groups are that novices provide reasoning based on the manipulation of a mental model of divided quantity whereas masters provide reasoning based on numerical relationships. Novices used the part-whole model (i.e., a fractional quantity is formed by selecting the parts out of the equally divided whole) to support the use of a variety of specific strategies. On the other hand, master’s reasoning involves three types of numerical relationships (Smith III et al., 1993, p. 135) to some extent:
(a) relations within and between the whole-number components,
(b) numerical benchmarks, or
(c) numerical conversions such as common denominator and decimal.

Benchmarks are generally used by masters in solving problems with fraction order and equivalence (Behr et al., 1985). They refer proper fractions to relevant points such as 0, $\frac{1}{2}$, and 1 to determine whether the fractions are bigger or smaller than the benchmark. The numerical relations within and between the whole number components are considered in determining that fractions like $\frac{4}{5}$ are close to 1. The strategies used by masters often have limited application to specific fractions, yet produce rapid and reliable solutions. Until numerical situations without an obvious relationship are encountered, masters will avoid time consuming computations of converting fractions into a common denominator or decimals.
The numerical strategies of masters are more accurate, direct and flexible yet connected to the divided quantity diagrams drawn by novices. As shown in Figure 2.15 and Figure 2.16 (p. 23 and p. 24), diagrams drawn to create equivalent fractions establish meaning to the development of the common denominator strategy. The underlying divided quantity knowledge of numerical strategies becomes visible when masters justify their numerical reasoning in terms of divided quantity. For instance, they explained that “the quantities expressed by the fractions were invariant under the subdivisions of their parts” (Smith III et al., 1993, p. 135) for the conversion of fractions into a common denominator. The continuity between novices and masters is represented by the changing role of divided quantity knowledge. Novice’s model of part-whole fractions provides a foundation to develop numerical relationships and operations of fractions by masters. The prior knowledge of divided quantity is retained and reused in learning general strategies included in the curriculum.

2.3.3.3 Reconceptualisation of students’ misconceptions of fractions

Fractions are different from natural numbers not only in symbolic representation but also in the way they are ordered. Fractions are not increasing in the way that natural numbers are and the values of fractions may become smaller when they involve bigger natural numbers. “For students, fractions represent a significant shift in their thinking from working with ones and groupings of one to working with parts of one whole” (Ministry of Education, 2008d, p.5). Previous research studies have shown that there is interference between the conventional counting system and the conceptual development of fractions (Stafylidou & Vosniadou, 2004; Ni & Zhou, 2005).

The difficulty most students experience in understanding fractions is due to the conflict between new information and prior whole number knowledge (Stafylidou & Vosniadou, 2004; Ni & Zhou, 2005). Misconceptions are inevitably generated in the process of assimilating new information to existing knowledge (Vosniadou, 2003). Inappropriate use of children’s prior knowledge about whole number in the instruction leads to persistent misconceptions of elementary students on fractions (Ni & Zhou, 2005). According to Stafylidou and Vosniadou (2004), the presuppositions of natural numbers not only inhibit students’ acquisition of fraction knowledge but also cause students’ systematic
misconceptions. They found two almost contradictory beliefs developed by the students in the interpretation of fractions as consisting of two independent numbers.

- The value of fractions increases when the value of the numerator (or denominator) increases or when the numbers that comprise them (either the numerator or the denominator) increase. This is directly related to the presupposition the unit is the smallest of all fractions, which is consistent with the base-theory of natural numbers.

- The value of the fraction increases as the value of the numerator (or denominator) decreases or when the numbers that comprise it (either the numerator or the denominator) decrease. This is directly related to the presupposition the unit is bigger than all fractions, which is created as a transitional phase in the process of understanding fractions in terms of a unit divided into parts and it could be argued that the more parts the less value a fraction has.

In addition, whole number dominance is noted as an invalid strategy that is frequently used by children to compare fractions (Behr et al., 1984; Newstead & Murray, 1998). Children consider the numerators and the denominators separately for comparing fractions by the method of the ordering of whole numbers. For example, three fifths (i.e., $\frac{3}{5}$) is less than six tenths (i.e., $\frac{6}{10}$) because “3 is less than 6, and 5 is less than 10.” This dominance is noted when misapplications or overgeneralisations of whole-number knowledge cause children to be unable to see a fraction as a quantity (i.e., a number). Instead, they see it as two separate and unrelated whole numbers in the counting strategy (Newstead & Murray, 1998; Empson, 1999; Miura, Okamoto, Vlahovic-Stetic, Kim & Han, 1999; Paik & Mix, 2003; Olive & Vomvoridi, 2006).

The students’ misconceptions are generally viewed as students’ mistakes. As such, misconceptions’ research has consistently focused on overcoming flawed ideas of students until the role of novices’ conceptions, including common misconceptions, were reviewed and reassessed by Smith III et al. (1993). They reconceptualise students’ misconceptions by highlighting the useful and productive nature, as well as their limitations, in science and
mathematics learning. They critique that “casting misconceptions as mistakes is too narrow a view of their role in learning” (p. 151). Instead, they recognise the validity of misconceptions as preconceptions that can be extended in an instructional context as how the novice knowledge is converted into expert reasoning. “Misconceptions are faulty extensions of productive prior knowledge” (p. 152) which can lead to useful applications, despite erroneous conclusions (Smith III et al., 1993). As such, some misconceptions can be changed with appropriately designed interventions and provision of plausible alternatives. Else, they are viewed as part of a conceptual system comprising many useful elements that can be productively engaged and developed. Therefore, replacing misconceptions by adding new expert knowledge and eliminating flawed ideas oversimplifies the conceptual changes required in learning complex subject matter. Instead of confronting the misconceptions of students, instruction can provide the experiential basis to support the gradual processes of conceptual change.

Smith III et al. (1993) noted important continuities between flawed ideas of students and concepts of experts as students build more advanced knowledge from prior understanding based on the basic premise of constructivism. The interrelationships among diverse knowledge are characterised while the understanding students bring to instruction involves a deep and complex change in acquiring expertise. In the process of learning, students construct their understanding by refining and reorganizing knowledge rather than simply replacing misconceptions that have been viewed as mistakes which impede learning.

In summary, the development of students’ understanding of fractions is shown in the continuities of fraction concepts from novices to masters which highlights the growth of mathematical understanding. Novices’ misconceptions can be changed and converted into masters’ concepts. Novices apply concepts in a constrained or inaccurate manner whereas masters apply fractions flexibly and spontaneously. There are shared characteristics of the knowledge system of novices and masters as detected from the use of the same strategy between them. Frameworks of students’ strategies are found to be helpful in providing instruction for particular students at particular stages in their development.

Summary

This chapter focuses on the knowledge and strategies required for understanding fraction’s sizes. The part-whole construct is fundamental to learning this. Divided quantity
diagrams showing the part-whole relation explicitly are commonly used to represent sizes of fractions. Students can use various strategies to compare fractions including drawing divided quantity diagrams, finding a common denominator and using benchmarks. The strategies used involve different levels of mathematical understanding. Novices depend directly on their mental model of divided quantity and their strategies involved lower levels of understanding. Masters are able to use number properties by coordinating the numerator and denominator of fractions and their strategies involve higher levels of understanding. It is shown that frameworks of students’ strategies which describe students’ mathematical understanding are useful for teachers in the instruction.
CHAPTER 3

COMPUTER GAMES

This chapter looks at the literature surrounding the various aspects of computer games. Section 3.1 discusses previous definitions of computer games by other researchers in order to determine a definition that will be used in this study. Section 3.2 explains the characteristics of computer games that make games engaging and appropriate for learning. Section 3.3 investigates important factors in developing computer games for learning mathematics, particularly with regard to fractions.

3.1 Definition of computer games

Based on previous definitions given by other researchers on computer games, this section intends to identify the basic components used in defining computer games and provide the definition of computer games for this study. In the literature on computer games, there are no standardised terms and different definitions are used. The terms computer games; video games; digital games; and even educational games are used interchangeably (Dempsey, Haynes, Lucassen & Casey, 2002; Asgari & Kaufman, 2004; Hays, 2005; Panagiotakopoulos, 2011). Table 3.1 shows the different definitions adopted by researchers within the literature regarding computer games.

<table>
<thead>
<tr>
<th>Source</th>
<th>Definitions of computer games</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dempsey, Lucassen, Haynes and Casey (1996) and Dempsey et al. (2002)</td>
<td>“A game is a set of activities involving one or more players. It has goals, constraints, payoffs and consequences. A game is rule-oriented and artificial in some respects. Finally a game involves some aspects of competition, even if that is competition with oneself.’ (p. 2, p.159)</td>
</tr>
<tr>
<td>Juul (2003)</td>
<td>“A game is a rule-based formal system with a variable and quantifiable outcome, where different outcomes are assigned different values, the player exerts effort in order to influence the outcome, the player feels attached to the outcome, and the consequences of the activity are optional and negotiable.” (para. 15)</td>
</tr>
</tbody>
</table>
Asgari and Kaufman (2004) “A *game* as a set of voluntary activities which has participants, goals, rules, and some kind of competition (physically or mentally)” (p.2).

Beasley (2004) For something to be a game, it must:  
• Be played for fun;  
• Contain a challenge or competition;  
• Have a goal;  
• Have rules;  
• Have an outcome. (p. 42)

Hays (2005) “A game is an artificially constructed, competitive activity with a specific goal, a set of rules and constraints that is located in a specific context” (p. 15).

Balasubramanian, and Wilson (2006) “We define a game as an engaging interactive learning environment that captivates a player by offering challenges that require increasing levels of mastery” (p. 2).

More commonalities than differences are found in these definitions and the basic components used in defining computer games can be identified. Since these definitions describe the different aspects of games and express things differently, modification or clarification on their similarities is needed. For example, some of the definitions described the game as an activity while some referred to a game as a system or learning environment. Although one writer states “rules” and another states “constraints”, both terms actually mean rules. Table 3.2 summarises the key defining characteristics of computer games: game rules (or constraints), goals and objectives, outcomes and feedback, challenge (or competition), conflict and interaction (or activity).

<table>
<thead>
<tr>
<th>Source</th>
<th>Game rules/constraints</th>
<th>goals and objectives</th>
<th>outcomes and feedback</th>
<th>competition/challenge</th>
<th>Interaction/activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Juul (2003)</td>
<td>√</td>
<td>√</td>
<td></td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Beasley (2004)</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Hays (2005)</td>
<td>√</td>
<td>√</td>
<td></td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Balasubramanian and Wilson (2006)</td>
<td>√</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Among the definitions found in the literature, only Beasley (2004) includes the concept of fun with the reason that it is the motivation for playing games. In fact, Dempsey et al. (2002) also promote fun as a possible motivation for game playing even though this has not been mentioned in their definition of computer games.

Prensky (2001a, 2007) relates fun and play in games to learning. He explains “fun - in the sense of enjoyment and pleasure – puts us in a relaxed, receptive frame of mind for learning” (Prensky, 2001a, p. 10) and “play, in addition to providing pleasure, increases our involvement, which also helps us learn” (Prensky, 2001a, p. 10). He suggests that games provide “a more formal and structured way to harness and unleash all the power of fun and play in the learning process” (Prensky, 2001a, p. 10). Consistently, Quinn (1994, 1997) points out games benefit educational practice and learning by combining fun elements with aspects of instructional design and system design such as motivational, learning and interactive components. Hard fun is gained without disputing the effort and energy involved in learning especially when learning really hard subjects that are not intrinsically motivating to anyone. Learning multiplication tables, typing and spelling are examples of this. These subjects are often called dry and technical and learning these subjects is unavoidably boring, but necessary to be learned (Prensky, 2001a, 2001c).

Based on the above discussion, the following definition of computer game will be used throughout the remained of this study.

*A computer game is an interactive type of fun play using a computer that is governed by a set of rules where players aim to win the game by achieving specific goals and outcomes under certain limitations such as time, space and opponents.*

In addition to the above definition, the following is required if the computer game is used for learning.

*Subject matter is integrated with the game where the players use particular knowledge to achieve content-specific goals and outcomes in order to win the game.*

### 3.2 Computer games for learning

This section discusses new learning in new technology specifically on students as digital natives and learning through playing and designing computer games. This section considers key design of educational computer games. The six key elements of educational computer games proposed by Prensky (2001a) are selected and elaborated in detail. These are
rules, goals and objectives, outcomes and feedback, conflict/competition/challenge/opposition, interaction, and representation or story.

3.2.1 New technology, new learning

The advancement of digital technology has changed and enhanced people’s lives in various ways including education. In response to technological change, students, the tools, and the requisite skills and knowledge, have changed too (Prensky, 2001b, 2006). Young people today who are natives of technology, fluent in the digital language of computers and the Internet are referred to as digital natives; while their parents who adopt many aspects of the technology later in life are referred as digital immigrants (Prensky, 2006). Having grown up in different media cultures and experiences (Fromme, 2003), natives and immigrants have different intellectual styles and preferences (Prensky, 2001b). Children today are more media literate; they are able to access, understand and create communications in a variety of contexts (Buckingham, 2005). They are connected to the entire world through digital media and a myriad of personal devices such as the Internet and cell phones (Prensky, 2008b). Most importantly, they become active participants through communicating (instant messaging, e-mails), sharing (blogs), searching (Google), coordinating (wikis), collecting (downloads), socializing (chat rooms, social networking such as facebook and twitter), buying and selling (eBay) and even learning (Web surfing) (Prensky, 2006).

In playing online games, players take up a significant role of ‘creating’ the texts they play through message boards and chat facilities. Games are effective learning tools in the flexible settings or student-centred world where children are allowed to play in guided after-school settings, or on their own (Prensky, 2008a). The computer gaming environments provide role play experiences which engage players in deep learning as described in the following (Lim, 2008):

Based on the information they have collected and the instantaneity of the feedback, players analyse, synthesise, and use critical thinking skills to play and execute moves to appropriate the roles the players enact in a game. Various features and key characters in the games provide scaffolds for players to learn and grow within these roles. The scaffolds may direct players’ attention to visual cues, promote their knowledge integration, and guide them to elaborate upon their thinking.

As players progress in the game, they become more knowledgeable and skilful, and hence, more equipped to take on challenges at higher levels.

(p. 1000)
The participatory and ‘hands-on’ pedagogical approaches are more motivating to students than the traditional approaches based solely on discussion and writing (Buckingham, 2005). Students are involved not only as users but as designers of games for learning (Kafai, 2006; Prensky, 2008a). Many powerful yet easy to use game-building tools have been created for young people to build simple computer games such as Game Maker, Click & Play and Stagecast Creator. The games can be upgraded into complex and sophisticated levels using Flash, modding tools (i.e., modification making tools are used to add content to computer games such as new items, weapons, characters, enemies, levels, story lines, music, and game modes), C++, game engines and graphics tools. These tools allow college and high school students and even primary and intermediate school children to create more games in organized programs including good educational games. Game creation provides a truly engaging learning environment in school through the process of modelling, designing, testing and meta-learning in between and within phases (Prensky, 2008a).

The cultural aspect of teachers’ perceptions of teaching and technologies is indicated by Lim (2008) as one of the practical problems in the failure of computer games as educational tools in schools. For instance, the paradigm of encouraging students learning on their own with guidance using computer games is in contrast to the prevailing teacher lecturing culture of schools. In fact, “game-play disturbs traditional distinctions between producers and consumers” (Buckingham, 2005, p. 28). Nevertheless, computer games do provide benefits to students. For example, research shows a positive effect of computer games-based learning that supports active participation by students’ in their achievement and motivation (Prensky, 2001a; Fromme, 2003; Van Eck, 2007). Digital game based learning not only requires mental effort and focus, but also engages students in constant iterative cycles of thought, action and feedback, hence inspiring dreams of deeper learning and greater roles in learning environments (Van Eck, 2007). In addition, playing computer games actively and critically accompanies the children’s process of growing up and developing their cognitive, social and physical abilities (Fromme, 2003).

3.2.2 Key design of educational computer games

There has been much literature discussing the use of educational computer games. Most of them describe the structural components and essential elements of games (e.g., goals and rules) and concern the instructional values of the games. Hays (2005) provides the
empirical data on the effectiveness of instructional games. In order to meaningfully integrate games into classrooms, Balasubramanian and Wilson (2006) propose guidelines that can inform the design of educational games. Asgari and Kaufman (2004) recommend guidelines for creating effective computer games especially fantasy games that enhance both motivation and learning. Far fewer researchers have proposed details regarding how to design computer games that provide instructional values.

Prensky (2001a, 2007) presents six key structural elements of game which have been used and adopted in many research studying educational computer games. These elements are rules, goals and objectives, outcomes and feedback, conflict/competition/challenge/opposition, interaction and, representation or story. In research conducted by others, the six game elements of Prensky are referred to when converting an existing training program into a digital game-based learning format (Floeter, 2009), developing a framework for assessing the quality of mobile learning (Parsons & Ryu, 2006) and constructing a game evaluation index for assessing the educational values of digital games (Hong, Cheng, Hwang, Lee & Chang, 2009).

Prensky is an expert in the use of games in education and he has created over 50 software games for learning (e.g., Fraction Junction). He highlights high learning which can be achieved though high engagement in digital-game-based learning rather than using innovative technologies to deliver the same boring content and same old fashioned strategy like traditional education. Through the six game elements, Prensky (2001a) promotes games that engage students in learning. These elements are in line with Prensky’s (2007) ‘rules of engagement’ that have been extracted from the best and most engaging commercial games for designing educational computer games (or any kind of learning) or selecting engaging classroom games. The engaging factors of educational games include goals, decision and discussion, emotional connection, cooperation and competition, personalization, review and iteration, and fun.

Many papers investigate game characteristics from students’ perspectives and the findings have confirmed the key structural elements of Prensky. Rieber, Davis, Matzko and Grant (2001) document the important game characteristics identified by children such as storyline or context, challenge and competitive affordances. Saxton and Upitis (1995) reveal the components identified by adolescents such as sounds, changing speed, detailed graphics, action, problem-solving and different levels as features of good computer games. Sedighian
and Sedighian (1996) found eight elements of computer games that satisfy children’s learning needs and motivate them to learn mathematics, namely meaningful learning, a goal, success, a challenge, cognitive artefact (man-made things that aid cognitive capabilities such as to-do-lists), association through pleasure, attraction and sensory stimuli.

This study referred to Prensky’s game elements in designing the computer game of *Tower Trap* to ensure the game was engaging and well suited for learning. Research findings from other researchers were used to validate each Prensky’s game element as discussed in the following.

### 3.2.2.1 Rules

As a rule-based formal system, a game is governed by a set of rules and has a clear underlying structure (Oldfield, 1991; Juul, 2003; Hays, 2005). The rules make games become organized play. For computer games, the rules are built right into the game. This is unlike other non-computer games where the rules are written down and managed by the players or a third party. The rules of games have to be sufficiently well defined that the player does not have to argue about the rules when playing the games. Nevertheless, the player accumulates the knowledge of how to play the game through observation and active participation in the gaming process rather than solely reading rules and instructions (Dempsey et al., 2002). Rules impose limits on the game so that all players take specific paths to reach goals inside the game world and each player has the same chance of winning the game. This makes games both fair and exciting (Prensky, 2001a). As described by the computer game players, “games are fair if they don’t kill you off without giving you a chance and they don’t require resources you cannot get (although surviving or finding the resources may not be easy)” (Prensky, 2001a, p. 30).

### 3.2.2.2 Goals and objectives

“Educational games are fun but purposeful” (Brandyberry & Pardue, 2001, p. 110). Goals turn free play into games and goal-oriented learning occurs in educational games. Achieving goals motivates players to go through repeated failure until winning the game (Neal, 1990). Normally the goals are clearly stated right at the beginning before the game is started. The goals can be to get the highest score, to get to the end, to beat the enemy, to save the princess, to collect gold, etc (Prensky, 2001a).
Mathematics games contain specific mathematical cognitive objectives for students to achieve (Oldfield, 1991; Booker, 2004). Integrating mathematical content with game ideas, mathematics games are played not simply for fun, but for the motivation and engagement of students in the development of mathematical concepts and procedures. An interesting finding about using games to promote mathematical learning was found by Bragg (2007) that for some students, “the fun derived from the game was in the challenge of learning a concept, not in the challenge of beating an opponent” (p.39). The aspiration to win the game encourages students to develop strategies for winning, including learning new mathematical concepts.

3.2.2.3 Outcomes and feedback

In the classic games that have a win-lose state or at least a goal state, outcomes and feedback are used as a measurement of progress against the goals (Prensky, 2001a). Juul (2003) explains that the rules of the game must provide different possible outcomes (e.g., win or lose) to make something work as a game. Since variable outcomes always depend on who plays the game, the game is commonly created to fit the skills of the players. This is a subjective aspect to the game as a game remains interesting to the player if the plays are mentally challenging. The features for ensuring a variable outcome can be found in fighting games and car racing games. On the other hand, quantifiable outcomes consider the points or scores obtained from playing the game. Players feel attached to the outcomes which are determined by their efforts (Juul, 2003). The consequences of the game that are optional and negotiable always stimulate players’ curiosity and make them care about what happens next (Juul, 2003). The outcome of either winning or losing has strong emotional and ego-gratification implications, and such an emotional connection makes for stronger learning (Prensky, 2001a, 2007).

The immediate feedback given to players enables them to know their performance and hopefully improve later. This makes computer games interactive (Prensky, 2001a, 2007). Feedback informs players whether their act is positive or negative, whether they are staying within or breaking the rules, moving closer to the goal or further away. Reimer and Moyer (2005) found that immediate and specific feedback provided in written form on computers helps students learn more about fractions. Such feedback corrects and highlights students’ errors, making them more aware of their misconceptions of fractions. Knowing whether what they did was wrong or right, and being able to try again engages students in the game (Prensky, 2001a, 2007). Van Eck (2006) advocates the use of advisement in computer-
mediated lessons to coach students who need additional help in understanding complex concepts. Students increase their performance in computer-based lessons using learner control with advisement (Johansen & Tennyson, 1983). However, it is problematic to get students to use advisement even though some advisement strategies are effective to remind the learners regarding important issues during instruction (Van Eck, 2006). The game designer needs to consider the placement and ease of use of the advisor so that more passive learners can access the advisor easily when needed (Lee & Lehman, 1993).

Feedback is given in a variety of forms through numerical scores, graphics and spoken messages. Interestingly, computer technology allows feedback through other senses as well. An example is the tactile rumble felt in “force feedback” joysticks or other controllers when played (figuratively or literally) on bumpy ground in the game world. Feedback can be spectacularly dramatic (e.g., crash landings, whole galaxies blowing up) or amusing (e.g., the response given out by a character called Jim Raynor in StarCraft (i.e., a military science fiction real-time strategy video game) “What’s your problem, man?” when he is repeatedly selected by the player in the game) (Prensky, 2001a).

3.2.2.4 Conflict/Competition/Challenge/Opposition

Value is assigned to possible outcomes of the game to create conflict and challenge in the game. If positive outcomes are harder to reach than negative outcomes, the game becomes challenging to the player. The levels of effort given by the player determine how “conflicting”, “challenging” and “interactive” the game is (Juul, 2003). Despite the fact that it is the problem that players are trying to solve in a game, conflict, competition, challenge and opposition give excitement to the game (Prensky, 2001a). “Balancing” the game is a key skill in game design by keeping the level of conflict/competition/challenge or opposition in synchronisation with the player’s skills and progress. Play by luck with no control of outcomes makes the game meaningless to players. A balance needs to be sought between skills and luck in the game by involving adequate skills to make the game challenging and providing enough luck to give everyone a chance (Parsons, 2008). “Good game design is balanced. Balance leaves the player feeling that the game is challenging but fair, and neither too hard nor too easy at any point.” (Prensky, 2001a, p. 23).

Difficulty is an important aspect of games. This means that if the game is too easy to win, the player will become bored and unmotivated to complete the goals (Miller, 1998).
Games should be made challenging enough to be interesting (Juul, 2003), but not so challenging as to be impossible and too hard to continue (Csikszentmihalyi, 1990). Giving students challenges at just the right level of difficulty provides an ideal learning state (Csikszentmihalyi, 1990). This is called the zone of flow. Craig, Graesser, Sullins and Gholson (2004) point to a link between learning and the affective states of confusion, flow and boredom. The affective state of confusion and flow needs to be positive in the learning process. In the context of their paper, confusion refers to the situation where students are struggling to understand the material and flow is the situation where students are paying attention and responding quickly. Both confusion and flow will get students involved in deep learning while playing games. On the other hand, learning gains are negatively correlated with boredom when students are not interested in the activity or respond slowly to the system and do not appear motivated.

3.2.2.5 Interaction

Two kinds of interaction exist in playing computer games. The basic interaction occurs between the player and the computer through visualization, manipulation of objects and the immediate feedback given following the players’ acts (Leutner, 1993; Amory, Naicker, Vincent & Adams, 1999). Rieber et al. (2001) found children care less about game’s production values such as the high-quality graphics and sound of commercial video games. They indicate this finding is different from the game characteristics emphasized in the literature. Instead, the children appreciate interesting interactive computer games even though these games have the amateur-like quality.

The interaction with other people is the social aspect of the game (Prensky, 2001a). There is a tendency for most computer games today to become multi-player games and more than one player is allowed to play the game. The player needs to interact with other players when playing in pairs or groups to explain his or her thinking and reasoning for each play. For instance, in online games such as Dota (i.e., Defence of the Ancient which involves teamwork), the players from the same team need to communicate with each other via the game’s text message tool on the strategy to win over the opponents’ camp. The opportunities to discuss, negotiate, resolve and reflect on the construction of ideas, processes and understanding in playing mathematics games benefit students’ mathematical understanding and thinking (Booker, 2004).
Play promotes the formation of social groupings as it is much more fun to play with others. When many participants sit side-by-side and playing on separate computers simultaneously, the atmosphere becomes exciting and competitive (Rieber et al., 2001). Fromme (2003) found 7 to 14 year old children prefer to play computer games with a peer group of the same gender. Game competition produces social connectivity which makes games enjoyable, while competitive affordances promote social interaction which makes games compelling (Rieber et al., 2001). Many online games are created to be played by thousands of simultaneous users such as *Ultima Online* (UO) (a fantasy role-playing game) and *Half-Life Counter Strike* (combat role-playing game).

### 3.2.2.6 Representation or story

Representation or story is proposed by Prensky (2001a) as the sixth and final element of games. In this context, representation or story means the game is about something; either it is abstract or concrete, direct or indirect. Representation includes any narrative or story elements in the game. Representation is viewed either as the essence of what makes a game or just as the “candy” around the game. The context becomes meaningful and forms a deep structure to the game if the rules for playing the game are tied with the context or the story of the game (Rieber et al., 2001). Hence, children emphasise that the context and storyline are more than surface features such as graphics and sound in the game. Narrative or story is considered as an important game design element which connects scenes in the games with the educational content (Kafai, 1996), and provides shared meaning between the designer and the user (Rieber et al., 2001). Representation or story also stimulates players’ emotions and engages players with the game. Such an emotional connection makes for stronger learning (Prensky, 2007).

This section has shown that using new technology in education leads to a new way of learning. As natives of technology, young people today are comfortable in using and even designing computer games for learning. Contrary to the conventional teaching method, computer games promote students’ learning on their own as well as with guidance. The key structural elements that make games engaging and well suited for learning as proposed by Prensky (2001a) have been adopted for designing the computer game of fractions (i.e., *Tower Trap*) in this study. This will be discussed in Section 5.1.
3.3 Computer games for learning mathematics

This section discusses the factors that need to be considered in developing computer games for learning mathematics. These factors are: 1) approach of mini games that are appropriate for instruction, 2) construction of mathematical concepts from concrete to abstract and 3) criteria for evaluating computer games particularly for learning fractions.

3.3.1 Mini games for instruction

Prensky (2008a) proposes mini game and complex game based curriculum as two approaches that are different educationally and physically. ‘Mini games’ or ‘casual games’ focus on a single subject and specific concept (Illanas, Gallego, Satorre & Llorens, 2008; Prensky, 2008a; Kickmeier-Rust, 2009). Multiple levels are developed for a mini game to include more difficult examples of the same basic game action. Mini games have basic rules that make the games easy to play and the rules do not change while playing the game (Illanas et al., 2008). They can be developed based on Flash or Shockwave technology and distributed through online platforms (Kickmeier-Rust, 2009).

A collection of related mini games makes up a ‘complex game’ (Illanas et al., 2008; Prensky, 2008a). Through learning a specific concept or skill in each separate but related mini game, a broad piece of subject matter is mastered in a complex game. Students apply a single skill in playing a mini game, which normally takes less than an hour. On the other hand, multiple skills are required in a complex game to achieve multiple levels of complex goals, challenges and quests which mostly take 20 to 60 hours (or even more) to complete. The games found in game stores normally are complex games designed in various genres such as action, adventure, role-playing and simulation. Examples of these are given in Table 3.3.
Table 3.3: Complex games

<table>
<thead>
<tr>
<th>Complex games</th>
<th>Descriptions</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action game</td>
<td>Typical arcade or home video console games (e.g., PS3, X-Box) which provide physical challenges (e.g., shooting falling objects, going through mazes) to the players through gameplay and require good reflex and hand-eye-coordination from players.</td>
<td>Maze games (e.g., Pac man), jumping games (e.g., Super Mario), shooter games (e.g., Half-Life).</td>
</tr>
<tr>
<td>Adventure games</td>
<td>Games focus on the interactively experiencing narratives and the gameplay is dominated by cognitive aspects.</td>
<td>In Myst V: End of Ages, the player travels across several worlds known as &quot;Ages&quot;, solving various puzzles by interacting with people or the environment.</td>
</tr>
<tr>
<td>Simulation game</td>
<td>Simulates aspects of a real or fictional reality to serve various purposes including training, analysis, or prediction.</td>
<td>F1 Challenge is the simulation-style racing game based on Formula One motorsport. Sim City is a popular simulation game for building worlds.</td>
</tr>
<tr>
<td>Role-playing games</td>
<td>Games with a fictional setting which allow players to choose from a few characters in the story setting and role playing to reach game objectives.</td>
<td>Role-playing video games descend from pen-and-paper role-playing games such as Dungeons &amp; Dragons. Ultima Online (UO) is a fantasy multiplayer role-playing game which can be played online by thousands of simultaneous users.</td>
</tr>
</tbody>
</table>

Most of the educational games, especially those found on the Internet, are mini games because they can be created with small cost by involving small teams taking a couple of months at most. Mini games that can often be played to completion within a single class period can be used in classrooms for instruction or put on the Internet as homework. Kickmeier-Rust (2009) indicates that mini games for preschool age and primary education level children are the most common and the most successful form of educational games. These games attempt to help young children learning basic skills such as numbers, letters, simple maths, and reading. In selecting games for instructional purposes, Dempsey et al. (2002) promote games that are simple to play and not overly complex. Klawe (1998) supports the use of simple games as classroom activities because the educational approaches of simple games are more focused than complex games. Students can increase their computational speed and accuracy by simply practicing on drill-and-practice games (Becker, 1990). Teachers can easily evaluate the progress and achievement of students and determine the educational value of mini games (Illanas et al., 2008). In contrast, a period of time is needed to learn the rules of a complex game before curriculum learning takes place (Parsons, 2008).
The existing mini games teach the same or similar curricular units are used around the world. For instance, *Coolmath-Games* (2010) is an online game that can be played by students from New Zealand or Malaysia to learn mixed numbers. A website called *Games2train.com* provides a catalogue of the growing number of *serious games* (i.e. video and computer games whose primary purpose is something other than to entertain). The aim is to help “people who want to locate serious games to find them” and “people who want to create serious games to see what others have done” (Games2train.com, 2005a, para. 2). The purpose is “to unite projects, information, and people in a single place” (Games2train.com, 2005b, para. 2). The concepts or subject matter areas that are the most boring or the most difficult to students can be focused on to develop mini games. These are, perhaps, the most critically needed (Prensky, 2008a).

### 3.3.2 Constructing mathematical concepts in computer games

Rather than identifying a game as a single best practice or ideal approach, Squire and Jenkins (2004) promote the versatility of games as a pedagogical medium. Learning by doing, goal-oriented learning, role playing and constructivist learning are the pedagogical approaches that could be used in a game. However, lack of pedagogical design and game-based learning principles limits educational value of games (Prensky, 2001). Specific instructional design is necessary for mathematics games that focus on problem solving, rules, and concepts in mathematics (Van Eck, 2007). Traditionally, the main use of mathematics games for teachers is to give students additional practice in applying mathematical ideas or processes (Booker, 2000), especially for arithmetical computation (Bragg, 2007). Woodward (2006) indicates the effectiveness of timed practice drills at helping students achieve automaticity in multiplication facts. Wright (2007) includes practice as one of the effective intervention for students to master newly taught mathematics skills and build computational fluency. Games create an enjoyable environment and meaningful situation for students to apply and practice their knowledge and skills of mathematics after learning in the classroom (Bragg, 2007; Booker, 2004). In teaching primary mathematics, Rowland (2008) promotes the use of examples and one of the purposes is to provide exercises for student to practice. This allows students to rehearse the newly learned mathematical procedures on several such ‘exercise’ examples. Repeating the same procedures helps students to retain and then develop the fluency of the procedures. Such practice-oriented exercises may lead to different kinds of awareness and comprehension.
In addition to maintaining the previously acquired skills through motivating practice, the discovery and use of strategies in mathematics games can improve problem solving skills (Booker, 2004). To play games based on mathematical ideas, students acquire an initial understanding of the target mathematics concept, skill and strategy. During a game, students manipulate materials, use diagrams, detect patterns and verbalise their actions, thoughts and interpretations. These activities assist students to construct mathematical concepts. Furthermore, through being fully involved in the game and caring about the outcome, students are motivated to keep playing and learning mathematics. Therefore, games provide a background in which mathematical concepts can be developed and constructed (Booker, 2000, 2004). Constructivist learning emphasizes that all knowledge has to be constructed by the individual and students gain their mathematical understanding through the continual organization of self-built knowledge structures (Pirie & Kieren, 1992).

The use of manipulatives is recognized as one of the classroom methods that promotes constructivist learning (Pirie & Kieren, 1992). Manipulatives help students’ construction of meaningful mathematical ideas if they are used in the context of educational tasks where students’ thinking is actively engaged with teacher guidance (Clements, 1999). Computers enable students to explore mathematical ideas through the power of record and replay sequences of their actions on manipulatives. Students gain an active experience by connecting objects that they make, move, and change to symbolic representations such as numbers and words. Clements (1999) recommends using concrete manipulative before formal symbolic instruction, such as teaching algorithms. The engagement with concrete materials is important as the first step in abstracting mathematical concepts. Linsell (2005) explains “many mathematical concepts may be inaccessible to most learners unless they have physical experiences to form images of, notice similarities between, and reflect upon” (p. 43).

The learning involving transition from working with concrete materials to abstraction is consistent with the teaching model of Pirie and Kieren (1989). This model was discussed in Section 2.3.1. In this model, students’ understanding progresses from operating on concrete materials, to visualisation, to abstraction. Initially, students model the mathematical concept using physical resources and then visualise the previously used concrete materials by forming images. Abstraction occurs when students notice properties of the images and think consciously about the noticed properties until the common structure is formulated and symbolised (Linsell, 2005). Pirie and Kieren (1989) assume an ongoing and dynamic process of mathematical understanding as a basis of a constructivist environment for learning.
mathematics (Pirie & Kieren, 1992). This model has been used extensively in the Numeracy Projects (Higgins, 2003; Linsell, 2005), emphasising the use of equipment to help students think about mathematical ideas rather than following the more traditional approach (i.e., using procedural models to solve a problem) (Higgins, 2003). Students construct the mathematical knowledge in meaningful ways rather than directly transmitting behaviours or skills from textbooks and teachers (Fosnot, 1996; Moseley, 2005). Likewise, learning in educational games becomes effective when an individual is given opportunities to construct knowledge through play, exploration and social discourse with others (Amory, 2001).

3.3.3 Evaluation of mathematical learning for rational number software

With the consideration of ensuring the success of computer games for learning, tools are developed for assessing and evaluating the educational values of computer games. For instance, Hong et al. (2009) suggest the important values that need to be considered by game designers and educational psychologists such as mentality change, emotional fulfilment, knowledge enhancement, thinking skill development, interpersonal skills, spatial ability and bodily coordination. Many researchers recognize the importance of students’ motivation gained from playing computer games in learning mathematics (Klawe, 1998; Koirala & Goodwin, 2000; Panagiotakopoulos, 2011). Kafai, Franke and Battey (2002) are concerned with mathematical learning more than the motivation of students in playing computer games. They analysed 95 reviews of rational number software for primary classrooms and propose software review criteria based on the principles of mathematical inquiry. The criteria provide guidelines and better information to teachers in selecting and integrating educational software in the instruction. Kafai et al. (2002) employed a mix of content-specific and content-general criteria, namely:

- topic
- students’ strategies
- representations
- context
- content integration

Topic describes different rational number operations; students’ strategies and representations can be applied to review other content areas of mathematics; context and integration are the general categories that are applicable to all subject areas.
This study referred to the above criteria when developing the computer game *Tower Trap* to ensure the game really benefits mathematical learning. In addition, the criteria proposed by Kafai et al. (2002) focus on the area of rational number and this is relevant to the topic of fraction sizes developed in *Tower Trap*.

Based on the results of evaluation of the 95 software reviews, Kafai et al. (2002) identify the percentage of software reviews falling into those categories. However, some of the software reviews do not provide any information relevant to the categories. As such, in the following discussions, the percentages reported may not add up to 100%.

### 3.3.3.1 Topic

“Any review of mathematical software should attend to the particular mathematical content being addressed, the problems used to support the development, the opportunities for students to use their informal knowledge”.

(Kafai et al., 2002, p. 167)

Considering the particular mathematical content being addressed in educational software, the following three topic areas are highlighted:

1. Operating on fractions  
   (including addition, subtraction, multiplication and division)  
2. Developing equivalence  
   (identifying and comparing fractions, fair sharing and equivalence)  
3. General fractions

*Topic* is the content specific category proposed by Kafai et al. (2002) to describe different rational number operations. A rational number is a number that can be expressed in the form \( \frac{p}{q} \) where \( p \) and \( q \) are integers and \( q \neq 0 \) (Wells, 2008; Weisstein, 2010). Usually rational numbers are referred to as fractions in the primary level of mathematics. As an example, *MathsIsFun* website introduces “a rational number is a number that can be written as a simple fraction” (2010, para. 1). “A rational number is a fraction” (MathsIsFun, 2010, para. 3) and students can refer to adding, subtracting, multiplying and dividing fractions as a way of using rational numbers.
The three topic areas also built on “substantial research efforts over the last 25 years, identifying the fundamental constructs underlying students’ understanding of rational numbers” as shown by Hiebert and Behr (1988), Kieren (1988, 1993), Behr, Harel, Post and Lesh (1992, 1993) (in Kafai et al., 2002). Hence, the ideas of partitioning, equivalence, unit, or ratio are distinguished as critical for understanding fractions (Kafai et al., 2002). Nevertheless, operating on fractions is identified as the main focus in the majority of software reviews (54%). 31% of the software reviews address the development of equivalence, in a combination of topics such as identifying and comparing representations, creating fair shares and identifying equivalent fractions. 17% of software reviews involve general fractions and decimals.

3.3.3.2 Students’ strategies

In the review on students’ strategy, Kafai et al. (2002) look at the opportunities given to engage students at the various levels of understanding and strategy use. They define three review categories with respect to strategy use: algorithmic, multiple, and connected. The majority of the software reviewed (65%) focus on a single, traditional fraction algorithm. The multiple strategy category includes 15% of software reviews which allow the player to use one of a range of strategies. As an example, the player can compare fractions either by using diagrams or by using numerical conversions. The connected strategy only occurs in 13% of software reviews which provide opportunities to use different strategies and make connections across strategies. For instance, the pictorial representations of fractions can be linked to their symbolic representations.

3.3.3.3 Representations

The critical role of representations with fraction software is taken into account by Kafai et al. (2002) because the opportunities given on forming and connecting different mathematical representations influence students’ understanding of fractions (Moseley, 2005). Educational software is more advantaged than traditional media in terms of using and linking multiple representational formats such as symbolic, graphic, spoken, dynamic or written forms (Kafai et al., 2002). Three review categories of representation are defined: single, combination, and manipulable (i.e., a choice of multiple representations that can be manipulated).
Forty-seven percent of the reviewed software provides only a single symbolic representation, where the user has access to only one representation (symbolic or pictorial) of rational numbers. Thirty percent of the reviewed software has a combination of multiple representations of rational numbers such as pictorial and symbolic representations. Unfortunately, there is not much software with manipulable representations where users can control or manipulate the representations (11% of the reviews) (Kafai et al., 2002). Providing opportunities for students to engage with varied representations and manipulate these representations are keys to assisting children in constructing conceptual knowledge of fractions (Kafai et al., 2002).

3.3.3.4 Context

Kafai et al. (2002) found “none”, “fantasy” and “real-life” contexts for fraction content in educational software. According to Linsell (2005), the concept of context not only includes the physical situation in which the mathematics is embedded but also the meanings that the learner ascribes to the situation. The context where the learning is taking place is important in understanding the activity or the actions of learners (e.g., playing computer games). Problem context is useful for motivating and supporting students’ development of mathematical understanding, especially for authentic problems that connect to students’ informal knowledge and provide real-life events and situations (Kafai et al., 2002; Linsell, 2005). However, no context is used in 36% of the software reviews many of which are just operation tutorials. Only 12% of the software reviews use a real-life context that is related to the player’s experience. Fantasy game context is the largest context category (48%) where the game is designed in a make-believe world. In investigating the efficacy of different contexts for mathematical word problems, Wiest (2001) found the variety fantasy contexts not only attract students’ interest but also yield cognitive benefits. Fantasy contexts are linked with abstract and creative thinking and enable students to work with more varied and less familiar types of mathematical problems. Through working with unfamiliar problems, students engage with contexts more fully and process problems more deeply to establish underlying mathematical relationships and identify questions posed.

3.3.3.5 Content integration

Kafai, Franke, Ching and Shih (1998) note that extrinsic, intrinsic and constructivist integrations are different levels of content integration between software activities and
mathematical content and these afford different opportunities for the player to engage in mathematical thinking. *Extrinsically integrated* games are typically centred on arcade-style action. The player hardly engages with the content when fraction questions are asked occasionally and unrelated to the game’s theme or objective. *Intrinsic integration* of fractions occurs in games that centre on real-life activities. The context is merged with the fraction content and so context and content enhance each other. As an example, Beach Fractions is a game which illustrates fraction ideas without the addition of a game narrative or context, instead using various fractions of people moving in and out of the water (Table 3.4). The further level of *constructivist integration* supports the player generation of fraction content. Manipulation is used by the player to actively create his or her own fraction ideas. For example, the player can change the fraction problem at any time in Color the Mouse by changing the animal to be coloured and determine the fraction that the animal is shaded. This provides multiple solution path possibilities when the player can create the representation and choose from different possible strategies as they manipulate the representations. As such, this game is more advanced than Beach Fractions, where the player’s strategy choices are fixed because the representations (i.e., people) are provided in a fixed way without choices. Table 3.4 provides examples of extrinsic, intrinsic and constructivist integrations games (Kafai et al., 1998, p. 160-163).

---

**Table 3.4: Extrinsic, intrinsic and constructivist integrations games (Kafai et al., 1998, p. 160-163)**

<table>
<thead>
<tr>
<th>Fraction integration</th>
<th>Games</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extrinsic integration game</td>
<td><em>Chase</em></td>
<td>A maze game in which different things were chasing the player character and if he/she bumped into them, the user had to answer a question</td>
</tr>
<tr>
<td></td>
<td><em>Maze</em></td>
<td></td>
</tr>
<tr>
<td>Intrinsic integration game</td>
<td><em>Beach</em></td>
<td>A beach in which various fractions of people move in and out of the water</td>
</tr>
<tr>
<td></td>
<td><em>Fractions</em></td>
<td></td>
</tr>
<tr>
<td>Constructivist integrations</td>
<td><em>Color the</em></td>
<td>The player could create his or her own fractions by coloring in different pieces of a given animal and then telling what fraction of the animal is shaded.</td>
</tr>
<tr>
<td>game</td>
<td><em>Mouse</em></td>
<td></td>
</tr>
</tbody>
</table>

---

Among the rational number software reviewed (Kafai et al., 2002), 41% of them employ extrinsic integration and 32% of them are made like tutorials without integration at all due to the lack of context. Intrinsic integration accounts only for 13% of all reviewed software,
and after 1990 10% of reviewed software is constructivistically integrated. The fraction games with the intrinsic and constructivist integration represent “a sophisticated departure from both the drill-and-practice routine of traditional education and the non-mathematical focus of most computer games” (Kafai et al., 2002, p. 173).

In the reviews of the educational software on rational numbers, Kafai et al. (2002) show that less focus is given to the development of equivalence as compared with operating on fractions. Therefore, the part-whole construct and relative amount, which have been identified as the basic yet critical concept for learning fractions (refer to Section 2.1) will be developed into a mini game in this study. This will be discussed in Chapter 5. The above criteria will be used to evaluate the Tower Trap designed in this study. This will be discussed in Section 5.1.

3.3.4 Examples of computer mathematics games

Research on computer mathematics games was conducted using internet search engine and the search terms mathematics games and fraction games. The pages of related links were shown with either fractions as one of the components or fractions as the main focus. Three examples of computer mathematics games were selected for review.

Table 3.5 shows three examples of computer mathematics games which provide more attractive interactive practice than traditional print media. Game 1 is about comparing fractions where the player chooses the mathematics symbols (i.e., <, > and = ) that make a true equation (e.g. $\frac{1}{2} = \frac{3}{6}$) or inequality ($\frac{1}{10} < \frac{2}{9}$). In Game 2, the player selects mixed numbers (e.g., $9\frac{7}{8}$) from a number of moving bubbles while avoiding other fractions (e.g., $\frac{1}{3}$). Game 3 requires the player to select answers to the number operations of addition ($3 + 4$), subtraction ($9 - 0$), multiplication ($6 \times 6$) and division ($2 \div 1$).
Table 3.5: Example of computer mathematics games

<table>
<thead>
<tr>
<th>Game elements</th>
<th>Game 1</th>
<th>Game 2</th>
<th>Game 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rules</strong></td>
<td>To choose the mathematics symbol that makes a true equation or inequality</td>
<td>To click on the mixed numbers while avoiding the others.</td>
<td>To select answers to the number operations of addition, subtraction, multiplication and division</td>
</tr>
<tr>
<td><strong>Objectives and goals</strong></td>
<td>To get the highest score</td>
<td>To get the highest score</td>
<td>To keep viruses and bacteria away from the body of patients</td>
</tr>
<tr>
<td><strong>Outcomes and feedback</strong></td>
<td>“Excellent” if correct and “That’s not right” if incorrect.</td>
<td>The bubble splat if a mixed number is selected</td>
<td>The viruses and bacteria are killed if a correct answer is selected to the number operations</td>
</tr>
<tr>
<td><strong>Conflict/Competition/Challenge/Opposition</strong></td>
<td>To answer questions correctly and avoid answering questions incorrectly</td>
<td>Select all mixed numbers within the time limit of 1 minute.</td>
<td>Select answers to the number operations within the time limit of 2 minutes. The questions will become harder and the germs will move faster as the player progresses through the game.</td>
</tr>
<tr>
<td><strong>Interaction</strong></td>
<td>Text and immediate feedback</td>
<td>Text, animation, manipulation of objects and immediate feedback</td>
<td>Text, animation, manipulation of objects and immediate feedback</td>
</tr>
<tr>
<td><strong>Representation or story</strong></td>
<td>NA</td>
<td>Fraction bubbles</td>
<td>The player acts as a doctor who can kill germs in the patient’s body by clicking on the germ with the correct answer to simple mathematical questions that are shown in the mouth of the white blood cell.</td>
</tr>
</tbody>
</table>
These three games focus on the algorithmic processes of comparing fractions, operating numbers and selecting mixed numbers. In fact, these are the categories of students’ strategies for which the majority of software has been developed (Kafai et al., 2002). Instead of giving lessons, this software contains sets of computational exercises. Such activities provide effective practice for improving mathematical skills but the focus is too much on mathematical algorithms for them to be good games.

Game 1 and Game 2 lack a game story which is identified by Prensky (2001a) as one of the elements that makes games engaging. Neither the rule of “choosing the mathematics symbol that made a true equation or inequality” nor “clicking on the mixed numbers while avoiding the others” is tied to any story in Game 1 and Game 2, respectively. Stories play a role of connecting scenes in the games with the educational content (Kafai, 1996). In addition, linking the rules for playing the game with the story of the game makes the context meaningful and forms a deeper structure to the game (Rieber et al., 2001).

The story of Game 3 which is about a doctor who kills germs in the patient’s body stimulates players’ emotions and this motivates learning (Prensky, 2007). However, the story is unrelated to the simple mathematical questions posed in the game. Kafai et al. (1998) suggest this type of game is extrinsically integrated, where a story is added into the game without merging with the mathematical content. On the other hand, a story provides a meaningful problem context to mathematical content if the content is intrinsically integrated with the context. In this way, the context and content enhance each other (Kafai et al., 1998).

Interaction between the player and the computer is one of the kinds of interactions that exists in playing computer games. This is an important game element that ensures the player engages in learning while playing a game (Prensky, 2001a). Such interaction occurs in the three games through text, animation, manipulation of objects and immediate feedback. However, these games provide only a single symbolic representation of fractions. They do not make use of the advantages of digital media by using and linking multiple representational formats such as symbolic and pictorial (Kafai et al., 2002). Students’ understanding of fractions is influenced by their experience of forming and connecting different mathematical representations (Moseley, 2005). By engaging with varied representations and manipulating these representations students’ construction of conceptual knowledge of fractions is assisted (Kafai et al., 2002).
Game 1 to Game 3 were provided as examples to show that many computer mathematics games were lacking in the game elements to make them engaging. Based on this understanding of the problems and practices of current computer mathematics games, insight into designing an educational game that engages and enhances mathematics learning was obtained and incorporated in the structure of Tower Trap (See Section 5.1).

Although complex and mini games are available to support learning, most online educational games are mini games that focus on a single subject and specific concept. A pedagogical approach is an important factor that ensures students engage in learning from playing games. Computers allow students to construct mathematical understanding from manipulating objects to forming images to noticing properties. The criteria proposed by Kafai et al. (2002) to review educational software on rational numbers include topic, students’ strategies, representations, context and content integration. The criteria are relevant to the computer game of fractions (i.e., Tower Trap) help to ensure the mathematical learning of Tower Trap is achievable.

Summary

This chapter has focused on the use of computer games for learning particularly in mathematics. First of all, the definition of computer games is provided after extensive literature review. Computer games play a significant role in the era of using new technology for learning. For instance, young people today who are digital natives are capable to adopt computer games for learning on their own with guidance. To ensure computer games are designed to be engaging and well suited for learning, the six key elements of educational computer games proposed by Prensky (2001a) can be incorporated in computer games design. It is noted that the approach of mini game which focuses on a single subject is the most suitable for instruction. It is also worthwhile to note that a pedagogical approach such as creation of a concrete to abstract learning environment is necessary to ensure students are constructing their mathematical understanding while playing computer games. In addition, Kafai et al. (2002) have suggested criteria for reviewing rational numbers software which can be referred to when evaluating the mathematical learning of computer games. The problems and practices of computer mathematics games are shown through the examples of computer mathematics games. The purpose is to identify the important considerations for designing an educational game that engages and enhances mathematics learning and focuses on a critical area of mathematics.
CHAPTER 4

RESEARCH METHODOLOGY

This chapter discusses the process involved in conducting this study. The purpose of the study is first elaborated in Section 4.1. Section 4.2 explains the research paradigm of mixed methods adopted in this study. Section 4.3 discusses data collection which involves research sample, research process and research tools. Section 4.4 describes the analysis of quantitative and qualitative data.

4.1 Purpose of the research

This study was conducted to investigate the use of mathematics games for school students to learn mathematics. The aim was to develop a computer game of fractions that would improve the learning of fractions. The area of fractions was focused on because of the fundamental role of fractions and because it is an area that students find difficult. The topic of comparing sizes of fractions was particularly selected because of the important role this plays in the understanding of fractions.

There are three key issues related to the effectiveness of the computer game of *Tower Trap* for learning fractions which led to the research questions in this study. The three issues are:

- **Pedagogical approach of the computer game**
  It described how the fractions concept was taught, expressed and integrated with the game context in the game for the students to learn fractions.

- **Students’ strategies for ordering fractions**
  It showed students’ mathematical thinking involved in the use of different strategies that could be ranked from less to more sophisticated.

- **Students’ enhancement in their knowledge of fractions**
  This reflected the effect of the game on students’ achievement in fractions. The aspects of fractions on which the students improved were of concern. Since the game was about comparing sizes of fractions, any improvement in the use of the strategies by the students to compare and order fractions was of particular interest.
The research questions are:

1. What needs to be considered in developing a computer game that would enable students to compare and order fractions?
2. What strategies are used by students in ordering fractions? How can these strategies be classified as in a framework?
3. What improvements, if any, were there in students’ ability as a result of playing the game?

4.2 Research paradigm

In order to address the complex nature of this study, I decided to use a mixture of qualitative and quantitative methods in order to address the research questions.

Qualitative and quantitative methods are contrasting approaches. Tashakkori and Teddlie (2009) associate quantitative techniques with “the gathering, analysis, interpretation, and presentation of numerical information” and qualitative techniques with “the gathering, analysis, interpretation, and presentation of narrative information” (p.5). Mertler (2006) explains “quantitative research methods require the collection of numerical data and utilize a deductive approach to reasoning” while “qualitative research methods require the collection of narrative data and utilize an inductive approach to reasoning” (p. 19).

Both qualitative and quantitative methods are combined in the definitions of mixed methods research. As defined by Johnson and Onwuegbuzie (2004), mixed methods research is “the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study” (p.17).

Tashakkori and Teddlie (2003) elaborate the use of qualitative and quantitative approaches and Tashakkori and Creswell (2007) discuss using both approaches in a single study. They defined mixed methods research as “a type of research design in which qualitative and quantitative approaches are used in types of questions, research methods, data collection and analysis procedures, and/or inferences” (Tashakkori & Teddlie, 2003, p. 711) and “research in which the investigator collects and analyses data, integrates the findings, and draws inferences using both qualitative and quantitative approaches or methods in a single study or program of inquiry” (Tashakkori & Creswell, 2007, p. 4).
The definition is consistent with other researchers (e.g. Creswell, 2003; Johnson, Onwuegbuzie & Turner, 2007). Researchers also consider mixed methods research in education generally (Punch, 2009). According to Mertler (2006), a better understanding of a research problem can be obtained from the combination of both qualitative and quantitative data. In this study, the effectiveness of Tower Trap for learning fractions could be determined by looking at the quantitative data of students’ attempts in ordering fractions in the computer game and the qualitative data of students’ strategies for ordering fractions.

Mixed methods designs are more advanced and complex than simply combining qualitative and quantitative approaches and techniques. The researchers need to consider the relevant characteristics of traditional qualitative and quantitative research in order to mix research in an effective manner (Johnson & Onwuegbuzie, 2004). Each method has its strengths and weakness and the researcher can capitalize on the strengths of each method and should avoid possible unimethod bias when using more than one method in the same research study (Patton, 2002; Wiersma & Jurs, 2005). In this study, the improvement students gained from playing Tower Trap was investigated from their ordering of fractions before and after playing the game. Nevertheless, the quantitative data of right and wrong ordering of fractions provided limited information on what aspect of fractions students had improved. The qualitative data of students’ reasoning were collected to understand the initial strategies used by the students and any changes of strategies after playing the game.

Yin (2006) indicates a challenge of maintaining the integrity of the single study without permitting the study to decompose into two or more parallel studies. He argues “the stronger the “mix” of methods throughout these procedures the more the researchers can derive the benefits from using mixed approaches” (p. 41). Johnson et al. (2007) highlight the mixing within a single study for a mixed methods study when defining mixed methods research as a type of research. They added “a mixed method program would involve mixing within a program of research and the mixing might occur across a closely related set of studies” (p. 123).

Mixing can occur in the data collection stage, at the data collection and data analysis stages or at all of the stages of research (Johnson et al., 2007). In a mixed approach methodology proposed by Tashakkori and Teddlie (2003), multiple approaches are incorporated in all stages of the study (i.e., problem identification, data collection, data analysis, and final inference).
Although mixed method studies use both qualitative and quantitative data collection and analysis, they are often marginally mixed in that they are frequently either qualitative or quantitative in the type of questions they ask and the type of inferences they make at the end of the studies. (Tashakkori & Teddlie, 2003, p. 11)

This study referred to the process for designing and conducting a mixed methods research proposed by Johnson and Onwuegbuzie (2004). First of all, this study elaborated the research purpose of developing a computer game that will improve the learning of fractions. This research purpose led to three research questions as discussed in Section 4.1.

The basic design of the research is described in Figure 4.1. The computer game of *Tower Trap* designed to improve student achievement took on the form of an experimental treatment. In an attempt to better understand the success (or lack thereof) of the game, the students were given pre test and pre maths tasks before playing the game and post test and post maths task afterwards. The tests and maths tasks were used to collect quantitative and qualitative data, respectively. The tests were multiple choice questions and the pre test scores were used to generate gain scores. The maths tasks were written questions which asked students to order fractions and write down their reasoning for the order of fractions made. The pre maths task solutions were used to inform changes of students’ strategies. The students played the computer game individually for about half an hour and their game play was recorded on computers. They evaluated the game afterwards using questionnaires.

![Figure 4.1: Research design](image-url)
The students involved in the study were selected by the teachers based on their mathematical abilities and were not selected at random. Such research took on the characteristics of a quasi-experiment (Wiersma & Jurs, 2005). An experimental group which was made up from three mathematical ability groups (i.e., high achievers, average students and cause-for-concern students), and a control group which also consisted of the three ability categories, were formed in this study. The categories of high achievers, average students and cause-for-concern students are discussed in the next section on research sample. The majority or approximately 80% of the students were selected from each ability group to participate in the experimental group. The minority or the remaining students from each ability group participated in this research as the control group. The small number of students in the control group was chosen because the purpose of forming the control group was to reaffirm the effectiveness of the computer game (if there was any) on the experimental group. Since the students from experimental and control groups were from the same school, it was deemed that there were no pre-existing differences between the two groups.

Both the experimental and control groups were tested and dependent variables were compared to identify the game effect. First of all, both groups were pretested on the pre test and pre maths task; only students from the experimental group played the game individually for about half an hour and evaluated the game in questionnaires; and then both groups were post tested on the post test and post maths task. There was a concern that while the experimental group had additional experiences with fractions through the game, the control group should also have been given other activities in order to provide equal experiences. Nevertheless, other activities were not included in the scope of this study. Therefore, the differences between the results from the experimental group and the control group might not be solely due to the effects of the game but might also be due to the fact that additional time working with fractions had been given to the students in the experimental group.

This study considered the rationales proposed by Greene, Caracelli and Graham (1989) to determine whether a mixed design is appropriate. These include triangulation, complementarity, initiation, development and expansion (Table 4.1).
Table 4.1: Mixed methods research (Greene et al., 1989)

<table>
<thead>
<tr>
<th>Purposes or rationales</th>
<th>Elaborations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangulation</td>
<td>Using more than one method to offset biases in assessing a phenomenon and enhance findings through convergence and corroboration of results from different methods</td>
</tr>
<tr>
<td>Complementarity</td>
<td>Combining qualitative and quantitative methods and using the results from one method to clarify, enhance, or illustrate the results from the other method</td>
</tr>
<tr>
<td>Initiation</td>
<td>Using qualitative and quantitative methods with an intention to discover paradoxes and contradictions that help in reframing research question</td>
</tr>
<tr>
<td>Development</td>
<td>Using the results obtained from the first method to inform the other method</td>
</tr>
<tr>
<td>Expansion</td>
<td>Using different methods to address different research components in order to expand the scope and breadth of the research</td>
</tr>
</tbody>
</table>

Complementarity was the main reason the mixed methods research was selected in this study, where the results from one method were used to elaborate, enhance, illustrate and clarify the results from the other method (Greene et al., 1989). In this study, qualitative data of students’ reasoning written on the pre and post maths tasks elaborated students’ enhancement in the knowledge of fractions and clarified the quantitative data of students’ achievement differences between the pre and post tests. The quantitative data of computer play was used to examine the pedagogical approaches of the computer game, which were further scrutinized by the data of students’ evaluations given in the questionnaires about learning fractions using the game.

Nudzor (2009) discredits mixed methods research and the claim that this is the best methodology for addressing complex social issues. The first reason given is that qualitative and quantitative approaches are opposed to each other. Combining both approaches attenuates the crucial issue of objectivity, validity and reliability in social research. The second reason is mixed methods research gives a conflicting and contradictory suggestion that “on the one hand that there is objectivity in social science research, and on the other, that there is no such thing as ‘objective reality’ as far as social science research is concerned” (Nudzor, 2009, p.125). Nudzor (2009) stresses that an appropriate methodological approach to social research
lies in the pre-eminence given to the research purpose before considering other issues (e.g., skills base of the researcher and research’s contributions to wider political discourse). In this study, the primary purpose was to investigate the different aspects relevant to the effectiveness of the computer game of *Tower Trap* for learning fractions. The mixed methods research was found to be appropriate approach to achieve the research purpose where findings from varied research tools (e.g., the pre and post tasks, pre and post tests, computer game play and questionnaires) could be included to complement each other.

This study used *mixed-method* design which emphasised qualitative and quantitative paradigms equally and conducted the qualitative and quantitative phases concurrently. Two primary decisions are required in constructing a mixed-method design, namely paradigm emphasis decision and time order decision (Johnson & Onwuegbuzie, 2004). Paradigm emphasis decision requires the researcher to decide whether to operate largely within one dominant paradigm (quantitatively or qualitatively) or not; while time order decision requires the researcher to consider whether to conduct the phases concurrently or sequentially.

![Image removed for copyright reasons](image)

**Figure 4.2: Mixed-method design (Johnson & Onwuegbuzie, 2004)**

This study had an equal status and concurrent use of qualitative and quantitative paradigms. Quantitatively, students’ game play was recorded on computer, their feelings and perceptions towards the game were obtained through questionnaires, and the achievement differences were computed by comparing the pre test and post test scores. The quantitative
data of students’ game play and feedback provided students’ perspective about the use of the computer game of *Tower Trap* for students to compare fractions. This aimed to answer research question “What needs to be considered in developing a computer game that would enable students to compare and order fractions?” The quantitative data of achievement differences showed the positive and negative changes after playing the game. An example of positive changes was being wrong in the pre test but right in the post test. This aimed to answer research question “What improvements, if any, were there in students’ ability as a result of playing the game?” However, the quantitative data provided limited information about students’ knowledge and understanding of fractions. The researcher would be able to identify the area of fractions or type of question asked in tests that the students had improved but not students’ thinking that caused the changes. In order to answer research question “What strategies are used by students in ordering fractions?”, the qualitative data of students’ strategies was collected by asking students to write their reasoning on the pre and post maths tasks. This qualitative approach was able to provide more insight into students’ thinking. Any changes of students’ strategies were examined to identify if there were any improvements in students’ ability as an effect of playing the game.

The usefulness of *Tower Trap* for learning fractions could be investigated using the quantitative data of students’ game play and feedback. Meanwhile, students’ learning of fractions could be investigated using both the quantitative data of achievement differences and qualitative data of students’ strategies for ordering fractions. Both quantitative and qualitative data were important in this study to investigate the effectiveness of *Tower Trap* on students’ strategies for ordering fractions.

The data collection and data analysis were conducted based on the mixed methods research designed above. These are discussed in Section 4.3 and Section 4.4, respectively.

Based on the data collected, the data were analysed and interpreted in the discussion chapters of this thesis with particular focus on:

- designing and developing the computer game of *Tower Trap* for learning fractions (refer to Chapters 5 and 6)
- students’ strategies for ordering fractions (refer to Chapter 7)
- impacts of *Tower Trap* computer game on learning fractions (refer to Chapter 8)
The quality of the game and the use of the game for learning fractions could be informed by students’ evaluation in the questionnaires. The game play record captured for each student while playing the computer game could provide insights into students’ understanding of fractions. The sophistication of students’ strategies for ordering fractions could be determined from the involvement of their understanding levels based on the Pirie-Kieren model (1989, 1994a, 1994b). Students’ strategies could be framed from less to more sophisticated to present students’ progression of fractional thinking. The Tower Trap was deemed to have helped students learning fractions if they showed an improvement in the knowledge of ordering fractions. This maximizes the legitimacy of the data as per the research paradigm (Johnson & Onwuegbuzie, 2004).

4.3 Data collection

Three main parts relevant to collecting data are research sample, research process, and research tools. One hundred and sixty-eight students participated in this study. The research process consisted of a pilot study and a testing of the developed computer game followed by the main study.

4.3.1 Research sample

This section presents the research sample, which are Year 8 students. The research sample is divided into the categories of high achievers, average students and cause-for-concern students.

4.3.1.1 Year 8 students

The sample in this study consisted of 168 Year 8 students (11 to 13 years old) from 3 intermediate schools (i.e., middle schools) in a New Zealand town. The schools involved in the study were selected at random and labelled as School A, School B and School C. The school teachers provided the list of Year 8 students and their stage levels in the Number Framework. The students were categorised into mathematical ability groups based on their achievement on the New Zealand Number Framework Stages. This will be discussed in Section 4.3.1.3. Among the 168 students, 9 students from School A participated in the pilot study. They were 5 males and 4 females. Six students (i.e., 2 males and 4 females) from School B were involved in the testing of the developed computer game. A class of 27 Year 8
students from School C participated in the main study. They were 15 males and 12 females. One hundred and twenty-six students from School A also participated in the main study. The Year 8 students involved in the pilot study from School A were different from the Year 8 students involved in the main study. Ninety-eight of them took part in the experimental group where 58 were males and 40 were females. Twenty-eight of them (i.e., 16 males and 12 females) took part as the control group.

Table 4.2: Number of students from Schools A to C

<table>
<thead>
<tr>
<th>Schools</th>
<th>Process</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>Pilot study</td>
<td>9</td>
</tr>
<tr>
<td>School B</td>
<td>Game testing</td>
<td>6</td>
</tr>
<tr>
<td>School C</td>
<td>Main study</td>
<td>27</td>
</tr>
<tr>
<td>School A</td>
<td>Main study</td>
<td>98</td>
</tr>
<tr>
<td>School A</td>
<td>Control group</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>168</strong></td>
</tr>
</tbody>
</table>

The students had given their consent to participate in this study. Prior to the start of this study, the researcher had also obtained ethical consent from the University of Otago and consent from the Board of Trustees of the schools for conducting this study at the schools. The researcher then explained verbally each step involved in the study to the school principals. A teacher was assigned to help the researcher in each school. An information sheet about the study was distributed to the students’ parents/guardians. Another information sheet was prepared in simpler language to make sure the students understand the purpose of this study. The students signed the given consent form if they understood and accepted the contents of the form and returned the signed form to the researcher before they started taking part in the study.

4.3.1.2 Knowledge of fractions

The Year 8 students who participated in this study had learned fractions as a key aspect in the number strand in the New Zealand Mathematics Curriculum. The instruction included halves, quarters, thirds and fifths; finding fractions of whole number and decimal amounts; naming equivalent fractions; and converting between fractions, decimals and percentages. Therefore, the students had some knowledge of fractions before taking part in the
research. However, the level of understanding of fractions varied across the sample. It is normal in a class that some students learn mathematics better than the others due to factors such as interest, fundamental knowledge, personal experience and expectation.

The NDP uses a diagnostic tool called Numeracy Project Assessment (NumPA) to obtain information about students’ knowledge and mental strategies and this information is aligned to the Number Framework. The NumPA is administered in an individual interview to find out how students solve number problems. In addition to determining whether or not a student is getting correct answers to the pencil and paper assessment tasks, an interview helps teachers to uncover students’ mental strategies that they use with the pencil and paper assessment. Teachers can identify the learning needs of students and develop targeted intervention programmes (Ministry of Education, 2008c, 2010e).

According to the Ministry of Education (2010d) curriculum, Year 8 students are expected to reach Stage 7 at Level 4 before furthering their studies at secondary level. Table 4.3 shows the Number Framework stages as mapped to the achievement objectives of the New Zealand Mathematics Curriculum through a “best fit” approach.

<table>
<thead>
<tr>
<th>Levels in the New Zealand Curriculum</th>
<th>Stages in the Number Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 0: Emergent</td>
<td></td>
</tr>
<tr>
<td>Level 1 i: Counting from one</td>
<td>Stage 1: One-to-one counting</td>
</tr>
<tr>
<td></td>
<td>Stage 2: Counting from one on materials</td>
</tr>
<tr>
<td></td>
<td>Stage 3: Counting from one by imaging</td>
</tr>
<tr>
<td>Level 1 ii: Advanced counting</td>
<td>Stage 4: Advanced counting</td>
</tr>
<tr>
<td>Level 2: Early additive</td>
<td>Stage 5: Early additive part-whole</td>
</tr>
<tr>
<td>Level 3: Advanced additive and early multiplicative</td>
<td>Stage 6: Advanced additive part-whole</td>
</tr>
<tr>
<td>Level 3-4: Advanced multiplicative</td>
<td>Stage 7: Advanced multiplicative part-whole</td>
</tr>
<tr>
<td>Level 4-5: Advanced proportional</td>
<td>Stage 8: Advanced proportional part-whole</td>
</tr>
</tbody>
</table>

The New Zealand Number Framework consists of nine stages which describe numerical development progressing from Stage 0 to Stage 8 as shown above. There are two main sections in the framework: knowledge and strategy. Comparing and ordering fractions involve
the knowledge of “identifying symbols for the most common fractions” and “ordering fractions with like denominators” as included in Stage 5; “identifying symbols for any fractions” and “ordering unit fractions” as included in Stage 6; “ordering fractions including halves, thirds, quarters, fifths, and tenths” as included in Stage 7 (refer to Table 4.4). The corresponding strategies are “equal sharing” in Stage 2 to Stage 4; “finding a fraction of a number by addition and multiplication” in Stage 5 to Stage 6; “solving problems with fractions, proportions, and ratios” in Stage 7 to Stage 8 (refer to Table 4.5).

### Table 4.4: Knowledge of the Number Framework (Ministry of Education, 2008a)

<table>
<thead>
<tr>
<th>Stages</th>
<th>The knowledge section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage Five: Early Additive</td>
<td>Number Identification:</td>
</tr>
<tr>
<td></td>
<td>The student identifies symbols for the most common fractions, including at least halves, quarters, thirds, fifths, and tenths.</td>
</tr>
<tr>
<td></td>
<td>Number Sequence and Order:</td>
</tr>
<tr>
<td></td>
<td>The student orders fractions with like denominators.</td>
</tr>
<tr>
<td>Stage Six: Advanced Additive</td>
<td>Number Identification:</td>
</tr>
<tr>
<td></td>
<td>The student identifies symbols for any fractions, including tenths, hundredths, thousandths, and improper fractions.</td>
</tr>
<tr>
<td></td>
<td>Number Sequence and Order:</td>
</tr>
<tr>
<td></td>
<td>The student orders unit fractions for halves, thirds, quarters, fifths, and tenths.</td>
</tr>
<tr>
<td>Stage Seven: Advanced</td>
<td>Number Sequence and Order:</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>The student orders fractions including halves, thirds, quarters, fifths, and tenths.</td>
</tr>
<tr>
<td>Stage Eight: Advanced</td>
<td>Number Sequence and Order:</td>
</tr>
<tr>
<td>Proportional</td>
<td>The student orders fractions, decimals and percentages.</td>
</tr>
</tbody>
</table>
Table 4.5: Strategies of the Number Framework (Ministry of Education, 2008a)

<table>
<thead>
<tr>
<th>Stages</th>
<th>The strategy section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage Two: Counting from One on Materials</td>
<td>Equal Sharing:</td>
</tr>
<tr>
<td></td>
<td>The student is able to divide a region or set into given equal parts using materials.</td>
</tr>
<tr>
<td>Stage Three: Counting from One by Imaging</td>
<td>Equal Sharing:</td>
</tr>
<tr>
<td></td>
<td>The student is able to share a region or set into given equal parts using materials or by imaging the materials for simple problems.</td>
</tr>
<tr>
<td>Stage Four: Advanced Counting</td>
<td>Equal Sharing:</td>
</tr>
<tr>
<td></td>
<td>The student is able to share a region or set into given equal parts using materials or by imaging the materials for simple problems.</td>
</tr>
<tr>
<td>Stage Five: Early Additive Part-Whole</td>
<td>Fraction of a Number by Addition:</td>
</tr>
<tr>
<td></td>
<td>The student finds a fraction of a number and solves division problems with remainders mentally using halving, or deriving from known addition facts.</td>
</tr>
<tr>
<td>Stage Six: Advanced Additive (Early Multiplicative Part-Whole)</td>
<td>Fraction of a Number by Addition and Multiplication:</td>
</tr>
<tr>
<td></td>
<td>The student uses repeated halving or known multiplication and division facts to solve problems that involve finding fractions of a set or region.</td>
</tr>
<tr>
<td>Stage Seven: Advanced Multiplicative (Early Proportional Part-Whole)</td>
<td>Fractions, Ratios, and Proportions by Multiplication:</td>
</tr>
<tr>
<td></td>
<td>The student uses a range of multiplication and division strategies to estimate answers and solve problems with fractions, proportions, and ratios.</td>
</tr>
<tr>
<td></td>
<td>The students can also find simple equivalent fractions and rename common fractions as decimals and percentages.</td>
</tr>
<tr>
<td>Stage Eight: Advanced Proportional Part-Whole</td>
<td>Fractions, Ratios, and Proportions by Re-unitising:</td>
</tr>
<tr>
<td></td>
<td>The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve using common factors, re-unitising of fractions, decimals and percentages, and finding relationships between and within ratios and simple rates.</td>
</tr>
</tbody>
</table>

The Year 8 students are supposed to build a solid foundation in fundamental mathematics as a preparation for high school works. However, the students’ achievement on proportion and ratio has been found to be rather disappointing (Young-Loveridge, 2005, 2006;
Tagg & Thomas, 2007). By the end of the year, only about a tenth of Year 8 students reach stage 8, advanced proportional reasoning (Young-Loveridge, 2005) and one-third of the Year 8 students reach stage 7, advanced multiplicative reasoning (Young-Loveridge, 2006, 2007). This is rather worrying as the Year 8 students need to be multiplicative (i.e., at least reaching Stage 7) in order to succeed with algebra (Lamon, 2007). Based on the above, it may appear that a very high standard has been set by the Ministry of Education. Many students do not reach the levels aimed by the Ministry of Education (2010d) in the curriculum expectations. Nevertheless, by expecting Year 8 students to reach Stage 7 at Level 4 before furthering their studies at secondary level, the Ministry of Education expects the students at that age will eventually meet the expectations.

4.3.1.3 Categories of high achievers, average and cause-for-concern students

In order to focus on the different needs of students, this study investigated the uses of the Tower Trap computer game for students in three mathematical ability categories. The students were categorised based on their achievement on the New Zealand Number Framework Stages. The framework stages allowed their knowledge of fractions to be determined. This was important in order to find any enhancement gained from playing the game. Figure 4.3 shows end of Year 8 curriculum expectations (Ministry of Education, 2010d). Students are rated via the NumPA interviews as “At Risk” if they are at Stage 0 to Stage 5, “Cause for Concern” at Stage 6, and “Achieving at or above expectation” at Stage 7 or Stage 8. The terminologies defined by the Ministry of Education are tabulated in Table 4.6.

![Image removed for copyright reasons](image)

Figure 4.3: End of Year 8 Curriculum Expectations (Tagg & Thomas, 2007)
Table 4.6: Categories of students (Ministry of Education, 2010d)

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>At Risk</td>
<td>Students rated as “At Risk” are those who are sufficiently below expectations and their future learning in mathematics is in jeopardy. Students rated “At Risk” require special teaching, modified classroom programmes and extra support to continue their development and maintain positive attitudes. The support required is likely to be beyond what can be reasonably expected from their classroom teacher alone.</td>
</tr>
<tr>
<td>Cause for Concern</td>
<td>Students rated as “Cause for Concern” are those who are below expectations, although at a stage where it is reasonable to expect that classroom teachers would be able to move them to the expected stage.</td>
</tr>
<tr>
<td>Achieving at or above expectation</td>
<td>“Achieving at or above expectation” are those students who are sufficiently above expectations that they may require special teaching, modified classroom programmes and extra support to continue their development and maintain positive attitudes. The support required could be beyond what can be reasonably expected from the classroom teacher alone.</td>
</tr>
</tbody>
</table>

The data of student level on the Number Framework which provided an indication of the number knowledge and strategies used by the students were considered in this study to categorise students into different mathematical ability groups. However, rather than adopting the categories defined by the Ministry of Education (i.e., at risk, cause for concern and achieving at or above expectation), this study considered the percentages of students being at the framework stages as revealed by Tagg and Thomas (2007) in a longitudinal study and subsequently defined the student categories as cause-for-concern students, average students and high achievers as shown in Figure 4.4. Since most or 69% of the students are in Stage 6 and Stage 7, they are considered as average students. The 14% of students who achieved Stage 8, which is the highest stage in the Number Framework, are categorized as high achievers. The other 18% of students who reached Stage 5 and below are categorised as cause-for-concern students. This approach seems to be more appropriate for grouping students’ achievement into a normal distribution with high achievers and cause-for-concern students at the two ends and average students in the middle. The normal distribution generally exists in classroom achievements (e.g., tests, scores and grades) (Brown & Hudson, 2002). It is shaped like a bell curve where most of students’ scores or average scores would fall in the middle two-thirds of the curve (Sayler, 2002). The scores of the academically able students will be above the average scores while the scores of the less able will be below the average scores (Hargis, 2003). In addition, the normal distribution of students’ achievement is produced when the same level of instruction and same amount of instructional time is provided to a group of students (Hargis, 2003).
Nine students from three mathematical ability groups, with 3 in each, participated in the pilot study. Six students from three mathematical ability groups, with 2 in each, participated in the testing of the developed computer game. The class of 27 Year 8 students from School C that participated in the main study had a distribution of 26% of high achievers, 52% of average students and 22% of cause-for-concern students. For School A, 98 students (i.e., 58 males and 40 females) participated in the experimental group of the main study while 28 students (i.e., 16 males and 12 females) participated in a control group. Among the 98 students, 9 (9%) were high achievers, 41 (42%) were average students and 48 (49%) were cause-for-concern students. Among the 28 students, 2 (7%) were high achievers, 9 (32%) were average students and 17 (61%) were cause-for-concern students. Table 4.7 shows the population of the sample with three mathematical ability categories.

Table 4.7: Schools and number of students involved in the study

<table>
<thead>
<tr>
<th>Schools</th>
<th>Process</th>
<th>Data collection</th>
<th>High achievers</th>
<th>Average students</th>
<th>Cause-for-concern students</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>Pilot study</td>
<td>Nov 2007</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>School B</td>
<td>Game testing</td>
<td>April 2008</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>School C</td>
<td>Main study</td>
<td>June 2008</td>
<td>7</td>
<td>14</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td>School A</td>
<td>Main study</td>
<td>June 2009</td>
<td>9</td>
<td>41</td>
<td>48</td>
<td>98</td>
</tr>
<tr>
<td>School A</td>
<td>Control group</td>
<td>June 2009</td>
<td>2</td>
<td>9</td>
<td>17</td>
<td>28</td>
</tr>
</tbody>
</table>

In the main study, the researcher intended to collect data 100 Year 8 students from School C. However, only one class managed to participate in the study due to constraints of school programme and schedule. After several attempts to continue with the data collection from School C, it was decided that data collection from another school was required due to persistent delays. Therefore, the study was continued at School A. The research tools used to collect data from School A were the same as the research tools used at School C. School C is located about 3km from School A. There were little possibilities that the students from both
schools would discuss the study and students from School A had no information about the research tools before participating in this study.

This study also employed qualitative purposive sampling or purposeful sampling with the objective of increasing understanding of the impact of the game. One high achiever, one average student and one cause-for-concern student were selected. These students were selected because their data in all of the assessments were typical of many other students in that ability group. Purposive sampling is used “to reach a targeted sample quickly and where sampling for proportionality is not the primary concern” (William M.K. Trochim, 2006, para. 3). Through purposive sampling, the performance of the three individual students in the tests, maths tasks, questionnaires and computer game play were integrated for individual analysis (refer Section 4.4.6). “Qualitative inquiry typically focuses on relatively small samples, even single cases (N=1) such as Anna or Isabelle, selected purposefully to permit inquiry into and understanding of a phenomenon in depth” (Patton, 2002, p. 46). As remarked by Onwuegbuzie and Leech (2007), if the goal is not to generalize to a population but to obtain insights into a phenomenon, individuals, or events, then individual students are purposefully selected for this phase that increases understanding of phenomena.

This research referred to the characteristics of a quasi-experimental research as proposed by Gersten, Fuchs, Compton and Coyne (2005) for research sampling (Table 4.8).

**Table 4.8: Research sampling according to the characteristics of quasi-experiment**

<table>
<thead>
<tr>
<th>Characteristics of a quasi-experiment (referring to Gersten et al., 2005)</th>
<th>Characteristics of samples selected in this research</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparability of samples</td>
<td>Appropriate procedures are used to ensure that participants are comparable across intervention conditions on relevant characteristics. Students in the same group had similar prior knowledge of fractions to ensure that study effects were caused by the game rather than by pre-existing differences.</td>
</tr>
<tr>
<td>Learning difficulties of participants</td>
<td>Sufficient information is provided to determine (or confirm) whether the participants demonstrated the disability(ies) or learning/social learning difficulties Sufficient information from the Number Framework was acquired to confirm students’ knowledge of fractions in order to categorise them in different ability groups.</td>
</tr>
<tr>
<td>Interventionists will be comparable</td>
<td>Appropriate procedures are used to increase the probability that teachers or interventionists will be comparable across conditions All of the experimental groups played the game and the effects across categories were identified. Playing the computer game was a way to ensure the students received the intervention constantly and consistently.</td>
</tr>
</tbody>
</table>
4.3.2 Research process

The research process includes conducting the pilot study, computer game testing and main study as described in Figure 4.5.

![Diagram of research processes]

**Figure 4.5: Research processes**

4.3.2.1 Pilot study

The pilot study was conducted to test the preliminary design of *Tower Trap* using a manipulative of fraction cards. Similar to the computer game to be designed for comparing sizes of fractions, fractions are represented using rectangles and ordered from the smallest to the largest using fraction cards. Testing and enhancing a computer game would be time consuming and expensive. Therefore, testing the game using fraction cards is an effective way to validate (in terms of cost and time) the approach of the game for learning fractions before developing the tested game on computers.

The pilot study was conducted during school hours, with approximately half an hour for each student. Before the study started, the researcher gave a briefing to the teacher and students regarding the objectives and the procedures of the study such as the number of students involved, the required time, the relevant mathematics topic and how to play the game.

The game involved forming a two-dimensional staircase by arranging fraction cards from smallest to largest up to a unit. An example is shown in Figure 4.6 where the fraction card $\frac{1}{3}$ was arranged first and this was followed with the fraction cards $\frac{2}{3}$ and 1.
The symbols of fractions were written on one side of the card while the sizes of the fractions were coloured on the reverse side. For example, the mathematical notation $\frac{1}{2}$ was written on one side of the fraction card while half of the other side of the card was coloured (Figure 4.7). Note that all fraction cards were the same size. The fractions were represented by the coloured areas of the cards.

The game included twelve fraction questions: four were fractions with like denominators and eight were fractions with unlike denominators as summarised in Table 4.9.
Table 4.9: Set of fractions used in the pilot study

<table>
<thead>
<tr>
<th>Level</th>
<th>Fraction questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{3}, \frac{2}{4}$, $\frac{3}{4}$, $\frac{1}{1}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{7}, \frac{4}{7}, \frac{1}{7}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99}, \frac{1}{1}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{1}{5}, \frac{1}{3}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{2}{5}, \frac{2}{5}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{1}{3}, \frac{1}{2}, \frac{1}{3}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{1}{7}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{8}{17}, \frac{1}{2}, \frac{7}{12}$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}$</td>
</tr>
</tbody>
</table>

The first level of the game asked the students to order $\frac{1}{2}$ and $\frac{2}{2}$. This was used to demonstrate the game to the players. The prior knowledge required to play the game includes the ability to identify symbols of fractions. Each student played the game individually and gave reasons for each order of fraction cards given. After playing the game, the students were asked to give their opinions about the game and the learning of fractions using the game.

Since the fraction cards represented the rectangular blocks developed in *Tower Trap*, the finding about students’ understanding on visual representation and ordering of fractions obtained from the pilot study provided valuable information to the subsequent development of the game on computers. This will be discussed in chapter 5. The results showed that students with different mathematical abilities benefited from the game in different ways and had different expectations of the game. Therefore, different needs were incorporated into the development of *Tower Trap* to create an instructional computer game for all abilities. The
game included fraction questions with a range of difficulty and provided teaching of fractions by giving specific and detailed explanations on the methods for comparing and ordering fractions. Collecting data using fraction cards in the pilot study was important to the researcher so that she was informed about students’ needs and this was useful in computer design.

4.3.2.2 Computer game testing

Based on the idea of ordering fraction cards in the pilot study, a game context of forming fraction staircase using fraction bricks was created for the computer game of Tower Trap. The developed computer game was tested in another school (i.e., School B). This was aimed to detect if there was any issues of conducting this study for a school which was new to this study. The issues included students’ understanding of the computer game instruction, detection of bugs in the computer game and whether students were able to complete the game. Six students took part in the testing to make sure the game could run properly on computers and no data were collected. It was found that the students played the game individually for about half an hour each. This helped to estimate the time needed when more students (i.e., approximately 100 students) were involved in the main study. During game testing, the students were able to complete the game on computers with little guidance from the researcher. No bugs that disturbed the running of the computer game were detected. This showed Tower Trap was ready to be used in the main study and played by more students.

4.3.2.3 Main study

The main study was conducted at two intermediate schools. One of the schools had been involved in the pilot study. Nevertheless, the Year 8 students who participated in the pilot study had left the intermediate school by the time the main study was conducted. Every participant took about an hour of school time to play the computer game of Tower Trap (refer to Chapter 5) and complete the assessment tools of the pre and post maths tasks, pre and post tests, and questionnaires. Note that no formal classroom teaching took place during this hour of the students’ time.

4.3.3 Research tools

This study employs four types of research tools: pre and post tests, pre and post maths tasks, computer game play and questionnaires.
4.3.3.1 Pre and post tests

The pre and post tests were two sets of multiple choice questions of fractions of similar difficulties. The two tests were called Test A and Test B (Figure 4.8). Half of the students took Test A as the pre test and Test B as the post test while the other half of the students took Test B as the pre test and Test A as the post test. The difficulty level of the two tests was similar in terms of the knowledge of fractions tested and the strategies used to answer the fraction questions. For example, question 1 in Test A asked students to select the picture which was \( \frac{3}{5} \) shaded blue while question 1 in Test B asked students to select the picture which was \( \frac{2}{5} \) shaded blue. Both \( \frac{3}{5} \) and \( \frac{2}{5} \) were proper fractions and the representation of fraction given were rectangular divided quantity diagrams. As shown in Figure 4.8, the similar incorrect representations are given. In one of the choices, numerator is unshaded that 3 parts are unshaded for \( \frac{3}{5} \) and 2 parts are unshaded for \( \frac{2}{5} \); in two other choices, the number of parts divided are equal to the total of numerators and denominator.

Another example is that question 6 in Test A asked for the number that was closest to the sum \( \frac{6}{7} + \frac{9}{10} \) while question 6 in Test B asked for the number that was the closest to the sum \( \frac{10}{11} + \frac{6}{7} \). The fractions \( \frac{6}{7} \) and \( \frac{9}{10} \) to be added in Test A were both close to one. Similarly, the fractions \( \frac{10}{11} \) and \( \frac{6}{7} \) to be added in Test B were also close to one. The results of adding fractions in both Test A and Test B were similar, which would be close to 2. The incorrect choices included the addition of numerators (i.e., addition of 10 and 6 in Test A and addition of 6 and 9 in Test B) and the addition of denominators (i.e., addition of 11 and 7 in Test A and addition of 7 and 10 in Test B).

It was expected that the achievement differences measured from the first and second halves of students would not be much different. This increased the validity of the research method when the achievement differences measured between the pre and post tests reflected the game effect rather than the influence of the questions in the tests. In this context, validity refers to the ability of an instrument in measuring what is intended to be measured (Polit & Hungler, 1995).
Each test consisted of 6 questions in which the first three questions were about representations of fractions and the other three questions were about operating on and ordering fractions. The questions were developed from items used in the Trends in Mathematics and Science Study (TIMSS) and National Education Monitoring Project (NEMP) tests. These tests had previously been used to assess the mathematics achievement of New Zealand students and referred to in an NDP longitudinal study (Thomas & Tagg, 2006). In addition, these questions were relevant to the knowledge of fractions focused in the computer game as shown in Table 4.10.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Knowledge of fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Represent fractions using divided quantity diagrams</td>
</tr>
<tr>
<td>2</td>
<td>Name the fractions</td>
</tr>
<tr>
<td>3</td>
<td>Find the range of fractions</td>
</tr>
<tr>
<td>4</td>
<td>Compare two fractions that were close to a half</td>
</tr>
<tr>
<td>5</td>
<td>Order fractions</td>
</tr>
<tr>
<td>6</td>
<td>Estimate fractions</td>
</tr>
</tbody>
</table>

Below are the lists of questions asked in Test A and Test B. The two sets of fraction questions are made to be equally difficult by focusing on the same knowledge and strategies of fractions. The fractions involved in the two sets of questions are slightly different in terms of numbers and a slight change in the numbers does not affect the difficulty of the question.

See Figure 4.8 for Test A which includes questions like the following:

- Question 1 asked students to select the picture which was $\frac{3}{5}$ shaded blue
- Question 2 asked about the fraction of the rectangle which was shaded blue
- Question 3 asked about the fraction range of the rectangle which was shaded blue
- Question 4 asked students to determine whether fraction $\frac{5}{9}$ or $\frac{6}{13}$ was larger
- Question 5 asked students to determine order from the smallest to the largest for fractions with unlike denominators $\frac{3}{18}, \frac{2}{9}, \frac{4}{6}$ and the unit 1
- Question 6 asked for the number that was closest to the sum $\frac{6}{7} + \frac{9}{10}$
See Figure 4.8 for Test B which includes questions like the following:

- Question 1 asked students to select the picture which was $\frac{2}{5}$ shaded blue
- Question 2 asked about the fraction of the rectangle which was shaded blue
- Question 3 asked about the fraction range of the rectangle which was shaded blue
- Question 4 asked students to determine whether fraction $\frac{5}{9}$ or $\frac{6}{13}$ was smaller
- Question 5 asked students to determine order from the largest to the smallest for fractions with unlike denominators $\frac{3}{18}, \frac{2}{9}, \frac{4}{6}$ and the unit 1
- Question 6 asked for the number that was the closest to the sum $\frac{10}{11} + \frac{6}{7}$

![Figure 4.8: Tests A and B](image-url)
4.3.3.2 Pre and post maths tasks

Since the context of the game was on the ordering of fractions, the pre and post maths tasks were designed specifically for students to order given sets of fractions from the smallest to the largest and explain their methods. The impact of the game could be measured more directly from the maths tasks which focused on the ordering of fractions. Similar to the pre and post tests, the maths tasks were administrated twice: the pre task before the game was played and the post task after playing the game. The pre and post maths tasks provided complementary information to the quantitative data of students’ scores obtained from the pre and post tests, especially to when analysing the data if there was any increase and decrease of scores.

This part of the research was a qualitative research design, which was not as prescriptive and structured as, but more flexible than, the quantitative design (Wiersma & Jurs, 2005). By identifying students’ methods or strategies when comparing and ordering fractions through the open question “How do you know?”, a measure of change of strategies could be obtained for the students who played the game. The maths tasks were set in written form so that every participant had an equal chance to complete the maths task individually at their own pace. It was like a structured interview, however, with minimum control and involvement of the interviewer.

Each task consisted of 5 questions (Table 4.11) which included a set of fractions for students to order and a section to give their reasoning. The questions on each task increased in difficulty by including more fractions, fractions with bigger numbers and fractions closer to each other. For example, it was easier to compare sets of two fractions in the first two questions than sets of three fractions in the last three questions. Unit fractions in question 1 were simpler than fractions with like numerators in question 2. The last two questions comparing fractions that were close to each other were relatively difficult.
Table 4.11: Sets of fractions used in the pre and post maths tasks

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre maths task</th>
<th>Post maths task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{5\cdot3}$</td>
<td>$\frac{1}{5\cdot4}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{3\cdot5}$</td>
<td>$\frac{3}{4\cdot5}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2}{5\cdot10\cdot2}$</td>
<td>$\frac{1}{2\cdot14\cdot7}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{6}{7\cdot3\cdot5}$</td>
<td>$\frac{7}{8\cdot4\cdot6}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2\cdot12\cdot17}$</td>
<td>$\frac{4}{7\cdot15\cdot2}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
</tbody>
</table>

4.3.3.3 Computer game attempt

Only students from the experimental group played the game, *Tower Trap*, which involved forming staircases using fraction bricks. The game consisted of 26 game levels and players had to order fraction bricks correctly in each level before they were allowed to move to the next floor of the tower. The game play data of every individual were recorded on computer, including the number of attempts to order fractions correctly, the number of additional questions chosen and scores of the game. Only the number of attempts made at each game level was used for data analysis. The data analysis is discussed in Section 4.4.4. The scores the students obtained were only displayed on the game page to motivate players.

4.3.3.4 Questionnaire

A questionnaire (refer to Appendix 3) was developed to measure students’ evaluations of the game. The questionnaire started by collecting background information of the students. They were asked the following questions to understand their background:

(1) How often do you play computer games?
(2) How good do you think you are at fractions?
The main part of the questionnaire consisted of 21 items in four categories, namely: playing the game, game features, learning fractions and teaching aids. The questionnaire was reviewed by education experts who were experienced in teaching school children mathematics. They focused on the content and language that suited Year 8 students. This maximised the validity of the questionnaire. The questionnaire was administered to the students from the experimental group as a measure of their perceptions of the game after playing it.

The questionnaire used a 5 point Likert-scale for students to respond to a choice of strongly agree, agree, neither agree nor disagree, disagree and strongly disagree in points 1 to 5. Five-point scales are commonly used by researchers in mathematics education to measure such things as students’ attitude towards mathematics (Van Eck, 2006), teachers’ pedagogical and instructional skills (McKinney & Frazier, 2008), and student teachers’ perceptions of mathematics learning (Sezer, 2008).

The Likert scale typically contains an odd number, usually three to seven, of points on a scale, (Wiersma & Jurs, 2005). The midpoint placed between the agree and disagree responses is used for a neutral or undecided selection. As defined by Hodge and Gillespie (2003), the midpoint is a level which bridges the levels signified by agreement and disagreement in a categorical continuity. However, it is possible that the midpoint is perceived as a “don’t know” or “not applicable” option when respondents are not sure how to answer a question due to lack of information or knowledge. In this research, the five-point Likert was adopted to provide an option for a neutral or undecided selection. This was to avoid incorrect information being collected if only “agree” or “disagree” options were given.

4.4 Analyse the data

This section discusses data analysis for the qualitative data obtained from the pre and post tasks and, the quantitative data obtained from the pre and post tests, computer game play and questionnaires.

4.4.1 Data analysis

Table 4.12 shows the number of students who completed the pre and post tasks, pre and post tests, computer game play and questionnaires. The data were used for the analysis in this study.
### Table 4.12: School A and C participants’ populations

<table>
<thead>
<tr>
<th></th>
<th>High achievers</th>
<th>Average students</th>
<th>Cause-for-concern students</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>C</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>Schools</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre and post tasks</td>
<td>8</td>
<td>7</td>
<td>37</td>
<td>13</td>
</tr>
<tr>
<td>Pre and post tests</td>
<td>8</td>
<td>7</td>
<td>37</td>
<td>12</td>
</tr>
<tr>
<td>Computer game play</td>
<td>8</td>
<td>7</td>
<td>37</td>
<td>13</td>
</tr>
<tr>
<td>Questionnaires</td>
<td>8</td>
<td>7</td>
<td>37</td>
<td>12</td>
</tr>
</tbody>
</table>

The data obtained from the samples of the two schools were initially analysed separately to determine if there were differences in outcomes based on inter-school factors (e.g., teacher approach). Since differences between schools in New Zealand were not expected as the school curricular, student populations and teaching methodologies used were similar, the data from School C and School A were then combined for an overall analysis of the outcomes.

#### 4.4.2 Pre and post maths tasks

Students’ written reasoning given to their order of fractions was analyzed using the qualitative methodology of grounded theory. This methodology involves three stages:

- an initial attempt to develop categories which illuminate the data
- an attempt to ‘saturate’ these categories with many appropriate cases in order to demonstrate their relevance
- developing these categories into a more general analytic framework with relevance outside the setting  
  (Silverman, 2005, p. 179).

The students’ answers were first subjected to open coding, where the categories of strategies were freely generated to capture the phenomenon observed in the students’ own words. This allowed potential categories to emerge from the data that would describe students’ strategies for ordering fractions.

Wiersma & Jurs (2005) noted some specific categories may only emerge from the data when patterns of strategies for comparing and ordering fractions start to appear with regularity. If patterns are evident, these could become the categories for coding students’ strategies.
As stated by Kazemi and Franke (2004),

In the analytic process, we made initial conjectures while analyzing existing data and then continually revisited and revised those hypotheses in subsequent analyses. The resulting claims and assertions are thus empirically grounded and can be justified by tracing the various phases of the analysis. (p.211)

As remarked by Wiersma and Jurs (2005), coding categories may be informed by the methods or by strategies noted in research prior to data review. This study considered students’ strategies indicated by Behr, Wachsmuth, Post and Lesh (1984), Smith III (2002), Gould (2005) and, Ministry of Education (2008d). These strategies were:

- considering denominator only
- considering numerator and denominator
- drawing divided quantity diagrams
- finding a common denominator
- converting fractions into percentages or decimals
- using benchmarks

This study also considered the whole number dominance as an invalid strategy for comparing fractions noted by Behr et al. (1984), Newstead and Murray (1998) and Stafylidou and Vosniadou (2004). These categories were

- big numbers are equal to big fractions
- big numbers are equal to small fractions.

In addition to the above categories, a category “$\frac{1}{2}$ is the biggest fraction” was a new category that had not been considered prior to data analysis. It was later identified and created to describe the misconception that led $\frac{1}{2}$ to be ordered incorrectly as the biggest fraction based on students’ reasoning such as “$\frac{1}{2}$ is usually the bigger than any fraction other than 1”.

Table 4.13 provides definitions and examples to the categories of strategies.
<table>
<thead>
<tr>
<th>Categories of strategy</th>
<th>Definitions/ Descriptions</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big numbers are equal to big fractions</td>
<td>Students incorrectly reason that fractions $\frac{1}{m} &gt; \frac{1}{n}$ if denominators $m &gt; n$ for unit fractions and $\frac{p}{m} &gt; \frac{q}{n}$ if denominators $m &gt; n$ and (or) numerators $p &gt; q$ for non-unit fractions.</td>
<td>- 3 is less than 8. Therefore, $\frac{1}{3} &lt; \frac{1}{8}$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- 2 comes before 5 and 3 comes before 11. Therefore, $\frac{2}{3} &lt; \frac{5}{11}$.</td>
</tr>
<tr>
<td>Big numbers are equal to small fractions</td>
<td>Students correctly reason $\frac{1}{m} &gt; \frac{1}{n}$ if denominators $n &gt; m$ for unit fractions and incorrectly reason $\frac{p}{m} &gt; \frac{q}{n}$ if denominators $n &gt; m$ and (or) numerators $q &gt; p$ for non-unit fractions.</td>
<td>- 9 is larger than 5. Therefore, $\frac{1}{9} &lt; \frac{1}{5}$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- The smaller the numbers in fractions the bigger the fractions. Therefore, $\frac{11}{17} &lt; \frac{3}{8}$.</td>
</tr>
<tr>
<td>$\frac{1}{2}$ is the biggest fraction</td>
<td>Students order fractions as though $\frac{1}{2}$ was the biggest fraction.</td>
<td>- $\frac{1}{2}$ is usually bigger than any fraction other than 1. Therefore, $\frac{2}{5} &lt; \frac{7}{13} &lt; \frac{1}{2}$.</td>
</tr>
<tr>
<td>Drawing divided quantity diagrams</td>
<td>Students draw circular or rectangular diagrams that represent fractions for comparison.</td>
<td>-----------------------------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
| Big denominators are equal to small fractions | Students correctly reason $\frac{1}{m} > \frac{1}{n}$ if denominators $n > m$ for unit fractions and sizes of denominators are considered for comparison. | • If a pizza is cut into 12 each piece will be smaller than if it was cut into 9. Therefore, $\frac{1}{12} < \frac{1}{9}$.  
• Fifths are smaller than quarters  
  Therefore, $\frac{1}{5} < \frac{1}{4}$. |
|-----------------|---------------------------------|---|
| Numerators and Denominators | Students correctly reason $\frac{p}{m} > \frac{q}{n}$ if denominators $n > m$ and numerators $p = q$. | • One third is bigger than one fifth because if the numerators are the same, the fraction with the smaller denominator is larger.  
  Therefore, $\frac{1}{5} < \frac{1}{3}$.  
• Three sevenths is bigger than three tenths because if the numerators are the same, the fraction with the smaller denominator is larger. Therefore, $\frac{3}{10} < \frac{3}{7}$. |
| Parts to make a whole | Students correctly reason $\frac{p}{m} > \frac{q}{n}$ if denominators $n > m$ and numerators $p = m - 1$ and $q = n - 1$. | • $\frac{4}{5}$ needs $\frac{1}{5}$ to make 1. $\frac{6}{7}$ needs $\frac{1}{7}$ to make 1. $\frac{6}{7}$ needs a smaller bit than $\frac{4}{5}$ to make one.  
  Therefore, $\frac{6}{7} > \frac{4}{5}$. |
| Finding a common denominator | Students find a common denominator for two (or more) fractions for comparison. | • $\frac{2}{3} = \frac{8}{12}$; $\frac{3}{4} = \frac{9}{12}$. Therefore, $\frac{5}{12} < \frac{8}{12} < \frac{9}{12}$.  
• A common denominator of 15 and $\frac{1}{3}$ is 15 so that $\frac{1}{3} = \frac{5}{15}$ and $\frac{1}{5} = \frac{3}{15}$. Therefore, $\frac{1}{5} < \frac{1}{3}$. |
| Converting fractions into percentage | Students convert fractions into percentages for comparison. | \[
\begin{align*}
\frac{3}{10} &= 30\% ; \quad \frac{2}{5} = 40\% ; \\
\frac{1}{2} &= 50\% . \text{ Therefore, } \frac{3}{10} < \frac{2}{5} < \frac{1}{2} .
\end{align*}
\]  
\[
\begin{align*}
\frac{3}{5} &= 60\% \text{ and } \frac{3}{4} = 75\% . \\
\text{Therefore, } &\frac{3}{5} < \frac{3}{4} .
\end{align*}
\]

| Converting fractions into decimals | Students convert fractions into decimals for comparison. | \[
\begin{align*}
\frac{3}{5} &= 0.6 \text{ and } \frac{3}{5} = 0.75 . \\
\text{Therefore, } &\frac{3}{5} < \frac{3}{4} .
\end{align*}
\]  
\[
\begin{align*}
\frac{3}{4} &\approx 0.7 ; \quad \frac{7}{8} \approx 0.87 ; \quad \frac{5}{6} \approx 0.83 . \\
\text{Therefore, } &\frac{3}{4} < \frac{5}{6} < \frac{7}{8} .
\end{align*}
\]

| Using benchmarks | Students use benchmarks such as \( \frac{1}{2} \) and decide whether the fractions are smaller or larger than \( \frac{1}{2} \). | \[
\begin{align*}
\frac{8}{17} &\text{ is smaller than } \frac{1}{2} \text{ because } \frac{8}{16} = \frac{1}{2} \left( \frac{8}{17} < \frac{8}{16} \right) \text{ and } \frac{7}{12} \text{ is bigger than } \frac{1}{2} \text{ because } \frac{7}{14} = \frac{1}{2} \left( \frac{7}{12} > \frac{7}{14} \right) . \text{ Therefore, } \frac{8}{17} < \frac{1}{2} < \frac{7}{12} .
\end{align*}
\]  
\[
\begin{align*}
\frac{1}{5} &\text{ is smaller than } \frac{1}{4} ; \text{ and } \frac{1}{3} \text{ is bigger than } \frac{1}{4} . \text{ Therefore, } \frac{1}{5} < \frac{1}{3} .
\end{align*}
\]

These categories of strategies could be ordered based on the level of sophistication of the strategies. Pirie-Kieren’s model (1989, 1994a, 1994b) of analysing students’
understanding levels was referred to for identifying the mathematical understanding involved in students’ strategies. The strategies involving lower level of understanding were less sophisticated than the strategies involving higher level of understanding. The levels of understanding were useful to understand the changes of strategies from lower to higher level caused by the game.

In this study, it is found that many students were unable to provide reasoning to the order made in the maths tasks and no specific strategy could be identified. A category of no reasoning was an initial conjecture made to describe such reasoning. After continuously revisiting this hypothesis, this category was split into three groups, which were partial reasoning, incorrect reasoning and no reasoning (Table 4.14). These categories are explained in the following using the example of ordering fraction from the smallest $\frac{1}{3}$ to the largest $\frac{1}{2}$.

**Table 4.14: Partial reasoning, incorrect reasoning and no reasoning**

<table>
<thead>
<tr>
<th>Categories</th>
<th>Order of fractions</th>
<th>Students’ reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial reasoning</td>
<td>“$\frac{1}{3}$ , $\frac{1}{2}$”</td>
<td>One-half is bigger than one-third</td>
</tr>
<tr>
<td>Incorrect reasoning</td>
<td>“$\frac{1}{2}$ , $\frac{1}{3}$”</td>
<td>There were more to go between them both</td>
</tr>
<tr>
<td>No reasoning</td>
<td>“$\frac{1}{3}$ , $\frac{1}{2}$”</td>
<td>I just know</td>
</tr>
</tbody>
</table>

Partial reasoning referred to the repetition of the order of fractions. Incorrect reasoning included incomplete reasoning. This kind of reasoning normally led to incorrect ordering of fractions. No reasoning often referred to non-mathematical reasoning given by the students. The resulting categories of partial reasoning, incorrect reasoning and no reasoning were then empirically grounded.

Through coding students’ strategies, the collected qualitative data were quantitized. This is a process to translate data into numerical codes for statistical analysis (Tashakkori & Teddlie, 1998, 2003). In this study, qualitative data from the pre and post maths tasks was subjected to this coding process to accurately capture the information in the data relative to the students’ strategies.
The following are the quantitative data relevant to students’ strategies:

- The number of strategies used by the students
  The data are presented in pie charts to show groups of strategies in proportion to all groups of strategies used by the students. In this way the strategies commonly used by the high achievers, average and cause-for-concern students can be identified.

- The number of types of strategies used by the students
  The data are shown in column graphs. The relationship between the number of types of strategies used and the number of correct orderings of fractions given (none to ten) in the maths tasks was also shown in column graphs.

- The strategies used by the students for the 5 questions in the pre maths tasks and the 5 questions in the post maths tasks.
  The data are presented in 100% stacked column graphs to compare the percentage that each strategy contributes to a total across the 10 questions in the pre and post maths tasks.

- The number of students who improved in the maths tasks (i.e., wrong in the pre task but right in the post task, which is denoted by WR)
  Students’ ordering of fractions was identified either as wrong (i.e., W) or right (i.e., R). The data of WR are shown in column graphs.

  In this study, the students ordered fractions in 10 fraction questions in the pre and post maths tasks (5 each) and provided reasoning for the 10 questions. If the students used different strategies for each question, they could use up to 10 strategies for ordering fractions.

  Counting the number of times a qualitative code occurs (e.g., the number of strategies used by the students) is one of the common strategies in quantitizing qualitative data (Driscoll, Yeboah, Salib & Rupert, 2007). Such quantitized frequencies are useful to indicate influential codes (e.g., the strategy used most by the students) but possibly confounded by repetitive respondents who fix on a certain concept or theme (Driscoll et al., 2007). For example, a student might rely on drawing diagrams in answering 10 questions in the tasks. This produced a high percentage of students using diagrams among a number of students.

  Driscoll et al. (2007) indicated a disadvantage of quantitizing qualitative data which is the loss of depth and flexibility. In addition, quantitized data like the number of strategies used by the students are fixed and one-dimensional. Nevertheless, the qualitative data of students’ reasoning that provided rich information of students’ thinking was examined in this
study. This was especially for the students who were right in the pre and post tests (denoted by RR) yet changed their strategies between the pre and post tasks. They were investigated to identify the relevant reasons and advantages of the changes of the strategies.

4.4.3 Pre and post tests

Students made selections in the pre and post tests which had a multiple choice format. Their selections that led to right and wrong answers were recorded as R and W, respectively. The right or wrong choices were used to measure achievement differences between the pre and post tests in the following way:

- wrong in the pre test but right in the post test (is denoted by WR),
- right in the pre test but wrong in the post test (is denoted by RW),
- wrong in the pre and post tests (is denoted by WW), and
- right in the pre and post tests (is denoted by RR).

The validity of the quantitative data of achievement differences was enhanced by using two sets of tests. Half the students received Test A then Test B and the other half Test B then Test A. We have noted above that these tests are equivalent so there is no inbuilt bias from this use of the tests. It was necessary to use the tests this way so that the achievement differences measured between the pre and post tests reflected the game effect rather than the influence of the questions in the tests. Since limited information of being right or wrong were collected and analysed, the qualitative data of students’ strategies were collected to provide more information for analysis. All students ordered the same set of fraction questions in the pre and post maths tasks and provided different reasoning of the way of ordering these fractions.

The achievement differences between the pre and post tests were presented using 100% stacked column graphs to compare the contribution of each achievement difference RR, WR, RW and WW to the total score of a question. The selections made by the students who were wrong in the pre task but right in the post task (i.e., WR) were particularly examined to identify particular reasons which caused the improvement.

There were 6 questions tested in each test. Students’ achievement differences between the pre and post tests were measured on an individual question basis. Different aspects of students’ knowledge of fraction were assessed in the tests (as highlighted in Table 4.10). The highest percentage of WR for a particular question was identified to show the positive impact
of the game on learning a specific aspect of fractions. Students’ selections in the multiple choice questions were further examined to identify the mistakes that had been corrected by the WR students.

This study used descriptive statistics including frequencies to analyse the achievement differences between the pre and post tests. Statistical tests were considered to compare the results between the experimental and control groups including standard mixed model repeated measures analysis, Chi square test and, Cochran’s Q test. However, in consultation with a statistics advisor, these were deemed inappropriate because:

- Standard mixed model repeated measures analysis could not be used as the students’ responses were measured on a continuous scale in this study.
- Chi square test was not suitable for this study because it could only be tested to compare each question separately and not between questions.

Therefore, in this study, direct comparison between the experimental and control group using statistical indicators was not possible.

### 4.4.4 Computer game attempts

The mean attempts were computed for the number of trials that high achievers, average and cause-for-concern students took to order fraction bricks at each game level. The data of game play were categorised into four groups: introduction, visible bricks, broken bricks and hidden bricks (Note that these are categories that are related to the game and will be explained in full later).

The mean attempts made by high achievers, average and cause-for-concern students at introduction, visible bricks, broken bricks and hidden bricks were presented using line graphs. This is to display the data of mean attempts that changes continuously (i.e., increase or decrease) over mathematical ability groups. In this way the trends of how mathematics ability affects the number of attempts made for different types of bricks can be shown. The highest mean number of attempts was identified to show the type of bricks and fractions that were the most difficult to play and order.

The Tukey HSD (Honestly Significant Difference) post hoc test was used to identify any significant difference of mean attempts between high achievers, average students and cause-for-concern students. The Tukey test was used after an ANOVA had been completed.
The one-way analysis of variance (ANOVA) could only show there were at least two group mean attempts that were significantly different from each other but it could not decide which specific ability group differed (Lund Research Ltd, 2010). Post hoc analysis was used to find patterns in subgroups of the sample. For example, high achievers took less mean attempts than cause-for-concern students. The Tukey's HSD test is a single-step multiple comparison procedure and statistical test generally used in conjunction with an ANOVA to find which means are significantly different from one another (Zar, 1984).

The Wilcoxon signed ranks test was used to identify any differences between two types of bricks among high achievers, average and cause-for-concern students. This test was used since the data were paired on respondents from individual scores on 1-5 ordinal scales. The Wilcoxon signed ranks test is “a statistical comparison of the average of two dependent samples” (Statistics Solutions, 2010a) or “a non-parametric statistical hypothesis test for the case of two related samples or repeated measurements on a single sample” (Zar, 1984). The test assumed that differences in the dependent variables of attempts taken at visible, broken and hidden bricks were caused by the independent variables of mathematical abilities.

4.4.5 Questionnaires

The background information of experimental group students was shown prior to presenting their results of evaluating the game (will be discussed in Chapter 6). The frequency they play computer games was shown in percentage in the range from never, rarely, once a week, a few times a week to everyday. To reply to the question “How good do you think you are at fractions”, the students selected a scale of 1 to 10 in which the higher the scale was the better the students were with fractions. The percentage of the scale that most students selected was shown.

The mean of students’ rates from points 1 to 5 meaning strongly agree, agree, neither agree nor disagree, disagree and strongly disagree were computed. The qualitative feedback about the game given by the students provided additional information to the relevant topics such as playing the game, game features, learning fractions and teaching aids. The Mann-Whitney U test (also called the Mann-Whitney-Wilcoxon (MWW) or Wilcoxon rank-sum test) was used to identify any differences between School A and School C on the evaluation of the game. This test is a non-parametric statistical hypothesis test for comparing two population means that come from the same population and assessing whether two population
means are equal or not (Statistics Solutions, 2010b). Differences between the three ability groups at School A and School C were analysed using the Kruskal-Wallis test (see Chapter 6) because an ordinal scale had been used for the data. The data were not normally distributed so that an analysis of variance could not reliably be used. The Kruskal-Wallis test is the non-parametric (distribution free) alternative to ANOVA when the sample cannot be assumed to be normally distributed (Zar, 1984; Statistics Solutions, 2010c).

4.4.6 Data integration

According to Johnson and Onwuegbuzie (2004), the findings of a mixed-method design must be “mixed or integrated at some point”. In this study, the quantitative data obtained from the tests, maths tasks, questionnaires and computer game play was integrated with the qualitative data of students’ reasoning obtained from the maths tasks. One high achiever, one average student and one cause-for-concern student were purposefully selected for individual analysis. These students’ achievement and performances in the maths tasks and tests were related to their responses in the computer game and questionnaires. The improvement in the knowledge of fractions was shown using quantitative data of WR in the tests and maths tasks, and qualitative data of the use of strategies in the maths tasks. Such improvement might be relevant to students’ appreciation of the game and attempts taken for ordering fraction bricks in the computer game of Tower Trap. By simultaneously analysing and interpreting students’ performances in the tests and maths tasks, students’ evaluation towards the game and students’ game play on computers, both quantitative and qualitative data were integrated into a coherent whole for a better understanding of the effectiveness of Tower Trap on learning of fractions for individual students.

4.4.7 Limitation of design

There are challenges of combining qualitative and quantitative research methods in social sciences. In answering the research question “What improvements, if any, were there in students’ ability as a result of playing the game?”, both qualitative and quantitative research methods were employed. The quantitative data of achievement differences were computed by comparing the pre test and post test scores. The qualitative data of students’ reasoning given on the pre and post maths tasks were examined to obtain more insight into students’ thinking. In the analysis of the quantitative data of achievement differences, this study did not detect a significant difference between the experimental and control groups. Various statistical tests
had been considered but none of the tests were suitable and the data could only be analysed using descriptive statistics. In addition, there was a big difference between the number of students in the experimental group and the number of students in the control group. Nevertheless, the comparisons between the experimental and control groups were not the main focus of this study. The results of achievement difference between the pre and post tests only showed the area of fractions or type of question that the students improved or not. The main focus of this study was to investigate any changes of students’ strategies between the pre and post tasks. As such, the failure in adopting experimental approach in this part of study did not affect the conclusions regarding the effects of the game on students’ learning of fractions. The mixed methods design was necessary to address the above research question, but better planning should have been employed so that appropriate statistical tests could have been used.

This chapter discusses the research method employed in this study. Nevertheless, the Tower Trap computer game used in the main study will be discussed in the next chapter. The design and development of Tower Trap will be discussed in detail in Chapter 5 which includes the structure of Tower Trap, game features and game pedagogy.
Chapter 5 describes the process of designing the computer game of fractions called "Tower Trap." Section 5.1 elaborates on the structure of "Tower Trap" which integrates instructional factors and game elements as the fundamentals in designing a digital game. The mathematical learning of the game is further assured by cross-checking with the criteria outlined in educational software reviews. The structure of "Tower Trap" was tested in a pilot study using manipulatives of fraction cards to identify students’ needs that can be incorporated in the development of the computer game. Section 5.2 elaborates the game features of game world and game play. Section 5.3 explains the game pedagogy through the use of key ideas pages, specific instructions, cognitive oriented feedback, tips pages, alternate questions and game user data.

The "Tower Trap" computer game described in this chapter was used to collect data in the main study as discussed in Chapter 4. Apart from discussing the research method used in this study, Chapter 5 also present findings to answer research question 1 in conjunction with discussion in Chapter 6:

*What needs to be considered in developing a computer game that would enable students to compare and order fractions?*

This research question was addressed from two perspectives. Chapter 5 describes the design of the computer game for learning fractions while Chapter 6 examines students’ evaluation of the game and their attempts in playing the game.

### 5.1 Structure of "Tower Trap"

The effectiveness of the game "Tower Trap" in achieving educational values and mathematical learning was ensured by considering the instructional factors suggested by Booker (2000, 2004) which included conceptual analysis of fractions, students’ difficulties and misconceptions on fractions, instructional strategies and students’ strategies. To create an engaging game for learning fractions, the six key structural elements of computer games proposed by Prensky (2001a) were adopted to include rules, goals and objectives, outcomes and feedback, conflict/competition/challenge/opposition, interaction and story. The criteria of
educational software reviews (on rational numbers) listed by Kafai, Franke and Battey (2002) were also referred to. This list included topic, students’ strategies, representations, context and content integration. In the structure of Tower Trap, the instructional factors were integrated with the game elements to design digital outputs and the benefit of the game for mathematical learning was further assured using the criteria of educational software. The digital outputs refer to what the player can see and do (interact) in the computer game of Tower Trap. These were shown to the player on the computer through text, graphic, animation and interactivity. For example, the player could see different types of fraction bricks that could be ordered from the smallest to the largest in the game. Figure 5.1 is given to provide an overview of the structure of Tower Trap.

![Figure 5.1: Structure of Tower Trap](image-url)
5.1.1 Learning objectives of ordering fractions from the smallest to the largest and vice versa

Order and equivalence are noted as the most basic topics in fractions yet they are difficult for students to understand even after learning fractions for a few years in school (Behr, Wachsmuth, Post & Lesh, 1984; Smith III, 2002). The order of fractions is also a critical aspect of fractions to school children and a key area of knowledge in the New Zealand Number Framework (Young-Loveridge, 2006, 2007; Ministry of Education, 2008a). In addition, more educational software focuses on the operations on fractions than the development of equivalence (Kafai et al., 2002). Hence, the topic of comparing sizes of fraction was identified as the mathematical topic to focus on in the computer game I was to develop. The learning objectives of the game were to order fractions from the smallest to the largest and vice versa.

5.1.2 Problem context of forming fraction brick staircases

A problem context is needed for students to apply their knowledge of fractions meaningfully in the game. A fantasy context for fraction content was created in the Tower Trap through the story of a boy who was lost in the woods and wanted to climb up to the top of a tower to see the way home. This gave a goal to students to move from one floor to the next using the staircases until reaching the top of the tower. The fantasy context attracts student interest and promotes abstract and creative thinking. Students are likely to engage with the context more fully and process problems more deeply to establish underlying mathematical relationships if they are allowed to work with more varied and less familiar problem types in a fantasy context (Wiest, 2001).

5.1.3 Visualize and manipulate representations of fractions

The main rule of the game was to drag the fraction bricks that represented fractions, and then drop them one by one, on a floor of the building to form staircases. Once the fraction bricks were ordered, the staircases that were made up from the fraction bricks were shown, and this allowed students to compare fractions side by side in a concrete way and for a purpose. This created an intrinsic integration where the game context was merged with the fraction content so that the context and content could enhance each other (Kafai et al., 2002). The graphics of fraction bricks were used to capture the students’ interest. Visualizing the representations of fractions enabled students to gain a concrete meaning to abstract
mathematical notations of fraction symbols. Manipulating fraction bricks engaged students actively in making sense of fraction sizes and so enhance their understanding of fractions.

### 5.1.4 Fraction bricks

The main components of the fraction game were rectangular fraction bricks which represented sizes of fractions and were labelled with the symbols of the fractions that the bricks represented. Various types of fractions bricks were created for the context of forming fraction brick staircases in the game. Through playing with different types of fraction bricks, students were given opportunities to manipulate a combination of multiple representations of fractions. This was important to assist students constructing their conceptual knowledge of fractions (Kafai et al., 2002).

The following describes different types of fraction bricks: *visible bricks*, *broken bricks* and *hidden bricks*. These brick types were designed to enable the students to experience fractions from concrete to abstract. This will be discussed in Section 5.1.6. The bricks are further subdivided into tall and long to enable *tall fraction bricks* to be ordered from the smallest to the largest and *long fraction bricks* to be ordered from the largest to the smallest. Therefore, six types of fraction bricks are generated, namely, *tall visible bricks*, *tall broken bricks*, *hidden tall bricks*, *long visible bricks*, *long visible bricks*, *hidden long bricks* (Figure 5.2).

![Figure 5.2: Types of fraction bricks](image)

---

125
5.1.4.1 Tall and long bricks

Fraction bricks came in two different types, which were tall bricks and long bricks. *Tall bricks* were used to order fractions from the smallest to the largest. An example of *tall fraction bricks* $\frac{1}{2}$, 1 and $\frac{1}{3}$ is shown in Figure 5.3.

![Figure 5.3: Tall bricks](image)

Long bricks were used to order fractions from the largest to the smallest. An example of *long fraction bricks* $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$ is shown in Figure 5.4.

![Figure 5.4: Long bricks](image)

In order to form staircases using *tall fraction bricks*, the shortest brick is ordered first and then followed by the next tallest brick and finally the tallest brick. Figure 5.5 shows that *tall fraction bricks* $\frac{1}{3}$, $\frac{1}{2}$ and 1 are dragged using a hand cursor and dropped at the indicated location. Note that the cursor of an arrow only changes to a hand when the player is dragging fraction bricks. This way of arranging *tall fraction bricks* represents the ordering of fractions from the smallest to the largest. The instruction “Place Brick Here” shows the location where the brick is to be placed but not the order of the bricks. Figure 5.6 shows a staircase is formed by the *tall fraction bricks* $\frac{1}{3}$, $\frac{1}{2}$ and 1.
In order to form staircases using the *long bricks*, the longest brick is ordered first and then followed by the next shortest brick and finally the shortest brick. Figure 5.7 shows *long fraction bricks* $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$ being dragged using a hand cursor and dropped at the indicated location with the instruction “Place Brick Here”. This way of arranging long fraction bricks represents the ordering of fractions from the largest to the smallest. Figure 5.8 shows a staircase is formed by the *long fraction bricks* $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{3}$. 

Figure 5.5: *Tall fraction bricks* were dragged and dropped to form a staircase

Figure 5.6: A staircase of *tall fraction bricks*

Figure 5.7: The *long fraction bricks* were dragged and dropped to form a staircase
The purpose of inventing the two types of fraction bricks in *Tower Trap* was to enable fractions to be ordered in two opposite ways: increasing and decreasing ordering of fractions. Students in New Zealand are familiar with the increasing ordering of fractions as they learn to put fractions in order from the smallest to the largest in the NDP programme (Ministry of Education, 2008d). However, ordering of *long bricks* from the largest to the smallest in *Tower Trap* required students to apply their knowledge of fractions in a different condition rather than memorising the order of familiar fractions. For example, “\(\frac{1}{2}\) is smaller than \(\frac{2}{3}\)” is the example that teachers like to use in teaching fractions. In *Tower Trap*, students arranged \(\frac{1}{2}\) first and then \(\frac{2}{3}\) if \(\frac{1}{2}\) and \(\frac{2}{3}\) were *tall bricks* that had to be ordered from the smallest to the largest. On the other hand, \(\frac{2}{3}\) was arranged first and then \(\frac{1}{2}\) if \(\frac{1}{2}\) and \(\frac{2}{3}\) were *long bricks* that had to be ordered from the largest to the smallest.

### 5.1.4.2 Visible bricks

*Visible bricks* were the physical representations of fractions that connected symbols and sizes of fractions. Figure 5.9 and Figure 5.10 show the *tall visible bricks* \(\frac{1}{3}\), \(\frac{2}{3}\) and \(\frac{3}{3}\) and, *long visible brick* \(\frac{1}{4}\), \(\frac{2}{4}\), \(\frac{3}{4}\) and 1. Labelled with mathematical notations, the bricks enabled students to connect symbols of fractions with the sizes of fractions (e.g., symbol of \(\frac{1}{3}\) refers to size of third).
The advantage of educational software in linking multiple representational formats such as symbolic and graphic is recognised to develop students’ understanding of mathematics (Kafai et al., 2002). Nevertheless, students could play the visible bricks without thinking hard because this type of brick allowed them to compare fractions in a concrete way.

5.1.4.3 Broken bricks

Broken bricks represented fractions in divided quantity or partitioned quantity forms.

Figure 5.11 and Figure 5.12 show the tall broken bricks $\frac{2}{4}$, $\frac{3}{4}$, $\frac{1}{4}$ and 1 and long broken bricks $\frac{6}{7}$, $\frac{2}{3}$ and $\frac{4}{5}$.

Students were given limited control to manipulate broken bricks where the blue parts could be selected from the whole and the symbol of each part changed according to the size of the parts selected. As shown in Figure 5.11 and Figure 5.12, the given fractions $\frac{2}{4}$, $\frac{3}{4}$, $\frac{1}{4}$ and 1 and $\frac{6}{7}$, $\frac{2}{3}$ and $\frac{4}{5}$ were represented by the blue parts of tall broken bricks and long broken bricks.
bricks, respectively. Therefore, all of the blue parts needed to be selected and dragged from the whole to form a fraction brick staircase.

Figure 5.13 (a) to (d) show specifically the fraction \( \frac{2}{4} \) is obtained by separating the blue parts of two quarters. In (a), the symbol \( \frac{1}{4} \) is shown once the hand cursor touches the quarter part. In (b) and (c), the symbol changes to \( \frac{2}{4} \) when two quarter parts are selected. In (d), the fraction \( \frac{2}{4} \) (i.e., the blue parts) can be split from the whole.

Figure 5.13: Selecting parts of tall broken bricks

In Figure 5.14, the selected fractions \( \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \) and 1 are dragged and dropped to the indicated locations.
Figure 5.14: Drag and drop tall broken bricks

An example of *long broken bricks* is shown in Figure 5.15 (a) to (d). The symbol $\frac{1}{7}$ is shown once the hand cursor touches the seventh part (a). When two, three, ... and six seventh parts are selected, the symbol changes to $\frac{2}{7}$, $\frac{3}{7}$, ..., $\frac{6}{7}$, respectively (b & c). The fraction $\frac{6}{7}$ (i.e., the blue parts) can be split from the whole (d).

Figure 5.15: Selecting parts of *long broken bricks*

As the given fractions $\frac{6}{7}$, $\frac{2}{3}$ and $\frac{4}{5}$ are represented by the blue parts of *long broken bricks*, all the blue parts need to be selected and dragged from the whole. In Figure 5.16, the selected fractions $\frac{6}{7}$, $\frac{4}{5}$ and $\frac{2}{3}$ are dragged and dropped to the indicated locations to form a staircase (d).
Splitting the selected blue parts from the whole provided an opportunity for children to engage with representation. The wholes of the bricks were in the same size to highlight the important concept of equal whole when comparing sizes of fractions. This provision was especially important to remind students of their typical error in drawing fractions where many of them thought the size of the whole for each fraction was in proportion to the size of the denominator (Yoshida & Kuriyama, 1995).

*Broken bricks* not only provided a mechanism for players to connect the divided quantity strategy with the task of comparing fraction sizes, it also functioned as a method of teaching fractions to cause-for-concern children. A divided quantity diagram is often used in the classroom to show on paper that an object is divided into equal parts. Fractions can be represented using divided quantity diagrams, the sizes of the whole for fractions with different denominators are equal, and the sizes of the parts for a given denominator have to be equal too (Yoshida & Sawano, 2002). Therefore, fractions are comprehended as the relationship of parts of a whole, rather than as mathematical notations consisting of ‘top’ and ‘bottom’ numbers. By highlighting the part-whole concept of fractions using *broken bricks*, students’ potential difficulties with the mental picture of fractions and the misconception of fractions as two independent numbers could be resolved.
5.1.4.4 Hidden bricks

Figure 5.17 shows only fraction symbols but no sizes are displayed on hidden bricks.

![Figure 5.17: Hidden bricks](image)

The physical sizes of these bricks did not become visible until after players had placed them in an order. An example of ordering tall hidden bricks $\frac{1}{5}$ and $\frac{1}{3}$ is shown in Figure 5.18 (a) to (c). The actual relevant sizes of $\frac{1}{5}$ and $\frac{1}{3}$ are only shown when the hidden brick is dropped at the indicated location to form a staircase.

![Figure 5.18: Ordering tall hidden bricks](image)

Ordering hidden bricks was harder than previous brick types because students had to interpret the fraction symbols and judge the sizes of the bricks solely from these symbols. Conflicts arise from students’ difficulties with fractions such as the limited understanding of the mental picture of fractions and misinterpretations of fractions as two independent numbers. Due to the inability to formulate a mental picture of fractions, students rely merely on the algorithm to find a common denominator to compare fractions (Ward, 1999). Students always find it hard to understand the relationship between the numerator and the denominator and regard fractions as consisting of two independent numbers due to their prior knowledge of natural numbers (Stafylidou & Vosniadou, 2004).
The set of fractions which had been used in broken bricks was also asked in hidden bricks. Comparing these fractions with only symbols of fractions without showing the sizes of fractions required advanced thinking of fractions (Pirie & Kieren, 1989, 1994b). Students gained the idea of fractions in divided quantity from manipulating broken bricks. This can be applied in interpreting symbols of fractions in hidden bricks.

### 5.1.4.5 Sequence of brick types in Tower Trap

Figure 5.19 shows the sequence for students to play Tower Trap: tall bricks (visible and broken), long bricks (visible and broken) and hidden bricks (tall and long). Tall bricks were ordered first and long bricks were ordered later. Students have been learning to put fractions in order from the smallest to the largest (Ministry of Education, 2008d). They would be more familiar with ordering tall bricks from the smallest to largest than ordering long bricks from the largest to the smallest.

![Figure 5.19: Game sequence](image-url)
In *tall bricks*, the concrete sizes of fractions were shown using *tall visible bricks*. This was followed by representing fractions in divided quantity using *tall broken bricks*. Once the parts of the *broken bricks* were dragged and dropped to form a staircase, the parts of the *broken bricks* changed into the actual sizes of fractions. Since *broken bricks* show how *visible bricks* were formed by taking parts out of the divided parts of the whole, *tall broken bricks* were arranged to be played right after *tall visible bricks* were ordered in the game. Similarly, in *long bricks*, students visualize fractions through ordering *long visible bricks* and construct an understanding of part-whole fraction through ordering *long broken bricks*. The students were expected to learn more about fraction sizes and order in these game levels. Finally, *hidden bricks* which showed only the symbols of fractions were played by the students. *Hidden tall bricks* required students to order fractions from the smallest to largest while *hidden long bricks* required students to order fractions from the largest to the smallest. The students were expected to apply their understanding of fraction sizes and order in ordering *hidden bricks*.

Referring to Figure 5.19, an introduction level was given when a brick type was played for the first time. “Tall” and “visible” types of fraction bricks were introduced using Introduction 1; the “broken” type of fraction bricks was introduced using Introduction 2; “long” type of fraction bricks was introduced using Introduction 3; “hidden” type of fraction bricks was introduced using Introduction 4. The figure also shows that each type of bricks consists of a number of levels of fraction questions as follows:

- *Tall visible bricks* consist of Levels 1 and 2 (i.e., 2 fraction questions).
- *Tall broken bricks* consist of Levels 1 to 4 (i.e., 4 fraction questions).
- *Long visible bricks* consist of Levels 1 and 2 (i.e., 2 fraction questions).
- *Long broken bricks* consist of Levels 1 to 5 (i.e., 5 fraction questions).
- *Hidden tall bricks* consist of Levels 1 and 4 (i.e., 4 fraction questions).
- *Hidden long bricks* consist of Levels 1 to 5 (i.e., 5 fraction questions).

### 5.1.4.6 Difficulty levels of fractions

Because of their simple nature, \( \frac{1}{2} \) and \( \frac{2}{2} \) were used at the introductory levels for players to warm-up and get used to the rules of the game before involving them in the learning of fractions. Only fractions less than or equal to the unit 1 were used in the game as a whole because the whole of brick had a size of unit 1 and the fraction bricks were smaller.
than the whole. Two to five fractions were used at every level for students to compare and order. Difficulty levels to compare and order fractions were increased in accordance with the involvement of numerators and denominators as shown in Figure 5.20.

<table>
<thead>
<tr>
<th>Difficulty levels</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Like denominators</td>
<td>$\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$</td>
</tr>
<tr>
<td>Unit fractions</td>
<td>$\frac{1}{5}, \frac{1}{3}$</td>
</tr>
<tr>
<td>Unlike denominators with magnitude of numerators that are equal</td>
<td>$\frac{2}{5}, \frac{2}{3}$</td>
</tr>
<tr>
<td>Fractions with unlike denominators and numerators</td>
<td>$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}$</td>
</tr>
<tr>
<td>Fractions that are close to 1</td>
<td>$\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$</td>
</tr>
<tr>
<td>Fractions that are close to a half</td>
<td>$\frac{8}{17}, \frac{1}{2}, \frac{7}{12}$</td>
</tr>
</tbody>
</table>

**Figure 5.20: Difficulty levels in the fraction game**
Table 5.1 shows the set of fractions used in the game levels.

<table>
<thead>
<tr>
<th>Fraction brick types</th>
<th>Levels</th>
<th>Fraction questions</th>
<th>Alternate questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tall visible bricks</strong></td>
<td>Level 1</td>
<td>( \frac{1}{3}, \frac{2}{3}, \frac{3}{3} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td>( \frac{1}{3}, \frac{2}{3}, \frac{1}{3} )</td>
<td>( \frac{1}{4}, \frac{2}{3}, \frac{1}{3} )</td>
</tr>
<tr>
<td><strong>Tall broken bricks</strong></td>
<td>Level 1</td>
<td>( \frac{1}{4}, \frac{3}{4}, \frac{1}{4} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td>( \frac{1}{8}, \frac{3}{4}, \frac{1}{2} )</td>
<td>( \frac{1}{16}, \frac{8}{4}, \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td>Level 3</td>
<td>( \frac{1}{5}, \frac{3}{3} )</td>
<td>( \frac{1}{7}, \frac{5}{3} )</td>
</tr>
<tr>
<td></td>
<td>Level 4</td>
<td>( \frac{2}{5}, \frac{2}{3} )</td>
<td>( \frac{3}{7}, \frac{5}{3} )</td>
</tr>
<tr>
<td><strong>Long visible bricks</strong></td>
<td>Level 1</td>
<td>( \frac{1}{4}, \frac{3}{4}, \frac{1}{4} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Level 2</td>
<td>( \frac{1}{5}, \frac{2}{3}, \frac{2}{3} )</td>
<td>( \frac{1}{5}, \frac{2}{3}, \frac{5}{3} )</td>
</tr>
<tr>
<td></td>
<td>Level 3</td>
<td>( \frac{3}{10}, \frac{2}{5}, \frac{1}{2} )</td>
<td>( \frac{3}{15}, \frac{5}{5}, \frac{3}{2} )</td>
</tr>
<tr>
<td></td>
<td>Level 4</td>
<td>( \frac{2}{3}, \frac{4}{6}, \frac{7}{3} )</td>
<td>( \frac{3}{4}, \frac{5}{6}, \frac{8}{3} )</td>
</tr>
<tr>
<td></td>
<td>Level 5</td>
<td>( \frac{8}{17}, \frac{1}{2}, \frac{7}{12} )</td>
<td>( \frac{6}{13}, \frac{5}{2}, \frac{9}{9} )</td>
</tr>
</tbody>
</table>

Fractions with like denominator (e.g., \( \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \) at level 1 of tall visible bricks and \( \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{4} \) at level 1 of long visible bricks) needed to be ordered before fractions with unlike denominator (e.g., \( \frac{1}{3}, \frac{1}{2}, \frac{1}{3} \) at level 2 of tall broken bricks and \( \frac{1}{3}, \frac{2}{3}, \frac{1}{3} \) at level 2 of long broken bricks). Ordering simple fractions with like denominators is the knowledge developed at the lower stages (i.e., to move students from Stage 4, Advanced Counting to Stage 5, Early
Additive Part-Whole) and is required to develop the knowledge of ordering unit fractions at the higher stages (i.e., to move students from Stage 5, Early Additive Part-Whole to Stage 6, Advanced Additive) (Ministry of Education, 2008d). The size of fractions with like denominators increases when numerators increase like counting numbers. In contrast, the size of unit fractions with unlike denominators increases when denominators decrease. Students progress through the stages in order to expand their knowledge and strategies. They build new strategies on their existing strategies and always revert to previous strategies when faced with unfamiliar problems, or when the mental load gets high (Ministry of Education, 2008a).

Similar to ordering unit fractions with unlike denominators, fractions with unlike denominators that have equal magnitudes of numerators (>1) increase, when the number in the denominators decreases. Nevertheless, fractions $\frac{2}{3}$ and $\frac{2}{5}$ which require more thinking were ordered at Level 4 to complement unit fractions $\frac{1}{3}$ and $\frac{1}{5}$ at Level 3 of tall broken bricks.

Fractions with unlike denominators can be converted into equivalent forms for comparison. Table 5.2 shows numerical conversions involving common denominators are getting complicated when the magnitudes of the fractions become larger.

**Table 5.2: Strategies for comparing fractions**

<table>
<thead>
<tr>
<th>Fraction question</th>
<th>Common denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$</td>
<td>$\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$, $\frac{1}{2} = \frac{1 \times 4}{2 \times 4} = \frac{4}{8}$ Therefore, $\frac{1}{8} &lt; \frac{2}{8} &lt; \frac{4}{8}$</td>
</tr>
<tr>
<td>$\frac{2}{10}$, $\frac{2}{5}$, $\frac{1}{2}$</td>
<td>$\frac{2}{5} = \frac{2 \times 2}{5 \times 2} = \frac{4}{10}$, $\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10}$. Therefore, $\frac{3}{10} &lt; \frac{4}{10} &lt; \frac{5}{10}$</td>
</tr>
<tr>
<td>$\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$</td>
<td>$\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$, $\frac{4}{3} = \frac{4 \times 5}{3 \times 5} = \frac{20}{15}$. This shows $\frac{10}{15} &lt; \frac{12}{15}$</td>
</tr>
<tr>
<td>$\frac{4}{5}$, $\frac{7}{5}$</td>
<td>$\frac{4}{5} = \frac{4 \times 7}{5 \times 7} = \frac{28}{35}$, $\frac{6}{5} = \frac{6 \times 7}{5 \times 7} = \frac{42}{35}$. This shows $\frac{28}{35} &lt; \frac{30}{35}$</td>
</tr>
<tr>
<td>Therefore, $\frac{2}{3} &lt; \frac{4}{5} &lt; \frac{6}{7}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{8}{17}$, $\frac{1}{2}$, $\frac{7}{12}$</td>
<td>$\frac{8}{17} = \frac{8 \times 12}{17 \times 12} = \frac{96}{204}$, $\frac{1}{2} = \frac{1 \times 102}{2 \times 102} = \frac{102}{204}$, $\frac{7}{12} = \frac{7 \times 17}{12 \times 17} = \frac{119}{204}$</td>
</tr>
<tr>
<td>Therefore, $\frac{96}{204} &lt; \frac{102}{204} &lt; \frac{119}{204}$</td>
<td></td>
</tr>
</tbody>
</table>
5.1.5 Strategies for comparing and ordering fractions

The Tower Trap facilitated multiple strategies of ordering fractions through the use of different types of fraction bricks. The game became increasingly challenging through comparing fractions concretely using visible bricks, visually using broken bricks and abstractly using hidden bricks. When comparing and ordering hidden bricks, students could use a range of strategies taught in school such as finding a common denominator, using a benchmark and converting fractions into percentages or decimals. As discussed in Section 5.3.4, tips pages are provided in the computer game. As such, the above strategies were displayed in the game for students to refer to, especially if they faced a difficulty in ordering the fraction bricks correctly. By showing different strategies for ordering a particular set of fractions in game levels, students were given the opportunity to identify the most simple strategy for certain numerical situations. This also provided a mechanism for students to make connections across strategies of ordering fractions.

5.1.6 Visible, broken to hidden bricks: Concrete to abstract

Visible, broken and hidden bricks were created for students to experience fractions from concrete to abstract. Such transition allows constructivist learning in the game where students reconstructed and enhanced their knowledge of fractions through the interconnection of physical, pictorial and symbolic modes of fractions (Pirie & Kieren, 1989, 1992). Playing with visible fraction bricks enabled students to be familiar with the physical sizes of fractions. Manipulating fraction bricks encouraged students to connect symbols and representations of fractions. On the pictorial representation of broken bricks that were partitioned into parts, students constructed the part-whole concept of fractions by splitting blue parts from the whole to form fractions. On hidden fraction bricks, they were required to make sense of fraction symbols before the physical sizes of fraction bricks appeared. Using manipulatives (i.e., fraction bricks) in the educational task (i.e., ordering fractions) is a way to engage students’ thinking actively with guidance. Computers allow more flexible action on manipulatives and this helps students to explore mathematical ideas (Clements, 1999).

Moving across different fraction bricks, students’ understanding of fractions progressed from operating on concrete materials (i.e., actual sizes of visible bricks), to visualisation (i.e., partitioned figure of broken bricks), to abstraction (i.e., symbols of fractions on hidden bricks) (Pirie & Kieren, 1989). After working with concrete blocks that represented fractions in physical form at the initial visible bricks stage, and visualizing the
images of fractions in divided quantity diagrams at the visualisation *broken bricks* stage, students were expected to abstract the number properties without visualising the initial situation at the final, *hidden bricks*, stage. This model of abstraction had been used extensively in the Numeracy Projects in New Zealand (Ministry of Education, 2003) to provide physical experiences of forming images of, noticing similarities between, and reflecting upon mathematical concepts that are inaccessible to most learners (Linsell, 2005).

5.1.7 **Incorporation of students’ needs into the development of Tower Trap**

The preliminary design of *Tower Trap* was tested by asking the students to order fraction cards from the smallest to the largest in the pilot study before the more advanced design was developed on computers including different brick types (i.e., *visible*, *broken* and *hidden bricks*) and ordering sequence (i.e. order *tall bricks* from the smallest to the largest and *long bricks* from the largest to the smallest). The findings of the pilot study showed that students with different mathematical abilities were advantaged by the game in different ways and had different expectations of the game. In order to cater for different needs, the following students’ concerns were incorporated into the development of *Tower Trap* to create an instructional computer game for all abilities.

The high achievers were too comfortable and familiar with a particular strategy. They referred to the denominators of fractions, used benchmarks and converted fractions into percentages, respectively. Unfortunately, their preferred strategy also produced constraints for them in comparing certain types of fractions. The strategy appropriate for one ordering task was not necessarily appropriate for others. Students should know a variety of ways of ordering fractions and be able to evaluate and determine the strategy that is most reasonable to use. Learning to use different strategies is a way to gain the strongest and more durable knowledge of fractions (Smith III, 2002). *Tower Trap* which is about the representation of fractions and comparing fractions using divided quantity diagrams (i.e., *broken bricks*) would facilitate the high achievers’ computing procedures. High achievers can justify their numerical strategies using divided quantity diagrams and be aware of the magnitude relations between fractions when focusing on the computing procedures. As suggested by Sophian and Madrid (2003b), students’ conceptual understanding of fractions helps their learning in computational procedures.
The high achievers required harder tasks that would allow them to learn more about fractions in the game. When the cause-for-concern students found some questions that were hard to compare, the high achievers might found these fractions were too easy for them. As such, *Tower Trap* included sets of fractions with a range of difficulty so that the game was also challenging to the high achievers. Fractions could be posed in increasingly complicated and difficult ways to maintain the challenge of the game. For fractions with like denominators, the numbers become bigger and more fractions are involved. For example, \( \frac{1}{7}, \frac{4}{7}, 1 \) in Level 1 and \( \frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99}, 1 \) in Level 2 (refer to Table 5.1). Fractions with unlike denominators started with unit fractions, and were followed by non-unit fractions that involve the same numerators and different numerators and denominators. For example, \( \frac{1}{5}, \frac{1}{3} \) in Level 3 and \( \frac{2}{5}, \frac{2}{3} \) in Level 4 (refer to Table 5.1).

The average students used various strategies such as divided quantity diagrams, numerical conversion and benchmarks, but they had yet to construct a firm knowledge of these strategies. They were not confident using the strategies to order fractions. They were more convinced after verifying their answers in symbols of fractions with the images of fractions shown on the cards. This shows manipulatives help students think about mathematical ideas (Higgins, 2003) so that students can construct the mathematical knowledge in meaningful ways (Fosnot, 1996; Clements, 1999; Moseley, 2005). Visualising the sizes of the fractions allowed these students to make better sense of the characteristics of fractions. The students think consciously about the noticed properties of the images until the common structure is formulated and symbolised (Linsell, 2005).

All of the “average” students struggled with the complicated fractions that they were not familiar with such as \( \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{17}, \frac{7}{12} \). For the fractions of \( \frac{2}{3}, \frac{4}{5} \), they relied on the pictorial image to lead them to the thinking based on the divided quantity diagram. Comparing fractions using the representations of fractions in the game promoted students’ ability to construct the mental images of fractions and further enhanced their strategies in solving problems involving fractions. In this case, pictures and images are used by students to help them understand mathematical situations and construct mathematical concepts (Sharp & Adams, 2002).
The average students agreed that the game provided an opportunity to visualize the concrete sizes of fractions, connect symbols and representations of fractions, relate to the divided quantity thinking and construct a mental picture of fractions. Such learning experiences were important for students to understand order and equivalence as the central characteristics of fractions. *Tower Trap* showed the concrete sizes of fractions using visible bricks, linked the symbols and representations of fractions by labelling the fraction bricks, and encouraged divided quantity thinking using broken bricks. The design of *visible, broken and hidden bricks* enabled students to move through fractions represented in physical form, images of divided quantity diagrams and mathematical notations. This provided an opportunity for students to construct a mental picture of fractions.

Two cause-for-concern students believed that big numbers were equal to small fractions and ordered unit fractions using the faulty reasoning. For example, \( \frac{1}{5} \) and \( \frac{1}{3} \) were ordered as \( \frac{1}{5} > \frac{1}{3} \). The inability to see a fraction as a quantity (i.e., a number) causes students to interpret it as two separate whole numbers in the counting strategy (Newstead & Murray, 1998; Empson, 1999; Olive & Vomvoridi, 2006). The numerators and the denominators were considered separately for comparing fractions by the method of the ordering of whole numbers. After the relationship between numerators and denominators of fractions were explained to them using the representations of fractions on the fraction cards, they finally realised that it was a misconception to consider fractions as two independent numbers. Indeed, concrete materials are helpful for students to check their thinking (Ministry of Education, 2010b).

Through manipulating the fraction cards where the same sized card represented a whole, and visualizing coloured parts of cards that represented sizes of fractions, the students acquired the knowledge of equal whole and constructed the knowledge of fractions as parts of a whole. It would appear that comparing fractions in a concrete way using the fraction cards helped to resolve students’ misconceptions, which had been shown could be changed in the process of learning (Stafylidou & Vosniadou, 2004).

The cause-for-concern students expected more teaching of fractions from the game. One of them needed help from the researcher at the beginning of the game but was able to catch up later. She applied the divided quantity strategy taught by the researcher but kept
forgetting and required guidance throughout the game. Based on this finding, it was decided that some teaching of fractions should be provided in the game for students who failed to order fraction bricks due to their misconceptions and difficulties with fractions. In *Tower Trap*, cognitively oriented feedback (see Section 5.3.3) was given for the player who ordered fractions incorrectly and needed help to continue the game. Subsequently, tips pages (see Section 5.3.4) were shown to present the steps of comparing fractions using several strategies such as finding a common denominator, numerical conversions and using benchmarks. Specific feedback and detailed explanations on the strategies or methods for comparing and ordering fractions are helpful not only to play the game but also to learn more about fractions. In addition, correcting and highlighting students’ errors through the specific instances of feedback in written form on the computer makes them more aware of their misconceptions (Reimer & Moyer, 2005). Besides, alternate questions (see Section 5.3.5) were posed in the *Tower Trap* as similar questions and they were useful to assess any newly constructed knowledge of fractions.

Although fraction cards were found to be effective for learning fractions, digitalization of the fraction cards is more advantageous to students. At least two players were required to play the fraction cards, where the first player showed the side of cards with fraction symbols and the second player ordered the fraction cards based on the symbols. The students needed to take turns to ask and order fraction cards. On the other hand, the computer game of ordering fraction bricks could be played even by only one player because the player could interact with the computer and sets of fractions were given randomly by the computer. This could be the reason why many traditional card games are developed into digital forms.

Many interesting factors could be found on computer games to enhance the learning. The goal of forming fraction brick staircases made ordering fraction bricks meaningful. The story of climbing up to the tower to see the way home motivated students to keep playing until they completed the game. The opportunities to visualize and manipulate representations of fractions engaged students with constructing fraction concept. The advancement of multimedia like animations, graphics and interactivity attracted the players to play the game. Without the story, goal and multimedia, students would order fraction cards in the same way that they would order fractions in textbooks. There was inadequate motivation for them to continue playing the fraction cards for prolong period.
5.2 Computer game features

In this section, the concepts of game world and game play are elaborated.

5.2.1 Game world

The game world (Figure 5.21) occurred in a tower, which had a brick wall and floors as the background. The rectangular fraction bricks, boy and creatures were objects for play in the game. The learning objectives “Place bricks from smallest to largest” for tall bricks and “Place bricks from largest to smallest” for long bricks were highlighted and displayed on the top of the game page.

On the left top corner, a “home” icon linked the game page to the game menu; on the right top corner, game scores showed the levels that had been completed and creatures that had been avoided. To enhance the excitement level of the game, creatures appeared and moved towards the boy once the staircases were formed. The play included avoiding being hit by the creatures that were blocking the boy’s way to the next floor, by jumping up or ducking down. One point was obtained for avoiding a creature. As noted by Prensky (2001a), conflict, competition, challenge and opposition increase the excitement of the game.

Scores were displayed and updated for both ordering fraction bricks and avoiding creatures. Note that the scores were not used in data analysis and the sole purpose of the scores being displayed was to motivate players to continue playing the game. Players feel attached to the scores of the game if they put their efforts into determining the outcomes of the game (Juul, 2003). Only the number of attempts made to order fractions, which were recorded automatically by the computer game (i.e., Tower Trap), was used in data analysis.
The creatures vanished when they crashed into the boy and no points for avoiding creatures were gained by the player (Figure 5.22).

Figure 5.22: Crashing the creature

5.2.2 Game play

The game involved both educational and entertaining play by manipulating the fraction bricks and a boy, which reacted to input from the player (i.e., using keyboard and mouse). Players interact with the computer through visualisation, manipulation of objects and receiving immediate feedback (Leutner, 1993; Amory, Naicker, Vincent & Adams, 1999).
Moving the cursor allowed the player to point to commands or screen positions. The cursor of an arrow would change to a hand when the fraction bricks could be dragged or hyperlinks (e.g., Home link) could be clicked by the player. In Figure 5.23, the bricks are dragged and dropped at the location indicated by an arrow to form a staircase to the next floor.

![Figure 5.23: Drag and drop fraction bricks](image)

The boy could be moved to left and right, go up and down and jump up (Figure 5.24) and duck down by using keyboard control (Figure 5.25).

![Figure 5.24: Jumping up the staircase](image)
There were two possible outcomes here - that fraction bricks could be ordered correctly or incorrectly. Juul (2003) suggests the optional consequences of the game stimulate players’ curiosity and made them care about what happens next. Such an emotional connection contributes to a stronger desire for learning (Prensky, 2001a, 2007). When fraction bricks were ordered correctly, the boy could be moved to climb up the staircase to reach the next floor and the positive feedback, “Well done” was displayed (Figure 5.26).

If the bricks were not ordered correctly, the bricks would return to the background for the player to try again until a correct order was made. For example, an incorrect ordering of *hidden bricks* was made from the largest $\frac{3}{10}$ through $\frac{2}{5}$ to the smallest $\frac{1}{2}$. The sizes of
*hidden bricks* would become visible after an incorrect staircase was formed (Figure 5.27). The bricks returning to the original location could be dragged and dropped to construct a new staircase. Specific feedback on fractions was given to remind and highlight the errors. The player then received a suggestion to “find the common denominator (bottom number) of the fractions or convert the fractions into percentages” (Figure 5.28). These strategies were displayed on the tips pages (see Section 5.3.4) to help the player to obtain the correct ordering of fractions (Table 5.3). The player was not allowed to proceed to the next floor until a correct ordering of fractions was given.

Figure 5.27: Incorrect order of fractions

Figure 5.28: Specific feedback on fractions helped the player to try again
### Table 5.3: Strategies for comparing fractions

<table>
<thead>
<tr>
<th>Fraction question</th>
<th>Numerical conversion</th>
<th>Common denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{10}, \frac{2}{5}, \frac{1}{2} )</td>
<td>( \frac{3}{10} = 0.3 \times 100 = 30% )</td>
<td>( \frac{2}{5} \times \frac{2}{5} = \frac{4}{10} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{2}{5} = 0.4 \times 100 = 40% )</td>
<td>( \frac{1}{2} \times \frac{5}{5} = \frac{5}{10} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} = 0.5 \times 100 = 50% )</td>
<td>( \frac{3}{10} &lt; \frac{4}{10} &lt; \frac{5}{10} )</td>
</tr>
</tbody>
</table>

### 5.2.3 Game flows

The flow of the game began from a front page (Figure 5.29) where players could “start” or “exit” the game. The access to “credits” enabled the players to get more information about the game development (Figure 5.30) and development team (Figure 5.31).

![Figure 5.29: Front page](image-url)
Once started, the players were required to key in a particular username (Figure 5.32). Only then were they linked to the game menu or homepage (Figure 5.33) to access the game.
The game menu page listed the levels with a tick sign box to show players what had been completed and the score obtained (Figure 5.34). At the end of the game, all game levels were ticked and the total scores gained from the game were shown (Figure 5.35). A key icon was displayed at the lower-left corner of the game menu for players to access the key idea pages (Figure 5.38).
The story of the game was about a boy who was lost in the woods and wanted to climb up to the top of a tower to see the way home. Completing all levels in order to reach the top of the tower (Figure 5.36) was a long way for young children. Two checkpoints were used in the middle of the game to remind them of the goal of the game. The checkpoints (Figure 5.37) showed that the boy made it to a window but it was not high enough to see over the trees and was asked to get higher in the tower. The players met the checkpoints after completing *tall bricks* and *long bricks* sections, respectively, and had to return to the game menu to continue.
to the subsequent sections. Students engage with the game if they know when they are moving closer to the goal or further away (Prensky, 2001a, 2007).

5.3 Pedagogy of the game

This section explains the game pedagogy through the use of key ideas pages, specific instructions, cognitive oriented feedback, tips pages, alternate questions and game user data.
5.3.1 Key ideas pages

The knowledge and strategies of the New Zealand Number Framework were the key ideas for playing the computer game of fractions. The framework consists of nine stages which describe numerical development progressing from Stage 0 to Stage 8 (refer to Table 1.1 in Chapter 1, p.3). Ordering fraction bricks in the game required the knowledge of “identifying symbols for the most common fractions” and “ordering fractions with like denominators” as included in Stage 5; “identifying symbols for any fractions” and “ordering unit fractions” as included in Stage 6; “ordering fractions including halves, thirds, quarters, fifths, and tenths” as included in Stage 7. The corresponding strategies are “equal sharing” in Stage 2 to Stage 4; “finding a fraction of a number by addition and multiplication” in Stage 5 to Stage 6; “solving problems with fractions, proportions, and ratios” in Stage 7 to Stage 8.

The knowledge and strategies were displayed on the key ideas pages (see Table 4.4 and Table 4.5 in Chapter 4) to provide a guideline to parents and teachers about the learning objectives that could be achieved by students from playing the game. This also related the game to the Number Strand of the New Zealand Mathematics Curriculum. Since mathematics games have specific mathematical objectives, students can develop relevant mathematical concepts and procedures from playing the games (Oldfield, 1991; Booker, 2004).

![Figure 5.38: Key idea page](image-url)
5.3.2 Specific instructions

Specific instructions were given in the game world to guide players to play with different types of bricks. This was especially helpful for those who were playing the game for the first time. At the first level, players were instructed that “To move the bricks, you could click with the left mouse button to pick them up. Or click and hold the left mouse button to start dragging the bricks. Build the staircases from smallest to largest” (Figure 5.39).

![Figure 5.39: Instructions to move the bricks](image)

Once the bricks were ordered, “Climb to the top of the staircase and try to avoid the creatures. Move with the left and right arrow keys, jump up and duck down. Each creature you avoid will give 1 point” (Figure 5.40).

![Figure 5.40: Instructions to climb up the staircase](image)
If the incorrect order was made, the comment “That staircase doesn’t look safe. You should try again” appeared (Figure 5.41). Then, the fraction bricks returned to their starting position and the player could try another order of fractions.

![Figure 5.41: Instructions to try again](image)

When moving to different bricks:
“These bricks need to be placed from largest to smallest. Move them in the same way as the previous bricks” (Figure 5.42).
“Select the blue part of each brick to build a staircase. Move the pieces in the same way as the previous bricks” (Figure 5.43).
“These bricks are hidden and won’t become visible until after you have placed them. Move them in the same way as the previous bricks” (Figure 5.44).
Figure 5.42: Order *long bricks* from the largest to the smallest.

Figure 5.43: Select the blue parts of the *tall broken bricks*.
5.3.3 Cognitively oriented feedback

“Well done” and “That staircase doesn’t look safe, you should try again” were the usual feedback given to correct and incorrect moves, respectively. The failure of completing the order task at game levels indicated poor understanding by students of comparing fractions and needed to be resolved immediately. Therefore, specific feedback was provided to remind players how they might correct their incorrect orderings. Immediate feedback is given not only to inform students’ performance but also to help them to improve later (Prensky, 2001a, 2007). The feedback for different sets of fractions is summarised in Table 5.4. These fractions were used in broken and hidden bricks and the player received the following feedback for the incorrect ordering of fractions made at the broken and hidden bricks.
Table 5.4: Feedback on fractions

<table>
<thead>
<tr>
<th>Fraction questions</th>
<th>Feedback to students if fractions are ordered incorrectly</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{8}, \frac{1}{4}, \frac{1}{2} )</td>
<td>The value of a fraction does NOT increase when the numbers that comprise it increase!</td>
</tr>
<tr>
<td>( \frac{1}{5}, \frac{2}{3} )</td>
<td>Think about the relationship between numerator (top number) and denominator (bottom number).</td>
</tr>
<tr>
<td>( \frac{2}{5}, \frac{3}{5} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{7}, \frac{4}{7}, \frac{1}{1} )</td>
<td>A fraction is smaller than the unit (1) when its numerator (top number) is smaller than its denominator (bottom number).</td>
</tr>
<tr>
<td>( \frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99}, \frac{1}{1} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{3}{10}, \frac{2}{5}, \frac{1}{2} )</td>
<td>Find the common denominator (bottom number) of the fractions or convert the fractions into percentage.</td>
</tr>
<tr>
<td>( \frac{2}{3}, \frac{4}{5}, \frac{6}{7} )</td>
<td>The value of a fraction increases as the size of the numerator (top number) approximates the size of the denominator (bottom number)</td>
</tr>
<tr>
<td>( \frac{8}{17}, \frac{1}{2}, \frac{7}{12} )</td>
<td>What is half of 17? What is half of 12?</td>
</tr>
</tbody>
</table>

Students’ incorrect ordering of fractions might be caused by their belief that the numerical value of a fraction was represented by two independent natural numbers.
(Stafylidou & Vosniadou, 2004). For example, “The value of a fraction is NOT increased when the numbers comprising it increase! Think about the relationship between numerator (top number) and denominator (bottom number)” were the feedback appropriate for ordering fractions which had unlike denominators such as \(\frac{1}{8}, \frac{1}{4}\) and \(\frac{1}{2}, \frac{1}{5}\) and \(\frac{1}{3}, \frac{2}{5}\) and \(\frac{2}{3}\). For questions involving unit 1 such as \(\frac{1}{7}, \frac{4}{7}\) and 1, players were reminded that “A fraction is smaller than the unit (1) when its numerator (top number) is smaller than its denominator (bottom number)”. Players were told to “Find the common denominator (bottom number) of the fractions or convert the fractions into percentages” (Figure 5.45) to order fractions \(\frac{3}{10}, \frac{2}{5}\) and \(\frac{1}{2}\). However, numerical strategies made \(\frac{2}{3}, \frac{4}{5}\) and \(\frac{6}{7}\) became more complicated. For example, in finding the common denominator, \(\frac{4}{5} = \frac{28}{35}\) and \(\frac{6}{7} = \frac{30}{35}\) or converting fractions into percentages, \(\frac{2}{3} = 67\%\), \(\frac{4}{5} = 80\%\) and \(\frac{6}{7} = 85.71\%\). Therefore the relationship was stressed in the feedback that “The value of a fraction increases as the size of the numerator (top number) approximates the size of the denominator (bottom number)”. Another feedback “What is half of 17? What is half of 12?” encouraged the player to take half as a benchmark in ordering fractions of \(\frac{8}{17}, \frac{1}{2}\) and \(\frac{7}{12}\).

5.3.4 Tips pages

The tips pages were set to pop up automatically once an incorrect order was made so that students could get help and continue with the game (Figure 5.46). Several strategies for comparing sizes of fractions were displayed on the tips pages, including common denominators, numerical conversions and benchmarks. Although having control over a help facility increases player’s performance (Johansen & Tennyson, 1983), it is noted that getting students to use advice or help has been problematic (Van Eck, 2006), particularly for more passive learners (Lee & Lehman, 1993). Therefore, an automatic setting of tips pages would help the players to refer to other strategies of comparing sizes of fractions.
5.3.5 Alternate questions

In playing the game, players had to give a correct ordering of fraction brick so that a fraction brick staircase was formed for the current level before moving to the next level. If the players took more than one attempt to order fraction bricks correctly, an alternate question which had fractions similar to the original question would be offered to the players before moving to the next level. The players could decide to accept or decline the alternate question. The alternate question provided students with one more chance to play at the same game level. Practice on similar questions helps students to learn mathematics skills (Woodward, 2006; Wright, 2007). Rowland (2008) focuses on the use of examples in primary mathematics teaching. One of the uses of example in teaching is to provide exercises for student to practice. The practice-oriented exercises assist students’ retention of the procedure by repetition, then development of fluency and eventually may lead to different kinds of awareness and comprehension. Table 5.5 shows sets of fractions used in original and alternate questions.
Table 5.5: Set of fractions used in original and alternate questions

<table>
<thead>
<tr>
<th>Original questions</th>
<th>Alternate questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dfrac{1}{3}, \dfrac{1}{2} )</td>
<td>( \dfrac{1}{4}, \dfrac{1}{2} )</td>
</tr>
<tr>
<td>( \dfrac{1}{8}, \dfrac{1}{4}, \dfrac{1}{2} )</td>
<td>( \dfrac{1}{16}, \dfrac{1}{8}, \dfrac{1}{4} )</td>
</tr>
<tr>
<td>( \dfrac{1}{5}, \dfrac{3}{5} )</td>
<td>( \dfrac{1}{7}, \dfrac{3}{5} )</td>
</tr>
<tr>
<td>( \dfrac{2}{5}, \dfrac{3}{5} )</td>
<td>( \dfrac{3}{7}, \dfrac{3}{5} )</td>
</tr>
<tr>
<td>( \dfrac{1}{3}, \dfrac{2}{3} )</td>
<td>( \dfrac{1}{5}, \dfrac{2}{5} )</td>
</tr>
<tr>
<td>( \dfrac{12}{99}, \dfrac{34}{99}, \dfrac{45}{99}, \dfrac{78}{99} )</td>
<td>( \dfrac{2}{9}, \dfrac{4}{9}, \dfrac{5}{9}, \dfrac{8}{9} )</td>
</tr>
<tr>
<td>( \dfrac{3}{10}, \dfrac{2}{5}, \dfrac{1}{2} )</td>
<td>( \dfrac{3}{15}, \dfrac{2}{5}, \dfrac{1}{3} )</td>
</tr>
<tr>
<td>( \dfrac{2}{3}, \dfrac{5}{7} )</td>
<td>( \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8} )</td>
</tr>
<tr>
<td>( \dfrac{8}{17}, \dfrac{1}{2}, \dfrac{12}{13} )</td>
<td>( \dfrac{6}{13}, \dfrac{1}{2}, \dfrac{5}{9} )</td>
</tr>
</tbody>
</table>

5.3.6 Game user data

By keying in username, the game was considered to belong to the particular player. In this way a record of every individual player’s progress was created. Students’ attachment to the game is increased by relating their effort to the outcome of the game (Juul, 2003). User data was produced from the game played and contained attempts at every level of question and alternate questions (if selected), and scores of the game. The attempts were the numbers of trials to obtain a correct ordering. For example, 3 attempts meant that the orderings were wrong for the first and second attempts and right for the third attempt. The score of the game was comprised of the sub scores obtained from completing levels and avoiding creatures. The perfect score was 315 points made up from two sub scores: 260 points for correct ordering of 26 levels of questions and 55 points for play (1 point for avoiding a creature). Players could gain the maximum 10 points by taking one trial only to order fractions correctly in a level. These points were reduced to 5 points or less if 2 trials or more were taken. Therefore, the more attempts that were taken to achieve a correct ordering the less points were gained for that level. Table 5.6 shows the scores gained from taking various attempts. All user data was
saved under the username (refer to Appendix 4) and players were allowed to stop when they wanted and continue the game at any later time (Figure 5.47).

<table>
<thead>
<tr>
<th>Attempts</th>
<th>Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>&gt; 6</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.6: Scores for various attempts

Figure 5.47: Players could stop and continue the game

Summary

The characteristics of good games, educational values, and mathematical inquiry as well as students’ concerns were included in the structure of Tower Trap to provide a theoretical framework for its development. The computer game of Tower Trap was designed based on the integration of the instructional factors and game elements, where mathematical learning were ascertained by referring to the criteria of educational software review on rational numbers. The game focused on the mathematical topic of comparing sizes of
fractions with the learning objectives of ordering fractions from the smallest to the largest and vice versa. Through a fantasy context of forming fraction brick staircases for a boy to climb to the top of a tower, a meaningful context of fractions and a playing goal were provided to motivate students to keep ordering fractions until winning the game (i.e., completing the game by reaching the top of the tower). The rule of dragging and dropping fraction bricks involved instructional strategies of visualizing and manipulating representations of fractions and intrinsically integrated the game context with the fraction content.

Various types of fraction bricks were created in the game to provide a combination of multiple and manipulable representations of fractions to students. Tall bricks were ordered from the smallest to the largest and long bricks were ordered from the largest to the smallest; visible bricks represented sizes of fractions that could be seen; broken bricks were blocks divided in parts that could be separated into visible fractions; hidden bricks showed only symbols of fractions. The game facilitated multiple strategies where fractions were compared in a concrete way using visible bricks, by divided quantity diagrams using broken bricks and by mathematical notation using hidden bricks. Different numerical strategies were included in the game to encourage students making connections across the strategies of ordering fractions. The game promotes constructivist learning where students reconstruct their knowledge of fractions from concrete to image to abstract by moving from visible and broken to hidden bricks.

The findings of the pilot study which revealed the benefits of and the expectations on the game for students with different mathematical abilities were included in the development of the computer game in order to cater for different learning needs. While cause-for-concern students found some fractions were hard to compare, high achievers required harder task. Sets of fractions with a range of difficulty were included in Tower Trap so that the game was challenging to students with different abilities. Since cause-for-concern students expected more teaching of fractions while playing the game, cognitively oriented feedback, tips pages and alternate questions were provided in Tower Trap to help students to continue playing the game, learn several strategies for comparing fractions and review any newly constructed knowledge of fractions.
CHAPTER 6

STUDENTS’ EVALUATION AND GAME PLAY IN TOWER TRAP

This chapter and Chapter 5 present findings to answer research question 1: 
*What needs to be considered in developing a computer game that would enable students to compare and order fractions?*

This chapter examines the effectiveness of *Tower Trap* for learning fractions from students’ perspectives. There are two main sections discussed in this chapter: students’ evaluation of the game and their attempts in playing the game. Section 6.1 discusses the quality of the game and the effectiveness of the game in teaching and learning fractions as evaluated by students using questionnaires. In Section 6.2, the number of attempts taken by students at game levels is recorded to determine the difficulty levels of game levels that, as we shall see, are affected by sets of fractions and types of fractions bricks. The computer generated the data of students’ attempts while the students were playing *Tower Trap*. The data from the pre and post tests and maths tasks is analysed in the subsequent chapters.

6.1 Students’ evaluation of the game

After playing *Tower Trap*, students were asked to evaluate the game using a questionnaire (refer to Appendix 3). Prior to the evaluation, the students’ background information were collected. It was found that the frequency of these students playing computer games ranged from never (3%), rarely (28%), once a week (11%), a few times a week (31%) to everyday (26%). When they were asked “How good do you think you are at fractions” using a scale of 1 to 10 in which the higher the scale was the better the students were with fractions, most or 75% of them rated themselves as 5 and above.

The students rated the twenty-one items in the questionnaire using numbers 1 to 5, meaning strongly agree, agree, neither agree nor disagree, disagree and strongly disagree. The lower the rating, the happier the students were with the game. School A and School C in general agreed on the evaluation of the game and no significant differences were found between the two schools on almost all items in the questionnaire using the Mann-Whitney U test. Table 6.1 shows the mean and standard deviation of students’ evaluation of game for each school and significant differences between School A and School C. The Kruskal-Wallis
test was used to identify the different evaluations of students between mathematical ability groups from School A and School C on some of the aspects of the game (as shown in Table 6.2 & Table 6.3, respectively). A score below 2.5 is good as far are the game goes and significant differences are printed in red.

Apart from answering the questionnaires, 42% (34/81) of the students from School A and 52% (13/25) of the students from School C provided qualitative feedback about the game (refer to Appendix 5). They made useful comments about the game and learning fractions using the game. Some of their qualitative feedback is presented in the following discussion.

Table 6.1: Students evaluation of game, by school

<table>
<thead>
<tr>
<th>Questionnaire items</th>
<th>School A</th>
<th>School C</th>
<th>Mann-Whitney U test</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Playing the game</strong></td>
<td>Mean (Standard deviation)</td>
<td>Sig. Diff. p values</td>
<td></td>
</tr>
<tr>
<td>1 I like playing the game.</td>
<td>2.22 (0.85)</td>
<td>2.08 (0.80)</td>
<td>0.474</td>
</tr>
<tr>
<td>2 I like playing the game because I can learn more about fractions.</td>
<td>2.57 (0.96)</td>
<td>2.46 (0.76)</td>
<td>0.666</td>
</tr>
<tr>
<td>3 I like playing the game because I like to play computer games.</td>
<td>2.75 (1.27)</td>
<td>2.65 (0.94)</td>
<td>0.946</td>
</tr>
<tr>
<td><strong>Game features</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 I like the boy in the game.</td>
<td>2.98 (1.06)</td>
<td>2.58 (1.10)</td>
<td>0.106</td>
</tr>
<tr>
<td>5 I like the creatures in the game.</td>
<td>2.69 (1.07)</td>
<td>2.46 (1.07)</td>
<td>0.400</td>
</tr>
<tr>
<td>6 I like to drag and drop the bricks.</td>
<td>2.72 (1.04)</td>
<td>2.50 (0.95)</td>
<td>0.385</td>
</tr>
<tr>
<td>7 I like to make the boy move.</td>
<td>2.22 (1.04)</td>
<td>2.69 (1.12)</td>
<td><strong>0.044</strong></td>
</tr>
<tr>
<td>8 The instructions for the game are clear.</td>
<td>1.9 (1.08)</td>
<td>1.92 (1.02)</td>
<td>0.800</td>
</tr>
<tr>
<td>9 The story makes the game interesting.</td>
<td>2.43 (1.01)</td>
<td>2.42 (1.10)</td>
<td>0.887</td>
</tr>
<tr>
<td>10 I find it easy to see my progress.</td>
<td>2.16 (0.83)</td>
<td>2.23 (0.77)</td>
<td>0.708</td>
</tr>
<tr>
<td>11 I find forming staircases is interesting.</td>
<td>3 (1.01)</td>
<td>3.15 (0.83)</td>
<td>0.443</td>
</tr>
<tr>
<td><strong>Learning of fractions</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 After playing the game, I want to learn more about fractions.</td>
<td>2.94 (0.89)</td>
<td>3.12 (0.95)</td>
<td>0.453</td>
</tr>
<tr>
<td>13 I use knowledge of fractions I learned in school when playing the game.</td>
<td>2.42 (1.04)</td>
<td>2.38 (0.94)</td>
<td>0.991</td>
</tr>
<tr>
<td>14 The game helps me to imagine the sizes of fractions.</td>
<td>2.17 (0.82)</td>
<td>2.42 (0.86)</td>
<td>0.203</td>
</tr>
<tr>
<td>15 I learned how to write fraction symbols from playing this game.</td>
<td>3.09 (0.98)</td>
<td>3.46 (0.86)</td>
<td>0.070</td>
</tr>
<tr>
<td>16 The game helped me to put fractions into order.</td>
<td>2.47 (0.98)</td>
<td>2.35 (1.02)</td>
<td>0.491</td>
</tr>
<tr>
<td>17 This game helped to fix mistakes I was making with fractions.</td>
<td>2.44 (1.05)</td>
<td>2.19 (0.94)</td>
<td>0.288</td>
</tr>
<tr>
<td>18 I have learned more about fractions.</td>
<td>2.41 (0.97)</td>
<td>2.27 (0.87)</td>
<td>0.648</td>
</tr>
<tr>
<td><strong>Teaching aids</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19 I would like to play the game at school as a part of learning fractions.</td>
<td>2.05 (0.99)</td>
<td>2.00 (0.98)</td>
<td>0.872</td>
</tr>
<tr>
<td>20 I had to think hard to play the game.</td>
<td>3.35 (1.16)</td>
<td>3.85 (0.83)</td>
<td><strong>0.050</strong></td>
</tr>
<tr>
<td>21 I learned more from the game than I do from my teacher in the classroom.</td>
<td>3.05 (1.05)</td>
<td>2.84 (1.14)</td>
<td>0.528</td>
</tr>
</tbody>
</table>
Table 6.2: Evaluation of high achievers, average and cause-for-concern students from School A

<table>
<thead>
<tr>
<th>Questionnaire items</th>
<th>High achievers N = 8</th>
<th>Average students N = 37</th>
<th>Cause-for-concern students N = 36</th>
<th>Kruskal-wallis test</th>
<th>Sig. Diff. p values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Playing the game</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 I like playing the game.</td>
<td>2.13 0.35</td>
<td>2.19 0.78</td>
<td>2.28 1.00</td>
<td>0.948</td>
<td></td>
</tr>
<tr>
<td>2 I like playing the game because I can learn more about fractions.</td>
<td>2.63 0.52</td>
<td>2.70 1.00</td>
<td>2.42 1.00</td>
<td>0.500</td>
<td></td>
</tr>
<tr>
<td>3 I like playing the game because I like to play computer games.</td>
<td>2.63 1.51</td>
<td>2.92 1.21</td>
<td>2.61 1.29</td>
<td>0.487</td>
<td></td>
</tr>
<tr>
<td><strong>Game features</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 I like the boy in the game.</td>
<td>3.00 1.20</td>
<td>3.03 1.07</td>
<td>2.92 1.05</td>
<td>0.825</td>
<td></td>
</tr>
<tr>
<td>5 I like the creatures in the game.</td>
<td>2.38 0.92</td>
<td>2.76 1.06</td>
<td>2.69 1.12</td>
<td>0.760</td>
<td></td>
</tr>
<tr>
<td>6 I like to drag and drop the bricks.</td>
<td>2.38 0.52</td>
<td>2.70 0.91</td>
<td>2.81 1.24</td>
<td>0.634</td>
<td></td>
</tr>
<tr>
<td>7 I like to make the boy move.</td>
<td>2.00 1.07</td>
<td>2.32 1.18</td>
<td>2.17 0.88</td>
<td>0.752</td>
<td></td>
</tr>
<tr>
<td>8 The instructions for the game are clear.</td>
<td>2.00 1.07</td>
<td>2.00 1.22</td>
<td>1.78 0.93</td>
<td>0.803</td>
<td></td>
</tr>
<tr>
<td>9 The story makes the game interesting.</td>
<td>2.13 0.83</td>
<td>2.49 0.96</td>
<td>2.44 1.11</td>
<td>0.497</td>
<td></td>
</tr>
<tr>
<td>10 I find it easy to see my progress.</td>
<td>2.00 0.76</td>
<td>2.19 0.91</td>
<td>2.17 0.77</td>
<td>0.877</td>
<td></td>
</tr>
<tr>
<td>11 I find forming staircases is interesting.</td>
<td>3.38 0.52</td>
<td>3.08 1.04</td>
<td>2.83 1.06</td>
<td>0.283</td>
<td></td>
</tr>
<tr>
<td><strong>Learning of fractions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 After playing the game, I want to learn more about fractions.</td>
<td>3.00 0.76</td>
<td>3.11 0.81</td>
<td>2.75 0.97</td>
<td>0.170</td>
<td></td>
</tr>
<tr>
<td>13 I use knowledge of fractions I learned in school when playing the game.</td>
<td>2.13 1.25</td>
<td>2.11 0.88</td>
<td>2.81 1.04</td>
<td>0.007</td>
<td></td>
</tr>
<tr>
<td>14 The game helps me to imagine the sizes of fractions.</td>
<td>2.50 0.76</td>
<td>2.14 0.75</td>
<td>2.14 0.90</td>
<td>0.456</td>
<td></td>
</tr>
<tr>
<td>15 I learned how to write fraction symbols from playing this game.</td>
<td>3.63 0.74</td>
<td>3.11 0.81</td>
<td>2.94 1.15</td>
<td>0.173</td>
<td></td>
</tr>
<tr>
<td>16 The game helped me to put fractions into orders.</td>
<td>2.13 0.64</td>
<td>2.30 0.85</td>
<td>2.72 1.11</td>
<td>0.185</td>
<td></td>
</tr>
<tr>
<td>17 This game helped to fix mistakes I was making with fractions.</td>
<td>3.00 1.07</td>
<td>2.41 0.93</td>
<td>2.36 1.15</td>
<td>0.173</td>
<td></td>
</tr>
<tr>
<td>18 I have learned more about fractions.</td>
<td>2.75 1.28</td>
<td>2.54 0.80</td>
<td>2.19 1.04</td>
<td>0.110</td>
<td></td>
</tr>
<tr>
<td><strong>Teaching aids</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19 I would like to play the game at school as a part of learning fractions.</td>
<td>2.25 0.89</td>
<td>1.97 1.04</td>
<td>2.08 1.00</td>
<td>0.576</td>
<td></td>
</tr>
<tr>
<td>20 I had to think hard to play the game.</td>
<td>3.88 0.83</td>
<td>3.59 1.17</td>
<td>2.97 1.13</td>
<td>0.028</td>
<td></td>
</tr>
<tr>
<td>21 I learned more from the game than I do from my teacher in the classroom.</td>
<td>3.25 1.04</td>
<td>3.11 1.20</td>
<td>2.94 0.89</td>
<td>0.782</td>
<td></td>
</tr>
</tbody>
</table>
### Table 6.3: Evaluation of high achievers, average and cause-for-concern students from School C

<table>
<thead>
<tr>
<th>Questionnaire items</th>
<th>High achievers $N = 7$</th>
<th>Average students $N = 12$</th>
<th>Cause-for-concern students $N = 6$</th>
<th>Kruskal-Wallis test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td><strong>Playing the game</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 I like playing the game.</td>
<td>2.00</td>
<td>1.00</td>
<td>2.38</td>
<td>0.65</td>
</tr>
<tr>
<td>2 I like playing the game because I can learn more about fractions.</td>
<td>2.43</td>
<td>0.98</td>
<td>2.69</td>
<td>0.63</td>
</tr>
<tr>
<td>3 I like playing the game because I like to play computer games.</td>
<td>2.71</td>
<td>1.11</td>
<td>2.62</td>
<td>0.87</td>
</tr>
<tr>
<td><strong>Game features</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 I like the boy in the game.</td>
<td>2.29</td>
<td>0.76</td>
<td>2.69</td>
<td>1.03</td>
</tr>
<tr>
<td>5 I like the creatures in the game.</td>
<td>2.00</td>
<td>1.29</td>
<td>2.85</td>
<td>0.99</td>
</tr>
<tr>
<td>6 I like to drag and drop the bricks.</td>
<td>2.57</td>
<td>0.98</td>
<td>2.62</td>
<td>0.96</td>
</tr>
<tr>
<td>7 I like to make the boy move.</td>
<td>2.71</td>
<td>1.70</td>
<td>2.77</td>
<td>1.01</td>
</tr>
<tr>
<td>8 The instructions for the game are clear.</td>
<td>1.71</td>
<td>1.25</td>
<td>2.08</td>
<td>1.12</td>
</tr>
<tr>
<td>9 The story makes the game interesting.</td>
<td>2.14</td>
<td>0.90</td>
<td>2.15</td>
<td>0.90</td>
</tr>
<tr>
<td>10 I find it easy to see my progress.</td>
<td>2.57</td>
<td>1.27</td>
<td>2.00</td>
<td>0.41</td>
</tr>
<tr>
<td>11 I find forming staircases is interesting.</td>
<td>2.43</td>
<td>0.79</td>
<td>3.31</td>
<td>0.63</td>
</tr>
<tr>
<td><strong>Learning of fractions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 After playing the game, I want to learn more about fractions.</td>
<td>3.14</td>
<td>0.69</td>
<td>3.15</td>
<td>0.69</td>
</tr>
<tr>
<td>13 I use knowledge of fractions I learned in school when playing the game.</td>
<td>1.71</td>
<td>0.76</td>
<td>2.38</td>
<td>0.65</td>
</tr>
<tr>
<td>14 The game helps me to imagine the sizes of fractions.</td>
<td>2.57</td>
<td>0.98</td>
<td>2.31</td>
<td>0.75</td>
</tr>
<tr>
<td>15 I learned how to write fraction symbols from playing this game.</td>
<td>3.57</td>
<td>0.53</td>
<td>3.62</td>
<td>0.87</td>
</tr>
<tr>
<td>16 The game helped me to put fractions into orders.</td>
<td>1.71</td>
<td>0.49</td>
<td>2.54</td>
<td>1.05</td>
</tr>
<tr>
<td>17 This game helped to fix mistakes I was making with fractions.</td>
<td>2.00</td>
<td>0.58</td>
<td>2.46</td>
<td>1.20</td>
</tr>
<tr>
<td>18 I have learned more about fractions.</td>
<td>2.29</td>
<td>0.95</td>
<td>2.38</td>
<td>0.96</td>
</tr>
<tr>
<td><strong>Teaching aids</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19 I would like to play the game at school as a part of learning fractions.</td>
<td>1.57</td>
<td>1.13</td>
<td>2.38</td>
<td>0.87</td>
</tr>
<tr>
<td>20 I had to think hard to play the game.</td>
<td>3.57</td>
<td>0.98</td>
<td>4.00</td>
<td>0.71</td>
</tr>
<tr>
<td>21 I learned more from the game than I do from my teacher in the classroom.</td>
<td>3.29</td>
<td>0.76</td>
<td>2.92</td>
<td>1.24</td>
</tr>
</tbody>
</table>
6.1.1 Playing the game

Questions 1 to 3 in Table 6.1 were asked in the questionnaires to determine whether students liked playing the game for the reason of learning fractions or they just liked to play computer games. Students from both schools agreed that they liked playing the game (School A, mean = 2.22; School C, mean = 2.08, Table 6.1, Question 1) with the reason being more about learning fractions (School A, mean = 2.57; School C, mean = 2.46, Table 6.1, Question 2) than merely playing computer games (School A, mean = 2.75; School C, mean = 2.65, Table 6.1, Question 3). Students find the game fun mainly because of the challenge involved in learning a mathematical concept, rather than in the challenge of game play such as beating an opponent (Bragg, 2007). At School C, cause-for-concern students strongly agreed that they liked playing the game (mean = 1.50) more than high achievers (mean = 2.00) and average students (mean = 2.38) ($p = 0.040$) (Table 6.3, Question 1).

The following positive feedback about the game was given by some of the students from all ability groups. They liked the game and agreed that the game was good.

- I thought that the fractions game was pretty good. I already knew heaps about fractions and most of it was easy. I liked how it was a sort of platform game, they are the ones I enjoy most. Thanks for letting me participate in your study.
- I enjoyed the game because we were learning while having fun at the same time.
- I like the game and it helps me learn fractions thanks now I know more about fractions.

A few students also commented negatively on the game:

- It was not fun.
- The game was fun but it gets boring really easily so yea.

6.1.2 Game features

Regarding the quality of the game, students from both School A and School C evaluated the game similarly. They favoured the general features of the game. Specifically, they agreed that the instructions for the game were clear (School A, mean = 1.9; School C, mean = 1.92, Table 6.1, Question 8), the story made the game interesting (School A, mean = 2.43; School C, mean = 2.42, Table 6.1, Question 9) and the game score helped them see their progress in the game (School A, mean = 2.16; School C, mean = 2.23, Table 6.1, Question 10).

In *Tower Trap*, specific instructions were given in the game like moving different bricks, climbing up staircases, avoiding creatures and trying again if the incorrect order was
made. The game is fair and exciting if the rules of the game are well defined and every player is led to specific paths to reach goals inside the game world (Prensky, 2001a). The story of losing in the woods and climbing up to the top of a tower was illustrated in *Tower Trap*. In such a context, the players had a goal to move from one floor to the next using the staircases until reaching the top of the tower. The story becomes meaningful and forms a deep structure to the game by linking the story to the rules for playing the game (Rieber, Davis, Matzko & Grant, 2001). The scores of ordering fraction bricks and avoiding creatures that were displayed and accumulated on the game page enabled players to be informed about their progress. Players feel attached to the game if the outcomes achieved are affected by their efforts (Juul, 2003). Winning a game by gaining a high score has a strong emotional implication especially for learning (Prensky, 2001a, 2007).

The game of dragging and dropping fraction bricks was simple so that students could focus on the visual representations of fractions. However, the students were indifferent to the question of forming staircases being interesting (School A, mean = 3; School C, mean = 3.15, Table 6.1, Question 11). They might get bored with the simple game as a cause-for-concern student noted: “the game was fun but it gets boring really easily”. At School C, average (mean = 3.31) and cause-for-concern students (mean = 3.67) did not find forming staircases as interesting as to high achievers (mean = 2.43), ($p = 0.019$) (Table 6.3, Question 11). Mini games are suitable for instructional purposes because they are simple to play and not overly complex (Dempsey, Haynes, Lucassen & Casey, 2002). The basic rules make the games easy to play and the rules do not change while playing the game (Illanas, Gallego, Satorre & Llorens, 2008). The simplicity of mini games might be the reason for students to feel bored after playing for a while. Therefore mini games should not be designed to be played for a long duration of time. However, it is worthy to note that educational approaches of simple games are more focused than complex games (Klawe, 1998). Mini games help young children learning basic skills and being recognised as the most common and the most successful form of educational games at primary education level (Kickmeier-Rust, 2009).

The students somewhat agreed about liking the boy, the creatures and dragging and dropping the bricks in the game (School A, means ranging from 2.69 to 2.98; School C, means ranging from 2.46 to 2.58, Table 6.1, Questions 4 to 6). Some students showed a particular interest in the creatures that blocked the way of the boy in the game. They thought the creatures were “cool” but hard to handle. They commented:

- I found the creatures were moving too fast for my reaction time.
• I found it hard to dodge the creatures
On the other hand, a student expected more challenging gameplay with the creatures and suggested:
• I think that the creatures in the game should become harder in harder levels.

Some students criticised the use of graphics and manipulation in the game.
• The boy shouldn't be able to jump higher than the three bricks put together.
• The game had bad graphics!
• The graphics could be a bit better.

A student suggested better graphics and manipulation would attract children to learn fractions.
• I think that if the game had better graphics, cooler characters, crazy aliens and weapons, I reckon it will attract kids to learn fractions and other mathematics. When the kid jumps he will take a long time to get down which I found annoying.

In contrary, Rieber et al. (2001) found children care less about the high-quality graphics and sound of commercial video games. The children still appreciate the games with amateur-like quality if they are interesting interactive computer games.

Some student gave suggestions to improve the game features. Two preferred to be shown the upper levels or floors reached by the character boy so that the game was connected to the story:
• “I think that the creatures in the game should become harder in harder levels. You should be able to see how many staircases you have made.”
• “The game got a bit boring because it was the same level over and over so it got boring if you changed the level a bit each time it would be good thanx.”

One female student strongly stressed the option of having a girl character for a girl player.
• “Why can't it be a girl with pretty clothes! For a girl it’s a girl and for a boy it’s a boy”

An average and a cause-for-concern student gave a long comment. They suggested the game to include a timer, more different activities than just the stairs, a girl character, more levels, more colours, a penalty if the boy hit the creatures and different endings.

6.1.3 Learning fractions

The effectiveness of the game for learning fractions was evaluated by students using the questionnaire items from 12 to 18 (Table 6.1). The students acknowledged that they used
the knowledge of fractions learned in school to play the game (School A, mean = 2.42; School C, mean = 2.38, Table 6.1, Question 13). Through the key ideas pages, students were informed about the knowledge and strategies of the New Zealand Number Framework that could be used for playing the game. This is important because mathematics games contain specific mathematical cognitive objectives and students are motivated and engaged with developing relevant mathematical concepts and procedures from playing these games (Oldfield, 1991; Booker, 2004).

There was a significant difference between high achievers, average and cause-for-concern students on the awareness of using their knowledge of fractions learned in school to play the game. At School A, high achievers (mean = 2.13) and average students (mean = 2.11) agreed favourably more than cause-for-concern students (mean = 2.81) \((p = 0.007)\) (Table 6.2, Question 13). At School C, high achievers (mean = 1.71) agreed more favourably than average students (mean = 2.38), while cause-for-concern students (mean = 3.17) were indifferent when they were asked about using the knowledge of fractions they had learned in school to play the game \((p = 0.028)\) (Table 6.3, Question 13). It was suggested that the students with higher ability are more motivated and engaged in developing fractional concepts than the lower ability groups while playing Tower Trap.

The students also agreed favourably that they learned more about fractions (School A, mean = 2.41; School C, mean = 2.27, Table 6.1, Question 18). A fantasy context of forming fraction brick staircases was created for fraction content in Tower Trap. Students have an interest in the fantasy context and such context promotes students’ abstract and creative thinking. Students engage with the fantasy context more fully and process fantasy problems more deeply to establish underlying mathematical relationships (Wiest, 2001). Merging the fantasy context of forming fraction brick staircases with the fraction content of ordering fractions produced an intrinsic integration of fractions. Context and content enhance each other in the intrinsic integration (Kafai, Franke, Ching & Shih, 1998).

The game helped the students to imagine the sizes of fractions (School A, mean = 2.17; School C, mean = 2.42, Table 6.1, Question 14) and put fractions into order (School A, mean = 2.47; School C, mean = 2.35, Table 6.1, Question 16). Fractions were represented using rectangular fraction bricks in the game. The bricks were labelled with the symbols of the fractions to link the pictorial representation to symbolic representation of fractions. Manipulatives help students to gain a concrete meaning to abstract mathematical ideas.
Three types of fraction bricks were used in the game. *Visible bricks* showed the physical representations of fractions; *broken bricks* represented fractions in divided quantity or partitioned quantity forms; *hidden bricks* gave fraction symbols without displaying their physical sizes. The opportunities to engage with varied representations and manipulate these representations help students to construct conceptual knowledge of fractions (Kafai, Franke & Battey, 2002).

After playing the game, the students fixed mistakes they had made with fractions (School A, mean = 2.44; School C, mean = 2.19, Table 6.1, Question 17). Represented in divided quantity forms, *broken bricks* had the same size of wholes to highlight the concept of equal whole for comparing sizes of fractions. By emphasising on the equal whole concepts, students are reminded of their mistakes in drawing fractions where the misconception of the size of the whole for a fraction is in proportion to the size of the denominator persist (Yoshida & Kuriyama, 1995).

The following students’ feedback showed that they were aware of the need of using knowledge of fractions and acknowledged the usefulness of the game in learning fractions.

- Well…the game was a bit easy and hard in some areas of fractions that I didn’t realise
- We don’t do fractions too much anymore in class so I learn some more
- I like the game and it helps me learn fractions thanks now I know more about fractions.
- I think it is a great way to learn your fractions.

A negative response was given by the students about the learning of writing fraction symbols from playing the game (School A, mean = 3.09; School C, mean = 3.46, Table 6.1, Question 15). The students compared fractions based on symbols of fractions on *hidden bricks*. However, they were not informed explicitly about the learning of mathematical notations of fraction symbols on *hidden bricks*. Therefore, they might not aware about the learning of writing fraction symbols while playing the game. Before *hidden bricks*, the students visualized the representations of fractions on *visible bricks* and manipulated the fractional parts on *broken bricks*. The growth of mathematical understanding is developed from concrete materials, to visualisation, to abstraction (Pirie & Kieren, 1989). Abstraction requires advanced thinking where students notice properties of the previous images, think consciously about the noticed properties and formulated and symbolised the properties into the common structure (Pirie & Kieren, 1989, 1994b; Linsell, 2005).
6.1.4 Teaching aids

In the questionnaires, students also evaluated the use of the game in classroom instruction. The students agreed favourably that they liked to play the game at school as a part of learning fractions (School A, mean = 2.05; School C, mean = 2, Table 6.1, Question 19). Some of them strongly recommended the game should be used in school.

- I really liked this game because at the moment at school we are learning about fractions I think this will help me a lot.
- I think it would be a good game for schools to use to help those who are having trouble with fractions.

Prensky (2008a) suggests games as effective learning tools in the flexible settings or student-centred world which enable children to play in guided after-school settings, or on their own. Such participatory and ‘hands-on’ pedagogical approaches are more motivating to students than the traditional approaches based solely on discussion and writing (Buckingham, 2005). Since only a single subject and specific concept is focused in mini games (Illanas et al., 2008; Prensky, 2008a; Kickmeier-Rust, 2009), teachers can easily evaluate students’ progress and achievement and determine the educational value of mini games (Illanas et al., 2008).

On the other hand, they neither agreed nor disagreed that they learned more from the game than from their teachers in the classroom (School A, mean = 3.05; School C, mean = 2.84, Table 6.1, Question 21) and disagreed that they had to think hard to play the game (School A, mean = 3.35; School C, mean = 3.85, Table 6.1, Question 20). This was particularly true for visible bricks that could be ordered without thinking hard. As commented by one average student that “the brick size made it so that you did not have to look at the fraction because you could just look at the size and order them by size”. Hence, a suggestion was given by one average student to include “a different way of ordering fractions” into the game. For example, “Maybe make a bridge using the different sizes?”

It is worthwhile to note that a significant difference ($p = 0.028$) (Table 6.2, Question 20) between mathematical ability groups was found at School A where high achievers tended to disagree more (mean = 3.88) while average students (mean = 3.59) were more indifferent than cause-for-concern students (mean = 2.97) when they were asked whether they had to think hard to play the game. This showed the ability to order fractions in the game was affected by students’ mathematical abilities.
6.2 Number of attempts taken at game levels

The data of game play was categorised into four brick type categories:

- Introduction (4 levels)
- Visible bricks (4 levels)
- Broken bricks (9 levels)
- Hidden bricks (9 levels)

Data were analysed by the number of attempts needed to order visible bricks, broken bricks and hidden bricks. The number of attempts was not stratified by tall and long bricks to avoid possible small numbers in each category (i.e., introduction, visible bricks, broken bricks and hidden bricks) in subsequent analysis. The introduction to the game was intended to familiarise students with the different bricks types. There were 4 levels in the introduction and all utilised the same set of fractions $\frac{1}{2}$ and $\frac{2}{2}$.

Figure 6.1 displays the mean attempts made at different types of fraction bricks by high achievers, average and cause-for-concern students from School A and School C, respectively. The mean attempts were obtained from the number of trials of students at each game level.

![Figure 6.1: Mean attempts across all types of fraction bricks](image_url)
6.2.1 Introduction levels

From Figure 6.1, it was shown that more than one attempt was made by the students (School A, mean game levels = 1.35; School C, mean game levels = 1.43) to learn to order new types of fraction bricks. This is comprehensible as players need to observe and participate actively in the gaming process to accumulate the knowledge of how to play the game (Dempsey, Haynes, Lucassen & Casey, 2002).

6.2.2 Visible bricks

Visible bricks represented sizes of fractions that could be seen and therefore enabled students to connect symbols and representations of fractions, and to compare fractions in a concrete way. This study found that minimum mean attempts were made at the game levels of visible bricks (School A, mean attempt = 1.03; School C, mean attempt = 1.02). This type of bricks could be compared in a concrete manner as the way the equipment (e.g., paper circles or commercial fraction kits) used in the phase of Using Materials in the Numeracy Development Project (NDP) classroom. This involves the lowest phase of developing students’ strategies and students need to achieve success before proceed to the next phase of Using Imaging and Using Number Properties (Ministry of Education, 2008d).

6.2.3 Broken bricks

The mean attempt of broken bricks (School A, mean attempt = 1.12; School C, mean attempt = 1.12) was slightly higher than the mean attempt of visible bricks. Broken bricks were divided in parts that were manipulable and therefore allowed students to select parts from the whole and see the consequent changes of symbols of fractions. The divided blocks of broken bricks showed the comparison of fractions with unlike denominators in the same whole. This challenges some students who misunderstand the size of the whole for each fraction was in proportion to the size of the denominator (Yoshida & Kuriyama, 1995).

At broken bricks, the students made a smaller mean number of attempts (in the range of 1 to 1.06) at the game levels which had two fractions ($\frac{1}{5}$ and $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{2}{3}$) than the other game levels (in the range of 1.04 to 1.07) which had three or more fractions ($\frac{1}{8}$, $\frac{1}{4}$ and $\frac{1}{2}$; $\frac{1}{7}$,
4\text{7}, \frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99} \text{ and } 1; \frac{3}{10}, \frac{2}{5} \text{ and } \frac{1}{2} \text{). The high means of 1.21 and 1.29 were made to order } long \text{ broken bricks } \frac{2}{3}, \frac{4}{5} \text{ and } \frac{6}{7} \text{ and, } \frac{8}{17}, \frac{1}{2} \text{ and } \frac{7}{12} \text{, respectively from the largest to the smallest.}

The following describes an example of tall broken bricks \(\frac{1}{5}\) and \(\frac{1}{3}\) (Figure 6.2) and the two choices of ordering these fraction bricks. In Figure 6.3 (a), the bricks are ordered correctly from the smallest \(\frac{1}{5}\) to the largest \(\frac{1}{3}\); in (b), the bricks are ordered incorrectly from the smallest \(\frac{1}{3}\) to the largest \(\frac{1}{5}\).

\[\text{Figure 6.2: Tall broken bricks } \frac{1}{5} \text{ and } \frac{1}{3}\]

\[\text{Figure 6.3: The possible ordering of tall broken bricks}\]
6.2.4 Hidden bricks

Hidden bricks were only labelled with symbols of fractions and required students to interpret fraction symbols and judge the sizes of fractions independently from the mathematical notation. Abstracting mathematical concepts involves the advanced level of mathematical understanding (Pirie & Kieren, 1989). This resulted in the highest number of attempts being made with hidden bricks (School A, mean attempt = 1.52; School C, mean attempt = 1.54). In the game levels of hidden bricks, comparing fractions close to 1 such as \( \frac{2}{3} \), \( \frac{4}{5} \) and \( \frac{6}{7} \), and fractions close to \( \frac{1}{2} \) such as \( \frac{8}{17} \) and \( \frac{7}{12} \) was hard. As a result, high achievers made 1.3 and 1.8 mean attempts, average students made 1.81 and 2.54 mean attempts, cause-for-concern students made 2.38 and 2.95 mean attempts, respectively. The mean number of attempts up to about 3 was the highest among all levels.

Average and cause-for-concern students at both School A and School C made about the same mean attempts in the game except for the hidden bricks. The mean number of attempts made at the hidden bricks increased from high achievers (School A, mean attempts = 1.1; School C, mean attempts = 1.25), to average (School A, mean attempts = 1.44; School C, mean attempts = 1.59) and then to cause-for-concern students (School A, mean attempts = 1.68; School C, mean attempts = 1.76). Differences between the three ability groups were analysed using the Kruskal-Wallis tests. The effect of mathematical ability groups on the number of attempts of ordering fractions is only significant in hidden bricks \( (p = 0.003) \).

Another difference between mathematical ability groups was found in the number of attempts needed for hidden long brick specifically \( (p = 0.002) \). Tukey HSD post hoc test was performed revealing that high achievers required less attempts than average students \( (p = 0.032) \) and cause-for-concern students \( (p = 0.001) \).

6.2.5 Increasing difficulty across different types of bricks

The attempts increased monotonically from visible bricks (i.e., School A, mean attempt = 1.03; School C, mean attempt = 1.02) to broken bricks (School A, mean attempt = 1.12; School C, mean attempt = 1.12) and then to hidden bricks (School A, mean attempt = 1.52; School C, mean attempt = 1.54). This meant the difficulty level of play increased from visible bricks, broken bricks to hidden bricks. Broken bricks were harder than visible bricks.
because representing fractions in divided quantity diagrams is more complicated than
representing fractions in sizes that could be seen. Although the fractions used in the broken
bricks were repeated at the hidden bricks levels, ordering fractions based on mathematical
notation without showing the actual sizes of fractions was much harder. Confusions about the
conventional symbols are noted as the main cause to students’ poor performance on fraction
tasks (Paik & Mix, 2003).

The mean attempts at game levels of hidden bricks were higher than the mean
attempts at game levels of broken bricks although the same fractions were used at both levels.
For example, the fractions \( \frac{1}{8}, \frac{1}{4}, \text{ and } \frac{1}{2} \) were used for both tall broken bricks and tall hidden
bricks. Differences between the two types of bricks were analysed using Wilcoxon signed
ranks test among high achievers, average and cause-for-concern students. The test assumed
that differences in the dependent variables of attempts taken at visible, broken and hidden
bricks were caused by the independent variables of mathematical abilities. Overall, statistical
differences between broken bricks and hidden bricks were found for average (\( p < 0.001 \)) and
cause-for-concern students (\( p < 0.001 \)).

According to Pirie and Kieren (1989), students’ mathematical understanding involves
an ongoing and dynamic process. They advocate a constructivist environment to cope with the
students’ learning transition from working with concrete materials to visualisation, to
abstraction. Students are encouraged to use materials or images for comparisons of fractions
in the NDP instruction before they gain a good understanding of coordinating the numerator
and denominator of fractions so that they are able to make their justifications relating to
relationships within the symbols (Ministry of Education, 2008d).

Summary

The effectiveness of Tower Trap for learning fractions was identified from
investigating students’ evaluation of the game and their attempts in playing the game. In the
questionnaires, students evaluated the game positively in terms of game play and learning
fractions. They liked the game play of dragging and dropping the fraction bricks and moving
the boy. They were also attracted to the game story of climbing up to the top of the tower
illustrated through the game world. The specific instructions of the game appeared to be clear
for the students to complete the game. Meanwhile, students’ progress in the game was
provided through the scores obtained from ordering fraction bricks and avoiding creatures. To
improve the game, students suggested the uses of enhanced graphics, animation and interactivity to provide more advanced experiences on the computer.

As regards to learning and teaching fractions using the game, students were aware of the use of the knowledge of fractions in playing the game and learned more about fractions especially on imaging sizes of fractions and ordering fractions. The game helped students to correct their mistakes with fractions and this may explain why the students supported the notion of including the game in school activities.

In the Tower Trap, students played with visible bricks which were easy to compare, learned the divided quantity thinking of fractions with broken bricks, and applied knowledge of fractions with hidden bricks which were labelled with symbols of fractions only. The monotonically increasing attempts showed that the game was getting challenging for students to play from visible bricks to broken bricks to hidden bricks. More attempts were taken to order broken bricks and hidden bricks that consisted of more fractions and increased difficulty. It was noted that high achievers made significantly smaller mean attempts than average and cause-for-concern students at hidden bricks.

There were also significant differences between the attempts made at hidden bricks and attempts made at broken bricks although the same fractions were used at both levels. Average students and especially cause-for-concern students made more attempts to order hidden bricks than broken bricks. This made sense as applying knowledge of fractions with hidden bricks was more difficult than learning representations of fractions in divided quantity with broken bricks. From the increasing attempts with the increasing difficulty level, it can be deduced that students are using their mathematical knowledge to order fraction bricks rather than just guessing. If students were just guessing, there would not be the trend of increasing attempts with increasing difficulty level.
CHAPTER 7

STUDENTS’ STRATEGIES TOWARDS A FRAMEWORK

This chapter presents findings to answer research question 2:

*What strategies are used by students in ordering fractions? How can these strategies be classified as in a framework?*

This chapter focuses on students’ strategies for comparing sizes of fractions in the pre and post maths tasks. Section 7.1 provides an overview of strategies used by high achievers, average and cause-for-concern students and their abilities to use a number of different types of strategies such as finding a common denominator, converting fractions into percentages or decimals, using benchmarks and drawing divided quantity diagrams. Some students used only one type of strategy, while some used up to 5 types of strategy throughout the 10 questions in the maths tasks. The relevance of the sets of fractions in the maths tasks with the strategy used is considered too. For example, why is it common for a common denominator approach to be found in question 3 and a benchmark approach in question 5 (Appendix 2). Section 7.2 shows levels of mathematical understanding reflected from students’ strategies by referring to Pirie and Kieren’s model (1989, 1994b). The advantages of using certain strategies for determining the order of a particular set of fractions are identified by comparison with other strategies. The strategies that are most appropriate for comparing certain types of fractions are found. A range of strategies used by students with different mathematical abilities are integrated in this study to describe a progression of students’ fractional thinking through the uses of strategies that involve different levels of mathematical understanding.

7.1 Students’ strategies for comparing and ordering fractions

Each of the pre and post maths tasks consisted of 5 questions and each question included 1) a set of two to three fractions for students to order from the smallest to the largest and, 2) a section for students to give their reasoning. Students’ methods or strategies for comparing and ordering fractions were identified through their written reasoning and similar strategies were classified into categories. It was found that students used a range of strategies. Some used numerical strategies such as using benchmarks, finding a common denominator and converting fractions into decimals or percentages, while others used representations of
fractions by drawing divided quantity diagrams, using big denominators were equal to small fractions, numerators and denominators, and parts to make a whole. On the other hand, some students were affected by the prior knowledge of counting numbers and ordered fractions based on the misconceptions of big numbers were equal to small fractions, big numbers were equal to big fractions, and \( \frac{1}{2} \) was the biggest fraction. There were also students who provided partial reasoning, no reasoning and incorrect reasoning and no specific strategies could be identified from these categories.

### 7.1.1 Main strategies of students with different mathematical abilities

The main strategies used by high achievers, average and cause-for-concern students from School C and School A are presented in percentages in the pie charts (Figure 7.2 to Figure 7.4). In order to identify the percentages of the strategies mostly used in the maths tasks, the total numbers of each strategy were computed and divided by the total number of questions answered by all students. As an example, Table 7.1 provides the data of strategies used by three particular students chosen simply to provide examples of these calculations. Every student uses three types or more strategies to answer the 10 questions of pre and post maths tasks. For example, Student 1 considers big denominators are equal to small fractions in 6 questions, uses benchmarks in 2 questions and provides partial reasoning in 2 questions. The percentage of ‘using benchmarks’, for example, is computed by comparing the total number of questions answered using benchmarks (i.e., 5) with the total questions answered by all the three students (i.e., 30), which is 5 out of 30 or 17%.

#### Table 7.1: Data of students’ strategies

<table>
<thead>
<tr>
<th>Students’ strategies</th>
<th>Student 1</th>
<th>Student 2</th>
<th>Student 3</th>
<th>Total</th>
<th>Percentages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big denominators are equal to small fractions</td>
<td>6</td>
<td>1</td>
<td>7</td>
<td></td>
<td>23%</td>
</tr>
<tr>
<td>Finding a common denominator</td>
<td>4</td>
<td>6.5</td>
<td>10.5</td>
<td></td>
<td>35%</td>
</tr>
<tr>
<td>Using benchmarks</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>17%</td>
</tr>
<tr>
<td>Numerators and denominators</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td></td>
<td>13%</td>
</tr>
<tr>
<td>Partial reasoning</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
<td>10%</td>
</tr>
<tr>
<td>Drawing divided quantity diagrams</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
<td></td>
<td>2%</td>
</tr>
<tr>
<td>Total questions in the maths tasks</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>30</td>
<td>100%</td>
</tr>
</tbody>
</table>
Student 3 used two strategies to compare fractions $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ in question 4 and each strategy was counted as 0.5. The fractions $\frac{3}{4}$ and $\frac{7}{8}$ were compared using the common denominator 8 and $\frac{5}{6}$ and $\frac{7}{8}$ were compared using divided quantity diagrams (Figure 7.1).

![Figure 7.1: Using two strategies to compare three fractions](image)

### 7.1.1.1 High achievers

Figure 7.2 shows the main strategies used by 15 high achievers in ordering fractions. Their main strategies are using benchmarks (25%), finding a common denominator (16%), converting fractions into percentages (16%), big denominators are equal to small fractions (13%) and, drawing divided quantity diagrams (11%).
Using benchmarks was one of the main strategies detected in the reasoning of the high achievers from both School A and School C. Behr, Wachsmuth and Post (1985) also notice the common use of a benchmark among high performers in solving problems with fractions. It was also noted that the high achievers preferred to use numerical strategies. Finding a common denominator was mostly detected in the reasoning of the high achievers from School A while converting fractions into percentages was detected in the reasoning of the high achievers from School C. Representations of fractions were referred to by the high achievers from both schools. High achievers from School C drew diagrams to represent fractions while the high achievers from School A believed big denominators were equal to small fractions and the relation between numerators and denominators. Students are encouraged to image or draw to show the representations of fractions in the Numeracy Development Project (NDP) instruction that develops students’ strategies through the phase of Using Imaging. Most important is to develop an understanding of the sizes of fractions that involve coordinating the numerator and denominator of fractions without much relying on materials or images in the phase of Using Number Properties (Ministry of Education, 2008b).
7.1.1.2 Average students

Figure 7.3 shows that no reasoning (23.8%) is the main feedback given by 50 average students to explain the order of fractions compared to drawing divided quantity diagrams (21.5%), big denominators are equal to small fractions (15.4%) and big numbers are equal to small fractions (12%). Numerical strategies such as using benchmarks (5.5%), finding a common denominator (4.6%), and converting fractions into percentages (5.2%) are noticed in a few of the students’ reasoning.

![Pie chart showing reasoning strategies](image)

**Figure 7.3: Strategies used by the 50 average students**

Representations of fractions were noted to be preferred by average students when comparing the given set of fractions, in which average students from School C drew diagrams while average students from School A used big denominators were equal to small fractions. Big numbers were equal to small fractions was also one of the main strategies found in the reasoning of the average students from both School A and School C. Some of the students’ reasoning was related to numerical strategies such as finding a common denominator, using benchmarks, and converting fractions into percentages or decimals.
7.1.1.3 Cause-for-concern students

Similar to the reasoning of the average students, Figure 7.4 shows that no reasoning (39.5%) is mainly the response given by 42 cause-for-concern students to explain the order of fractions made besides drawing divided quantity diagrams (15%) and considering big numbers are equal to small fractions (10.5%). Some of the students’ reasoning is related to considering fractions as parts to make a whole (7.7%), and big denominators are equal to small fractions (8.3%).

![Figure 7.4: Strategies used by the 42 cause-for-concern students](image)

The average and cause-for-concern students from both School A and School C gave no reasoning to most of the questions in the maths tasks. In fact, students are encouraged to give either written or verbal explanation to mathematical problems in the NDP classroom (Maguire, Neil, & Fisher, 2007). The inability to give reasoning can be related to the lack of practice in using a consistent procedure for recording thinking. In addition, the inability to describe thinking with fractions causes students to experience difficulty in using formal procedure (Sharp & Adams, 2002).
As shown in the above results, very little of the reasoning given by the cause-for-concern students was related to numerical strategies such as using benchmarks, finding a common denominator, and converting fractions into percentages or decimals. Some of the reasoning was related to big numbers were equal to small fractions. The cause-for-concern students from School C mainly drew diagrams, whereas the cause-for-concern students from School A drew diagrams and used parts to make a whole and big denominators were equal to small fractions.

It was found that the strategies used were different between students from School C and School A. Students from School C drew diagrams to represent fractions, whereas most of the students from School A used big denominators were equal to small fractions. More high achievers from School C converted fractions into percentages, while more high achievers from School A found a common denominator for comparisons. As instruction directs students’ construction of mathematical concepts, students’ mathematical thinking is often influenced by teachers’ explanations and reasoning. Research has noted a close relationship between teachers’ subject matter and pedagogical content knowledge and students’ achievement in mathematics (Ward & Thomas, 2007). So it is likely here that this difference reflects the teaching of fractions that the students have experienced.
### 7.1.2 Using a number of different types of strategies

Some of the students showed the ability to use up to 5 types of strategies when comparing fractions in the maths tasks. Throughout the 10 questions in the maths tasks, most of the high achievers used two to four types of strategies (Figure 7.5), while most of the average and cause-for-concern students used one to three types of strategies (Figure 7.6 and Figure 7.7).

**Figure 7.5: Number of types of strategies used by high achievers**

**Figure 7.6: Number of types of strategies used by average students**

**Figure 7.7: Number of types of strategies used by cause-for-concern students**
The relationship between the number of types of strategies used (T=0 to T=5, T represents number of types of strategies) and the number of correct ordering of fractions given (none to ten) in the maths tasks is shown in Figure 7.8. Some students gave 5 or fewer correct orderings of fractions by using at most 3 types of strategies. Most of the students used 1 to 2 types of strategies and gave 6 to 7 correct orderings of fractions. Nineteen students gave 10 correct ordering of fractions using from 1 to 5 types of strategies. They were 5 high achievers, 4 average students and 3 cause-for-concern students from School A, and 4 high achievers and 3 average students from School C.

![Figure 7.8: Number of correct ordering of fractions and types of strategies](image)

Most of the students who were right in all questions in the maths tasks used 3 to 5 types of strategies. Almost all tended to find a common denominator to compare fractions that contained denominators that were the factors of the other denominators in question 3, referred to the representations of fractions that were close to one in question 4 and used a benchmark to compare fractions that were close to a half in question 5. The use of appropriate strategies contributed to the correct ordering of fractions given in the maths tasks.

### 7.1.3 Relevance of the sets of fractions to be compared to the strategies adopted

Some strategies were particularly used for certain questions in the maths tasks. It seemed that strategies were selected depending on the sets of fractions that were to be compared. In order to identify the relevance of the sets of fractions to be compared with the
strategies adopted by students, the use of different strategies in every question in the maths tasks is presented in column charts for high achievers, average and cause-for-concern students.

7.1.3.1 High achievers

Figure 7.9 shows the strategies used by the high achievers for the 5 questions in the pre maths tasks (i.e., PreQ1 to PreQ5) and the 5 questions in the post maths tasks (i.e., PostQ1 to PostQ5). To compare simple fractions in questions 1 and 2, the high achievers from both School a and School C considered big numbers were equal to small fractions, big denominators were equal to small fractions, and the relation between numerator and denominator. Approximately 50% of the high achievers used the common denominator strategy in question 3. Most importantly, almost all high achievers (13 out of 15) used a benchmark in question 5.

![Figure 7.9: Strategies used by the 15 high achievers for the 10 questions in the maths tasks](image)

**Figure 7.9: Strategies used by the 15 high achievers for the 10 questions in the maths tasks**
7.1.3.2 Average students

Figure 7.10 shows the strategies used by the average students for the 5 questions in the pre maths tasks (i.e., PreQ1 to PreQ5) and the 5 questions in the post maths tasks (i.e., PostQ1 to PostQ5). The number of students who gave no reasoning increased from question 1 to question 5, whereas the number of students who compared fractions based on big denominators were equal to small fractions and big numbers were equal to big fractions, reduced from question 1 to question 5. It seemed as if comparing fractions using these strategies was inadequate for the average students to compare fractions in questions 4 and 5. At both School A and School C, the average students used benchmarks in question 5 and considered fractions as parts to make a whole in question 4 more than in other questions. At School A, more students found a common denominator in question 3 than in other questions.

Figure 7.10: Strategies used by the 50 average students for the 10 questions in the maths tasks
7.1.3.3 Cause-for-concern students

Figure 7.11 shows the strategies used by the cause-for-concern students for the 5 questions in the pre maths tasks (i.e., PreQ1 to PreQ5) and the 5 questions in the post maths tasks (i.e., PostQ1 to PostQ5). At both schools, the strategy big numbers were equal to small fractions was replaced by other strategies in the post task. At School A, no reasoning increased from question 1 to question 5, whereas comparing fractions based on big denominators were equal to small fractions reduced from question 1 to question 5. It seems as though comparing fractions based on big denominators were equal to small fractions and big numbers were equal to small fractions was inadequate for cause-for-concern students to compare fractions in questions 4 and 5 and it was getting increasingly difficult to give reasoning from question 1 to question 5. Some students from School A used a benchmark in question 5, considered fractions as parts to make a whole in question 4 and found a common denominator in question 3 more than in other questions.

![Diagram showing strategies used by the 42 cause-for-concern students for the 10 questions in the maths tasks](image)

**Figure 7.11: Strategies used by the 42 cause-for-concern students for the 10 questions in the maths tasks**
7.2 Students’ strategies and mathematical understanding

This section investigates mathematical understanding and thinking involved in students’ strategies for ordering fractions in the maths tasks. Pirie and Kieren (1989, 1992, 1994a, 1994b) (refer to Chapter 2) develop a theoretical model for analysing the growth of students’ mathematical understanding. They consider understanding as a whole, dynamic, organizing, and re-organizing, levelled but non-linear, recursive process. Such consideration of understanding leads to the model of eight embedded rings or layers. The eight potential levels in the growth of mathematical understanding are Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring, and Inventising. This model provides a framework for the following analysis on how certain strategies related to the growth of students’ mathematical understanding. Examples of students’ reasoning are provided to elaborate more about students’ strategies and the features of the understanding levels.

7.2.1 Big numbers are equal to big fractions

Primitive Knowing is the background mathematics that the students have for building a new mathematical understanding (Pirie & Kieren, 1989, 1994b). Whole number knowledge is the Primitive Knowing that students have brought into learning fractions. However, students’ prior knowledge often causes the students to overgeneralize whole number knowledge in the learning of fractions and interpret fractions as two separate whole numbers (Behr, Wachsmuth, Post & Lesh, 1984; Newstead & Murray, 1998; Empson, 1999; Miura, Okamoto, Vlahovic-Stetic, Kim & Han, 1999; Paik & Mix, 2003; Olive & Vomvoridi, 2006). This study detected misconceptions generated in the interpretation of fractions as consisting of two independent numbers.

There were average and cause-for-concern students who believed that big numbers were equal to big fractions and fractions were ordered in the same way as the counting of numbers. Stafylidou and Vosniadou (2004) also note the similar belief that “the value of the fraction increases as the value of the numerator (or denominator) increases or when the numbers that comprise it (either the numerator or the denominator) increase”.

193
This misconception led to the incorrect order in all questions except for ordering $\frac{2}{3}$, $\frac{4}{5}$ and $\frac{6}{7}$; $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ in question 4. Below is some of their reasoning:

- I know my numbers
- $\frac{2}{3}$ is smaller than $\frac{2}{5}$ and because the five is bigger than three
- $\frac{7}{12}$ is more than $\frac{1}{2}$ because bigger numbers and $\frac{8}{17}$ has even bigger numbers
- $\frac{8}{17}$ is quite a large number compared to $\frac{7}{12}$

7.2.2 Big numbers are equal to small fractions

Stafylidou and Vosniadou (2004) found two almost contradictory beliefs developed by the students who treated fractions as a combination of two whole numbers. One was big numbers were equal to big fractions as discussed in the previous section. Big numbers were equal to small fractions was another misconception consistent with the belief “the value of the fraction increases as the value of the numerator (or denominator) decreases or when the numbers that comprise it (either the numerator or the denominator) decrease”.

This belief led to the correct order in the first three questions as in the following:

- “$\frac{2}{5}, \frac{2}{3}$” - the denominator 5 is bigger than 3 so $\frac{2}{3}$ would be bigger than $\frac{2}{5}$
- “$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}$” - three tenth is smaller because the number is bigger one half is bigger because the number is smaller

7.2.3 $\frac{1}{2}$ is the biggest fraction

The misconception “$\frac{1}{2}$ is the biggest fraction” was detected in a few students’ reasoning. One average and four cause-for-concern students mistakenly thought $\frac{1}{2}$ was the biggest among $\frac{8}{17}$, $\frac{7}{12}$ and $\frac{1}{2}$ in question 5. They provided the following reasons:
• \( \frac{1}{2} \) would always be the biggest with the fractions above

• \( \frac{1}{2} \) is only \( \frac{1}{2} \) of 1 so it would stay bigger

I know that \( \frac{1}{2} \) is biggest

The fractions \( \frac{1}{2} \) and \( \frac{2}{2} \) are always among the first few fractions introduced in the instruction considering the familiarity of half to children before school (Nunes & Bryant, 1996). One-half, \( \frac{1}{2} \) is commonly used in the NDP classroom in different contexts (Ministry of Education, 2008d). This caused some students to learn and memorize \( \frac{1}{2} \) as the largest fraction among other unit fractions (i.e., \( \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} \ldots \)) rather than thinking of the strategy of splitting one into two equal parts.

7.2.4 Drawing divided quantity diagrams

The students drew divided quantity diagrams to compare sizes of fractions in a concrete way. This involved the second understanding level of Image Making. The first recursion of Image Making occurs when images are formed out of the Primitive Knowing of Pirie and Kieren (1989). The effective action is related to the images of specific fractions such as \( \frac{1}{2} \) and \( \frac{1}{3} \) and action at this level may involve physical objects, figures, graphics or symbols. The students divided a number of parts equally on objects such as circles and rectangles to represent denominators (e.g., 2 for \( \frac{1}{2} \) and 3 for \( \frac{1}{3} \)) and shaded a number of parts to represent numerators of fractions (e.g., 1 for \( \frac{1}{2} \) and \( \frac{1}{3} \)). The part-whole concept underlies the representations of fractions by shading the parts from the divided or partitioned parts.

Circular divided quantity diagrams were commonly drawn and only a few students drew rectangular divided quantity diagrams to represent fractions. Pizza and cake sharing problems that are based on area models of circular shapes are commonly used in instruction to develop students’ part-whole concept of fractions (Moseley, 2005; Ministry of Education,
2008d). Similar to this study, Baturo (2004) found students normally represent fractions using circles more than other shapes such as squares and rectangles.

Drawing diagrams allows students to compare simple fractions in an easy and direct way. For example, fractions $\frac{1}{5}$ and $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{4}$, $\frac{2}{5}$ and $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{3}{4}$ involve small numbers and only a few parts of the whole are divided. This strategy becomes harder if fractions involve bigger numbers such as $\frac{3}{10}$, $\frac{2}{5}$ and $\frac{1}{2}$ and $\frac{5}{14}$, $\frac{3}{7}$ and $\frac{1}{2}$ when more wholes are drawn to represent more fractions and more parts are divided. Below are examples of the divided quantity diagrams drawn by the students (Figure 7.12).

![Figure 7.12: Divided quantity diagrams drawn by students](image)

The divided quantity diagrams drawn by the students to represent fractions that were close to one in question 4 had different sizes of whole and these wholes were unequally divided (Figure 7.13). Equal shares of fractional parts and equal-sized wholes are the critical mathematical ideas in representation of fractions (Lamon, 1996; Newstead & Murray, 1998; Yoshida & Shimanchi, 1999; Mack, 2001; Yoshida & Sawano, 2002; Olive & Vomvoridi, 2006; Ministry of Education, 2008d, 2010d). Nevertheless, Moseley (2005) highlights the underlying mathematical relation more than surface feature of visual appearance of the part-whole representations when interpreting representations of fractions meaningfully. Interestingly, the understanding of the part-whole relation that was embedded in the divided quantity diagrams helped the students to order fractions correctly from the smallest $\frac{2}{3}$
through $\frac{4}{5}$ to the largest $\frac{6}{7}$ in the pre task and from the smallest $\frac{3}{4}$ through $\frac{5}{6}$ to the largest $\frac{7}{8}$ in the post task.

![Divided quantity diagrams drawn in question 4](image)

Figure 7.13: Divided quantity diagrams drawn in question 4

Comparing fractions by diagrams is not as accurate as comparing fractions numerically or by using benchmarks. It is not easy to compare fractions that are close to each other on diagrams, especially for fractions that have big denominators as there are too many parts to be divided on the diagrams. Therefore, incorrect ordering was detected among students who drew diagrams to compare fractions that were close to a half in question 5. One of the examples was the fractions $\frac{1}{2}$, $\frac{8}{17}$, and $\frac{7}{12}$ were thought to be the same size based on the diagrams drawn (Figure 7.14). Consistently, Moseley (2005) found some students who focused more on the surface features of the part-whole representations tried to match surface features to show two fractions were the same without considering the relations between the quantities.

![Divided quantity diagrams with similar size](image)

Figure 7.14: Divided quantity diagrams with similar size

Some students ordered fractions from the smallest $\frac{1}{2}$ through $\frac{7}{12}$ to the largest $\frac{8}{17}$ based on the diagrams drawn (Figure 7.15). The size of whole for each fraction represented was getting bigger in accordance to the denominator of the fraction. The fraction $\frac{8}{17}$ had the
largest denominator and so had the largest size of whole. Other researchers attribute this error in representing fractions to students’ misunderstanding of the direct proportional relation between the size of the whole and the size of the denominator (Peck & Jencks, 1981; Yoshida & Kuriyama, 1995; Gould, 2005). Steffe and Olive (2010) promote partitive reasoning to develop students’ conceptions of fractional quantities in which students conceive fractions as lengths rather than solely as parts of wholes. The partitive fraction scheme enables students to generate fractional knowledge by partitioning, disembedding and iterating. For example, to represent $\frac{1}{2}$, $\frac{7}{12}$ and $\frac{8}{17}$ using circular diagrams, the diagrams can be partitioned into 2, 12 and 17 equal parts, respectively. The fractional parts $\frac{1}{2}$, $\frac{1}{12}$ and $\frac{1}{17}$ can then be disembedded from the partitioned whole ($\frac{2}{2}$, $\frac{12}{12}$ and $\frac{17}{17}$), and iterated into 1, 7 and 8 parts, respectively to make another fractional part of the partitioned whole (i.e., $\frac{1}{2}$, $\frac{7}{12}$ and $\frac{8}{17}$).

Figure 7.15: Divided quantity diagrams with unequal whole

Some students decided fraction $\frac{1}{2}$ was smaller than $\frac{8}{17}$ as shown in the diagrams drawn (Figure 7.16). Due to the unequally divided parts on the whole, the area shaded to represent $\frac{1}{2}$ ‘looked like’ it was smaller than the area shaded to represent $\frac{8}{17}$. Students’ limited concept of equal-partitioning prohibits them from partitioning a unit into parts for the representation of fractions (Newstead & Murray, 1998; Steinle & Price, 2008).

Figure 7.16: Divided quantity diagrams drawn in question 5
7.2.5  **Big denominators are equal to small fractions**

Based on the understanding of equal partitioning, the students know that two pieces of halves make up the whole while three pieces of thirds make up the whole. Therefore, one-half is bigger than one-third. When more unit fractions are compared, students find that

\[
\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} \ldots
\]

and can say further that big denominators are equal to small fractions. They have generalized these specific images and reach the next stage of Image Having. This stage of Image Having is a recursion on the stage of image making, where students have now replaced their actions on specific fractions with an image and can talk about it (Pirie & Kieren, 1989). Students can use a mental picture to construct sizes of fractions without having to perform particular physical actions in ordering fractions. This is the first level of abstraction in which students themselves recursively build on images based in action. Below are examples of students reasoning who ordered unit fractions \( \frac{1}{5} \) and \( \frac{1}{3} \) based on the belief of big denominators are equal to small fractions.

- if you cut an apple into a \( \frac{1}{3} \) you have more to eat than if you cut it into \( \frac{1}{5} \).
- if a circle is broken down into 5 pieces and another circle the same size is broken down into thirds. One piece from the thirds would be bigger than a piece from the fifths.
- if something is split into fifths they will be smaller than if it is split into thirds

Some students ordered non-unit fractions correctly by considering denominators only. When ordering fractions with like numerators in question 2, \( \frac{2}{5} \) was ordered smaller than \( \frac{2}{3} \) with the explanation “a fifth is smaller than a third”. Since the same numerator (i.e., 2) was counted in the fractions with like numerator (for \( \frac{2}{5} \) and \( \frac{2}{3} \)), only denominators (i.e., 5 and 3) needed to be compared. Although both the numerators and the denominators should be referred to when comparing fractions with the same numerators and denominators (Behr et al., 1984; Gould, 2005; Smith III, 2002), for the special case of fractions with the same number of parts (the numerators), the size of the parts (the denominators) could be compared directly in ordering fractions.

In this study, we differentiated big numbers were equal to small fractions from big denominators were equal to small fractions. Students’ reasoning relating to “the more parts
are divided on a whole, the smaller each part becomes” was categorized as big denominators were equal to small fractions. The students had usable fraction language to describe, justify and express their action using mathematical languages. For example, “one-third” refers to the relation $\frac{1}{3}$ in which one part is taken out of three equal parts. One-to-third shows the explicit numerical one-to-many comparison that deals with two numerals comprising a fraction (i.e., numerator 1 and denominator 3) (Steffe, 2002; Lee, 2008). This allows students to recognise the patterns in their informal activities with specific fractions and fosters their growth of mathematical understanding (Pirie & Kieren, 1994a).

The misconception of big numbers were equal to small fractions was rather affected by the interpretation of fractions as two independent numbers. These students did already have an image, albeit an erroneous one, for the order of fractions. Stafylidou and Vosniadou (2004) suggest this misconception is created as a transitional phase in the process of understanding fractions as a unit divided into parts. “When the number of the parts a unit is divided into increases, each part becomes progressively smaller in relation to the whole” (p. 515). They indicate “the more parts the less value a fraction has” is the shared belief for interpreting fractions as two independent numbers and fractions as parts of a whole.

The students' use of language makes it possible to distinguish between different strategies (Table 7.2).

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Strategies</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{5}$ and $\frac{1}{3}$</td>
<td>Big denominators are equal to small fractions</td>
<td>• the denominator is higher so $\frac{1}{5}$ would be smaller than $\frac{1}{3}$</td>
</tr>
<tr>
<td>$\frac{5}{14}$ and $\frac{3}{7}$</td>
<td>Big numbers are equal to small fractions</td>
<td>• the numbers 5 and 14 are bigger than 3 and 7 so $\frac{3}{7}$ would be bigger than $\frac{5}{14}$</td>
</tr>
</tbody>
</table>
7.2.6 Numerators and denominators

Advanced from the belief of big denominators were equal to small fractions, which was valid for ordering unit fractions, two high achievers considered numerators and denominators of fractions for ordering a wider range of fractions including non-unit fractions. Their reasoning was:

- the bottom number is bigger and numerator is the same so this fraction is smaller.
- if the numerators are the same, the fraction with the smaller denominator is larger.

Behr et al. (1984), Gould (2005) and Smith III (2002) documented the use of the numerator and denominator strategy for comparing fractions with the same numerators or fractions with the same denominators. If the numerators are the same, the denominators are compared. Students develop this strategy based on a mental image of their experience with manipulative aids. When a manipulative is not used, students’ thinking can still refer to the inverse relation between the number of equal parts of a whole and their size (Behr et al., 1984).

7.2.7 Converting fractions into equivalent form

In the previous strategies for ordering fractions, students thought of fractions in singular image related terms, and comparison was carried out dependent on these terms. Students’ idea of comparison was not standard and not applicable to many situations. For examples, the belief of big denominators were equal to small fractions was limited to ordering unit fractions. Drawing diagrams and referring to numerators and denominators were restricted by the complexity of fractions. Students’ previous ideas of comparison are transformed into a consciously-made mathematical definition or algorithm when fractions are converted into an equivalent form using a formal mathematical character at the level of Formalising. Converting fractions into a common denominator, decimals and percentages, allows comparison of fractions using only the number concepts and symbols related to fractions. This method which works for any set of fractions without reference to their more physical quantitative meaning (Pirie & Kieren, 1989) is also included in the NDP instruction (Ministry of Education, 2008b).

In this study, students from all categories preferred to find a common denominator for fractions that contained denominators as the factors of another denominator. Examples are the sets of fractions in question 3: \(\frac{1}{2}, \frac{2}{5}\) and \(\frac{3}{10}, \frac{1}{2}, \frac{3}{7}\) and \(\frac{5}{14}\). Most high achievers (9 out of
15) and some average students (7 out of 50) and cause-for-concern students (4 out of 42) found a common denominator and all of them ordered the fractions correctly. In the NDP classroom, students are reminded that the choice of common denominator is influenced by the denominators of the fractions that are considered (e.g. compared, added or subtracted) (Ministry of Education, 2008b, 2008c). Since \( \frac{1}{2} \), \( \frac{2}{5} \) and \( \frac{3}{10} \) contained the denominators 2 and 5 which were the factors of the denominator 10, it was easy for the students to convert the fractions here to the common denominator 10 so that \( \frac{1}{2} = \frac{5}{10}, \frac{2}{5} = \frac{4}{10} \), thus \( \frac{3}{10} < \frac{2}{5} < \frac{1}{2} \). In the post task, \( \frac{1}{2} \), \( \frac{3}{7} \) and \( \frac{5}{14} \) contained the denominators 2 and 7 which were the factors of 14, the common denominator 14 could be easily found so that \( \frac{1}{2} = \frac{7}{14}, \frac{3}{7} = \frac{6}{14} \), thus \( \frac{5}{14} < \frac{3}{7} < \frac{1}{2} \).

This appeared to be a relatively easy computational task. On the other hand, converting certain fractions to percentages or decimals could involve complex multiplication and divisions. For example, \( \frac{5}{14} = 0.3576... \).

In the NDP, students learn some standard simple fractions and their percentages equivalents as benchmarks to estimate percentages. For example, 10% = \( \frac{1}{10} \) and 20% = \( \frac{1}{5} \).

This helped the students to convert familiar fractions such as \( \frac{1}{3}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}, \frac{3}{10}, \frac{1}{2}, \frac{3}{4} \) and \( \frac{4}{5} \) to percentages 33.33%, 20%, 66.67%, 40%, 30%, 75% and 80%. Two high achievers were able to convert fractions as complex as \( \frac{6}{7}, \frac{5}{6}, \frac{7}{8}, \frac{7}{12}, \frac{8}{17}, \frac{4}{7} \) and \( \frac{7}{15} \) to percentages 84%, 83% and 87.5%. However, the average students had difficulty in converting complicated fractions such as \( \frac{5}{14}, \frac{3}{7}, \frac{6}{7}, \frac{5}{8}, \frac{7}{12}, \frac{8}{17}, \frac{4}{7} \) and \( \frac{7}{15} \) to percentages. For example, one student knew \( \frac{3}{4} = 75\% \) but not the percentage for \( \frac{7}{8} \) and \( \frac{5}{6} \) and another student knew \( \frac{2}{3} \) is 66.666% and \( \frac{4}{5} \) is 80% but was not sure about \( \frac{6}{7} \) (Figure 7.17).
Considering the difficulty of calculating the percentages of fractions such as \( \frac{3}{7}, \frac{5}{14}, \frac{6}{7}, \frac{5}{6}, \text{ and } \frac{7}{8} \) accurately, an estimation of the percentage equivalent of some fractions is worthwhile. One high achiever estimated fractions \( \frac{6}{7} \) as 86\% in the pre task and \( \frac{5}{6} \) as 80\% and \( \frac{7}{8} \) as 84\% in the post task of question 4 (Figure 7.18).

Numerical conversions became difficult and time consuming when comparing fractions that were close to one in question 4 and close to a half in question 5. The following shows three common denominators are found to compare fractions that are close to one:

- The fraction \( \frac{6}{7} \) was compared with \( \frac{4}{5} \) using the common denominator 35 so that
  \[
  \frac{6}{7} = \frac{30}{35} \quad \text{and} \quad \frac{4}{5} = \frac{28}{35}.
  \]
- The fraction \( \frac{6}{7} \) was compared with \( \frac{2}{3} \) using the common denominator 21 that
  \[
  \frac{6}{7} = \frac{18}{21} \quad \text{and} \quad \frac{2}{3} = \frac{14}{21}.
  \]
- The fraction \( \frac{2}{3} \) was compared with \( \frac{4}{5} \) using the common denominator 15 so that
  \[
  \frac{2}{3} = \frac{10}{15} \quad \text{and} \quad \frac{4}{5} = \frac{12}{15}.
  \]
The students compared \( \frac{30}{35} \) and \( \frac{28}{35} \) to decide \( \frac{6}{7} \) was bigger than \( \frac{4}{5} \); and then compared \( \frac{18}{21} \) and \( \frac{14}{21} \) to decide \( \frac{6}{7} \) was bigger than \( \frac{2}{3} \); and then compared \( \frac{10}{15} \) and \( \frac{12}{15} \) to decide \( \frac{2}{3} \) was smaller than \( \frac{4}{5} \). As explained by one of the students “I made two of them the same found the higher one then did it again with each number like this”.

As shown in the above, the procedures of finding more than one common denominator and conducting long divisions might accidentally introduce computational errors. Nevertheless, converting fractions into a common denominator, decimal or percentage was still preferred by some of the students although there were simpler yet equally effective alternative strategies that could be used to order fractions. As revealed by Ward (1999), students with limited understanding of the mental picture of fractions are inclined to rely on the common denominator method to compare fractions. The dominance of procedural over conceptual aspects of students’ knowledge of fractions is concerned by Sophian and Madrid (2003b). They emphasise the facilitation of conceptual understanding of fractions as opposed to complex computational procedures in finding a common denominator. For instance, by knowing that a common denominator can be found by using the lowest common multiple of both denominators can help students to avoid multiplying big numbers in getting equivalent fractions. Sophian and Madrid (2003b) also note that the magnitude relations between fractions with unlike denominators are easily neglected if students focus only on the computing procedures.

As an example in this study, one average student who used the common denominator strategy throughout the questions in the maths tasks conducted the common denominator strategy as a routine algorithm rather than understanding the strategy as a theory requiring the equivalence concept. The student compared \( \frac{1}{2} \) and \( \frac{7}{12} \) using the common denominator 12 and correctly decided that \( \frac{1}{2} = \frac{6}{12} \) was smaller than \( \frac{7}{12} \). Although this was inadequate to order the three fractions, the student simply ordered incorrectly from the smallest \( \frac{6}{12} \) through \( \frac{7}{12} \) to the largest \( \frac{8}{17} \) by guessing. The same student later changed to another incorrect
ordering $\frac{8}{17}$, $\frac{7}{12}$ and $\frac{6}{12}$ with the reason “the bigger the denominator is the smaller the fraction is” (Figure 7.19).

![Figure 7.19: Students’ incorrect ordering of fractions](image)

On the other hand, one cause-for-concern student showed a limited basic skill of multiplication and this prohibited the student from comparing fractions correctly using the common denominator strategy. The student converted incorrectly $\frac{1}{2}$, $\frac{7}{12}$ and $\frac{8}{17}$ into common denominator $34$: $\frac{1}{2} = \frac{17}{34}$, $\frac{7}{12} = \frac{14}{34}$ and $\frac{8}{17} = \frac{16}{34}$. This led to an incorrect ordering from the smallest $\frac{7}{12}$ through $\frac{8}{17}$ to the largest $\frac{1}{2}$ (Figure 7.20).

![Figure 7.20: Students’ incorrect ordering of fractions](image)

When students move to the next level of Observing from the level of Formalising, the method of converting fractions into equivalent form becomes “pieces of a possible theory and not simply techniques for computation as they might earlier have been” (Pirie & Kieren, 1989, p. 9). The students were able to observe fractions equivalent as a mathematical object and make another critical step in their theory of understanding. They were looking for an image for the conversion methods (e.g., converting fractions into a common denominator and percentages) which could be distinguished by certain features (i.e., fractions contained denominators as the factors of another denominator and standard simple fractions and their percentages equivalents) but which can still be characterized as belonging to the class of
equivalence. For instance, fractions could be easily converted into a common denominator if the fractions contained denominators as the factors of another denominator (e.g., $\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{2}$).

At the next level of Structuring, students observed their own thought structures on fractions equivalence and conversions and organised them consistently. They knew the consequences of using particular methods on fraction equivalence, hence they were aware of associations and sequences among numerical methods in comparing and ordering fractions. For instance, students used two strategies to compare three fractions in a question and a particular method was selected based on its features. In the comparison of $\frac{3}{7}$, $\frac{1}{2}$ and $\frac{5}{14}$, the fraction $\frac{3}{7}$ was decided smaller than $\frac{1}{2}$; meanwhile the fractions $\frac{3}{7}$ and $\frac{5}{14}$ were compared using the common denominator 14. If only the benchmark was used, the students could only determine that both $\frac{3}{7}$ and $\frac{5}{14}$ were smaller than $\frac{1}{2}$.

Another example was in comparing $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$. A high achiever could easily find the common denominator 8 for fractions $\frac{3}{4}$ and $\frac{7}{8}$ because 8 is the multiple of 4. However, the high achiever drew divided quantity diagrams to compare $\frac{5}{6}$ and $\frac{7}{8}$ to avoid big numbers multiplication (Figure 7.21). The high achiever ordered correctly from the smallest $\frac{3}{4}$ through $\frac{5}{6}$ to the largest $\frac{7}{8}$ although the above methods did not show the comparison between $\frac{3}{4}$ and $\frac{5}{6}$.

![Figure 7.21: Using two strategies to compare three fractions](image)
As shown in Section 7.1, there was a relationship between students’ strategies and sets of fractions compared in the maths tasks. The use of appropriate strategies contributed to the correct ordering of fractions. This was especially true for high achievers who used several types of strategies and were right in most or all of the questions. Generally, a common denominator was found for fractions that contained denominators as the factors of another denominator; diagrams and parts to make a whole were referred to for fractions that were close to 1; the benchmark of a half was used for fractions that were close to a half. These students had the understanding levels of Observing and Structuring as they were cognizant of the features and the consequences of numerical conversion methods, and were able to select specific strategies for ordering particular types of fractions.

Students who created their own ways of ordering fractions in this study had an understanding level of Inventising (Pirie & Kieren, 1989). “These students have a full structured understanding and may therefore be able to break away from the preconceptions which brought about this understanding” (Pirie & Kieren, 1994b, p. 67). Facing the difficulty of converting certain fractions into percentages, some students estimated percentages of fractions for comparisons. There was a student who computed the percentage of the fractional part that made up a whole as a way to compare and order fractions. This sequence or structure of thought was initiated based on the previous knowledge of converting fractions into percentages and fractional parts of fractions.

7.2.8 Using benchmarks

According to Behr et al. (1984, 1985), the method of using a third number as a benchmark in solving problems with fractions is not taught in classroom but invented by students, especially by high achievers. Nevertheless, the important benchmarks for “0”, “$\frac{1}{2}$”, and “1” are included in the NDP instruction to help students to compare the relative sizes of fractions through estimating, ordering and placing them on a number line (Ministry of Education, 2008c). Smith III (2002) suggests that some students even as early as the upper primary school years are able to relate complicated fractions to benchmarks such as “0”, “$\frac{1}{2}$”, and “1” as important markers.
In this study, the students used benchmarks involving different strategy stages to compare fractions. Students’ number knowledge of double was required to compare fractions that were close to a half at the stage of *Using Number Properties*. The students found the fractions that were equivalent to the benchmark of $\frac{1}{2}$ in order to determine whether the fractions were bigger or smaller than $\frac{1}{2}$. On the other hand, parts to make a whole was used to compare fractions that were close to one at the stage of *Using Imaging*. The students referred to the visual display and considered sizes of fractional parts for fractions that were progressively closer to 1.

Almost all high achievers (87% or 13 out of 15) used the benchmark of “$\frac{1}{2}$” to decide whether $\frac{7}{12}$, $\frac{8}{17}$, $\frac{4}{7}$ and $\frac{7}{15}$ were smaller or larger than $\frac{1}{2}$ when comparing $\frac{7}{12}$, $\frac{8}{17}$ and $\frac{1}{2}$ in the pre task and $\frac{4}{7}$, $\frac{7}{15}$ and $\frac{1}{2}$ in the post task in question 5. Among the students who used this strategy, the percentage the high achievers obtain a correct ordering was higher than other ability groups. Seventy-seven percent of high achievers (10 out of 13), 33% of 4 average students (4 out of 12) and 43% of cause-for-concern students (3 out of 7) ordered the fractions correctly from the smallest $\frac{8}{17}$ through $\frac{1}{2}$ to the largest $\frac{7}{12}$ in the pre task and from the smallest $\frac{7}{15}$ through $\frac{1}{2}$ to the largest $\frac{4}{7}$ in the post task.

The students used the number knowledge of doubles (e.g., $8.5 \times 2 = 17$ and $6 \times 2 = 12$) to benchmark fractions to a half. They found that half of 17 was 8.5, therefore $\frac{8}{17}$ was smaller than $\frac{8.5}{17}$, which was equal to $\frac{1}{2}$. Meanwhile half of 12 was 6, therefore $\frac{7}{12}$ was bigger than $\frac{6}{12}$, which was equal to $\frac{1}{2}$. Some students explained that “$\frac{8}{17}$ is smaller than $\frac{1}{2}$ because $\frac{8}{16}$ is $\frac{1}{2}$ ($\frac{8}{17} < \frac{8}{16}$) and $\frac{7}{12}$ is bigger than $\frac{1}{2}$ because $\frac{7}{14}$ is $\frac{1}{2}$ ($\frac{7}{12} > \frac{7}{14}$).”
The students used the ‘half of’ relationship within the components of fractions to find their equivalence. For example, the numerator 8.5 was half of the denominator 17 and so \( \frac{8.5}{17} \) was equivalent to \( \frac{1}{2} \). This strategy is generalized to establish the equivalence of other fractions (Smith III, diSessa & Roschelle, 1993). When comparing fractions, an amount was estimated either to attain or go beyond the benchmark of half. For example, \( \frac{8}{17} \) was 0.5 unit smaller than \( \frac{8.5}{17} \). Students need to acquire generalised and abstract thinking and estimating skill in using the benchmark strategy for comparison of fractions (Behr et al., 1984). The benchmarking process involved calculation rather than mental image. Students use benchmarks at the stage of Using Number Properties where they justify their calculation steps and estimate and use leveraging from known benchmark fractions (Ministry of Education, 2008d).

Behr et al. (1985) suggest high performers use a cognitive mechanism that is based on a flexible and spontaneous application of fraction order and equivalence concepts and the use of benchmarks. On the other hand, they point out low performers lack such cognitive mechanisms and apply concepts in a constrained or inaccurate manner. This was shown in the reasoning of some students especially for average and cause-for-concern students who faced difficulty in benchmarking fractions to a half. Their reasoning was characterized by a very uncertain or inaccurate use of an estimation process as shown in the following:

- \( \frac{8}{17} \) is more than \( \frac{1}{2} \) because 8 is more than half of 17.
- \( \frac{1}{2} \) is exactly half; \( \frac{8}{17} \) is slightly off and \( \frac{7}{12} \) is even more off.
- \( \frac{8}{17} \) is just bigger than 1 half and \( \frac{7}{12} \) is a bit bigger than \( \frac{1}{2} \) and \( \frac{8}{17} \).

Smith III et al. (1993) criticize the restriction of low performers to the manipulation of a mental model of divided quantity and their reliance on part-whole model to support the use of a variety of specific strategies. They highlight the numerical relationships involved in high performers’ reasoning, which include relations within and between the whole-number components and numerical benchmarks.
When comparing fractions that were close to one in question 4 such as $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$, the students from all ability groups considered the fractional parts that made up 1 and referred these fractions to the benchmark of 1. High achievers were able to provide specific fractional parts that were needed by the given fractions in order to make up 1 and decided the order of fractions by comparing the fractional parts. For example, “$\frac{2}{3}$ needs $\frac{1}{3}$ to make 1. $\frac{4}{5}$ needs $\frac{1}{5}$ to make 1. $\frac{6}{7}$ needs $\frac{1}{7}$ to make 1. Since $\frac{6}{7}$ needs the smallest bit to make one it means it is the largest”. The average students only refer generally to the gaps between the fractions and the whole that “the fraction has the smallest gap would have the most filled in and has the biggest gap would have the least filled in”. Very few cause-for-concern students used this strategy and their explanations were limited to a fraction such as “$\frac{6}{7}$ has a smaller gap so the shape is bigger” and “$\frac{6}{7}$ is the highest because you need to add the least on to get a whole”.

All of the students above noticed a specific and relevant property of fractions as parts to make a whole in the process they examined the images of fractions that were progressively closer to 1. This strategy requires students to fit each fraction into a whole and compare the fractional parts or the gaps between the fractions and the whole (Ministry of Education, 2010f). Students are assisted with visual display when using benchmarks for comparing sizes of fractions at the stage of Using Imaging (Ministry of Education, 2008d).

The above results showed that high achievers preferred to use the benchmark of $\frac{1}{2}$ to compare fractions that were close to a half even though they had an ability to use various types of strategies. Benchmarking fractions to $\frac{1}{2}$ saved the students’ time from spending in the process of converting fractions numerically and carrying out multiplications and divisions steps. Nevertheless, in certain situations, numerical conversion was useful and acted as a complementary strategy to using benchmarks. An example is in ordering three fractions $\frac{1}{2}$, $\frac{5}{14}$ and $\frac{3}{7}$. The students could only estimate $\frac{3}{7}$ being closer to $\frac{1}{2}$ than $\frac{5}{14}$ until a common
The denominator 14 was found to decide \( \frac{3}{7} = \frac{6}{14} \) was slightly bigger than \( \frac{5}{14} \). The ability to use the most suitable strategy (either numerical conversions or benchmarking) and take a proper step (either benchmarking or converting fractions into one common denominator or another) required the understanding levels of Observing and Structuring.

### 7.2.9 Partial reasoning, no reasoning and incorrect reasoning

Figure 7.22 shows the frequencies of partial reasoning, no reasoning and incorrect reasoning found in the works of high achievers, average, and cause-for-concern students in the maths tasks.

![Figure 7.22: Students’ reasoning in the maths tasks](image)

Some students struggled to provide significant reasoning for the ways fractions were ordered in the maths tasks. The following are examples of ordering of fractions given by the students and this was regarded as partial reasoning:
• \[ \frac{1}{5}, \frac{1}{3} \] (correct) - \[ \frac{1}{3} \text{ is bigger than } \frac{1}{5} \]

• \[ \frac{3}{10}, \frac{2}{5}, \frac{1}{2} \] (correct) - \[ \frac{3}{10} \text{ is smaller than } \frac{2}{5} \text{ and } \frac{1}{2} \text{ is the biggest} \]

No reasoning could be identified when the reasoning section was left blank or students just stated the following to the order of fractions given in the maths tasks:

• \[ \frac{1}{5}, \frac{1}{3} \] (correct) - \[ \text{“It’s not hard”} \]

• \[ \frac{5}{6}, \frac{7}{8}, \frac{3}{4} \] (incorrect) - \[ \text{“I’m not so sure”} \]

• \[ \frac{5}{14}, \frac{3}{7}, \frac{1}{2} \] (correct) - \[ \text{“I don’t know I’m just used to fractions”} \]

Below is students’ incorrect reasoning for their incorrect ordering of fractions in the maths tasks:

• \[ \frac{1}{3}, \frac{1}{5} \] (incorrect) - \[ \text{“there is (sic) less numbers between } \frac{1}{3} \text{”} \]

• \[ \frac{1}{2}, \frac{2}{5}, \frac{3}{10} \] (incorrect) - \[ \text{“} \frac{3}{10} \text{ is usually bigger than } \frac{1}{2} \text{ and } \frac{2}{5} \text{”} \]

• \[ \frac{2}{3}, \frac{4}{5} \] and \[ \frac{6}{7} \] are same (incorrect) – “They are all the same”

The above reasoning was detected in the works of students from all categories. With only partial reasoning or no reasoning, some of the high achievers could still give the correct ordering of fractions. In reality, these high achievers were able to use other valid strategies (e.g., using benchmarks and finding a common denominator) as detected in other questions that they completed in the maths tasks. These students might calculate and visualize everything mentally when comparing and ordering fractions. This can be related to students’ conceptual understanding of fractions that is too mentally quick to need a symbolic algorithm (Sharp & Adams, 2002). These students can consistently provide correct answers without giving any explanations.

The average and cause-for-concern students who gave partial reasoning also gave no reasoning in other questions and they were mostly wrong in ordering fractions that were close
to each other in questions 4 \( \left( \frac{2}{3}, \frac{4}{5} \text{ and } \frac{6}{7} \right) \) and 5 \( \left( \frac{8}{17}, \frac{1}{2}, \frac{7}{12} \right) \). More cause-for-concern students than average students gave no reasoning, partial reasoning and incorrect reasoning and a number of cause-for-concern students gave no reasoning to all the 10 questions in the maths tasks. The ordering of fractions given by these average and cause-for-concern students was also affected by the belief that big numbers were equal to small fractions and big numbers were equal to big fractions. This showed the students had limited knowledge of fractions and applied the invalid whole number strategy. For students who are unable to give any explanation to their correct answers, Young-Loveridge, Taylor, Hawera and Sharma (2007) suggest the students are unsure of their answers and impute their correct answers to be a lucky guess.

7.3 Discussion

Based on the above results, this section matches the most suitable strategies for certain fractions, frames the strategies used by high achievers, average and cause-for-concern, discusses the mathematical understanding underlying students’ strategies and compares the strategies’ framework developed in this study with the literature.

7.3.1 Matching strategies with sets of fractions

The results in Section 7.1 showed a relationship between the strategies used by high achievers and the sets of fractions that were to be compared in the maths tasks. Specifically, they found a common denominator in question 3 to compare fractions that contained denominators as the factors of another denominator; drew diagrams or consider fractions as parts to make a whole in question 4 to compare fractions that were close to one; and used a benchmark in question 5 to compare fractions that were close to half. Meanwhile, different strategies were used in questions 1 and 2 to compare unit fractions and fractions with like numerators and one of them was big denominators were equal to small fractions. Such relationship was detected among some average students and a few cause-for-concern students. The reasons for using particular strategies in certain situations were explored in Section 7.2 and it was found that some fractions can be easily compared using particular strategies. Hence, the identified students’ strategies are matched with different types of fractions in Figure 7.23.
The strategies of converting fractions into a common denominator, percentages and decimals are included in the curriculum because the numerical conversion method is adequate to solve all problems with fractions (Smith III et al., 1993). Consistently, this study also showed the usefulness of these strategies in ordering all types of proper fractions (i.e., fractions \( \leq 1 \)). On the other hand, other strategies like big denominators were equal to small fractions; parts to make a whole and benchmark of \( \frac{1}{2} \) were designed for ordering particular types of fractions.
Being capable of using different strategies and selecting the most appropriate strategies helps students to order fractions correctly in the simplest and fastest way. For instance, unit fractions could be compared using various types of strategies but the simplest strategy was by considering denominators only. A common denominator could be found easily if the denominator of one fraction was the factor of the denominator of the other fraction. Using parts to make a whole, was the suitable strategy for comparing fractions that were close to one and if the difference was a unit fraction (refer to Section 7.2.8). For fractions that were close to a half, the benchmark of a half could be used to determine either the fraction was smaller or bigger than a half. With the good knowledge and understanding of fraction knowledge, high performers often use flexible, spontaneous and specific tools to obtain rapid and reliable solutions for particular mathematical situations (Behr et al., 1985; Smith III et al., 1993; Smith III, 2002).

7.3.2 A framework of fractional strategies and thinking

There were some differences on the use of strategies and mathematical understanding between cause-for-concern students, average students and high achievers. Cause-for-concern students applied the same strategies across all problems. Most of them were unable to give reasoning and relied on big numbers were equal to small fractions for most of the questions in the maths tasks. Some of them drew diagrams or used big denominators were equal to small fractions and parts to make a whole. Average students had similar strategies as cause-for-concern students but could adapt their strategies for more complex problems. Average students mainly referred to the representations of fractions when comparing fractions in the maths tasks. Some of them drew diagrams and others used big denominators were equal to small fractions and parts to make a whole. They also believed big numbers were equal to small fractions and few used a benchmark, found a common denominator, and converted fractions into percentages.

High achievers had more strategies and selected appropriate strategies based on the complexity of the problem. High achievers ordered fractions correctly in almost all questions in the maths tasks. They were able to use several types of strategies and mainly compared fractions numerically, by using a benchmark, by finding a common denominator and by converting fractions into percentages. They also referred to the representations of fractions by drawing diagrams and using big denominators were equal to small fractions and as the
relation between numerators and denominators. Some of them even compared fractions based on big numbers were equal to small fractions and got the correct ordering for unit fractions.

In fact, high achievers were capable of using various types of strategies including sophisticated numerical conversions and benchmarking; average students mainly relied on representations of fractions (i.e., diagrams, big denominators were equal to small fractions, numerators and denominators and, parts to make a whole); and cause-for-concern students employed less sophisticated whole number knowledge. The strategies of cause-for-concern students, average students and high achievers are framed in the following figure to describe a developmental picture of students’ fractional thinking, which I call the strategies’ framework for ordering fractions.

![Figure 7.24: Strategies’ framework for ordering fractions](image)

Not many researchers investigating students’ strategies or method for comparing sizes of fractions have specified students’ mathematical abilities. Smith III et al. (1993) and Behr et al. (1985) show high achievers use numerical strategies including algorithmic procedures that are more accurate, direct and flexible; cause-for-concern students are restricted to an inaccurate manner with the manipulation of a mental model of divided quantity. Nevertheless,
an exception was noticed in this study that some students tended to use higher strategies in the maths tasks. The students were all assessed in an early year assessment. They participated in the study after being in the class for six months. It was possible that some of them might have already improved since then. Seven out of 50 average students (14%) and 4 out of 42 cause-for-concern students (9.5%) used the common denominator strategy in the maths tasks and were able to find a common denominator in question 3 for fractions that contained denominators as the factors of the other denominators. Three of the average students and one of the cause-for-concern students used this strategy of numerical conversion in most of the questions in the maths tasks but most of them gave incorrect orderings in questions 4 and 5 for fractions that had big numbers and were close to each other. Meanwhile, 12 out of 50 average students (24%) and 6 out of 42 cause-for-concern students (14%) used benchmarks in the maths tasks. All of them benchmarked fractions that were close to a half in question 5 to \( \frac{1}{2} \), but some of them gave an incorrect ordering of fractions due to the inability to determine how far the fractions were from \( \frac{1}{2} \). Only 3 out of 50 average students (6%) and 2 out of 42 cause-for-concern students (5%) converted fractions into percentages or decimals, but the successful conversion was limited to simple fractions in the first three questions in the maths tasks. It is the comparison situations that promote the use of numerical strategies even for students whose earlier thinking is primarily based on representations of fractions and whole number knowledge. Unfortunately, the average and cause-for-concern students are hampered by their numerical ability and so are unable to achieve a higher strategy that has been mastered by the high achievers.

### 7.3.3 Mathematical understanding underlying students’ strategies

Pirie and Kieren’s proposed model on the growth of mathematical understanding and this model sheds light on the levels of mathematical understanding underlying students’ strategies in ordering fractions (Table 7.3).
Table 7.3: Levels of mathematical understanding underlying students’ strategies

<table>
<thead>
<tr>
<th>Students’ strategies</th>
<th>Levels of mathematical understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Flexible choice</td>
<td>Inventising</td>
</tr>
<tr>
<td>• Using benchmarks</td>
<td>Structuring</td>
</tr>
<tr>
<td></td>
<td>Observing</td>
</tr>
<tr>
<td>• Finding a common denominator</td>
<td>Structuring</td>
</tr>
<tr>
<td>• Converting fractions into percentages and decimals</td>
<td>Observing</td>
</tr>
<tr>
<td></td>
<td>Formalizing</td>
</tr>
<tr>
<td>• Parts to make a whole</td>
<td>Property Noticing</td>
</tr>
<tr>
<td>• Numerators and denominators</td>
<td>Image Having</td>
</tr>
<tr>
<td>• Big denominators are equal to small fractions</td>
<td>Image Having</td>
</tr>
<tr>
<td>• Drawing divided quantity diagrams</td>
<td>Image Making</td>
</tr>
<tr>
<td>• Whole number knowledge</td>
<td>Primitive Knowing</td>
</tr>
</tbody>
</table>

Both Pirie and Kieren’s model and the strategies’ framework proposed in this study describe how students’ mathematical understanding grows. Such growth is seen as a dynamical and active process and involves a continual movement between different layers or ways of thinking. Pirie and Kieren’s model observes students’ mathematical understanding generally whereas the strategies’ framework focuses specifically on students’ fractional thinking and highlights the relevance of students’ mathematical abilities and their strategies in ordering fractions. In addition, Pirie and Kieren’s model divides 8 levels of mathematical understanding whereas strategies’ framework identifies 9 strategies and one or more strategies are associated with one or more levels of understanding.

Primitive Knowing was the first level of understanding where the students already knew the language and construction of individual fractions for the growth of initial understanding of fraction sizes. Through drawing divided quantity diagrams to compare fractions, the students began to form images out of this knowing at the second understanding level of Image Making. They generalized the specific images of unit fractions to big denominators were equal to small fractions and reached the next level of Image Having. This belief involved students’ understanding of equal partitioning of sets and shapes. On the other hand, big numbers were equal to small fractions was rather an erroneous image for the order of fractions due to the misinterpretation of fractions as two independent numbers. Big
numbers were equal to big fractions was another contradictory belief developed in the
evergeneralization of whole number knowledge to the acquisition of fractional knowledge and
fractions were ordered in the same way as counting numbers.

The students related fractions’ numerators and denominators indicating their Image
Having understanding of mathematics. This was a valid strategy for ordering a wider range of
fractions including non-unit fractions. At the fourth level of Property Noticing, the students
examined the images of fractions that were progressively closer to 1 (e.g., \( \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots \)) and
noticed a specific and relevant property of fractions as parts to make a whole. Hence, the
students fitted each fraction into a whole and considered fractional parts for comparisons.

Converting fractions into a common denominator, decimals and percentages involved
the fifth understanding level of Formalising. The students abstracted a method or common
quality from the previous image and formalized their understanding of fractions as
mathematical objects that could be transformed into a mathematical definition or algorithm.
Through numerical conversions, fractions were compared using only the number concepts and
symbols related to fractions without reference to their more physical quantitative meaning.

These conversion methods became pieces of a possible theory and not simply
techniques for computation to the students when the students could observe fraction
equivalence as a mathematical object at the next level of Observing. They noticed an image
for the conversion methods which could be distinguished by certain features (i.e., fractions
contained denominators as the factors of another denominator and standard simple fractions
and their percentages equivalents) but which can still be characterized as belonging to the
class of equivalence. Subsequently, the students observed and organised consistently their
own thought structures on fraction equivalence and numerical conversions at the level of
Structuring. The students knew the consequences of using particular methods of fraction
equivalence and this was crucial for them to associate and sequence numerical methods in
comparing and ordering fractions.

The understanding levels of Observing and Structuring were also required in the
strategy of using benchmarks. The students observed sets of fractions that were being
compared before leveraging fractions from known benchmark fractions and further justified
the order of fractions either using imaging or number properties.
Although numerical conversions were applicable to any set of fractions, the strategy of using benchmarks was considered more appropriate in certain situations. Achieving the understanding levels of Observing and Structuring assisted the students to select the most suitable strategy (either numerical conversions or benchmarking) and conduct the most proper step (either benchmarking to $\frac{1}{2}$ or 1; converting fractions into one or more common denominators) for comparison of fractions.

Only a few students gained the highest level of understanding – Inventising. They could create their own ways of ordering fractions and initiated the sequence or structure of thought based on their previous knowledge of order and equivalence of fractions.

### 7.3.4 Comparison of strategies’ framework with literature

The findings of this study were consistent with the list of strategies proposed by Darr and Fisher (2006) and Maguire, Neil and Fisher (2007) for comparing sizes of fractions. The list of strategies is increasingly sophisticated from attempting to use whole number knowledge, drawing pictures, identifying fractions with the same denominator or numerator, benchmarking fractions to well known fractions, to using equivalent fractions. In this study, cause-for-concern students were affected by their prior knowledge of whole numbers, average students mainly relied on the representations of fractions (e.g., drawing divided quantity diagrams, big denominators were equal to small fractions), and high achievers were able to use numerical strategies. The higher the mathematical abilities, the more sophisticated are the strategies used. This study also noted some different findings on students’ strategies (Table 7.4).
<table>
<thead>
<tr>
<th><strong>Table 7.4: Different opinions on students’ strategies</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Darr and Fisher (2006), Maguire et al. (2007) and Ministry of Education (2010)</strong></td>
</tr>
<tr>
<td><strong>The least sophisticated strategy:</strong></td>
</tr>
<tr>
<td>&quot;Attempting to use whole number knowledge&quot;</td>
</tr>
<tr>
<td>Fractions are ordered like counting numbers that big numbers are equal to big fractions.</td>
</tr>
<tr>
<td>Fractions are ordered in contrast with counting numbers.</td>
</tr>
<tr>
<td><strong>Second less sophisticated strategy:</strong></td>
</tr>
<tr>
<td>“Drawing pictures”</td>
</tr>
<tr>
<td>“Drawing divided quantity diagrams”</td>
</tr>
<tr>
<td>Considering the part-whole concept that underlies the divided quantity diagrams is more sophisticated than just comparing the drawing picture of fractions side by side in a concrete way.</td>
</tr>
<tr>
<td><strong>“Identifying fractions with the same denominator or numerator”</strong></td>
</tr>
<tr>
<td>Students use the part-whole understanding of fractions. The whole is broken into more parts and “the larger the denominator the smaller the fraction”.</td>
</tr>
<tr>
<td>“Numerator and denominators”</td>
</tr>
<tr>
<td><strong>The second sophisticated strategy:</strong></td>
</tr>
<tr>
<td>“Benchmarking fractions to well known fractions”</td>
</tr>
<tr>
<td>“Parts to make a whole” The numerators are close to the denominators to make up almost a whole.</td>
</tr>
<tr>
<td><strong>The most sophisticated strategy:</strong></td>
</tr>
<tr>
<td>“Using equivalent fractions” Finding a common denominator or converting fractions to percentages or decimals.</td>
</tr>
<tr>
<td>“Converting fractions to percentages or decimals”</td>
</tr>
<tr>
<td>“Using benchmarks”</td>
</tr>
<tr>
<td>The appropriateness of strategies for comparing fractions is greatly depending on the situations of fractions that are compared.</td>
</tr>
</tbody>
</table>

Students’ belief of big numbers were equal to small fractions was affected by their prior knowledge of whole numbers when learning new concept of fractions but this category of strategy contradicts big numbers were equal to big fractions. In drawing diagrams to represent fractions, the part-whole concept was taken into account in the representations of fractions for comparisons.

In the case of “identifying fractions with the same denominator or numerator”, big denominators were equal to small fractions was a specific strategy for ordering unit fractions. Numerator and denominator was advanced from big denominators were equal to small fractions. Parts to make a whole specifically focused on the images of fractions that were
progressively closer to 1 when the numerators were close to the denominators to make up almost a whole.

Although the numerical conversion method is considered as the most sophisticated strategy (Darr & Fisher, 2006; Maguire et al., 2007) and sufficient to solve all kinds of fraction problems (Smith III et al., 1993), the situations of fractions that are being compared actually determine the appropriateness of strategies to compare the fractions. Comparisons of fractions became perplexing and time consuming when more than one common denominator was needed to compare three fractions that contained denominators that were not the factors of the other denominators, and long division was required to convert complicated fractions into percentages or decimals. Using benchmarks could be equally sophisticated when students had to decide which benchmark to be used and to justify the order of fractions using number knowledge.

The involvement of students with different mathematical abilities in this study confirmed the list of students’ strategies from less to more sophisticated. Students with lower mathematical ability (e.g., cause-for-concern students) relied on less sophisticated strategies which involved lower level of mathematical understanding. On the other hand, students with higher mathematical ability (e.g., high achievers) were able to use more sophisticated strategies which involved higher level of mathematical understanding. Most importantly, this study found that the ability to use the most suitable strategies rather than the more sophisticated strategies was necessary to compare fractions in the simplest and fastest way.

Through investigating students’ reasoning, a more detailed perspective on students’ thinking in ordering fractions was revealed from this study. This adds to the body of knowledge on students’ thinking when comparing fractions as presented in the following. The understanding of the part-whole relation enabled students to order fractions correctly even though the divided quantity diagrams drawn had unequal whole and parts. The misconception of big numbers were equal to small fractions was shown to be different from big denominators were equal to small fractions. Students were affected by their prior knowledge of whole numbers when learning new concept of fractions. For fractions that were difficult to be converted into percentages, students estimated the percentage equivalent of these fractions to decide the ordering of fractions. Finding a common denominator for fractions that contained denominators as the factors of another denominator could be easier than converting certain fractions to percentages or decimals that involve complex multiplication and divisions.
Finding a common denominator using the lowest common multiple of both denominators is a way to avoid multiplying big numbers in getting equivalent fractions. The above students’ thinking provided practical and significant information to the teaching and learning of fractions and could be included in the NDP materials to develop students’ fractional thinking. Teachers can enrich their pedagogical content knowledge by focusing on students’ thinking and problem-solving processes (Young-Loveridge et al., 2007) and improve instructional strategies based on students’ possible conceptual schema (Olive & Vomvoridi, 2006).

Summary

This chapter shows the types of strategies used by the students are related to their mathematical abilities. High achievers were capable of using various types of strategies including the sophisticated numerical conversions and benchmarking; average students mainly relied on representations of fractions (i.e., drawing diagrams and using images of fractions) and cause-for-concern students employed less sophisticated whole number knowledge. Based on Pirie and Kieren’s model of the growth of students’ mathematical understanding, it was found that the strategies used by cause-for-concern students involved lower understanding levels while higher understanding levels were required in the strategies used by average students and high achievers. The strategies used by the students with different mathematical abilities that involved different levels of mathematical understanding were framed in this study to describe a progression of students’ fractional thinking. These findings about students’ strategies and the development of strategies’ framework have answered research question 2.

There were average and cause-for-concern students who tended to use higher strategies to order fractions but they showed a lack of expertise to achieve a higher strategy that had been mastered by the high achievers. The relationship between the strategies used by the students and the sets of fractions that were to be compared in the maths tasks (i.e., using particular strategies for particular fractions) was noticed more among high achievers than average and cause-for-concern students. By examining the reasons for using particular strategies in certain situations, it is found that some fractions can be easily compared using particular strategies. More complicated situations of fractions that are compared promotes the use of numerical strategies even for students whose earlier thinking is primarily based on representations of fractions and whole number knowledge. High achievers show the ability to select appropriate strategies based on the complexity of the problem.
In order to move the average and cause-for-concern students to more advanced strategies, the numerical situations of the sets of fractions can be highlighted during teaching. This will be discussed further in Chapter 9. Prior to that, the improvement of students’ strategies will be identified from the changes of strategies used between the pre and post maths tasks as presented in Chapter 8.
CHAPTER 8

IMPROVEMENT OF STUDENTS’ KNOWLEDGE OF FRACTIONS

This chapter presents findings to answer research question 3: *What improvements, if any, are there in students’ ability as a result of playing the game?*

This chapter discusses the effectiveness of playing *Tower Trap* on students’ learning of fractions. The performance of three individuals is elaborated first in Section 8.1. These students include a high achiever, an average student and a cause-for-concern student, which were selected using purposive sampling (See Chapter 4, Section 4.3.1.3). Their results in the maths tasks and tests are discussed in relation to their responses in the questionnaires and computer game play. The aim is to gain a rich understanding of these individuals on fractions.

After examining the performance of the three individual students, the subsequent sections analyse the performance in the maths tasks and tests of all the students in this study by grouping them into the three mathematical ability groups. Section 8.2 explains the changes of students’ strategies for ordering fractions between the pre and post maths tasks with special focus on the changes that lead to improvement in the task. In addition, the reasons and advantages of the changes of strategies are also noted in some of the students who are right in both maths tasks yet change their strategies. Finally, Section 8.3 analyses the achievement differences between the pre and post tests of the students, especially for the improvement of being wrong in the pre test but right in the post test (i.e., WR).

8.1 Individual performance

This section elaborates the performance of Sam, Mary and Peter from three mathematical ability groups (i.e., high achievers, average students and cause-for-concern student, respectively). These students are not the students used in Chapter 7 to demonstrate percentage calculations. Their results in the tests and maths tasks and responses given in the questionnaires and computer game play are presented in a detailed manner. A summary is given for each student to highlight the impact of *Tower Trap* on each individual where specific improvements have been detected.

The performance of three individuals is elaborated to provide an initial idea of the changes of strategies between the pre and post maths tasks and achievement differences.
between the pre and post tests. This demonstrates how the data is analysed in this study. After examining the performance of the three individual students, the subsequent sections analyse the performance in the maths tasks and tests of all the students in this study by grouping them into the three mathematical ability groups. Based on the case study of these individual students, the next chapter discusses instructional approaches that can be used by the teacher to move these students towards more advanced strategies.

8.1.1 Sam

Sam was a high achiever. He was 12 years old. He had never played computer games before. He believed he was excellent when asked about how well he thought he was at fractions.

8.1.1.1 Pre and post maths tasks

The students including Sam were asked to order the given sets of fractions from the smallest to the largest in the pre and post maths tasks (Appendix 2) and provide reasoning for the strategies used for comparing the sizes of fractions. The categories of students’ strategies identified in Chapter 4 are adopted in this chapter.

Sam gave correct ordering to all 10 questions in the pre and post maths tasks (5 questions each). An improvement in the use of strategies was noticed in his work though. Table 8.1 shows the ordering of fractions and relevant reasoning given by Sam in the pre and post maths tasks.

In questions 1 and 2, Sam compared fractions by referring to denominators of fractions such as “third”, “quarter” and “fifth”. Fractions were ordered based on the strategy that big denominators were equal to small fractions. When comparing unit fractions, only denominators were considered instead of the relation between numerators and denominators of fractions. Even for fractions with like numerators such as \( \frac{3}{5}, \frac{3}{4} \) in question 2, Sam stressed the denominators “5ths” and “quarters” for comparison and ignored the like numerator 3.

In questions 3 and 4, Sam gave partial reasoning in the pre task. In the post task, he referred to the “parts” of fractions and ordered fractions based on the strategy that big denominators were equal to small fractions. In question 5, Sam used the benchmark of a half
to compare fractions that were close to a half in both the pre and post tasks. He only stated a fraction was less or more than a half without using the number knowledge of double.

Table 8.1: Sam’s results in the pre and post maths tasks

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre maths task</th>
<th>Post maths task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ordering</td>
<td>Reasoning</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{5}$, $\frac{1}{3}$</td>
<td>Because a 5th is smaller than a third</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$, $\frac{2}{3}$</td>
<td>Because two 5th is a smaller amount than 2 thirds</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{2}$</td>
<td>Because $\frac{3}{10}$ is smaller than $\frac{2}{5}$ and $\frac{2}{5}$ is smaller than $\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$</td>
<td>Because $\frac{2}{3}$ is smaller than $\frac{4}{5}$ and $\frac{4}{5}$ is smaller than $\frac{6}{7}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{7}{12}$, $\frac{1}{2}$</td>
<td>$\frac{8}{17}$ = is less than half</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}$ = One half</td>
<td>$\frac{1}{2}$ = One half</td>
</tr>
<tr>
<td></td>
<td>$\frac{7}{12}$ = more than a half</td>
<td>$\frac{7}{12}$ = more than a half</td>
</tr>
</tbody>
</table>
8.1.1.2 Pre and post maths tests

Sam was right in all the 6 questions in the pre and post tests (Appendix 1) (Table 8.2).

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre test (B)</th>
<th>Post test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Questions</td>
<td>Selections</td>
</tr>
<tr>
<td>1</td>
<td>Which picture is ( \frac{2}{5} ) shaded blue?</td>
<td>![Selection]</td>
</tr>
<tr>
<td>2</td>
<td>What fraction of this rectangle is shaded blue?</td>
<td>( \frac{5}{8} )</td>
</tr>
<tr>
<td>3</td>
<td>What fraction of this rectangle is shaded blue?</td>
<td>between ( \frac{1}{2} ) and ( \frac{3}{4} )</td>
</tr>
<tr>
<td>4</td>
<td>Which is smaller: ( \frac{5}{9} ) or ( \frac{6}{13} )?</td>
<td>( \frac{6}{13} )</td>
</tr>
<tr>
<td>5</td>
<td>Which of these orders is from largest to smallest?</td>
<td>( \frac{4}{6}, \frac{2}{9}, \frac{3}{18} )</td>
</tr>
<tr>
<td>6</td>
<td>Which number is closest to the answer to ( \frac{10}{11} + \frac{6}{7} )?</td>
<td>2</td>
</tr>
</tbody>
</table>
8.1.1.3 Computer game play

Figure 8.3, Figure 8.4 and Figure 8.5 show the attempts taken by Sam for ordering various types of fraction bricks in the 26 game levels. The computer game of Tower Trap that is designed for ordering fractions was simple for Sam and he ordered almost all game levels using one attempt only. At the introductory levels of tall broken bricks, he had taken two attempts to learn to form a fraction brick of $\frac{1}{2}$ by dragging the part from the whole $\frac{2}{2}$.

Table 8.3: Sam’s attempts in ordering tall visible and broken bricks

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Sam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$ $\frac{2}{2}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Tall visible bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{3}$ $\frac{1}{2}$ $\frac{1}{1}$</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$ $\frac{2}{2}$</td>
<td>2</td>
</tr>
<tr>
<td><strong>Tall broken bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{1}{1}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{2}$ $\frac{1}{1}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{1}{5}$ $\frac{1}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{5}$ $\frac{2}{3}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 8.4: Sam’s attempts in ordering *long visible* and *broken bricks*

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Sam</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td>(\frac{1}{2}\div2)</td>
<td>(1)</td>
</tr>
<tr>
<td><strong>Long visible bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>(\frac{1}{4}\div4\div\frac{3}{4}\div\frac{4}{4}\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 2</td>
<td>(\frac{1}{3}\div2\div\frac{2}{3})</td>
<td>(1)</td>
</tr>
<tr>
<td><strong>Long broken bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>(\frac{1}{7}\div7\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 2</td>
<td>(\frac{12}{99}\div\frac{34}{99}\div\frac{45}{99}\div\frac{78}{99}\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 3</td>
<td>(\frac{3}{10}\div\frac{2}{5}\div\frac{1}{2})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 4</td>
<td>(\frac{2}{3}\div\frac{4}{5}\div\frac{6}{7})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 5</td>
<td>(\frac{8}{17}\div\frac{1}{2}\div\frac{7}{12})</td>
<td>(1)</td>
</tr>
</tbody>
</table>

Table 8.5: Sam’s attempts in ordering *hidden tall* and *long bricks*

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Sam</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td>(\frac{1}{2}\div2)</td>
<td>(1)</td>
</tr>
<tr>
<td><strong>Hidden tall bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>(\frac{1}{4}\div\frac{3}{4}\div\frac{1}{4}\div\frac{4}{4}\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 2</td>
<td>(\frac{1}{8}\div\frac{4}{4}\div\frac{1}{2})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 3</td>
<td>(\frac{1}{5}\div\frac{3}{3})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 4</td>
<td>(\frac{2}{5}\div\frac{2}{3})</td>
<td>(1)</td>
</tr>
<tr>
<td><strong>Hidden long bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>(\frac{1}{7}\div\frac{7}{7}\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 2</td>
<td>(\frac{12}{99}\div\frac{34}{99}\div\frac{45}{99}\div\frac{78}{99}\div1)</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 3</td>
<td>(\frac{3}{10}\div\frac{2}{5}\div\frac{1}{2})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 4</td>
<td>(\frac{2}{3}\div\frac{4}{5}\div\frac{6}{7})</td>
<td>(1)</td>
</tr>
<tr>
<td>Question 5</td>
<td>(\frac{8}{17}\div\frac{1}{2}\div\frac{7}{12})</td>
<td>(1)</td>
</tr>
</tbody>
</table>
8.1.1.4 Questionnaires

The students rated from 1 to 5 (i.e., strongly agree, agree, neither agree nor disagree, disagree and strongly disagree) about “playing the game”, “game features”, “learning of fractions” and “teaching aids” in the questionnaires. Table 8.6 shows the responses given by Sam to the 21 questionnaire items.

Table 8.6: Sam’s rating in questionnaires

<table>
<thead>
<tr>
<th>N.</th>
<th>Questionnaire Items</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Playing the Game</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>I like playing the game.</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>I like playing the game because I can learn more about fractions.</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>I like playing the game because I like to play computer games.</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td><strong>Game Features</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I like the boy in the game.</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>I like the creatures in the game.</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>I like to drag and drop the bricks.</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>I like to make the boy move.</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>The instructions for the game are clear.</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>The story makes the game interesting.</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>I find it easy to see my progress.</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>I find forming staircases is interesting.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td><strong>Learning of Fractions</strong></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>After playing the game, I want to learn more about fractions.</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>I use knowledge of fractions I learned in school when playing the game.</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>The game helps me to imagine the sizes of fractions.</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>I learned how to write fraction symbols from playing this game.</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>The game helped me to put fractions into order.</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>This game helped to fix mistakes I was making with fractions.</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>I have learned more about fractions.</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td><strong>Teaching Aids</strong></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>I would like to play the game at school as a part of learning fractions.</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>I had to think hard to play the game.</td>
<td>4</td>
</tr>
<tr>
<td>21</td>
<td>I learned more from the game than I do from my teacher in the classroom.</td>
<td>4</td>
</tr>
</tbody>
</table>
Sam liked playing the game but this was not because he liked to play computer games. He liked to make the boy move (e.g., jump up and duck down) in the game. Sam thought the story about the boy who was lost in the forest and wanted to find the way home from the top of the tower made the game interesting. However, he disagreed that forming staircases was interesting. He found the instructions for the game (e.g., guiding the player to order various types of fraction bricks) were clear. As the scores obtained by the player for ordering fractions and avoiding creatures were accumulated and displayed at the top right corner of the game screen, Sam agreed that it was easy for him to see his progress in the game.

Sam was aware of using the knowledge of fractions learned in school to play the game. He commented positively about the usefulness of the game for learning fractions. He had learned more about fractions from the game which helped him to imagine fraction sizes and order fractions.

Sam disagreed that he had to think hard to play the game and learned more from the game than he did from his teacher in the classroom. Nevertheless, he would like to play the game at school as a part of learning fractions.

Summary

Maths tasks and tests were easy for Sam and he was right in all questions. He was capable of using big denominators were equal to small fractions as well as the benchmark of a half for ordering fractions in the maths tasks. He mastered the knowledge of fractions including representing, ordering and operating fractions as asked in the tests.

Sam was not a computer game player and he had never played computer games before. However, his knowledge of fractions allowed him to complete the Tower Trap game easily. He enjoyed playing the game, recognised the use of the game for learning fractions and would be happy to play the game at school as a part of learning fractions.

8.1.2 Mary

Mary was an average student. She was 12 years old. She played computer games once a week. She believed she was above average when asked about how good she thought she was at fractions.
8.1.2.1 Pre and post maths tasks

Table 8.7 shows the ordering of fractions and relevant reasoning given by Mary in the pre and post maths tasks.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre maths task</th>
<th>Post maths task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ordering</td>
<td>Reasoning</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{5}$, $\frac{1}{3}$</td>
<td>Because if you cut a pie into 5 pieces, the slices will be smaller than if it was cut into 3</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$, $\frac{2}{3}$</td>
<td>Because if your pie is cut into 3 and you get 2 that’s more than if your pies is cut into 5 and you get 2</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{2}$</td>
<td>Because $\frac{3}{10}$ is less than half and $\frac{2}{5}$ is too.</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{6}{7}$, $\frac{4}{5}$, $\frac{2}{3}$ (Incorrect)</td>
<td>If you cut a pie into 7 pieces, another into 3 and another into 5, then take away the amount it says, see how much you have left</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2}$, $\frac{7}{12}$, $\frac{8}{17}$ (Incorrect)</td>
<td>Because 7 pieces of 12 is more than half and 8 pieces of 17 is more than half too.</td>
</tr>
</tbody>
</table>
Mary described $\frac{1}{5}$ and $\frac{1}{3}$ in terms of pie. This strategy focuses on the denominator as classified in Chapter 4 as big denominators are equal to small fractions. Based on this belief, Mary gave the correct ordering to unit fractions and fractions with like numerators in questions 1 and 2 but not to fractions that were close to one in question 4 (pre task). An incorrect ordering was given when Mary ordered fractions with a bigger denominator (e.g., $\frac{6}{7}$) as smaller than the fractions with smaller denominators (e.g., $\frac{2}{3}$). By drawing circular divided quantity diagrams in the post task for question 4, Mary could visualize the part left constructed in the mental picture of fractions (as mentioned in the pre task “see how much you have left”). She shaded the one part that left from the whole and found the bigger the one part that was left over the smaller the parts that were taken from the whole. Since $\frac{1}{4}$ was bigger than $\frac{1}{8}$, Mary determined $\frac{3}{4}$ was the smallest fraction while $\frac{7}{8}$ was the largest fraction. However, some errors were noticed in Mary’s diagrams. The wholes of circles were not in the same size and the parts were not equally divided on the whole. These errors were made by other students in their diagrams too and this was especially common for circular divided quantity diagrams. In this case, students’ understanding of part-whole relation embedding in the divided quantity diagrams was crucial to ensure that they ordered fractions correctly.

Mary also made a mistake by drawing a diagram of $\frac{6}{7}$ instead of $\frac{7}{8}$ and labelled it as the largest fraction and this is inconsistent with her ordering “$\frac{3}{4}, \frac{5}{6}, \frac{7}{8}$”.

Mary improved her strategies in question 3 from giving only partial reasoning in the pre tasks to benchmarking fractions to a half in the post task. She also used the benchmark in question 5 but gave the incorrect ordering in both the pre and post tasks. Her reasoning in the pre task “7 pieces of 12 is more than half” was right but “8 pieces of 17 is more than half too” was wrong. Her reasoning in the post task was too brief. She needed the number knowledge of double to show that the fractions were more or less than a half.
8.1.2.2 Pre and post maths tests

Mary improved in question 4 by making the following selections (Table 8.8).

<table>
<thead>
<tr>
<th>Question</th>
<th>Pre test</th>
<th>Post test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Questions</td>
<td>Selection</td>
</tr>
<tr>
<td>4</td>
<td>Which is smaller: $\frac{5}{9}$ or $\frac{6}{13}$?</td>
<td>Same (Incorrect)</td>
</tr>
</tbody>
</table>

In the pre test, Mary thought $\frac{5}{9}$ or $\frac{6}{13}$ had the same size. She needed the number knowledge of double to decide that $\frac{5}{9}$ was larger than $\frac{4.5}{9}$, which was equal to $\frac{1}{2}$; $\frac{6}{13}$ was smaller than $\frac{6}{12}$, which was equal to $\frac{1}{2}$.

8.1.2.3 Computer game play

Mary took one attempt for ordering fraction bricks at most of the game levels except for 5 game levels (Table 8.9, Table 8.10 & Table 8.11):

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Mary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>Tall visible bricks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>Tall broken bricks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{2}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 8.10: Mary’s attempts in ordering *long visible* and *broken bricks*

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Mary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td>$\frac{1}{2}, \frac{2}{2}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Long visible bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{3}{3}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Long broken bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{7}, \frac{4}{7}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 5</td>
<td>$\frac{8}{17}, \frac{1}{2}, \frac{7}{12}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8.11: Mary’s attempts in ordering *hidden tall* and *long bricks*

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Mary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td>$\frac{1}{2}, \frac{2}{2}$</td>
<td>2</td>
</tr>
<tr>
<td><strong>Hidden tall bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{8}, \frac{1}{4}, \frac{2}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{1}{5}, \frac{1}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{5}, \frac{2}{3}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Hidden long bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{7}, \frac{4}{7}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{12}{99}, \frac{34}{99}, \frac{45}{99}, \frac{78}{99}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 5</td>
<td>$\frac{8}{17}, \frac{1}{2}, \frac{7}{12}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Mary took two attempts to order $\frac{1}{2}$ and $\frac{2}{2}$ at the introductory levels, presumably to get herself familiar with *tall visual bricks*, *tall broken bricks* and *hidden bricks*. She also took two attempts to order *tall visual bricks*: $\frac{1}{3}$, $\frac{1}{2}$ and 1 and *tall broken bricks*: $\frac{1}{8}$, $\frac{1}{4}$ and $\frac{1}{2}$ at the earlier levels. After that, Mary could order other fractions at the later levels using only one attempt even for fractions that were close to each other and involved big numbers such as “$\frac{2}{3}$”, $\frac{4}{5}$ and $\frac{6}{7}$” and “$\frac{7}{12}$, $\frac{1}{2}$ and $\frac{8}{17}$”.

### 8.1.2.4 Questionnaires

Table 8.12 shows the responses given by Mary in the questionnaires.

#### Table 8.12: Mary’s rating in questionnaires

<table>
<thead>
<tr>
<th>N.</th>
<th>Questionnaire Items</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Playing the Game</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>I like playing the game.</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>I like playing the game because I can learn more about fractions.</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>I like playing the game because I like to play computer games.</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td><strong>Game Features</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I like the boy in the game.</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>I like the creatures in the game.</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>I like to drag and drop the bricks.</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>I like to make the boy move.</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>The instructions for the game are clear.</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>The story makes the game interesting.</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>I find it easy to see my progress.</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>I find forming staircases is interesting.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td><strong>Learning of Fractions</strong></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>After playing the game, I want to learn more about fractions.</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>I use knowledge of fractions I learned in school when playing the game.</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>The game helps me to imagine the sizes of fractions.</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>I learned how to write fraction symbols from playing this game.</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>The game helped me to put fractions into order.</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>This game helped to fix mistakes I was making with fractions.</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>I have learned more about fractions.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td><strong>Teaching Aids</strong></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>I would like to play the game at school as a part of learning fractions.</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>I had to think hard to play the game.</td>
<td>4</td>
</tr>
<tr>
<td>21</td>
<td>I learned more from the game than I do from my teacher in the classroom.</td>
<td>5</td>
</tr>
</tbody>
</table>
According to Mary, she liked playing the game because of her interest in playing computer games but not for learning more about fractions. About the general appeal of the game, Mary agreed that the instructions for the game were clear. She liked the boy and disliked the creatures. She liked to make the boy move but did not find forming the staircases interesting. In her additional comment, she was critical of the game in that “the game had bad graphics!”.

Mary was aware of the use of knowledge of fractions learned in school for playing the game. She agreed the game helped her to imagine the sizes of fractions but not to write fraction symbols. However, she disagreed that she had learned more about fractions and was motivated to learn more about fractions after playing the game.

Mary did not think hard to play the game and had learned less from the game than she did from her teacher in the classroom. Nevertheless, she was interested in playing the game at school as a part of learning fractions.

**Summary**

Mary mainly referred to the representations of fractions for comparing and ordering fractions in the maths tasks. The game focusing on pictorial images of fractions encouraged her to visualize fractions using diagrams and helped her to improve in the maths task. She had a limited ability in using the benchmark of a half for comparing fractions that were close to a half and was wrong in both the pre and post tasks. Nonetheless, in the post test, when comparing two fractions that were close to half, she managed to identify the larger fraction and improved in the tests overall.

Mary liked playing computer games and she played computer games once a week. She took a few attempts to order fraction bricks at the earlier level of *Tower Trap*. Since Mary was playing the game without the intention of learning fractions, she was not aware of the improvement in her achievement of fractions after playing the game.

**8.1.3 Peter**

Peter was a cause-for-concern student. He was 12 years old. He played computer games every day. He believed he was average when asked about how good he thought he was at fractions.
8.1.3.1 Pre and post maths tasks

Table 8.13 shows the ordering of fractions and relevant reasoning given by Peter in the pre and post maths tasks.

Table 8.13: Peter’s results in the pre and post maths tasks

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre maths task</th>
<th>Post maths task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ordering</td>
<td>Reasoning</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{5}$, $\frac{1}{3}$</td>
<td>I just knew it</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$, $\frac{2}{3}$</td>
<td>There are two 3 and two 5</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{2}$</td>
<td>Because I know</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{6}{7}$, $\frac{4}{5}$, $\frac{2}{3}$ (Incorrect)</td>
<td>Because I know</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{8}{17}$, $\frac{7}{12}$, $\frac{1}{2}$ (Incorrect)</td>
<td>Because I know</td>
</tr>
</tbody>
</table>

Peter improved in the use of strategies for ordering fractions in the maths tasks. He was unable to provide reasoning and described fractions $\frac{2}{5}$ and $\frac{2}{3}$ as “two 3 and two 5” in the pre task. In the post task, he used fraction language “fifths” and “quarters” and referred to parts to make a whole for comparison of fractions.

In the ordering given by Peter in questions 1 to 5, the fractions with bigger numbers were ordered smaller than the fractions with smaller numbers. Such ordering seemed to be affected by the belief of big numbers were equal to small fractions. This belief led to the correct ordering in the first three questions but not for the last two questions. Even though he
tried to change his strategy in the post task, he still failed to order fractions that were close to one in question 4 correctly. In question 5, he used the benchmark of $\frac{1}{2}$ and found that “4 is just over half of 7” and “7 is just under half of 15”. However, he ordered both $\frac{4}{7}$ and $\frac{7}{15}$ smaller than $\frac{1}{2}$. His reasoning was inconsistent with his ordering of fractions.

8.1.3.2 Pre and post maths tests

Table 8.14 shows the selections made by Peter in the tests. When representing the range of fractions in question 3 and ordering fractions in question 5, he was right in the pre test but was wrong in the post test. He was wrong in both the pre and post tests when estimating fractions in question 6. Nevertheless, he did not treat fractions as two independent numbers as he did in the pre test where he added up the numerators.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre test</th>
<th>Post test</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Questions</strong></td>
<td><strong>Selections</strong></td>
<td><strong>Questions</strong></td>
</tr>
<tr>
<td>3 What fraction of this rectangle is shaded blue?</td>
<td>between $\frac{1}{4}$ and $\frac{1}{2}$</td>
<td>What fraction of this rectangle is shaded blue?</td>
</tr>
<tr>
<td>5 Which of these orders is from smallest to largest?</td>
<td>$\frac{3}{18}, \frac{2}{9}, \frac{4}{6}, 1$</td>
<td>Which of these orders is from largest to smallest?</td>
</tr>
<tr>
<td>6 Which number is closest to the answer to $\frac{6}{7} + \frac{9}{10}$?</td>
<td>15 (Incorrect)</td>
<td>Which number is closest to the answer to $\frac{10}{11} + \frac{6}{7}$?</td>
</tr>
</tbody>
</table>
8.1.3.3 Computer game play

Peter took one attempt in the earlier game levels (Table 8.15 & Table 8.16) and did not need more attempts at the introductory levels. He took more than one attempt to order hidden long bricks from the largest to the smallest as shown in Table 8.17. At hidden bricks, only symbols of fractions were labelled and no physical sizes of fractions were shown. Since the knowledge of fractions was required for ordering hidden bricks, Peter, like the other cause-for-concern students, needed more attempts than average students and high achievers to order fractions until a correct ordering was obtained.

Table 8.15: Peter’s attempts in ordering tall visible and broken bricks

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Peter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1 $\frac{2}{2}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Tall visible bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{2}, \frac{2}{3}, \frac{2}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{3}, \frac{1}{2}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}, \frac{2}{2}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Tall broken bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$</td>
<td>1</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{8}, \frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{1}{5}, \frac{3}{5}$</td>
<td>1</td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{5}, \frac{2}{3}$</td>
<td>1</td>
</tr>
</tbody>
</table>
### Table 8.16: Peter’s attempts in ordering long visible and broken bricks

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Peter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{2}$</td>
</tr>
<tr>
<td><strong>Long visible bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{2}$ $\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{3}$ $\frac{2}{3}$</td>
<td></td>
</tr>
<tr>
<td><strong>Long broken bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{7}$ $\frac{1}{7}$</td>
<td></td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{12}{99}$ $\frac{34}{99}$ $\frac{45}{99}$ $\frac{78}{99}$</td>
<td>$\frac{1}{1}$</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{3}{10}$ $\frac{2}{5}$</td>
<td></td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{3}$ $\frac{4}{5}$</td>
<td></td>
</tr>
<tr>
<td>Question 5</td>
<td>$\frac{8}{17}$ $\frac{1}{17}$</td>
<td>$\frac{2}{12}$</td>
</tr>
</tbody>
</table>

### Table 8.17: Peter’s attempts in ordering hidden tall and long bricks

<table>
<thead>
<tr>
<th>Bricks types and questions</th>
<th>Fraction questions</th>
<th>Peter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{2}$</td>
</tr>
<tr>
<td><strong>Hidden tall bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{2}$ $\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{1}{8}$ $\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{1}{5}$ $\frac{3}{5}$</td>
<td></td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{5}$ $\frac{2}{3}$</td>
<td></td>
</tr>
<tr>
<td><strong>Hidden long bricks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question 1</td>
<td>$\frac{1}{7}$ $\frac{1}{7}$</td>
<td></td>
</tr>
<tr>
<td>Question 2</td>
<td>$\frac{12}{99}$ $\frac{34}{99}$ $\frac{45}{99}$ $\frac{78}{99}$</td>
<td>$\frac{1}{1}$</td>
</tr>
<tr>
<td>Question 3</td>
<td>$\frac{3}{10}$ $\frac{2}{5}$</td>
<td></td>
</tr>
<tr>
<td>Question 4</td>
<td>$\frac{2}{3}$ $\frac{4}{5}$</td>
<td></td>
</tr>
<tr>
<td>Question 5</td>
<td>$\frac{8}{17}$ $\frac{1}{17}$</td>
<td>$\frac{2}{12}$</td>
</tr>
</tbody>
</table>
In the pre task, Peter ordered incorrectly from the smallest \( \frac{8}{17} \) through \( \frac{7}{12} \) to the largest \( \frac{1}{2} \) without giving any reasoning. After taking three attempts to order similar fractions correctly in the computer game, he tried to use the benchmark of a half in the post task. His reasoning “4 is just over half of 7 and 7 is just under half of 15” was right but his ordering “\( \frac{4}{7}, \frac{7}{15}, \frac{1}{2} \)” was incorrect.

8.1.3.4 Questionnaires

Table 8.18 shows the responses given by Peter in the questionnaires.

Table 8.18: Peter’s rating in questionnaires

<table>
<thead>
<tr>
<th>N.</th>
<th>Questionnaire Items</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Playing the Game</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>I like playing the game.</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>I like playing the game because I can learn more about fractions.</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>I like playing the game because I like to play computer games.</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td><strong>Game Features</strong></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I like the boy in the game.</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>I like the creatures in the game.</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>I like to drag and drop the bricks.</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>I like to make the boy move.</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>The instructions for the game are clear.</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>The story makes the game interesting.</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>I find it easy to see my progress.</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>I find forming staircases is interesting.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td><strong>Learning of Fractions</strong></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>After playing the game, I want to learn more about fractions.</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>I use knowledge of fractions I learned in school when playing the game.</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>The game helps me to imagine the sizes of fractions.</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>I learned how to write fraction symbols from playing this game.</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>The game helped me to put fractions into order.</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>This game helped to fix mistakes I was making with fractions.</td>
<td>2</td>
</tr>
<tr>
<td>18</td>
<td>I have learned more about fractions.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td><strong>Teaching Aids</strong></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>I would like to play the game at school as a part of learning fractions.</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>I had to think hard to play the game.</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>I learned more from the game than I do from my teacher in the classroom.</td>
<td>4</td>
</tr>
</tbody>
</table>
Peter liked playing the *Tower Trap* game mostly because he liked to play computer games. He was indifferent to the boy as well as to the creatures in the game. He liked to make the boy move but not to drag and drop the bricks. He commented positively on the instructions for the game and the story of the game.

Peter knew that he was using knowledge of fractions learned in school for playing the game. He agreed the game helped him more to imagine fraction sizes and fix his mistakes with fractions than to write fraction symbols and order fractions. He did not show an interest to learn more about fractions after playing the game.

Peter neither agreed nor disagreed that he had to think hard to play the game. However, he disagreed that he had learned more from the game than he did at school. Yet, he liked to play the game at school as a part of learning fractions.

**Summary**

Peter improved in his strategies for ordering fractions from not giving any reasoning in the pre task to using big denominators were equal to small fractions in the post task. He gave the correct ordering for simple fractions such as unit fractions and fractions with like numerators in the first three questions. He used a benchmark of a half to compare fractions that were close to a half in the post task but this did not help him to order them correctly.

The computer game that focused on comparing sizes of fractions provided limited help to Peter on the aspects of fractions assessed in the tests, which included representing the range of fractions, estimating fractions, and ordering fractions. As such, there was no significant improvement on Peter’s achievement in the post maths task.

Peter was a computer game lover and he played computer games every day. Therefore, he could simply play the game without much thinking at the visible and broken bricks levels until his knowledge of fractions was required for long hidden bricks levels. With only symbols of fractions being given on hidden bricks, he needed a few attempts to order these bricks. Such experience in the game did provide opportunities for him to fix his mistakes with fractions.
8.1.4 Comparison of the responses of the three students

All students had a basic knowledge of fractions. They were right in ordering simple fractions in the first three questions in the maths tasks. These fractions were unit fractions, fractions with like numerators and fractions that contained denominators as the factors of another denominator. They were also able to represent fractions correctly in the first three questions in the pre and post tests. The students showed a significantly different performance in advanced knowledge of fractions. When ordering fractions that were close to each other in the last two questions in the maths tasks, only Sam gave the correct ordering. He was also capable of operating and ordering fractions in the tests while Mary performed poorer than Sam and Peter was less able.

The game on comparing and ordering fractions was suitable to be played by these Year 8 students. They had learnt fractions yet their advanced knowledge of fractions needed to be enhanced. After playing the game, Mary showed improvement in answering questions from the pre to the post maths tasks and tests but there was no improvement detected for Peter. Nevertheless, Peter who had not given any reasoning for his ordering of fractions in the pre task referred to big denominators were equal to small fractions in the post task. The game helped Mary, who had average understanding of fractions, more than Peter, with weaker understanding of fractions, to improve their knowledge and strategies of fractions especially on the representations of fractions. In the game, students compared fractions using physical, pictorial and symbolic representations. Despite the game effect on students’ representations of fractions, Peter required more learning of fractions while Mary needed to be enhanced with operation and ordering of fractions specifically.

Sam and Mary played computer games less frequently than Peter in their daily life. They needed more attempts than Peter at the introductory level to get familiar with the Tower Trap computer game. The attempts the students took in ordering different types of fraction bricks were affected by their fraction knowledge in which Sam used one attempt only whereas Mary and Peter took more attempts to order visible, broken and hidden bricks. Therefore, more difficult fraction sets could be included for Sam to order while Mary and Peter needed to play the game for a few times to strengthen their fraction concepts. Since students gain additional practice from applying mathematical ideas or processes in playing mathematics games (Booker, 2000), practicing on the newly taught mathematical procedures is one of the effective ways to help students retain, develop and master the fluency of the procedures (Woodward, 2006; Wright; 2007; Rowland, 2008).
In addition to providing practice, games can be used as a pedagogical medium (Squire & Jenkins, 2004). Mathematics games require specific instructional design (Van Eck, 2007) and the pedagogical design enhances the educational value of games (Prensky, 2001). The *Tower Trap* computer game promotes constructivist learning through the play from *visible* and *broken* to *hidden bricks*. Such transition is created to enable students’ understanding to progress from operating on concrete materials, to visualisation, to abstraction (Pirie & Kieren, 1989). This encourages students to construct their mathematical knowledge in meaningful ways (Fosnot, 1996; Moseley, 2005). The opportunities to construct knowledge through play and exploration make the learning in educational games more effective (Amory, 2001).

Regarding learning fractions, all three students were aware of the use of fractions concept learned in school for playing the game. They agreed the game helped them to imagine the sizes of fractions more than to write fraction symbols. Sam had the strongest knowledge of fractions yet he agreed more than the others that the game helped him to order fractions and he had learnt more about fractions. With the weakest knowledge of fractions, Peter appreciated more than others that the game helped him to fix mistakes with fractions.

The students did not agree that they had to think hard to play the *Tower Trap* game especially for Sam and Mary with their better knowledge of fractions. In fact, they liked to make the boy move more than to drag and drop the bricks. Moving the boy involved less thinking than ordering the fraction bricks. All students liked playing the game. Peter and Mary like the game mostly because they liked to play computer games. They found the instructions for the game were clear and the story of the game interesting. They were interested in playing the game at school as a part of learning fractions. Students were attracted to the fun way of learning in playing games (Young-Loveridge, 2005). Furthermore, games provide a pleasant environment and meaningful situation where students can apply and practice mathematical concepts and skills learnt in the classroom (Booker, 2004; Bragg, 2007). Fun learning is effective to engage students and improve their performance in mathematics (Lepper & Cordova, 1992; Bragg, 2003; Bragg, 2007; Sullivan, 2010).

In conclusion, the three individuals liked the game and they had learnt some fractions from playing the game. The following sections discuss further the positive impact of *Tower Trap* on more students by investigating their changes of strategies in the maths tasks and achievement difference of WR in the tests.
8.2 Changes of students’ strategies between the pre and post maths tasks

The previous section on individual performance showed that Sam (i.e., a high achiever) was right in all questions in the maths tasks and still changed his strategies for ordering fractions. Mary (i.e., an average student) improved in the use of representations of fractions which led her to a correct ordering of fractions in the post task. However, she did not improve her strategy in using the benchmark of a half and was wrong in both the pre and post tasks. Peter (i.e., a cause-for-concern student) did not give any reasoning in the pre task but referred to the representations of fractions in the post task.

This section extends the investigation on the changes of strategies between the pre and post maths tasks from individual students to mathematical ability groups of students. The change of strategies after playing Tower Trap which leads to a correct ordering of fractions in the post task from a wrong answer in the pre task was referred to as an improvement in this study. The sets of fractions used in the pre and post maths tasks are shown in Appendix 2.

Figure 8.1 shows the number of students who improved (i.e., WR) in the maths tasks. Students improved most in questions 4 and 5, especially for cause-for-concern and average students.

![Figure 8.1: The number of students who were WR in the maths tasks](image)

Figure 8.2 shows the numbers of students who are RR and use different strategies in the pre and post maths tasks. Among them, high achievers changed their strategies in all questions, average students changed their strategies in questions 1 to 4, while cause-for-concern students were only able to change their strategies in the first three questions.
The following sections examine the changes of students’ strategies particularly on the improvement that helped students who played *Tower Trap* to improve in the maths tasks (i.e., WR). These sections also identify the problems of using particular strategies and the important considerations for ordering fractions correctly. The reasoning of the students who change their strategies and give correct ordering in the pre and post tasks (i.e., RR) is examined in order to identify the relevant reasons and advantages of the changes of the strategies.

In the following discussions, high achievers are labelled as HA1 to HA8 (from School A) and HC1 to HC7 (from School C), average students are labelled as AA1 to AA37 (from School A) and AC1 to AC13 (from School C), cause-for-concern students are labelled as CA1 to CA36 (from School A) and CC1 to CC6 (from School C).

### 8.2.1 Use of representations of fractions

The representations of fractions referred to by the students in this study included drawing divided quantity diagrams, big denominators were equal to small fractions, numerator and denominator and, parts to make a whole.

Section 8.2.1.1 and Section 8.2.1.2 describe the students who improved in questions 4 and 5 by correcting their mistakes made in the representations of fractions in the pre tasks. The students ordered fractions with different numerators and denominators in question 4.
incorrectly based on big denominators were equal to small fractions in the pre task. They gave the correct ordering of fractions in the post task by interpreting fractions as relationships between numerators and denominators. Some students had difficulty in comparing fractions that were close to a half using circular divided quantity diagrams in question 5. A few of them managed to overcome this problem by drawing rectangular divided quantity diagrams.

Section 8.2.1.3 to Section 8.2.1.6 describe the students who got the correct ordering of fractions in both tasks, but changed their strategies between the pre and post tasks. The improvement of strategies was detected through the changes to the use of representations of fractions in the post task, especially for students who were unable to provide reasoning in the pre task (Section 8.2.1.3) and believed big numbers were equal to small fractions in the pre task (Section 8.2.1.4). Section 8.2.1.5 shows the diagrams drawn in the post task reflected more about the fractional thinking of the students who used big denominators were equal to small fractions in the pre task. Section 8.2.1.6 shows the ability of interpreting fractions differently by the students who compared fractions numerically in the pre task but referred to the representations of fractions in the post task.

**8.2.1.1 Big denominators are equal to small fractions**

Based on big denominators were equal to small fractions, students only considered denominators of fractions for comparisons even for fractions with different numerators and denominators. One average student (AF30) emphasised the one part left in the whole rather than the parts taken from the whole when comparing fractions $\frac{2}{3}$, $\frac{4}{5}$ and $\frac{6}{7}$. The student explained “If you cut a pie into 7 pieces, another into 3 and another into 5, then take away the amount it says, see how much you have left”. Since the more pieces a pie was cut, the smaller the one piece left from the whole. Unfortunately, the student considered denominators only and ordered incorrectly from the smallest $\frac{6}{7}$ through $\frac{4}{5}$ to the largest $\frac{2}{3}$ in the pre task.

In the post task, the same student (AF30) drew a divided quantity diagram. On the circular divided quantity diagrams drawn (Figure 8.3), the student shaded the one part left in the whole rather than the parts taken from the whole. Since students use symbols and pictures to communicate their thinking (Sharp & Adams, 2002) their ideas about fractions can be shown using the diagrams drawn by themselves (Darr & Fisher, 2006; Ministry of Education, 2008a, 2008d). Through the diagrams drawn, each student was given the opportunity to
visualize the size of the one part that was left in the whole and think about the parts taken from the whole. The bigger the one part left in the whole, the smaller the parts taken from the whole. Hence, the further the fraction was from a whole. This was close to the parts to make a whole. As the part $\frac{1}{4}$ taken from the whole was the biggest, the fraction $\frac{3}{4}$ became the smallest; meanwhile the part $\frac{1}{8}$ taken from the whole was the smallest, the fraction $\frac{7}{8}$ became the largest. The diagrams drawn helped the student to order fractions correctly from the smallest $\frac{3}{4}$ through $\frac{5}{6}$ to the largest $\frac{7}{8}$ in the post task despite the flaws (i.e., unequal parts and wholes) made in the diagrams.

![Diagram](image)

**Figure 8.3: Circular divided quantity diagrams**

8.2.1.2 **Circular divided quantity diagrams**

In this study, most of the students drew circular divided quantity diagrams. Similar results are found by Young-Loveridge, Taylor, Hawera and Sharma (2007), that only circular diagrams are drawn among the students who use diagrams. They highlight the over-reliance on drawing pictures of ‘pies’ and suggest that teachers should consider the relevant benefits of drawing circular divided quantity diagrams that may affect instruction.

One of the problems noticed in this study was the fractions $\frac{8}{17}$ and $\frac{7}{12}$ looked as if they covered the same area on circular divided quantity diagrams. Three students (HC1, AC1 and AC6) drew divided quantity diagrams in the pre and post tasks and all of them ordered “$\frac{1}{2}$, $\frac{8}{17}$ and $\frac{7}{12}$” in the pre task based on the diagrams drawn (Table 8.19). AC1 added that “one half takes up not as much room, eight seventeenths take up the same amount of room as seven twelfths”. It is difficult to compare fractions that are close to each other on circular divided quantity diagrams, especially when there are large numbers of parts that need to be divided on the wholes. Moreover, the circles are always not in equal size of wholes and the
parts divided on the circles are also not of equal size. Hence, an incorrect result can be obtained from the diagrams drawn. Students are unable to interpret representations of fractions meaningfully if they focus only on the surface feature of visual appearance of the representations and ignore the underlying part-whole mathematical relation (Moseley, 2005). Teachers should mention explicitly the need of partitioning a whole into equal-parts in the approximate representations used in the instruction to remind students of the important concept of equal partition requirement in dividing a unit into fractions (Olive & Vomvoridi, 2006).

Two students changed to rectangular divided quantity diagrams in the post task. In the diagrams drawn by HC1, the parts divided and shaded on rectangles enabled the student to compare fractions more precisely than circular divided quantity diagrams, even though these fractions were drawn slightly bigger or smaller than each other in size. AC1 drew circular divided quantity diagrams initially and found that “one half covered the least amount of the circle and four sevenths has covered the most of the circle”. The student made the correction from the order “$\frac{1}{2}$, $\frac{7}{15}$ and $\frac{4}{7}$” to “$\frac{7}{15}$, $\frac{1}{2}$ and $\frac{4}{7}$” after drawing the rectangular diagrams.

**Table 8.19: Diagrams drawn in the pre and post tasks of question 5**

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC1</td>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>AC1</td>
<td><img src="image3.png" alt="Diagram" /></td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
<tr>
<td>AC6</td>
<td><img src="image5.png" alt="Diagram" /></td>
<td><img src="image6.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

One half takes up not as much room, eight seventeenths take up the same amount of room as seven twelfths.

One half covered the least amount of the circle and four sevenths has covered the most of the circle.
The circular divided quantity diagrams drawn by the student did not represent the wholes as having equal sizes and also were unequally divided. As such, rectangular divided quantity diagrams give a better alternative because “the wholes” represented by rectangles can be more easily drawn as having equal sizes. Although a circle can be easily divided into equal parts using the repeated halving strategy (e.g., halves, quarters and eighths) (Figure 8.4), circular divided quantity diagrams that have odd numbers of pieces such as thirds and fifths are often unequally divided (Figure 8.5). This may add to student confusion or mistakes.

![Figure 8.4: Circular divided quantity diagrams that are equally divided by the students](image)

According to Baturo (2004), students normally represent fractions using circles and they decline to recognise the use of other familiar shapes such as squares and rectangles. He advocates partitioning a variety of wholes using different shapes as a way to challenge students’ understanding to a greater depth than before and help students to correct their mistakes made earlier with the representations of fractions.

![Figure 8.5: Circular divided quantity diagrams that are unequally divided by the students](image)

8.2.1.3 Alternative for students who are unable to provide good reasoning

Using representations of fractions was a better alternative for the students who were unable to provide good reasoning in the pre task, including high achievers. Students may be unsure of their answers and their correct answers may be just a lucky guess if they are unable to explain how to get them (Young-Loveridge et al., 2007). For the students who calculate everything mentally, they have difficulty in documenting their thinking using a consistent
procedure by describing and writing their method onto paper using mathematical concepts and expressions (Sharp & Adams, 2002). An improvement was noticed in the reasoning of these students by drawing diagrams or using big denominators were equal to small fractions in the post task (Table 8.20). Some of them such as AF24 and CF18 even showed the changes in more than one question.

Table 8.20: Improved arguments from the pre to the post tasks

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>AF24 (Question 1)</td>
<td>No reasoning</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>AF24 (Question 2)</td>
<td>No reasoning</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>CF18 (Question 1)</td>
<td>I just knew it</td>
<td>There are five fifths to a whole and four quarters in a whole. There are more fifths than quarters</td>
</tr>
<tr>
<td>CF18 (Question 2)</td>
<td>There are 2 3 and 2 5</td>
<td>There are five fifths to a whole and four quarters in a whole. There are more fifths than quarters</td>
</tr>
<tr>
<td>HF1 (Question 3)</td>
<td>$\frac{3}{10}$ is smaller than $\frac{2}{5}$ and $\frac{2}{5}$ is smaller than $\frac{1}{2}$</td>
<td>the bigger the bottom number is the smaller the parts are</td>
</tr>
<tr>
<td>HF4 (Question 4)</td>
<td>as percentages they are in an easy way to understand</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

As described by Pirie and Kieren (1989), students’ mathematical understanding is a dynamical and active process which involves continual movement between different layers or ways of thinking. They propose a model containing eight embedded layers of understanding to explain how the understanding grows. In the above cases, the improvement shown in the ability to provide reasoning involved the lower level of understanding. Drawing divided quantity diagrams for the given, specific fractions in ordering fractions involves the second level of Image Making. The students form images out of the previous knowing and capability and specific images are made by the students to convey the meaning of mental image. When more diagrams were drawn to represent specific images, the students realise that the bigger the denominator the more equal parts are divided on a whole. They can now replace their actions on representing fractions physically with an image of fractions and this involves the third level of Image Having (Pirie & Kieren, 1989).
8.2.1.4 Big numbers are equal to small fractions

In questions 2 and 3, some students ordered fractions based on big numbers were equal to small fractions in the pre task (Table 8.21). According to Stafylidou and Vosniadou (2004), big numbers are equal to small fractions is a misconception generated in the interpretation of fractions as two whole numbers. Despite the influence of whole numbers, drawing diagrams or using parts to make a whole was a way for students to gain a meaningful understanding of fractions that related denominators and numerators. Through drawing divided quantity diagrams for the specific fractions, students form images out of the previous knowing about fraction concept and convey the meaning of mental image of fractions as parts of a whole (Pirie & Kieren, 1989). Students examined the features or distinctions among the pictorial and mental images formed earlier about fractions and noticed the fractional parts or gaps each fraction needed to make a whole. Students construct a context-specific and relevant property (i.e., parts to make a whole) and this involves the fourth understanding level of Property Noticing (Pirie & Kieren, 1989).

Table 8.21: Referring to representations of fractions in the post task

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>CC1 (Question 2)</td>
<td>The bigger the bottom number, the smaller the fraction is</td>
<td></td>
</tr>
<tr>
<td>CC1 (Question 3)</td>
<td>The bigger the bottom number, the smaller the fraction is</td>
<td></td>
</tr>
<tr>
<td>CF32 (Question 2)</td>
<td>The bigger the number, the smaller it is</td>
<td>( \frac{3}{4} ) is one off being a whole and ( \frac{3}{5} ) is two off being a whole</td>
</tr>
<tr>
<td>CF32 (Question 3)</td>
<td>The bigger the number, the smaller it is</td>
<td>( \frac{5}{14} ) is 9 (sic) away from being a whole and ( \frac{3}{7} ) is 4 away from being a whole but not so sure.</td>
</tr>
</tbody>
</table>

8.2.1.5 Reflect fractional thinking

Some students used big denominators were equal to small fractions in the pre task and drew divided quantity diagrams in the post task and vice versa. The diagrams drawn reflected more about students’ fractional thinking. For instance, CF17 mentioned “5 fifths to make a
whole” and “3 thirds to make a whole” in the pre task to describe $\frac{1}{5}$ and $\frac{1}{3}$, which in fact involved unequal wholes that were partitioned unsystematically. Such thinking was only disclosed through the diagrams drawn to represent $\frac{1}{5}$ and $\frac{1}{4}$ in the post task. A similar finding was noticed in the reasoning given by CF17 from questions 1 to 3 (Table 8.22) where the diagrams drawn reflected more of the fractional thinking.

**Table 8.22: Diagrams drawn in the post task**

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF17 (Question 1)</td>
<td>You have to have 5 fifths to make a whole but only 3 thirds to make a whole</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>CF17 (Question 2)</td>
<td>5 fifths make a whole (1) and only 3 thirds make a whole (1)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>CF17 (Question 3)</td>
<td>Ten tenths make a whole, five fifths make a whole, and two halves make a whole. The one closer to making a whole is bigger</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

CF26 used the strategy of big denominators were equal to small fractions in the pre task and drew diagrams in the post task for questions 1 and 2 (Table 8.23). Apparently, the “piece” mentioned in the pre task referred to a similar part shaded on the rectangular divided quantity diagrams to those drawn in the post task.

**Table 8.23: “Piece” as shaded parts**

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF26 (Question 1)</td>
<td>The larger the denominator the smaller the piece</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
</tbody>
</table>

$\frac{1}{4}$ has fatter pieces
Drawing diagrams is a way students express their ideas about fractions (Sharp & Adams, 2002; Darr & Fisher, 2006; Ministry of Education, 2008a, 2008d). Teachers examine students’ drawing to gain a much richer insight into students’ fraction concept images and ensure students’ understanding of fractions reaches a deep and connected level (Gould, 2005; Young-Loveridge et al., 2007). Both concepts and misconceptions underlying students’ strategies can be revealed in their reasoning (Gould, 2005; Steinle & Price, 2008). There is a possibility students provide “correct” answers based on “incorrect” reasoning (Gould, 2005). Teachers need to consider students’ incomplete thinking of fractions in the instruction and tailor lessons according to students’ needs (Ward, 1999; Tirosh, 2000; Young-Loveridge et al., 2007; Steinle & Price, 2008).

### 8.2.1.6 Interpreting fractions differently

Some students showed the ability of interpreting fractions differently for comparisons by using numerical strategies in the pre task but referring to the representations of fractions in the post task (Table 8.24). For example, AC4 converted fractions into decimals in the pre task. In the post task, the student drew circular divided quantity diagrams in question 2.

#### Table 8.24: Refer to representations of fractions in the post task

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC4</td>
<td>[ \frac{4}{5} ]</td>
<td>[ \frac{2}{3} ]</td>
</tr>
</tbody>
</table>

| AC4 (Question 2) | \[ \frac{2}{3} \] | \[ \frac{4}{5} \] | \[ \frac{2}{3} \] | \[ \frac{4}{5} \] |

Referring to the representations of fractions was easier than using numerical strategies when comparing fractions that were close to one. Two students took a long procedure to find three common denominators in question 4:

1) common denominator 35 to compare \( \frac{4}{5} \) and \( \frac{6}{7} \) that \( \frac{6}{7} = \frac{30}{35} \) and \( \frac{4}{5} = \frac{28}{35} \), thus \( \frac{30}{35} > \frac{28}{35} \);

2) common denominator 21 to compare \( \frac{2}{3} \) and \( \frac{6}{7} \) that \( \frac{6}{7} = \frac{18}{21} \) and \( \frac{2}{3} = \frac{14}{21} \), thus \( \frac{18}{21} > \frac{14}{21} \);

3) common denominator 15 to compare \( \frac{2}{3} \) and \( \frac{4}{5} \) that \( \frac{2}{3} = \frac{10}{15} \) and \( \frac{4}{5} = \frac{12}{15} \), thus \( \frac{12}{15} > \frac{10}{15} \).
In the post task, one of the students, HF8 also referred to the representations of fractions rather than using the common denominator strategy only to compare \( \frac{7}{8}, \frac{3}{4} \) and \( \frac{5}{6} \).

- The fractions \( \frac{3}{4} \) and \( \frac{7}{8} \) were compared by finding the common denominator 8 so that \( \frac{3}{4} = \frac{6}{8} \) and \( \frac{6}{8} < \frac{7}{8} \).

- The fractions \( \frac{7}{8} \) and \( \frac{5}{6} \) were compared by drawing the following divided quantity diagrams (Figure 8.6) to decide \( \frac{7}{8} > \frac{5}{6} \).

\[
\begin{align*}
\frac{7}{8} &= \quad \includegraphics{diagram1.png} \\
\frac{5}{6} &= \quad \includegraphics{diagram2.png}
\end{align*}
\]

Figure 8.6: Circular divided quantity diagrams

The ability to use different strategies shows the students have gained a stronger and more durable knowledge of fractions (Smith III, 2002). Even though numerical conversion methods taught in the classroom are sufficient to solve all kinds of fraction problems, high performers rather use strategies that are not explicitly taught but are tailored for solving specific classes of problems (Smith III, diSessa & Roschelle, 1993). The numerical strategies that involve algorithmic procedures are more accurate and direct (Smith III et al., 1993) but there are equally effective but simpler alternative strategies for solving fraction magnitude problems (Sophian & Madrid, 2003b). As documented by other researchers, high performers are capable of using flexible, spontaneous and specific tools that suit particular mathematical situations to obtain rapid and reliable solutions (Behr, Wachsmuth and Post, 1985; Smith et al., 1993; Smith III, 2002).

### 8.2.2 Use of numerical strategies in the post task

The numerical strategies used by the students in this study were finding a common denominator, converting fractions into percentages or decimals and using a benchmark. These
strategies were also included in the tips pages of the computer game of *Tower Trap*, which were shown once an incorrect order was made. This ensured that students were able to refer to different numerical strategies and helped them to make the correct ordering of fractions. This section focuses on the changes to the numerical strategies in the post task to identify any indirect effect of the game. Section 8.2.2.1 discusses the changes from using representations of fractions to converting fractions into a common denominator, percentages or decimals, Section 8.2.2.2 discusses the changes between different numerical strategies and Section 8.2.2.3 discusses the improvement in using benchmarks.

### 8.2.2.1 Changes from using representations of fractions to converting fractions into a common denominator, percentages or decimals

In the pre task, some students compared unit fractions $\frac{1}{5}$ and $\frac{1}{3}$ in question 1 and fractions with like numerators $\frac{2}{5}$ and $\frac{2}{3}$ in question 2 by using big denominators were equal to small fractions. The fractions were sometimes compared numerically in the post task by finding the common denominator 20 that $\frac{1}{5} = \frac{4}{20}$, $\frac{1}{4} = \frac{5}{20}$ thus, $\frac{5}{20} > \frac{4}{20}$ in question 1 and $\frac{3}{5} = \frac{12}{20}$, $\frac{3}{4} = \frac{15}{20}$, thus $\frac{12}{20} < \frac{15}{20}$ in question 2.

Two high achievers compared fractions numerically by converting fractions into percentages in the post tasks. This was more accurate than using big denominators were equal to small fractions or drawing divided quantity diagrams to compare the fractions in the pre task (Table 8.25). Representing fractions that were close to one using divided quantity diagrams enabled HC7 to determine how far a fraction was from a whole for comparison and answered correctly in the pre task. Although HC7 changed his strategy to convert fractions into percentages in the post task, he only managed to convert $\frac{3}{4}$ to 75% but was unable to convert $\frac{5}{6}$ and $\frac{7}{8}$ accordingly, probably due to the difficulty in converting these fractions to percentages.
Table 8.25: Students’ reasoning given in the post task

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>HF7</td>
<td>the numerators are the same but the denominator is bigger so it is smaller</td>
<td>$\frac{3}{5}$ is 60% and $\frac{3}{4}$ is 75%</td>
</tr>
<tr>
<td>(Question 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HC7</td>
<td>$\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$</td>
<td>$\frac{3}{4}$ is 75%, $\frac{5}{6}$ is larger than $\frac{3}{4}$</td>
</tr>
<tr>
<td>(Question 4)</td>
<td></td>
<td>and $\frac{7}{8}$ is the largest of all of them</td>
</tr>
</tbody>
</table>

Based on mental image formed in the earlier understanding level, students recognise the patterns of the singular images of fractions and characterise the properties of these images using a formal character. Students achieve the fifth understanding level of Formalizing where they can compare fractions using symbols only. Numerical conversion is a way students formalize their understanding and transform the images of the concept into mathematical definitions or algorithms (Pirie & Kieren, 1989).

8.2.2.2 Changes between different numerical strategies

Some students changed to different numerical strategies when ordering similar sets of fractions between the pre to the post maths tasks. Table 8.26 shows the reasoning of students who convert fractions into percentages in the pre task but find a common denominator in the post task. In question 3, it was simple to convert fractions to percentages for $\frac{3}{10} = 30\%$,

$\frac{2}{5} = 40\%$, $\frac{1}{2} = 50\%$ in the pre task. However, fractions in the post task were difficult to be converted into percentages because $\frac{3}{7} = 0.4285...$ and $\frac{5}{14} = 0.3571...$. The similar problem in conversions was noticed in question 4. In the post task, converting fractions $\frac{5}{6}$ and $\frac{7}{8}$ to percentages 83% and 87.5% required complex computations. In contrast, students could easily convert fractions $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ into the common denominator 24 because these fractions contained denominators 4, 6 and 8 which were the factors of the denominator 24.
Table 8.26: Students’ change from converting fractions into percentages in the pre task to finding a common denominator in the post task

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC4 (Question 3)</td>
<td>$\frac{3}{10} = 30%$, $\frac{2}{5} = 40%$, $\frac{1}{2} = 50%$ to decide that 30% is smaller than 40% and 40% is smaller than 50%.</td>
<td>$\frac{25}{70} = \frac{5}{14}$, $\frac{30}{70} = \frac{3}{14}$, $\frac{3}{2} = \frac{35}{70}$. Thus, $25 &lt; 30$ and $30 &lt; 35$.</td>
</tr>
<tr>
<td>CF10 (Question 3)</td>
<td>$\frac{3}{10} = \frac{30}{100}$; $\frac{2}{5} = \frac{40}{100}$; $\frac{1}{2} = \frac{50}{100}$; $\frac{3}{7} = \frac{6}{14}$; $\frac{6}{14}$ is bigger than $\frac{5}{14}$ but not half.</td>
<td>$\frac{3}{10} = \frac{6}{14}$; $\frac{6}{14}$ is bigger than $\frac{5}{14}$ but not half.</td>
</tr>
<tr>
<td>HC4 (Question 4)</td>
<td>$\frac{2}{3} = 66%$, $\frac{4}{5} = 80%$, $\frac{6}{7} = 84%$ or 85%.</td>
<td>$\frac{3}{2} = \frac{18}{24}$, $\frac{5}{24}$; $\frac{6}{7} = \frac{20}{24}$ and $\frac{7}{24} = \frac{21}{24}$.</td>
</tr>
</tbody>
</table>

Table 8.27 shows the reasoning of students who use a benchmark in the pre task but find a common denominator in the post task. If the benchmark of a half was used, students could only determine that $\frac{3}{10}$, $\frac{2}{5}$, $\frac{5}{14}$ and $\frac{3}{7}$ were smaller than $\frac{1}{2}$ whereas $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ were bigger than $\frac{1}{2}$. They could not decide how much smaller $\frac{3}{10}$ was from $\frac{2}{5}$ and how much bigger $\frac{4}{5}$ was from $\frac{2}{3}$ using the benchmark of $\frac{1}{2}$. The difference sizes between these fractions could be determined accurately when they were converted into a common denominator for the comparison in the post task.
Table 8.27: Students’ use of a benchmark in the pre task and finding a common denominator in the post task

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC6 (Question 3)</td>
<td>$\frac{2}{5}$ is smaller than $\frac{2.5}{5}$ which is equal to half and $\frac{3}{10}$ is smaller than $\frac{2}{5}$</td>
<td>$\frac{3}{7}$ would have to be $\frac{3}{6}$ to be bigger than $\frac{1}{2}$ . If I multiplied $\frac{3}{7}$ by 2 it was $\frac{6}{14}$ which is bigger than $\frac{5}{14}$ .</td>
</tr>
<tr>
<td>HC2 (Question 4)</td>
<td>$\frac{2}{3}$ is just bigger than a half, $\frac{4}{5}$ is just bigger than $\frac{2}{3}$ and $\frac{6}{7}$ is almost a whole</td>
<td>$\frac{3}{4} = \frac{18}{24}, \frac{5}{6} = \frac{20}{24}$ and $\frac{7}{8} = \frac{21}{24}$ .</td>
</tr>
</tbody>
</table>

In other cases, finding a common denominator could be more difficult and time consuming especially when more than one common denominator was needed. For example, in the pre task of question 5, a high achiever found $\frac{1}{2} = \frac{12}{24}$ and $\frac{7}{12} = \frac{14}{24}$ to decide $\frac{14}{24} > \frac{12}{24}$ and $\frac{1}{2} = \frac{17}{34}$ and $\frac{8}{17} = \frac{16}{34}$ to decide $\frac{17}{34} > \frac{16}{34}$ . In the post task, the high achiever changed to another strategy by using the benchmark of $\frac{1}{2}$ . It was shown that finding the half of 7 and 15 to decide whether $\frac{4}{7}$ and $\frac{7}{15}$ were bigger or smaller than $\frac{1}{2}$ was simpler than converting these fractions into a common denominator.

The above changes of strategies showed the students had the ability to use different numerical strategies and select the most suitable strategy for comparing particular types of fractions. Students gain the understanding level of Formalizing when using numerical methods as techniques for computation. They can improve to higher understanding levels of Observing and Structuring if they are cognizant of the features and the consequences of numerical conversion methods. At these understanding levels, students are able to observe an image for the conversion methods and aware of the consequences of using particular methods on fraction equivalence (Pirie & Kieren, 1989).
8.2.2.3 Improvement in using benchmarks

Quite a number of students improved in ordering fractions that were close to a half in question 5 by using the benchmark of $\frac{1}{2}$. Some students changed from other strategies to the benchmark strategy in the post task. For example, CF11 and AC11 were unable to provide good reasoning in the pre task (Table 8.28). Some students improved their ordering of fractions by using the number knowledge of doubles in the post task. For example, HC2 simply decided that $\frac{8}{17}$ or $\frac{7}{12}$ was more or less than a half and they were unable to order the fractions correctly in the pre task (Table 8.28). In fact, a fraction is the same as $\frac{1}{2}$ when the denominator is exactly twice the numerator and the exact half can be found by halving the denominator. Finding the exact half is necessary to compare fractions that are close to each other even for a fraction that is just slightly different from a half. For example, half of 17 is 8.5, therefore $\frac{8}{17}$ is smaller than $\frac{8.5}{17}$ or a half; half of 12 is 6, therefore $\frac{7}{12}$ is bigger than $\frac{8.5}{17}$ or a half. By halving 15 and 7 in the post task, students were able to determine precisely whether $\frac{7}{15}$ and $\frac{4}{7}$ were smaller or larger than $\frac{1}{2}$.

Table 8.28: Reasoning given in the pre and post tasks of question 5

<table>
<thead>
<tr>
<th>Students</th>
<th>Pre task</th>
<th>Post task</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC2</td>
<td>$\frac{8}{17}$ is just bigger than $\frac{1}{2}$ and $\frac{7}{12}$ is a bit bigger than $\frac{1}{2}$ and $\frac{8}{17}$?</td>
<td>Half of 15 is 7.5, so it is just under half. Half of 7 is 3.5, so $\frac{4}{7}$ is over half</td>
</tr>
<tr>
<td>CF11</td>
<td>Not sure</td>
<td>3.5 is half of seven, so 4 must be bigger than 1 half</td>
</tr>
<tr>
<td>AC11</td>
<td>Smaller the denominator is the smaller the fraction</td>
<td>Half of 15 is 7.5, so it is just under half. Half of 7 is 3.5, so $\frac{4}{7}$ is over half</td>
</tr>
</tbody>
</table>
Summary

A positive effect of the *Tower Trap* was identified through some changes of strategies between the pre and post task among students who were RR in questions 1 to 3 and WR in questions 4 and 5. The divided quantity blocks of fraction *broken bricks* in the computer game highlighted the part-whole concept of fractions. With the understanding that fractions were related in terms of numerators and denominators and as part to make a whole, students realised that not only denominators, but also numerators, needed to be considered when comparing fractions using representations of fractions. This led to the correct comparison for fractions that were close to one in question 4. Representing fractions in rectangular blocks of fraction bricks gave an alternative method for the representations of fractions besides the typical pie diagrams. In comparison to circular divided quantity diagrams, the advantages of drawing rectangular divided quantity diagrams were that the parts could be easily divided equally on rectangles and the wholes of rectangles were more easily made the same size. A few students improved in comparing fractions that were close to a half in question 5 by changing from circular divided quantity diagrams to rectangular divided quantity diagrams.

The improvement in the use of fraction ordering strategies was noticed when the students who were RR changed their strategies to the representations of fractions in the post task. In the pre task, some of them gave the correct ordering of fractions without providing good reasoning or just explained their answers based on big numbers were equal to small fractions in questions 1 to 3, especially to the simple fractions such as unit fractions and fractions with like numerators. It was noted that diagrams drawn in the post task could reflect fractional thinking of the students who used big denominators were equal to small fractions in the pre task. Some students compared fractions numerically in the pre task but geometrically by using representations of fractions in the post task. They performed a flexible thinking of fractions that enabled them to comprehend fractions in different appearances. Numerical computations allowed for accurate comparisons, while visualization of fractions ensured meaningful comparisons. In fact, it was often easier to compare fractions that were close to one by referring to the representations of fractions as parts to make a whole than by using numerical strategies.

Representations of fractions were emphasised more in the game than the numerical strategies which were displayed on the tips pages of the game. Nevertheless, changes to numerical strategies were still found in this study especially among the high achievers. Some students referred to the representations of fractions in the pre task but found a common
denominator or converted fractions into percentages in the post task for accurate comparisons. Finding a common denominator was preferred more than other numerical strategies in the situations that fractions contained denominators as the factors of another denominator. In addition, some fractions required complex computations to be converted into percentages or decimals while some fractions were smaller or bigger than a half, hence they were not suitable to be compared using the benchmark of a half. When comparing fractions that were close to a half in question 5, quite a number of students improved in this question by using the benchmark of a half. Some found this strategy was easier than finding more than one common denominator in the pre task. Some managed to apply the number knowledge of doubles and improved in the use of benchmark strategy in the post task.

### 8.3 Achievement differences between the pre and post tests

The individual performances of Sam (i.e., a high achiever), Mary (i.e., an average student) and Peter (i.e., a cause-for-concern student) in the pre and post maths tests were discussed in Section 8.1. Among them, Mary improved in question 4 on comparing two fractions that were close to a half. In this section, the achievement differences between different mathematical ability groups will be investigated. The improvement of students’ achievement after playing the *Tower Trap* was determined by considering the achievement differences between the pre and post tests. The sets of questions asked in the pre and post maths tests are shown in Appendix 1. Students’ performances of being right or wrong for each question in the pre and post tests were analysed to determine their achievement differences: wrong in the pre test but right in the post test is denoted by WR, right in the pre test but wrong in the post test is denoted by RW, wrong in the pre and post tests is denoted by WW and right in the pre and post tests is denoted by RR. In the contexts of this write-up, WR is noted as an improvement in students’ achievement and only the positive impacts of the game noticed from the improved students (i.e., WR) are discussed in detail.

#### 8.3.1 School A and School C

The achievement differences between the pre and post tests from the 106 students at School A and School C are presented in Figure 8.7. The column graph shows that questions 1 and 2 are shown to be relatively easy as an average of 89% are RR, while question 6 is shown to be relatively difficult question with an average of 59% of students being WW. Students achieved better in the first two questions about representations of fractions, than the last three
questions about ordering and operating with fractions. The game had the most positive impact in question 5, which had the largest improvement of 25% (i.e., WR) as compared to other questions. However, the largest decrease of 27% (i.e., RW) was also found in students’ achievement in question 3.

Figure 8.7: Achievement differences between the pre and post tests

Figure 8.8, Figure 8.9 and Figure 8.10 present the achievement differences between the pre and post tests for 15 high achievers, 49 average students and 42 cause-for-concern students, respectively. Students’ performances were directly proportional to mathematical ability groups. Almost all high achievers were RR in the first four questions and 60% (or 9 of 15) were RR in the last two questions. Nearly 90% of average and cause-for-concern students were RR in questions 1 and 2 and average students had a higher percentage of RR than cause-for-concern students in the last four questions. Cause-for-concern students had a higher percentage of WW than average students in all questions in the tests. 20% (or 3 out of 15) or less high achievers were WW in questions 5 and 6. Up to 79% (or 33 out of 42) cause-for-concern students and 55% (or 27 out of 49) average students were WW in question 6.
High achievers were RR in almost all questions in the test and not more than 4 of them improved (i.e., WR) in questions 5 and 6. The achievement differences of WR and RW detected in the tests were noticed among average and cause-for-concern students for all questions. The percentage of cause-for-concern students RW in question 3 (40% or 17 out of 42) were double the percentage of average students (22% or 11 out of 49). More cause-for-concern students were WR in question 5 (31% or 13 out of 42) than average students (20% or 10 out of 49).
8.3.2 Comparison between experimental group and control group

Twenty-eight students in three mathematical ability categories participated as the control group and were tested without playing the *Tower Trap*. The results of the control group were compared with the results of the experimental group and similar results were found in questions 1, 2, 4 and 6, whereas different results were found in questions 3 and 5 (Figure 8.11).

![Figure 8.11](image)

**Figure 8.11: Comparison between experimental group and control group**

- a) Experimental group
- b) Control group

**Figure 8.10: Achievement differences between the pre and post tests for cause-for-concern students**
Questions 1 and 2 were relatively easy but the percentage of RR of the control group (80%) was slightly lower than the experimental group (88%). In fact, only average and cause-for-concern students were involved in WR and RW in questions 1 and 2 on representing and naming fractions. Question 6 appeared to be harder for the control group because the percentage of WW of the control group (68%) was slightly higher than the experimental group (59%), and more students from the experimental group (29%) were RR than the control group (14%). The control group performed similarly to the experimental group in question 4 where half of the students were RR, 11% of students were WW, 21% of students were WR, and 14% of students were RW.

For question 3 there were more students of WR (25%) than RW (14%) in the control group but more RW (27%) than WR (16%) in the experimental group (Figure 8.11). For question 5 there was the same number of WR and RW (21%) in the control group but more WR (25%) than RW (16%) in the experimental group (Figure 8.11). The experimental group had played the computer game of *Tower Trap* in between the pre and post tests. They ordered fraction bricks from the smallest to the largest and vice versa in the game and this experience appeared to help them on ordering fractions in question 5. These students’ selections in the multiple choice tests were examined to identify the problems and reasons causing achievement differences of WR and RW in the tests especially for RW in question 3 and WR in question 5.

It was found that there were some differences in the ability of the two groups at the start. Compared to the control group, a higher proportion of experimental group answered correctly in the pre test and this is especially obvious in questions 3 and 6. In question 6, 37% of the experimental group students were RR and RW while 63% were WW and WR. On the other hand, 14% of control group students were RR and RW and 86% were WW and WR. It is worthy to note that 61% of the control group students were cause-for-concern students whereas 49% of the experimental group students were cause-for-concern students. Therefore, it is likely that the experimental group students would perform better than the control group students initially. As shown in Figure 8.11, the experimental group students performed better than the control group initially in simple questions on the representations of fractions (questions 1 and 2) and the control group performed poorer than the experimental control group in the hard question on the estimation of fractions (question 6).
In question 3, 78% of the experimental group students were RR and RW while 22% were WW and WR. Since more experimental group students were right than wrong in the pre test, more students would likely be RW than WR. On the other hand, 57% of control group students were RR and RW and 43% were WW and WR. Slightly more control group students were right than wrong initially, yet there were more WR than RW students. These results showed the game had a limited effect in question 3. A lot more control group students were wrong than right initially and there were 18% of students WR and none was RW. These results showed the game had a limited effect in question 3.

In question 5, 56% of the experimental group students were RR and RW while 44% were WW and WR. Slightly more experimental group students were right than wrong in the pre test, yet more students were WR than RW. On the other hand, 46% of the control group students were RR and RW while 54% were WW and WR. Slightly more control group students were wrong than right initially, but there were the same numbers of WR and RW. These results showed the positive effect of the game in question 5.

The above results might be affected by the big difference between the number of students in the experimental group and the number of students in the control group. For instance, when a small number of students improved in the control group, this produced a big increase in the percentages.

Various tests were considered for the above data analysis to determine whether there was a significant difference between the experimental and control groups as described in Chapter 4, Section 4.4.3. The tests included standard mixed model repeated measures analysis, Chi square test and, Cochran’s Q test. However, these statistical approaches were later rejected as they were not suitable to be adopted in this study:

- Standard mixed model repeated measures analysis could not be used as the students’ responses were measured on a continuous scale in this study.
- Chi square test was not suitable for this study because it could only be tested to compare each question separately and not between questions.

Since none of the tests were suitable, the data were only analysed using descriptive statistics.
8.3.3 Students’ selections in the tests

Question 3 asked about the ranges of fractions that were shaded blue on the rectangles (Figure 8.12), which were between $\frac{1}{4}$ and $\frac{1}{2}$ in test A and between $\frac{1}{2}$ and $\frac{3}{4}$ in test B.

Eleven average students and 21 cause-for-concern students considered the unshaded parts in the representations of fractions and determined the ranges of fractions as “between $\frac{1}{4}$ and $\frac{1}{2}$” in test A and “between $\frac{1}{2}$ and $\frac{3}{4}$” in test B (Figure 8.12). Among them, 3 average students and 7 cause-for-concern students improved (i.e., WR) and determined the range of fractions correctly by referring to the blue shaded parts in the post test. However, more students (i.e., 8 average students and 14 cause-for-concern students) decreased (i.e., RW) although they had been right in the pre test. Almost all of the students who were RW selected the same answer - “between $\frac{1}{4}$ and $\frac{3}{4}$” for both test A and test B.
Among the 14 cause-for-concern students who were RW and appeared to confuse the shaded and unshaded parts in question 3, some of them also considered the unshaded parts when representing fractions in questions 1 and 2. In question 1, the students were asked “which picture is $\frac{3}{5}$ shaded blue?” in test A and “which picture is $\frac{2}{5}$ shaded blue?” in test B. Three of the cause-for-concern students considered the unshaded parts to represent $\frac{2}{5}$ (Figure 8.13) and $\frac{3}{5}$ (Figure 8.14).

![Figure 8.13: Representing $\frac{2}{5}$ in rectangular divided quantity diagrams](image)

![Figure 8.14: Representing $\frac{3}{5}$ in rectangular divided quantity diagrams](image)

In question 2, students were asked to determine the symbols of fraction for representations of fractions that were shaded blue in rectangular divided quantity diagrams (Figure 8.15). Three of the cause-for-concern students referred to the three unshaded parts as the numerator 3 and the five blue shaded parts as the denominator 5 and named the fraction “$\frac{3}{5}$”.

![Figure 8.15: Divided quantity diagrams in Question 2](image)
Divided quantity diagram is the area model or region drawing that is commonly used to present fraction information (Behr, Lesh, Post & Silver, 1983; English & Halford, 1995; Reeve & Pattison, 1996; Bulgar, 2009). In *Tower Trap*, the area model is developed on the rectangular divided quantity blocks of *broken bricks*. For example, the fraction $\frac{2}{5}$ is represented by two blue shaded parts out of five divided parts in a whole and the fraction $\frac{2}{3}$ is represented by two blue shaded parts out of three divided parts in a whole (Figure 8.16).

![Figure 8.16: $\frac{2}{5}$ and $\frac{2}{3}$ shaded blue on the broken bricks](image)

The *broken brick* shows that the number of parts coloured blue is given by the numerator of fractions. The same concept is applied to the fractions that are represented by other divided quantity diagrams. While playing the *Tower Trap*, students drag the blue shaded parts from the dividing parts of *broken bricks* to form the representations of fractions. For example, the whole of the broken brick is divided into five parts for denominator 5 and two of the parts are shaded blue to represent numerator 2. Therefore, the representation of the fraction $\frac{2}{5}$ is formed by shading two parts out of five divided parts comprising a whole (Figure 8.17).

![Figure 8.17: Representation of $\frac{2}{5}$ dragged from the broken brick](image)
Manipulating parts of broken bricks engages students with the relation of a part of a quantity to its total amount in the representations of fractions. This way students are given opportunities to construct the part whole concept explicitly. Most importantly, by splitting parts from a whole, students are led to understand fractions as being made up of related numbers for the numerators and denominators rather than as two independent natural numbers (Stafylidou & Vosniadou, 2004).

In the above question 3, finding the ranges of fractions that were shaded blue on the rectangles (Figure 8.12) (i.e., between $\frac{1}{4}$ and $\frac{1}{2}$ in test A and between $\frac{1}{2}$ and $\frac{3}{4}$ in test B) was related to the conceptual knowledge of fractions as parts of a whole, as had been emphasized with the broken bricks played in the game. However, the effect of the game was limited here because the learning outcomes of the game were only indirectly related to the continuous representations of fractions. In addition, the students might have been confused with the ranges of fractions that were different from the parts of factions that were shaded blue on broken bricks.

Question 5 required students to determine the set of fractions that was in the correct order (Figure 8.18).

<table>
<thead>
<tr>
<th>Test A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 5</td>
</tr>
<tr>
<td>Which of these orders is from smallest to largest?</td>
</tr>
<tr>
<td>$\frac{1}{6}$, $\frac{2}{9}$, $\frac{3}{18}$, $\frac{2}{9}$, $\frac{4}{6}$, $\frac{1}{9}$, $\frac{1}{6}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Test B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 5</td>
</tr>
<tr>
<td>Which of these orders is from largest to smallest?</td>
</tr>
<tr>
<td>$\frac{1}{6}$, $\frac{2}{9}$, $\frac{3}{18}$, $\frac{2}{9}$, $\frac{4}{6}$, $\frac{1}{9}$, $\frac{1}{6}$</td>
</tr>
</tbody>
</table>

**Figure 8.18: Comparing and ordering fractions with unlike denominators in Question 5**

Students seemed to be easily affected by the belief that big numbers were equal to big fractions when ordering fractions from the largest to the smallest. Fourteen students selected
“$\frac{3}{18}, \frac{2}{9}, \frac{4}{6}$” as the order from the largest to the smallest and 2 students selected “$1, \frac{4}{9}, \frac{2}{6}, \frac{3}{18}$” as the order from the smallest to the largest. Among them, 2 high achievers, 2 average students and 4 cause-for-concern students selected “$\frac{3}{18}, \frac{2}{9}, \frac{4}{6}$” as the order from the smallest to the largest in the post test and were WR in question 5.

This misconception of big numbers were equal to big fractions also affected some of these students in question 4. They selected “$\frac{6}{13}$” as the fraction that was larger than “$\frac{5}{9}$” in test A or “$\frac{5}{9}$” as the fraction that was smaller than “$\frac{6}{13}$” in test B.

With the belief that big numbers were equal to big fractions, some students interpreted fractions as two independent numbers. In question 6, they added numerators 6 and 9 to get “15” or denominators 7 and 10 to get “17” as the closest to $\frac{6}{7} + \frac{9}{10}$ in test A. In test B, they added numerators 10 and 6 to get “16” or denominators 11 and 7 to get “18” as the closest to $\frac{10}{11} + \frac{6}{7}$.

As unit 1 is the smallest unit in the natural number system, other fractions, $\frac{3}{18}, \frac{4}{6}$ and $\frac{2}{9}$, that had numbers bigger than 1 were misunderstood by the students as bigger than unit 1. The students were reminded of their misconceptions when dealing with the broken bricks in the game that had the wholes of “1” unit and moreover, the one whole was the biggest block of fractions among other blocks of proper fractions. One of the examples of broken bricks is shown in Figure 8.19.

![Figure 8.19: Broken bricks](image-url)
Since the bigger the numbers the denominators are, the more parts are divided on a unit, hence the smaller each part becomes. The students came to realize that fractions with bigger denominators had smaller pieces in each individual part than fractions with smaller denominators.

**Summary**

The effects of the *Tower Trap* on students’ achievement were minimal on questions 1, 2 and 6, average on question 4, substantially positive for question 5 but negative for question 3. Only a few average and cause-for-concern students improved from the pre to the post test in questions 1 and 2 as the Year 8 students were supposed to know about representations of fractions prior to playing the game. Estimating the addition of two fractions that were close to 1 in question 6 was too hard especially for average and cause-for-concern students. Very few of them improved in this question after playing the game, probably since this topic of fractions was not emphasised in the game. While some students improved in comparing fractions that were close to a half in question 4, others did not improve and some did worse from the pre test to the post test.

Students from all ability groups especially cause-for-concern students improved (i.e., WR) in question 5 on comparing and ordering fractions. Even for the high achievers who were RR in most of the questions, a few of them were also WR in question 5. On the other hand, the game appeared to have a negative impact on average and cause-for-concern students on question 3 where students had to find the ranges of fractions. The high percentage of RW in question 3 was mainly caused by cause-for-concern students.

The finding of WR in question 5 and RW in question 3 was verified by comparing the results between the control group and experimental group. It was found that the experimental group had more WR than RW while the control group had the same number of WR and RW in question 5. This suggests the game promotes students’ WR in question 5. In question 3, the RW of the experimental group was two times of the WR while the control group had the same number of WR and RW. This suggests the game did not help in question 3.

Students’ selections made in the tests with multiple choices were particularly examined to identify the reasons causing WR and RW in the tests. The students who were WR in question 5 corrected their mistakes of ordering big numbers as big fractions. The fraction bricks ordered in the game had the biggest unit of one as a whole. The bigger the
denominator a fraction brick had the more parts were divided on a whole and the smaller the parts became. Since the game was designed for comparing sizes of fractions, it had the most significant impact on question 5 relating to the ordering of fractions in the tests. On the other hand, the students might have been confused with the ranges of fractions that were shaded blue on the rectangles in question 3 as these fractions were different from the parts of fractions that were shaded blue on rectangular fraction bricks played in the game, which led to more RW in the tests.

**Summary to the chapter**

Students’ improvement in the knowledge and strategies for ordering fractions were investigated in this chapter to identify the effectiveness of *Tower Trap* computer game for learning fractions. Students’ performances in maths tasks and tests were analysed by individuals and by groups of different mathematical abilities (i.e., high achievers, average and cause-for-concern students).

The individuals selected showed an improvement in comparing sizes of fractions after playing the game. Sam who was a high achiever, changed his strategy from partial reasoning to big denominators were equal to small fractions. He was right in all questions in the maths tasks and tests. Mary who was an average student, changed from partial reasoning to benchmarking fractions to a half and big denominators were equal to small fractions to parts to make a whole (with the aid of divided quantity diagrams). Such improvement enabled her to achieve better in the post task and test. Peter who was a cause-for-concern student, gave no reasoning in the pre task and changed to big denominators were equal to small fractions. He tried to use the benchmark of a half in the post task but did not show a better achievement in the post task and test.

An improvement in ordering fractions in the maths tasks was observed among the students from different mathematical ability groups. A few high achievers improved (i.e., WR) in question 5 when comparing fractions that were close to a half using benchmark of a half. Average and cause-for-concern students improved in comparing different types of fractions in all of the questions asked in the maths tasks. They fixed their problems with the use of representations of fractions in the pre task and gave the correct ordering to fractions that were close to one in question 4 and fractions that were close to a half in question 5.
Among the students who were right in both the pre and post maths tasks, they showed an improvement in their strategies.

- Some students who were unable to provide reasoning or ordered fractions based on big numbers were equal to small fractions in the pre task had changed to representations of fractions in the post task.
- Some students ordered fractions in the pre task based on big denominators were equal to small fractions. Their drawings of divided quantity diagrams in the post task revealed more about their fractional thinking.
- Some students gained a flexible thinking in fractions by comparing fractions numerically in the pre task but geometrically using diagrams in the post task.

Changes to numerical strategies were also noticed in this study especially among the high achievers. Some students referred to the representations of fractions in the pre task but changed to numerical conversions in the post task for accurate comparisons. Some students improved in the use of benchmark strategy by applying the number knowledge of doubles in the post task. Some students changed the numerical strategies they used of converting fractions into percentages in the pre task to finding a common denominator in the post task. Such change might, however, be due to the numerical situation of the fractions rather than the effect of the game.

Among the various aspects assessed in the test, the game which was designed to compare sizes of fractions had the best result of WR in question 5. Most high achievers were right in almost all questions but a few of them were WR in question 5. For the average and cause-for-concern student, WR was noticed in all questions, but mostly for questions 3 to 5. Most of those who improved (i.e., WR) in question 5 were cause-for-concern students. These students had corrected their mistakes of ordering big numbers as big fractions in the post test.

In conclusion, the Tower Trap game did have a positive impact on students’ learning of fractions. The above findings about students’ improvements in the knowledge and strategies of fractions have answered research question 3.
CHAPTER 9

MOVING STUDENTS TOWARDS ADVANCED STRATEGIES

Based on the findings about students’ strategies for ordering fractions and improvement of students’ strategies in the previous chapter, this chapter aims to provide an instructional framework that can be used to move students towards more advanced strategies. Section 9.1 explains the identification of students’ current level of mathematical thinking and misconceptions through their strategies in order to plan for instruction that suits students’ needs. Section 9.2 highlights the continuities of mathematical thinking through the use of similar strategies between students from different ability groups. Subsequently, Section 9.3 discusses instructional approaches that suit particular strategies. For students who are affected by whole number knowledge, the understanding of the inverse or compensating relation between the size of unit fractions and the number of the parts of their whole can be reaffirmed using representations of fractions. For students who mainly rely on diagrams, a concrete meaning of the numerical strategies can be constructed by linking the representations of fractions to numerical procedures. For students who use only one type of numerical strategy, the problems and reasons of using particular strategies and the reasons to use other strategies can be stressed to encourage the use of the most appropriate strategies for certain numerical situations.

9.1 Identifying students’ strategies

By referring mainly to the work of three individual students (i.e., Sam, Mary and Peter) in this study, this chapter describes how a teacher can identify students’ strategies and limitations in using certain strategies. The students’ works in the maths tasks have been discussed in the previous chapter and this section considers their reasoning given in the pre maths task before playing the Tower Trap game (refer to Figure 9.1, Figure 9.2 & Figure 9.3).

First of all, the teacher can search for the keywords used in students’ reasoning. The use of “5th”, “third”, “cut a pie into pieces” in Sam and Mary’s reasoning was relevant to the denominators of fractions that fractions were ordered using the strategy big denominators were equal to small fractions. The repetitions of the order of fractions given by Sam and Mary in their reasoning were considered as partial reasoning. Finding whether the fractions were
less or more than a half by Sam and Mary involved the strategy of using the benchmark of a half. No reasoning could be identified when Peter just stated “I just knew it” and “because I know” to the order of fractions given in the maths tasks.

The categories of students’ reasoning developed in Chapter 4 (Table 4.8) which includes definitions or descriptions and examples of students’ reasoning can be useful to refer to in assessing students’ strategies by teachers.

Table 9.1: Sam’s reasoning and strategies

<table>
<thead>
<tr>
<th>Questions</th>
<th>Ordering</th>
<th>Reasoning</th>
<th>Keywords</th>
<th>Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{5}$, $\frac{1}{3}$</td>
<td>Because a 5th is smaller than a third</td>
<td>5th, third</td>
<td>Big denominators are equal to small fractions</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$, $\frac{2}{3}$</td>
<td>Because two 5ths is a smaller amount than 2 thirds</td>
<td>5ths, thirds</td>
<td>Big denominators are equal to small fractions</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{2}$</td>
<td>Because $\frac{3}{10}$ is smaller than $\frac{2}{5}$ and $\frac{2}{5}$ is smaller than $\frac{1}{2}$</td>
<td>Repetitions of the ordering of fractions</td>
<td>Partial reasoning</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$</td>
<td>Because $\frac{2}{3}$ is smaller than $\frac{4}{5}$ and $\frac{4}{5}$ is smaller than $\frac{6}{7}$</td>
<td>Repetitions of the ordering of fractions</td>
<td>Partial reasoning</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{7}{12}$, $\frac{1}{2}$, $\frac{8}{17}$</td>
<td>$\frac{8}{17}$ = is less than a half, $\frac{1}{2}$ = One half, $\frac{7}{12}$ = more than a half</td>
<td>Less than half, more than a half</td>
<td>Using the benchmark of a half</td>
</tr>
<tr>
<td>Questions</td>
<td>Ordering</td>
<td>Reasoning</td>
<td>Keywords</td>
<td>Strategies</td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
<td>-----------</td>
<td>----------</td>
<td>------------</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{5}, \frac{1}{3} )</td>
<td>Because if you cut a pie into 5 pieces, the slices will be smaller than if it was cut into 3</td>
<td>Cut a pie into 5 pieces</td>
<td>Big denominators are equal to small fractions</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{2}{5}, \frac{2}{3} )</td>
<td>Because if your pie is cut into 3 and you get 2 that’s more than if your pies is cut into 5 and you get 2</td>
<td>Pie is cut into 3</td>
<td>Big denominators are equal to small fractions</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{3}{10}, \frac{2}{5}, \frac{1}{2} )</td>
<td>Because ( \frac{3}{10} ) is less than half and ( \frac{2}{5} ) is too.</td>
<td>Repetitions of the ordering of fractions</td>
<td>Partial reasoning</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{6}{7}, \frac{4}{5}, \frac{2}{3} ) (Incorrect)</td>
<td>If you cut a pie into 7 pieces, another into 3 and another into 5, then take away the amount it says, see how much you have left</td>
<td>Cut a pie into 7 pieces</td>
<td>Big denominators are equal to small fractions</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{2}, \frac{7}{12}, \frac{8}{17} ) (Incorrect)</td>
<td>Because 7 pieces of 12 is more than half and 8 pieces of 17 is more than half too.</td>
<td>More than half</td>
<td>Using the benchmark of a half</td>
</tr>
</tbody>
</table>
Table 9.3: Peter’s reasoning and strategies

<table>
<thead>
<tr>
<th>Questions</th>
<th>Ordering</th>
<th>Reasoning</th>
<th>Keywords</th>
<th>Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{5}$ , $\frac{1}{3}$</td>
<td>I just knew it</td>
<td>NA</td>
<td>No reasoning</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{5}$ , $\frac{2}{3}$</td>
<td>There are two 3 and two 5</td>
<td>NA</td>
<td>No reasoning</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{10}$ , $\frac{2}{5}$ , $\frac{1}{2}$</td>
<td>Because I know</td>
<td>NA</td>
<td>No reasoning</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{6}{7}$ , $\frac{4}{5}$ , $\frac{2}{3}$ (Incorrect)</td>
<td>Because I know</td>
<td>NA</td>
<td>No reasoning</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{8}{17}$ , $\frac{7}{12}$ , $\frac{1}{2}$ (Incorrect)</td>
<td>Because I know</td>
<td>NA</td>
<td>No reasoning</td>
</tr>
</tbody>
</table>

Since students’ strategies reflect their thinking and problem-solving processes (Kafai, Franke & Battey, 2002; Young-Loveridge, Taylor, Hawera & Sharma, 2007), teachers can gain a rich insight into students’ fraction concept images by examining the reasoning given by students to justify their mathematical solutions and answers (Gould, 2005). Not only fractional thinking and concepts underlying the correct strategies of fractions, students’ misconceptions that cause the prevalence of various invalid strategies can be revealed too (Gould, 2005; Steinle & Price, 2008). In this study, students’ limited thinking and understanding of fractions was discovered in their reasoning and drawing when comparing different fractions in the maths tasks. Some students explained their ordering of fractions by drawing divided quantity diagrams even though their thinking was affected by the knowledge of natural numbers. For example, one cause-for-concern student was found trying to draw diagrams (Figure 9.1 (a)) to explain the following correct order of fractions but only stated “pie” without drawing any diagrams in other questions (Figure 9.1 (b)):

- $\frac{1}{5}$ and $\frac{1}{3}$;
- $\frac{2}{5}$ and $\frac{2}{3}$;
- $\frac{3}{10}$ , $\frac{2}{5}$ and $\frac{1}{2}$;
- and the incorrect order of $\frac{6}{7}$ , $\frac{4}{5}$ and $\frac{2}{3}$.
The student merely described that he was using diagrams to compare fractions. However, he did not show the ability to compare the fractions using diagrams. This is particularly peculiar in the post maths task. As noted by Shard and Adams (2002), not all pictures drawn by students match their answers, hence it can be inferred that students merely describe their thinking rather than do their thinking using diagrams.

Another cause-for-concern student also kept mentioning “I thought of a pie in my head” but ordered fractions with smaller numerators and denominators as smaller than fractions with bigger numerators and denominators. These students’ understanding of fractions was still affected by their prior knowledge of whole numbers. By interpreting fractions as consisting of two independent numbers, some students believe big numbers are equal to big fractions while others believe big numbers are equal to small fractions (Stafylidou & Vosniadou, 2004). These students needed to gain the ability of representing fractions using diagrams or imaging so as to enhance the part-whole concept of fractions. Considering students’ conceptions and
misconceptions is vital in the instruction as teacher tailor lessons according to students’ needs (Ward, 1999; Tirosh, 2000; Young-Loveridge et al., 2007) and remind students of their mistakes in the process of constructing the knowledge of fractions (Steinle & Price, 2008).

9.2 Continuities of mathematical thinking

In the above work of individual students, it was found that the same strategy was identified from the work of students with different mathematical abilities. Sam and Mary used the benchmark of a half to order fractions in question 5. However, among them only the high achiever, Sam got the right order of fractions. On the other hand, Sam and Mary also ordered fractions based on big denominators were equal to small fractions. With this strategy, the average student, Mary ordered non-unit fractions in question 4 incorrectly.

As shown in the framework of strategies for students with different mathematical abilities developed in Chapter 7 (Figure 9.2), high achievers were able to use various strategies whereas average students’ strategies were limited to representations of fractions. The average student who tried to use a higher strategy was developing her fractional thinking to a more advanced level.

![Figure 9.2: Strategies’ framework for ordering fractions](image-url)
The use of the same strategies with different levels of understanding reflects the continuities between the flawed ideas of novices and the concepts of experts (Smith III, diSessa & Roschelle, 1993). The high achiever built the knowledge of benchmarking from his prior understanding of fractions whereas the average and cause-for-concern students need to refine, reorganize and reuse rather than confront and replace their knowledge of representations of fractions. The following describes the fundamental knowledge that is needed but is lacking among some average and cause-for-concern students in order to apply numerical strategies.

9.2.1 Numerical strategies based on divided quantity thinking

Smith III et al. (1993) note that the numerical strategies of masters are connected to the divided quantity diagrams drawn by novices as masters justify their numerical reasoning in terms of divided quantity. On the other hand, some average students who used the common denominator strategy in this study were actually influenced by the misconception of big numbers equal to small fractions. For example, $\frac{1}{2}$ and $\frac{7}{12}$ were compared using the common denominator 12, in which $\frac{1}{2} = \frac{1\times6}{2\times6} = \frac{6}{12}$; thus $\frac{7}{12} > \frac{6}{12}$. However, the fractions were ordered from the smallest $\frac{8}{17}$ and $\frac{7}{12}$ to the largest $\frac{1}{2}$ as “the bigger the denominator is the smaller the fraction is” (Figure 9.3).

![Figure 9.3: Students’ reasoning](image)

An understanding of divided quantity representation would provide a concrete basis for the numerical conversions conducted by the student. Table 9.4 shows the conversion of fraction $\frac{1}{2}$ into $\frac{6}{12}$ which now can be compared with $\frac{7}{12}$.
Table 9.4: Relate diagrams to finding a common denominator

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$ and $\frac{7}{12}$</td>
<td>![Diagram 1]</td>
</tr>
<tr>
<td>$\frac{1}{2} = \frac{1 \times 6}{2 \times 6} = \frac{6}{12}$ and $\frac{7}{12}$</td>
<td>![Diagram 3]</td>
</tr>
</tbody>
</table>

The fraction $\frac{7}{12}$ could not be compared with $\frac{8}{17}$ because they had unlike denominator. Since the divided parts of two fractions were different, the 7 parts taken from one fraction could not be compared with the 8 divided parts taken from the other fraction.

Instead of only carrying out routine procedures, a solid foundation in the representations of fractions is necessary to employ numerical strategies meaningfully. According to Smith et al. (1993), the changing role of divided quantity knowledge moving from prior knowledge to numerical operations reflects the continuity between novices and masters. High achievers experience substantive conceptual and procedural changes on the way to building a solid understanding of numerical relationships.

### 9.2.2 Requirement of fundamental number knowledge in algorithmic procedures

This section emphasises the link and importance of fundamental mathematics skills to propel students to advanced mathematical skills.

- The basic skill of multiplication is required when converting fractions into a common denominator.
One cause-for-concern student found the common denominator 16 to order fractions incorrectly from the smallest $\frac{3}{4}$ through $\frac{7}{8}$ to the largest $\frac{5}{6}$ because of the incorrect multiplication that $\frac{5}{6} = \frac{15}{16}$ (Figure 9.4).

![Figure 9.4: Students’ reasoning](image)

- The number knowledge of double is helpful for the students to use the benchmark of a half.

The average student, Mary ordered fractions incorrectly from the smallest $\frac{1}{2}$ through $\frac{7}{12}$ to the largest $\frac{8}{17}$ with the explanation “7 pieces of 12 is more than half and 8 pieces of 17 is more than half too”. She needed the number knowledge of double to calculate that half of 17 was 8.5, so $\frac{8}{17}$ was smaller than a half.

### 9.3 Instructional approaches to move students to higher strategy

Based on the findings of students’ strategies and limitations, the teacher can plan for instruction that suits the particular student in order to move them to higher strategies. Teachers play a vital role in leading students to construct a proper understanding of fractions and to evolve students’ understanding and use of strategies. It is especially crucial to prevent the misconception of whole number knowledge from rooting deeply in students’ mathematical thinking and affecting the subsequent learning of more complicated concepts of fractions (Steinle & Price, 2008). As proposed by Smith III et al. (1993), students’ misconceptions reflect their preconceptions of fractions that can be extended in the instructional context. The misconceptions are changeable if an appropriately designed intervention and plausible alternatives are given. The following discussion on moving students to a higher strategy is primarily based on the case study of individuals (i.e., Sam, Mary and Peter) presented in Chapter 8.
9.3.1 Peter: Big numbers are equal to small fractions

Although Peter gave no reasoning, he was affected by the belief of big numbers were equal to small fractions and ordered fractions in contrast with counting numbers. This is a misconception generated in the process of assimilating the knowledge of fractions to the prior knowledge of whole numbers (Vosniadou, 2003). Students gain the knowledge of whole numbers prior to learning fractional concepts which differs from the whole number system in terms of symbols, order and operations (Stafylidou & Vosniadou, 2004).

Although this belief allowed Peter to order unit-fractions correctly, the teacher can remind Peter of his misconception by representing non-unit fractions using divided quantity diagrams. Taking fractions which are consistent with big numbers are equal to small fractions (i.e., \( \frac{1}{2} > \frac{1}{3} \) and \( \frac{1}{5} > \frac{1}{6} \)), the teacher can use the fractions \( \frac{1}{2} < \frac{2}{3} \) and \( \frac{1}{5} < \frac{2}{6} \) (as shown in Figure 9.5 and Figure 9.6, respectively) to highlight Peter’s misconception.

![Figure 9.5: Fractions \( \frac{1}{2} < \frac{2}{3} \)](image)

![Figure 9.6: Fractions \( \frac{1}{5} < \frac{2}{6} \)](image)
The part-whole relationship in fractions can be highlighted in representations of fractions. Through the process of drawing divided quantity diagrams to represent fractions, students come to understand that fraction sizes relate to numerators and denominators and as parts of a whole, rather than two independent numbers that are just presented symbolically. Therefore, the shift to more advanced knowledge of fractions can be built on the prior understanding of whole numbers via knowledge refinement and reorganization as advocated by Smith III et al. (1993).

Some of the cause-for-concern students believed big numbers were equal to big fractions and ordered fractions as if they were counting numbers. Such a mistake is due to the dominance of whole number knowledge, in which fractions are comprehended as two independent numbers. They adopt the whole number ordering for ordering fractions (Behr, Wachsmuth, Post & Lesh, 1984; Newstead & Murray, 1998; Stafylidou & Vosniadou, 2004).

Table 9.5 shows the rectangular divided quantity diagrams that are drawn to represent fractions. By comparing the fractions pictorially, \( \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \frac{1}{6} \). This is in contrast with the misconception of big numbers are equal to big fractions like counting numbers that 2 < 3 < 4 < 5 < 6.

Table 9.5: Rectangular divided quantity diagrams that represent fractions

| \( \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) |

9.3.2 Mary: Big denominators are equal to small fractions

Mary ordered fractions using big denominators were equal to small fractions. She ignored the relation between numerators and denominators and ordered non-unit fractions in question 4 incorrectly from the smallest \( \frac{6}{7} \) through \( \frac{4}{5} \) to the largest \( \frac{2}{3} \). The teacher can lead
students to construct the more advanced strategy of parts to make a whole by explaining to Mary that these fractions are progressively closer to a whole using divided quantity diagrams. The previous chapter discussed the improvement made by Mary in the post maths task. The divided quantity diagrams drawn allowed her to visualize the one part left from the whole and determined the fraction which had the largest part left was the smallest fraction.

Drawing diagrams is a way to develop students’ conceptual understanding of algorithms (Shard & Adams, 2002). According to Bulgar (2009), students can recognise, retrieve and apply the concepts constructed on representations of fractions rather than the algorithmic procedures memorized as meaningless routines such as finding a common denominator.

For students like Mary who mainly drew diagrams or used big denominators were equal to small fractions, the teacher can encourage building a concrete meaning on the numerical procedures by linking the representations of fractions to numerical strategies. The computational procedures can be explained clearly on divided quantity diagrams using simple fractions such as unit fractions and fractions with like numerators that have small numbers. In divided quantity diagrams, the knowledge of fractions is connected to the counting system as each fractional part is treated like an independent whole number quantity that can be counted, combined or removed arithmetically (Mack, 1990, 1995).

The teacher can demonstrate the strategy of finding a common denominator for comparisons on diagrams that are partitioned into 15 equal parts (Table 9.6). In the representation of \(\frac{1}{3}\), 5 out of 15 parts are shaded; in the representation of \(\frac{1}{5}\), 3 out of 15 parts are shaded. Thus, \(\frac{5}{15} > \frac{3}{15}\). Numerically, \(\frac{1}{3}\) and \(\frac{1}{5}\) are converted into the common denominator 15 by \(\frac{1\times5}{3\times5} = \frac{5}{15}\) and \(\frac{1\times3}{5\times3} = \frac{3}{15}\), thus \(\frac{5}{15} > \frac{3}{15}\) or \(\frac{1}{3} > \frac{1}{5}\).
Table 9.6: Relate diagrams to finding a common denominator

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{3}$ and $\frac{1}{5}$</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>$\frac{1 \times 5}{3 \times 5} = \frac{5}{15}$ and $\frac{1 \times 3}{5 \times 3} = \frac{3}{15}$</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

If a whole is 100%, the whole that is divided into 4 equal parts has 25% for each divided part; meanwhile, the whole that is divided into 5 equal parts has 20% for each divided part. Therefore, $\frac{1}{4}$ is 25% while $\frac{1}{5}$ is 20%. On the divided quantity diagrams, $\frac{1}{4} > \frac{1}{5}$; in percentage, 25% is bigger than 20%. Fractions that are compared on divided quantity diagrams can be compared numerically after being converted into percentages (Table 9.7).

Table 9.7: Relate diagrams to converting fractions into percentages

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$ and $\frac{1}{5}$</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>$\frac{1}{4} = 25%$ and $\frac{1}{5} = 20%$</td>
<td>25%</td>
</tr>
</tbody>
</table>
A half is drawn to compare with $\frac{2}{3}$ and $\frac{2}{5}$ on divided quantity diagrams (Table 9.8).

As shown on the diagrams, $\frac{2}{3}$ is more than a half while $\frac{2}{5}$ is less than a half. In fact,

$$\frac{1.5}{3} = \frac{1}{2}, \text{ thus } \frac{2}{3} > \frac{1.5}{3}; \quad \frac{2.5}{5} = \frac{1}{2}, \text{ thus } \frac{2}{5} < \frac{2.5}{5}.$$ 

Table 9.8: Relate diagrams to using the benchmark of a half

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{3}$ and $\frac{2}{5}$</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>$\frac{2}{3}$ is more than $\frac{1.5}{3}$, which is equal to $\frac{1}{2}$;</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>$\frac{2}{5}$ is less than $\frac{2.5}{5}$, which is equal to $\frac{1}{2}$.</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Simple fractions are selected for numerical comparisons to avoid complex computational procedures and difficulty in dividing diagrams for complicated fractions. Nevertheless, the NDP teaching model often proposes the use of large numbers to push students from imaging to using the number properties (although the numbers used do not seem very large). This can be shown in the following examples:

- The problems that involve simple fractions (e.g., $\frac{3}{5} + \frac{1}{2}$) are used to encourage students to image or draw to show what fraction strips would be used in solving these problems. In the problems students generalize the number properties rather than rely on images of the materials, the common denominators require students to go outside the pieces available in the fraction strips (e.g., $\frac{5}{7} + \frac{2}{3}$) (Ministry of Education, 2008d, p. 67).
• The problems such as $7.1 + 2.9 = 10$ are used in story context for students to think about how decimal ones, tenths, hundredths, etc. are partitioned and/or recombined. The students can demonstrate their understanding of the number properties involved by calculating and explaining the problems such as $2.31 + 1.99 = 4.3$ (Ministry of Education, 2008d, p. 46).

9.3.3 Sam: Numerical strategies

Sam ordered fractions correctly using big denominators were equal to small fractions and benchmark of a half. He also gave partial reasoning to two fraction questions. The teacher can encourage Sam to explain his ordering using other numerical strategies. Some of the high achievers used only one type of numerical strategy to compare all types of fractions in the study. In fact, there are limitations on using particular strategies in certain situations.

It is suggested that teachers choose sets of fractions that enable the students to compare fraction sizes with reference to benchmark numbers and equivalent fractions (Ministry of Education, 2008b). For example, the set of fractions $\frac{3}{5}, \frac{1}{3}, \frac{3}{4}$ includes fractions that are bigger and smaller than the benchmark of $\frac{1}{2}$. In question 3, the fractions $\frac{2}{5}, \frac{3}{10}, \frac{3}{7}$ and $\frac{5}{14}$ were smaller than a half, yet some students used the benchmark of $\frac{1}{2}$ for comparison. There are problems with the benchmark when two of the fractions being compared are both more than or less than the benchmark. By using the benchmark, the students could only estimate one fraction being closer to the benchmark than another fraction. For example, “$\frac{3}{7}$ was under $\frac{1}{2}$ and $\frac{5}{14}$ even more under $\frac{1}{2}$”. Numerical conversions (e.g., converting fractions into a common denominator) leading to a more accurate comparison became a better alternative for students to justify their order of fractions.

The problems of using a particular strategy and the reasons to use other strategies are tabulated in Table 9.9.
**Table 9.9: Problems and reasons of using particular strategies for ordering fractions**

<table>
<thead>
<tr>
<th>Numerical strategies</th>
<th>Difficulties with using the strategies</th>
<th>Alternative strategies</th>
</tr>
</thead>
</table>
| **a) Using a benchmark** | Fractions are smaller than \( \frac{1}{4} \)  
Example 1:  
\( \frac{1}{8} \) and \( \frac{3}{16} \) are smaller than \( \frac{1}{4} \) because  
\( \frac{1}{8} \) is smaller than \( \frac{2}{8} = \frac{1}{4} \) and \( \frac{3}{16} \) is also smaller than \( \frac{4}{16} = \frac{1}{4} \). | Recommended strategy:  
Use of common denominator would reduce confusion for comparison where both fractions are less than \( \frac{1}{2} \). |
| | Fractions are bigger than \( \frac{1}{4} \)  
Example 2:  
\( \frac{2}{3}, \frac{4}{5} \) and \( \frac{6}{7} \) are bigger than \( \frac{1}{4} \)  
because \( \frac{2}{3} \) is bigger than \( \frac{1.5}{3} \) (which is equal to \( \frac{1}{2} \)), \( \frac{4}{5} \) is bigger than \( \frac{2.5}{5} \) (which is equal to \( \frac{1}{2} \)) and \( \frac{6}{7} \) is bigger than \( \frac{3.5}{7} \) (which is equal to \( \frac{1}{2} \)). | Recommended strategy:  
Use of divided quantity diagrams can help students physically to determine which fraction is closer to the whole. |
| **b) Finding a common denominator** | More than one common denominator is needed.  
Example 1:  
Common denominator 24 is found that  
\( \frac{1}{2} = \frac{12}{24} \) and \( \frac{7}{12} = \frac{14}{24} \), thus \( \frac{14}{24} > \frac{12}{24} \)  
and common denominator 34 is found that  
\( \frac{1}{2} = \frac{17}{34} \) and \( \frac{8}{17} = \frac{16}{34} \), thus \( \frac{17}{34} > \frac{16}{34} \). | Recommended strategy:  
Use of benchmarks would reduce calculation error introduced in complex multiplication. |
c) **Converting fractions into percentages and decimals**

<table>
<thead>
<tr>
<th></th>
<th>Fractions are complicated to convert.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1:</td>
<td>( \frac{3}{7} = 0.4285... ) and ( \frac{5}{14} = 0.3571... )</td>
</tr>
<tr>
<td>Example 2:</td>
<td>( \frac{3}{4} = 75% ), ( \frac{5}{6} = 83% ) and ( \frac{7}{8} = 87.5% )</td>
</tr>
</tbody>
</table>

**Recommended strategy:**

Use of common denominator could reduce complicated conversions especially if fractions contain denominators as the factors of another denominator in a particular situation.

---

d) **Representations of fractions**

<table>
<thead>
<tr>
<th></th>
<th>Fractions that are too close to be compared on diagrams.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1:</td>
<td>Approximately the same area is covered by ( \frac{8}{17} ) and ( \frac{7}{12} ) on the divided quantity diagrams drawn.</td>
</tr>
</tbody>
</table>

**Recommended strategy:**

Use of benchmarks would reduce confusion for comparison where fractions are close to each other.

---

Table 9.10 lists students’ strategies from more to less sophisticated and provides the instructional approaches that are appropriate for moving students to higher strategies. After identifying students’ strategies as listed and as described, the teacher can plan for the instruction by referring to the relevant instructional approaches.
<table>
<thead>
<tr>
<th>Categories of strategy</th>
<th>Definitions/Descriptions</th>
<th>Instructional approaches to higher strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexible choice</td>
<td>Students are able to use a range of strategies to compare fractions.</td>
<td>• Compare fractions in different numerical situations.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Ask for the simplest and fastest way of comparing fractions.</td>
</tr>
<tr>
<td>Finding a common denominator</td>
<td>Students find a common denominator for two (or more) fractions for comparison.</td>
<td>• Represent two fractions with unlike denominators using divided quantity diagrams.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Divide parts on the diagrams until the same number of equally divided parts were obtained and this is the common denominator for the two fractions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Count the number of parts shaded on each diagram and they were the new numerators for each fraction.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• While explaining the process using diagrams, write the numerical expressions.</td>
</tr>
<tr>
<td>Converting fractions into percentage</td>
<td>Students convert fractions into percentages for comparison.</td>
<td>• Represent the whole of divided quantity diagrams as 100% and the equally divided parts as the division of the number of parts using percentages.</td>
</tr>
<tr>
<td>Converting fractions into decimals</td>
<td>Students convert fractions into decimals for comparison.</td>
<td>• Represent the whole of divided quantity diagrams as unit 1 and the equally divided parts as the division of the number of parts using decimals.</td>
</tr>
<tr>
<td>Using benchmarks</td>
<td>Students use benchmarks such as $\frac{1}{2}$ and decide whether the fractions are</td>
<td></td>
</tr>
</tbody>
</table>

Table 9.10: Instructional approaches to move students to higher strategies
| Parts to make a whole | Students correctly reason \( \frac{p}{m} > \frac{q}{n} \) if denominators \( n > m \) and numerator \( p \) needs one part to be \( m \) and \( q \) needs one part to be \( n \). | ▪ Compare fractions that were close to a half.  
▪ Using the number knowledge of double. |
| --- | --- | --- |
| Numerators and Denominators | Students correctly reason \( \frac{p}{m} > \frac{q}{n} \) if denominators \( n > m \) and numerators \( p = q \). | ▪ Compare fractions that were progressively closer to 1.  
▪ Highlight the one part left from the whole. |
| Big denominators are equal to small fractions | Students correctly reason \( \frac{1}{m} > \frac{1}{n} \) if denominators \( n > m \) for unit fractions and sizes of denominators are considered for comparison. | ▪ Compare non-unit fractions.  
▪ Show the comparison using divided quantity diagrams. |
| Drawing divided quantity diagrams | Students draw circular diagrams that represent fractions for comparison. | ▪ Imagine the divided quantity diagrams that represent fractions and describe the equally divided parts as “pieces”.  
▪ Compare unit fractions using denominators only. |
<p>| Big numbers are equal to small fractions | Students correctly reason ( \frac{1}{m} &gt; \frac{1}{n} ) if denominators ( n &gt; m ) for unit fractions and incorrectly reason ( \frac{p}{m} &gt; \frac{q}{n} ) if denominators ( n &gt; m ) and (or) numerators ( q &gt; p ) for non-unit. | ▪ Draw divided quantity diagrams to represent non-unit fractions. |</p>
<table>
<thead>
<tr>
<th>Fractions</th>
<th>Description</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Big numbers are equal to big fractions</td>
<td>Students incorrectly reason that fractions $\frac{1}{m} &gt; \frac{1}{n}$ if denominators $m &gt; n$ for unit fractions and $\frac{p}{m} &gt; \frac{q}{n}$ if denominators $m &gt; n$ and (or) numerators $p &gt; q$ for non-unit fractions.</td>
<td>- Draw divided quantity diagrams to represent unit fractions.</td>
</tr>
<tr>
<td>$\frac{1}{2}$ is the biggest fraction</td>
<td>Students order fractions as though $\frac{1}{2}$ was the biggest fraction.</td>
<td>- Draw divided quantity diagrams to represent non-unit fractions that are more than $\frac{1}{2}$.</td>
</tr>
<tr>
<td>Partial reasoning</td>
<td>Students repeat the order of fraction given.</td>
<td>- Draw divided quantity diagrams to represent unit fractions.</td>
</tr>
<tr>
<td>No reasoning</td>
<td>Students do not provide any valid reasoning.</td>
<td>- Draw divided quantity diagrams to represent unit fractions.</td>
</tr>
<tr>
<td>Incorrect reasoning</td>
<td>Students give incorrect reasoning.</td>
<td>- Draw divided quantity diagrams to represent unit fractions.</td>
</tr>
</tbody>
</table>

The high achiever, Sam, gave partial reasoning and used benchmark of a half in the pre maths task. The teacher could ask Sam to represent unit fractions using divided quantity diagrams and show the comparison using diagrams. Sam might have an idea about fractions but drawing diagrams would enable him to explain his ordering to others. Sam had an ability to use number properties when benchmarking fractions to a half. The teacher could link the representations of fractions with numerical conversions using the diagrams drawn by Sam earlier. In this way, Sam would be able to move to other numerical strategies such as converting fractions into percentages or decimals and finding a common denominator.

The average student, Mary, used big denominators were equal to small fractions. The teacher could ask Mary to compare non-unit fractions and explain the comparison using divided quantity diagrams. In this way, Mary would be led to the higher strategy numerators.
and denominators. Mary referred fractions to a half, but without using the number properties, she ordered fractions incorrectly. The teacher needs to emphasise the use of the number knowledge of doubling in using the benchmark of a half for comparisons of fractions.

The cause-for-concern student, Peter, was unable to give reasoning in the pre maths task. The teacher could ask him to draw divided quantity diagrams to represent unit fractions and compare fractions using the diagrams drawn. In this way Peter would be able to move to the high strategy - drawing divided quantity diagrams.

The teacher can also refer to students’ current stages in the Number Framework (see Table 1.1 in Chapter 1) to identify the strategies used by the students and the more advanced strategies that can be developed.

- At Stage 3 (Counting from One by Imaging), the student is able to visualise sets of objects. When ordering fractions, the student may be able to compare fractions using divided quantity diagrams.
- At Stage 6 (Advanced Additive/Early Multiplication Part-Whole), the student can estimate answers by choosing appropriately from a broad range of advanced mental strategies. The student uses a combination of known facts and a limited range of mental strategies to derive answers. When ordering fractions, the student may be able to compare fractions using big denominators are equal to small fractions.
- At Stage 7 (Advanced Multiplicative Part-Whole), the student is able to choose appropriately from a broad range of mental strategies to estimate answers. When ordering fractions, the student may be able to compare fractions using image of fractions such as numerators and denominators and parts to make a whole and benchmarks.
- At Stage 8 (Advanced Proportional Part-Whole), the student can estimate answers and solve problems involving the multiplication and division of fractions and decimals using mental strategies. These strategies involve recognising the effect of number size on the answer and converting decimals to fractions where appropriate. When ordering fractions, the student may be able to compare fractions numerically by finding a common denominator, converting fractions into percentages and decimals.

Summary

The fractional thinking framework proposed in this study informs teachers about students’ numerical development. This is useful for planning instruction that considers students’ mathematical thinking and misconceptions on fractions. There are continuities in
mathematical thinking between cause-for-concern students, average students and high achievers. The use of more sophisticated strategies by high achievers involving advanced knowledge that is built on prior understanding is connected with lower strategies commonly used by average and cause-for-concern students. For students who are adversely affected by whole number knowledge, visualizing fractions on diagrams or through mental images is a way to construct the part-whole concept as the fundamental knowledge of fractions. In addition, drawing divided quantity diagrams reaffirms students’ preconceptions of factions of the inverse relation between the size of unit fractions and the number of the parts of their whole (i.e., \( \frac{1}{2} > \frac{1}{3} \) and \( 2 < 3 \)) yet discloses the contradiction between the misconceptions of fractions and the representations of fractions.

For students whose main strategy is representations of fractions, it is noted that numerical strategies should be linked to diagrams in order to construct a concrete meaning of the computational procedures. If the students use only one type of numerical strategy such as finding a common denominator, different numerical strategies such as using a benchmark and converting fractions into percentages can be taught for ordering sets of fractions in diverse numerical situations. This provides opportunities to determine the easiest and fastest procedures for comparing fractions. It is important to move all the students to reach the higher strategies in order for them to have a strong foundation to handle problems with other mathematics domains which are related to fractions such as algebra, measurement and ratio concepts (Lamon, 2007).
CHAPTER 10

CONCLUSIONS

Considering fractions are difficult for students to learn and challenging for teachers to teach, this study aimed to investigate the use of computer games as one of the activities that help students in learning fractions. This study developed the computer game of Tower Trap for students to order fractions and the game effect was examined to determine the effectiveness of the game in improving students’ knowledge of fractions. The game was designed based on a theoretical framework that integrated instructional factors and game elements. The suitability of the game as a tool for learning fractions was evaluated by students’ responses in questionnaires and game play attempts captured on computers. In order to identify the game effect, pre and post tests and pre and post maths tasks were taken by the students before and after playing Tower Trap without any other intervention such as the teaching of fractions.

This chapter provides conclusions on the research findings which answer the research questions on the computer game of Tower Trap and the learning of fractions:

- Section 10.1 discusses the design of a good computer game for learning fractions (research question 1).
- Section 10.2 discusses students’ strategies for comparing sizes of fractions (research question 2).
- Section 10.3 discusses the improvements in students’ knowledge and strategies of fractions (research question 3).

Subsequently, Section 10.4 explains the limitations of this study before Section 10.5 ends this chapter by providing recommendations for future research.

This research has reinforced the body of knowledge on good game design for learning fractions and students’ strategies for comparing sizes of fractions. Specifically, it has contributed to the body of knowledge by providing insight into students’ fractional thinking and beneficial changes of students’ strategies for ordering fractions after playing Tower Trap.
10.1 Design of a good computer game for learning fractions

This section provides conclusions on the research findings to answer research question 1:

*What needs to be considered in developing a computer game that would enable students to compare and order fractions?*

This study identified the successful implementation of *Tower Trap* design which enables students to learn more about comparing sizes of fractions through playing the computer game. The following describes factors contributing towards designing a good computer game of *Tower Trap* and students’ evaluation of the game.

10.1.1 Theoretical framework of the game – The structure of *Tower Trap*

A theoretical framework of the structure of *Tower trap* was designed to incorporate the characteristics of good games, educational values, mathematical inquiry and students’ concerns into the development of the computer game of *Tower Trap*. The instructional factors were integrated with the game elements to design the digital outputs in *Tower Trap*. The benefit of the game for mathematical learning was further assured by referring to the criteria of educational software review on rational numbers. The preliminary design of *Tower Trap* was tested and the findings about the benefits of and the expectations on the game for students with different mathematical abilities were considered in order to create an instructional game that caters for different learning needs.

10.1.2 Intrinsic integration of content – Forming fraction brick staircases

Most of the rational number software which focuses on operating with fractions has an *extrinsic integration* of content where the fraction questions asked in the game are irrelevant to the game context (Kafai, Franke & Battey, 2002). In the computer game of *Tower Trap* that developed in this study, the learning content of fractions was intrinsically integrated with the game play in the game context of forming fraction brick staircases. Through dragging and dropping the fraction bricks, students visualise and manipulate representations of fractions. Moving the boy in the game story of climbing up to the top of the tower relates learning fractions to a fantasy situation and makes the game interesting. The specific instructions of the game and the scores obtained from ordering fraction bricks and avoiding creatures, keep students playing and learning in the game.
10.1.3 Pedagogical approach of the game - Fraction brick types

Three types of fraction bricks, namely visible bricks, broken bricks and hidden bricks were created for students to learn fractions in different ways from concrete to image to abstract. Visible bricks which represent sizes of fractions are easy to compare and students play by dragging and dropping the bricks. Students learn the divided quantity thinking of fractions with broken bricks when manipulating parts from the whole. Hidden bricks which are labelled with symbols of fractions require application of knowledge of fractions. The embedded constructivist learning approach in the game using different types of bricks allowed students to reconstruct and enhance their understanding of fractions (Pirie & Kieren, 1989; Pirie & Kieren, 1992). With such a pedagogical approach, the game becomes increasingly more challenging. This was evidenced by the monotonically increasing number of attempts of that students took to play the computer Tower Trap from visible bricks, broken bricks to hidden bricks.

Considering the influence of whole number knowledge on students’ misinterpretation of fractions as two independent numbers, the part-whole relation is made explicit by the divided quantity blocks of broken bricks. Divided into equal parts, broken bricks show the representations of fractions that are contrary to the misconception of big numbers are equal to big fractions as fractions with big denominators have smaller pieces of each individual part. In addition, the representations of other proper fractions which are formed by dividing the wholes of unit 1 on broken bricks show that the whole of unit 1 is the biggest proper fraction and this in contrast with the misconception that some students have that “unit 1 is the smallest”.

10.1.4 Students’ positive evaluation of the game for learning fractions

Students evaluated Tower Trap positively in terms of game play and learning fractions. They liked the game features such as the story, instructions and game scores. The play to move a boy and avoid creatures in the game amused the students. They found forming staircases less interesting but they did enjoy dragging and dropping fraction bricks to form staircases in the game. The game is simple so that students could focus on the visual representations of fractions. Simple games are suitable for instructional purposes because they are more focused than complex games (Klawe, 1998; Dempsey, Haynes, Lucassen & Casey, 2002; Kickmeier-Rust, 2009). Although some students criticised the use of less advanced
graphics and manipulation in Tower Trap yet most of the students liked playing the computer game and learned more about fractions especially on sizes of fractions and ordering fractions. This reaffirms the findings by Bragg (2007), which highlight a mathematics game is fun to students if there is a challenge of learning mathematical concepts rather than just beating an opponent. Designing a good game play is essential to enhance players’ experience through providing a lot of challenge and fun that makes games exciting (Prensky, 2001). Engaging game playing experience is important to lead to positive learning outcomes in educational games (Amory, 2001; Prensky, 2001).

10.2 Students’ strategies for comparing sizes of fractions

This section provides conclusions on the research findings to answer research question 2:

What strategies are used by students in ordering fractions? How can these strategies be classified as in a framework?

Considering that there is little research on computer games in schools (Buckingham, 2004) especially for developing the equivalence of fractions, this study aimed to investigate the effectiveness of the computer game of Tower Trap on students’ knowledge and strategies for ordering fractions using assessment tools such as the pre and post tests, pre and post maths tasks, questionnaires and computer game play record. Most of the findings were consistent with the previous work by other researchers. However, a more detailed perspective on students’ thinking in ordering fractions was identified through this study.

10.2.1 Students’ mathematical abilities and use of strategies

Students’ strategies for comparing fractions have been focused on in previous research though not in relation to using computer games in teaching fractions. Researchers indicate the common strategies for comparing fractions used by students included manipulative aids, reference points (or benchmarks), whole number knowledge and, conversions to common denominators, decimals or percentages (Behr, Wachsmuth, Post & Lesh, 1984; Smith III, 2002; Gould, 2005).

Focusing on learning fractions using Tower Trap, students in this study exhibited the above mentioned strategies for comparing sizes of fractions. High achievers mainly compared
fractions numerically, by using a benchmark, finding a common denominator and converting fractions into percentages. Average students mainly referred to the representations of fractions. Cause-for-concern students were unable to give significant reasoning and explained big numbers were equal to small fractions to most of their ordering of fractions. Other researches also relate students’ mathematical abilities to their strategies for comparing fractions. As documented by Behr, Wachsmuth and Post (1985), high performers use a flexible and spontaneous application of fraction order and equivalence concepts whereas low performers tend to apply concepts in a constrained or inaccurate manner. High performers also carry out algorithmic procedures including common denominators.

There was a relationship between the strategies used by high achievers and the sets of fractions that were to be compared in the maths tasks. For example, a common denominator was found in question 3, diagrams were drawn in question 4 and a benchmark was used in question 5. This means the strategies used by students are affected by the types of questions asked. Smith III, diSessa and Roschelle (1993) explain that masters used specific tools that are well suited to particular numerical situations while novices are restricted to the manipulation of a mental model of divided quantity.

Different findings were noted in this study, namely that some weaker students tended to use higher strategies in the maths tasks, where the comparison situations have promoted the use of numerical strategies even for students whose earlier thinking is primarily based on representations of fractions and whole number knowledge.

10.2.2 Continuities of mathematical thinking

This study showed that similar strategies have been used by different ability groups. This finding highlighted the continuity of thinking from novices to experts where flawed ideas of novices become ideal concepts of experts as advocated by Smith III et al. (1993). In this study, high achievers mostly referred to representations of fractions for the comparison of simple fractions whereas some average students were influenced by the misconception of big numbers were equal to small fractions when converting fractions to a common denominator. Since students build advanced knowledge from their prior understanding (Smith III et al., 1993), the numerical strategies (i.e., using benchmarks, converting fractions to a common denominator, percentages and decimals) of masters can be traced to the divided quantity
diagrams drawn by novices. Through knowledge refinement, reorganization and reuse, the masters actually justify their numerical reasoning in terms of divided quantity. This highlights the continuity of knowledge from novices to masters and in this numerical strategy the high achievers have exhibited changes in terms of conceptual and procedural knowledge to build a sound understanding of numerical relationships. This is consistent with the model of the growth of mathematical understanding proposed by Pirie and Kieren (1989, 1994b).

10.2.3 Students’ strategies toward a framework

This study identified the mathematical understanding underlying students’ strategies by referring to the model of the growth of mathematical understanding proposed by Pirie and Kieren (1989, 1992, 1994a, 1994b). The strategies used by high achievers, average students and cause-for-concern students required different levels of mathematical understanding. A framework is proposed to integrate these strategies to present a developmental picture of students’ fractional thinking (Figure 10.1).

![Figure 10.1: Strategies’ framework for ordering fractions](Image)

Using benchmarks
Finding a common denominator
Converting fractions into percentages and decimals
Parts to make a whole
Numerator and denominators
Big denominators are equal to small fractions
Drawing divided quantity diagrams
Big numbers are equal to small fractions

High achievers
Average students
Cause-for-concern students
The involvement of students with different mathematical abilities in this study provided the evidence to verify the list of less to more sophisticated strategies for ordering and comparing fractions proposed by Darr and Fisher (2006) and Maguire, Neil and Fisher (2007). The list of strategies is increasingly sophisticated from attempting to use whole number knowledge, drawing pictures, identifying fractions with the same denominator or numerator, benchmarking fractions to well known fractions, to using equivalent fractions.

This study also noted some different findings on students’ strategies. Previous research indicated fractions are ordered like counting numbers that big numbers are equal to big fractions when the students attempt to use whole number knowledge (Stafylidou & Vosniadou, 2004; Darr & Fisher, 2006; Maguire et al., 2007). This study found a contradict belief of big numbers are equal to small fractions is also affected by students’ prior knowledge of whole numbers when learning new concept of fractions.

As shown in previous research, drawing pictures is more sophisticated than whole number knowledge but less sophisticated than other strategies (Darr & Fisher, 2006; Maguire et al., 2007). However, this study showed drawing diagrams to represent fractions could be more sophisticated if the part-whole concept is taken into account in the representations of fractions for comparisons. This is discussed further in Section 10.2.4 where the importance of emphasizing the part-whole relation in the divided quantity diagrams drawn is shown.

In previous research, the most sophisticated strategy refers to the use of equivalent fractions such as finding a common denominator or converting fractions to percentages or decimals (Darr & Fisher, 2006; Maguire et al., 2007). Additionally, the numerical conversion method has also been considered to be sufficient to solve all kinds of fraction problems (Smith III et al., 1993). However, this study showed that the appropriateness of strategies for comparing fractions is rather determined by the situations of fractions that are being compared. This is consistent with Smith III et al. (1993) that specific tools are used by masters to compare fractions that involve particular numerical situations. Therefore, this study suggested that the use of the strategies is not necessarily dependent on the levels of strategy’s sophistication.
10.2.4 Part-whole concept

Previous research showed that the flaws of representing fractions in unequal wholes and parts in students’ diagrams are related to their incomplete thinking of fractions (Peck & Jencks, 1981; Yoshida & Kuriyama, 1995; Baturo, 2004; Gould, 2005; Olive & Vomvoridi, 2006; Steinele & Price, 2008). For instance, numerator and denominator are mapped onto the number of shaded and unshaded parts of the fractions, respectively (Paik & Mix, 2003); the size of the whole that each fraction represents is drawn directly proportional to the size of the denominator (Peck & Jencks, 1981; Yoshida & Kuriyama, 1995; Gould, 2005).

This study had similar findings with the above research on students’ mistakes in their drawing. Nevertheless, it is worthy to note that some students were still able to order fractions correctly based on the diagrams drawn although the representations of fractions were not in equal whole and equal parts. In contrast, with the limited understanding of the part-whole concept, other students made the improper comparison of fractions using divided quantity diagrams. For example, two cause-for-concern students gave the correct ordering from the smallest $\frac{3}{10}$ through $\frac{2}{5}$ to the largest $\frac{1}{2}$ although the diagrams drawn to represent fractions were in unequal parts and wholes (Figure 10.2). Most students were bound to draw circular divided quantity diagrams with different sizes of whole. However, many of them mentally probably thought of these circular diagrams as the same size. This approach together with the understanding on part-whole concept enabled some of the students to answer correctly.

![Figure 10.2: Diagrams drawn by cause-for-concern students in question 3](image)

Through manipulating parts of the divided quantity blocks of *broken bricks* in *Tower Trap*, students are given opportunities to enhance their understanding of the part-whole concept underlying the divided quantity diagrams. The whole is divided into a number of parts according to denominator and a number of parts are shaded blue according to numerators. In the game, students are asked to select the blue parts from the dividing whole of
*broken bricks* to form fractions. In this way students construct the knowledge of fractions that is related to numerators and denominators and as part to make a whole. Therefore, improvements on students’ performance were seen in their use of strategies in the post task.

### 10.3 Changes of strategies

This section provides conclusions on the research findings to answer research question 3:

*What improvements, if any, are there in students’ ability as a result of playing the game?*

The improvement of students’ fractional knowledge was identified through the investigation on the changes of students’ strategies between the pre and post maths tasks. Since the students did the pre and post tasks before and after playing *Tower Trap* and no other teaching intervention was given in between the tasks, any changes were attributed to the effect of playing the computer game. Among the research on students’ strategies for comparing sizes of fractions, none of them look at the changes of students’ strategies (Behr et al., 1984; Smith III et al., 1993; Smith III, 2002; Gould, 2005; Darr & Fisher, 2006; Maguire et al., 2007). Therefore, a new finding about the changes of students’ strategies for comparing sizes of fractions was identified in this study.

More than half of the average and cause-for-concern students and about a quarter of high achievers improved (i.e., WR) in at least one question (out of 5 questions) when ordering fractions in the maths tasks. Improvement of students’ strategies was noticed from the changes of strategies between the pre and post maths tasks. The ability to change the strategies is related to students’ mathematical abilities. High achievers improved mainly on the use of numerical strategies while average and cause-for-concern students improved on the use of representations of fractions.

The computer game focusing on visualizing and manipulating representations of fractions has the most positive effect on the use of the representations of fractions. The significant changes of strategies from big denominators are equal to small fractions to numerators and denominators have led to the correct ordering for fractions with different numerators and denominators. A few students who had difficulty in comparing fractions that were close to a half using circular divided quantity diagrams managed to overcome this
problem by changing to rectangular divided quantity diagrams. It was also found that some students used the better alternative of representations of fractions in the post task, rather than the insufficient reasoning and the misconception of big numbers were equal to small fractions that they used in the pre task.

10.3.1 High achievers: Improvement on the use of numerical strategies

After playing Tower Trap, high achievers started to change to use simpler strategies that do not need excessive computational efforts. For example, it was more tedious to compare factions especially when more than one common denominator needed to be found in the pre task. As such, in the post task, some high achievers started to use the benchmark of \( \frac{1}{2} \) in comparing fractions. Deciding whether the other fractions are bigger or smaller than \( \frac{1}{2} \) can be simpler than converting these fractions into a common denominator. Apart from adopting a simpler strategy for certain questions, high achievers also used more accurate strategies to compare fractions in the post task. As an example, the strategy of converting fractions into percentages adopted by some high achievers in the post tasks was more accurate than drawing diagrams to represent fractions in the pre task.

10.3.2 Average and cause-for-concern students: Improvement on the use of representations of fractions

The average and cause-for-concern students tried to change to better strategies for comparing simple fractions after playing Tower Trap. Some of the students had been ordering fractions based on the misconceptions of fractions (i.e., big numbers were equal to big fractions and big numbers were equal to small fractions) in the pre task. However, through the representations of fractions, these students gained a better understanding of fractions that related to denominators and numerators and as parts to make a whole after playing the game. This is demonstrated by the fact that some average and cause-for-concern students had improved in the maths task (i.e., wrong in the pre task but right in the post task or WR). These students had started to consider both numerator and denominator in comparing fractions whereby they actually ignored numerators and considered denominators only when using big denominators were equal to small fractions during the pre task. They thought the more parts
were divided the smaller each part became as shown in one of their comments “$\frac{6}{7}$ has more pieces so it has smaller pieces”. Finally, these students were also influenced by the *Tower Trap* to use rectangular divided quantity diagrams (i.e., similar to the *broken bricks* in the game) to compare fractions. Some of the average and cause-for-concern students were unable to compare fractions that were close to each other on circular divided quantity diagrams such as $\frac{8}{17}$ and $\frac{7}{12}$ which looked as if they covered the same area on circular divided quantity diagrams. This is overcome by using rectangular divided quantity diagrams because rectangles could easily be kept in the same whole.

### 10.4 Limitation of this study

This section explains two limitations of this study which are the allocated time to play the game and the design of the study.

#### 10.4.1 Allocated time for playing the game

In this study, every participant took about an hour of school time to play *Tower Trap* and complete the assessment tools. The participating students were given about half an hour to play the computer game individually. Nonetheless, the high achievers could spend approximately the minimum of 20 minutes to order the fractions bricks for 26 game levels by using only one attempt at each game level. This included the time for the player to read the story of the game, drag and drop the bricks, jump and duck down when avoiding creatures and climb up to the next floor. Most of the students especially the cause-for-concern students needed more time to order fraction bricks and some took more than one attempt to make a correct ordering of fractions. The game required the players to move to the next floor only if a correct fraction brick staircase was formed, hence the players had to try and take more attempts until a correct ordering of fractions was given.

During the one hour, the improvement of the knowledge of fractions was noticed from the changes to better strategies and the correct ordering of fractions in the post task. An even bigger effect might be expected if the students were allowed to play the game on more than one occasion, especially for the weaker students. However, the time to spend with each
student was limited due to school restraints. The schools did not want the students’ regular education to be interrupted too much.

10.4.2 Design of the study

This study was designed to use a mixed methods approach to address the research question “What improvements, if any, are there in students’ ability as a result of playing the game?” Unfortunately, the quantitative data of achievement differences between the pre and post test scores (i.e., WR, RR, RW and WW) could only be analysed using descriptive statistics. This study failed to detect a significant difference between the experimental and control groups using various statistical tests. Furthermore, in the quantitative data, the improvement in students’ ability that could be shown was limited to the area of fractions based on each of the questions that the students answered in the tests. The results showed the game had the best result of WR in question 5 on ordering fractions as compared to other aspects assessed in the tests through other questions such as representations of fractions, comparing and operating fractions. Nevertheless, the qualitative data of students’ reasoning given on the pre and post maths tasks was able to provide more information about students’ fractional thinking. Therefore, the primary focus of this study was to investigate the changes of students’ strategies between the pre and post tasks in order to identify any improvement on students’ learning of fractions as a result of playing Tower Trap.

10.5 Recommendations for future research

This section provides recommendations to improve Tower Trap by including a range of difficulty categories, fractions bigger than unit 1, decimal fractions, tall and long fraction bricks and numerical strategies. Future research can also extend strategies’ framework to develop assessment and instructional materials for teachers and enhance the understanding of students who use partial strategies.

10.5.1 A range of difficulty categories

The game could include fractions in a range of difficulties in the categories from easy and normal to hard to cater for the needs of students with different abilities (Figure 10.3). The
easy category could include unit fractions such as $\frac{1}{3}$ and $\frac{1}{5}$; the normal option could include non-unit fractions such as $\frac{3}{10}$ and $\frac{2}{5}$; the hard option could include fractions that are close to each other such as $\frac{8}{17}$ and $\frac{7}{12}$. The game levels could be designed to be increasingly difficult within a category by including more fractions for comparisons and involving big numbers in the fractions. Since similar fractions are compared within a category, a particular strategy could be used for the comparisons. Cause-for-concern students need to practice more on the easy level before proceeding to the normal level. High achievers only find the game challenging if they could skip the fractions that are far below their ability. The attempts taken by students for ordering fractions within a particular category could be examined. If the attempts taken are decreasing, this means the students improved in ordering fractions included in the category. The strategies used to order the similar fractions could be examined to identify if the same strategies or different strategies are used.

![Difficulty options of easy, normal and hard](image)

**Figure 10.3: Difficulty options of easy, normal and hard**

### 10.5.2 Improper fractions or fractions bigger than unit 1

Only proper fractions were compared in *Tower Trap* and the biggest fraction was the unit 1. If improper fractions are compared in the game, the conversion of the improper fractions into equivalent mixed number (the sum of a whole number and a proper fraction) could be shown using the divided quantity blocks of *broken bricks*. For example, several *broken bricks* which are divided into 2 equal parts and 3 equal parts, respectively could be
displayed for students to form the improper fractions $\frac{3}{2}$ and $\frac{5}{3}$ for comparison. Therefore, a whole of $\frac{2}{2}$ and a part of $\frac{1}{2}$ could be joined to form the improper fraction $\frac{3}{2}$ (Figure 10.4). Similarly, a whole of $\frac{3}{3}$ and a part of $\frac{2}{3}$ could be joined to form the improper fraction $\frac{5}{3}$ (Figure 10.5). By extending the game to include the topic of improper fractions, which is more advanced, the high achievers would benefit more from the game as they would be able to develop their fraction concepts.

![Figure 10.4: Forming the improper fraction $\frac{3}{2}$](image1.png)

![Figure 10.5: Forming the improper fraction $\frac{5}{3}$](image2.png)

Future research could investigate the differences in the strategies used to order improper fractions with those strategies used to order proper fractions identified in this study and the challenges faced by students in comparing improper fractions.
10.5.3 Decimal fractions

The game of forming fraction brick staircases could be extended to decimal fractions. A decimal fraction is a fraction where the denominator is a power of ten (i.e., a number such as 10, 100, 1000, etc). Decimal fractions are commonly expressed without a denominator. For example, \(0.5 = \frac{5}{10}\); \(0.25 = \frac{25}{100}\). A problem many students have with ordering decimals is that they think 0.25 is bigger than 0.5 because there are more digits (Ministry of Education, 2008d). Broken bricks could be utilised to present a pictorial image of decimals to students. For example, a broken brick (1) is divided into 2 equal parts and each part is 0.5. Each part of 0.5 can be divided into 2 equal parts to get smaller parts which are 0.25 each (Figure 10.6). This would help the students to construct a concrete meaning of the sizes of decimals (e.g., 0.5 and 0.25). It is believed that high achievers would be able to develop their knowledge on decimal fractions by playing the game that included the decimal fractions.

![Figure 10.6: Representations of decimal fractions using broken bricks](image)

10.5.4 Tall and long fraction bricks

This study focused on the design of visible, broken and hidden bricks. The effects of using tall and long fraction bricks could be investigated in the future research. The tall fractions bricks are used to order fractions from the smallest to the largest whereas the long fractions bricks are used to order fractions from the largest to the smallest. As such, the use of tall fractions bricks or long fractions bricks may have different effects on students’ learning. In order to do so, the same sets of fractions can be developed in both types of tall and long fraction bricks for students to order. By analysing the ordering of fraction and strategies used by the students to order these two types of bricks, the effects of tall and long fraction bricks to students’ learning can be examined. In fact, this study seems to suggest that long fraction bricks are more difficult to be ordered as compared with tall fraction bricks.
10.5.5 Integrate numerical strategies with the computer game context

Numerical conversion strategies which were shown only on the tips pages of Tower Trap could be integrated intrinsically with the game to focus on the use of numerical strategies. The conversions of a common denominator, percentages and decimals could be linked with the divided quantity block of broken bricks by dividing the blocks into a number of parts according to the common denominator or labelling with percentages and decimals. The link between numerical strategies and representations of fractions would provide a concrete meaning to the students before they merely employ the numerical procedures to order hidden bricks. Therefore, the improvement on the use of numerical strategies could be investigated too.

10.5.6 Instructional assessment and materials

This study provided a strategies’ framework which can be useful to teachers to improve students’ strategies. Further from here, assessment and instructional materials should be developed to be used by the teachers. The assessment could include a set of fraction questions where students’ strategies can be identified when they order fractions. For example, the 5 questions asked in the maths tasks that consisted of different types of fractions (i.e, unit fractions, fractions with like numerators, fractions with unlike denominators and numerators, fractions that were close to one, and fractions that were close to a half) could be used. The students who were affected by the misconception of big numbers were equal to small fractions got the correct ordering of fractions in the first three questions but this misconception could be detected in the last two questions.

The instructional materials could be specifically designed to provide approaches that suit students reaching particular understanding levels and using particular strategies for ordering fractions. For example, it is useful to include examples of fraction questions that link numerical conversions with divided quantity diagrams. This could be used by the teachers to lead the students who mainly draw divided quantity diagrams to compare sizes of fractions to utilise numerical conversions when comparing sizes of fractions.

10.5.7 Enhance the understanding of students who used partial strategies

Some students showed only partial strategies in their reasoning. The examples are:
- A cause-for-concern student tried to find a common denominator but was affected by the misconception of big numbers were equal to small fractions in the answer.
- Many students used the benchmark of a half to compare fractions that were close to a half but lacked the number knowledge of doubles.

In future, researchers could focus on these students by considering their current understanding of fractions which is only adequate to support their partial strategies. These students’ understandings should be further enhanced to enable them to master the strategy.

**Summary**

Many methods have been investigated, proposed and used for teaching and learning fractions. At this moment, there is no perfect method for teachers and students because different methods may suit different learning needs of students. Using computer mathematics games is one of the methods that can be used to provide learning activities in schools.

This study showed the positive effects of the *Tower Trap* computer game for learning fractions. This study highlighted the need for good computer game design to enable learning and to enhance student interest. Future research should further explore computer game methods for improving teaching and learning of fractions.
References


Retrieved from http://www.gamestudies.org/0301/fromme/


Juul, J. (2003). The game, the player, the world: Looking for a heart of gameness. In M. Copier & J. Raessens (Eds.), *Level Up: Digital Games Research Conference Proceedings* (pp. 30-45). Utrecht, the Netherlands: Utrecht University, DiGRA.


Kyriakides, A. O. (2010). Engaging everyday language to enhance comprehension of fraction multiplication. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Conference of European Research in Mathematics Education* (pp. 1003-1012). Lyon, France: CERME.


http://www.cs.ubc.ca/nest/egems/reports/authors.html


## APPENDIX 1

### Pre and post tests

<table>
<thead>
<tr>
<th>Test A</th>
<th>Test B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Question 1</strong>: Which picture is $\frac{3}{4}$ shaded blue?</td>
<td><strong>Question 1</strong>: Which picture is $\frac{3}{4}$ shaded blue?</td>
</tr>
</tbody>
</table>

<p>| | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Question 2**: What fraction of this rectangle is shaded blue? | **Question 2**: What fraction of this rectangle is shaded blue? |

<table>
<thead>
<tr>
<th>$\frac{3}{8}$</th>
<th>$\frac{5}{8}$</th>
<th>$\frac{7}{8}$</th>
<th>$\frac{9}{8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Question 3**: What fraction of this rectangle is shaded blue? | **Question 3**: What fraction of this rectangle is shaded blue? |

<table>
<thead>
<tr>
<th>Between $0$ and $\frac{1}{4}$</th>
<th>Between $\frac{1}{4}$ and $\frac{1}{2}$</th>
<th>Between $\frac{1}{2}$ and $\frac{3}{4}$</th>
<th>Between $\frac{3}{4}$ and $1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Question 4**: Which is larger, $\frac{5}{9}$ or $\frac{6}{13}$? | **Question 4**: Which is smaller, $\frac{5}{4}$ or $\frac{6}{11}$? |

<table>
<thead>
<tr>
<th>$\frac{5}{9}$</th>
<th>$\frac{6}{13}$</th>
<th>Same</th>
<th>Impossible to tell</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Question 5**: Which of these orders is from smallest to largest? | **Question 5**: Which of these orders is from largest to smallest? |

<table>
<thead>
<tr>
<th>$\frac{1}{1}$, $\frac{4}{6}$, $\frac{2}{18}$, $\frac{3}{18}$, $\frac{9}{18}$, $\frac{11}{18}$</th>
<th>$\frac{1}{1}$, $\frac{4}{6}$, $\frac{2}{18}$, $\frac{3}{18}$, $\frac{9}{18}$, $\frac{11}{18}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Question 6**: Which number is closest to the answer to $\frac{5}{9} + \frac{6}{10}$? | **Question 6**: Which number is closest to the answer to $\frac{10}{11} + \frac{6}{7}$? |

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>15</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 1 | 2 | 16 | 18 |
APPENDIX 2

Pre and post maths tasks

<table>
<thead>
<tr>
<th>Questions</th>
<th>Pre maths task</th>
<th>Post maths task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{5\cdot 3}$</td>
<td>$\frac{1}{5\cdot 4}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{3\cdot 5}$</td>
<td>$\frac{3}{4\cdot 5}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2}{5\cdot 10\cdot 2}$</td>
<td>$\frac{1}{2\cdot 14\cdot 7}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{6}{7\cdot 3\cdot 5}$</td>
<td>$\frac{7}{8\cdot 4\cdot 6}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2\cdot 12\cdot 17}$</td>
<td>$\frac{4}{7\cdot 15\cdot 2}$</td>
</tr>
<tr>
<td></td>
<td>How do you know?</td>
<td>How do you know?</td>
</tr>
<tr>
<td>Questionnaire items</td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------------------</td>
<td>------------------</td>
<td></td>
</tr>
<tr>
<td><strong>Playing the game</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>I like playing the game.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>I like playing the game because I can learn more about fractions.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>I like playing the game because I like to play computer games.</td>
<td></td>
</tr>
<tr>
<td><strong>Game features</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>I like the boy in the game.</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>I like the creatures in the game.</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>I like to drag and drop the bricks.</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>I like to make the boy move.</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>The instructions for the game are clear.</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>The story makes the game interesting.</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>I find it easy to see my progress.</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>I find forming staircases is interesting.</td>
<td></td>
</tr>
<tr>
<td><strong>Learning of fractions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>After playing the game, I want to learn more about fractions.</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>I use knowledge of fractions I learned in school when playing the game.</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>The game helps me to imagine the sizes of fractions.</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>I learned how to write fraction symbols from playing this game.</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>The game helped me to put fractions into order.</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>This game helped to fix mistakes I was making with fractions.</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>I have learned more about fractions.</td>
<td></td>
</tr>
<tr>
<td><strong>Teaching aids</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>I would like to play the game at school as a part of learning fractions.</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>I had to think hard to play the game.</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>I learned more from the game than I do from my teacher in the classroom.</td>
<td></td>
</tr>
</tbody>
</table>
## APPENDIX 4

### Game user data

<table>
<thead>
<tr>
<th>Name</th>
<th>Lee</th>
<th>Score</th>
<th>242</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Level</td>
<td>Attempts</td>
<td>Score 1</td>
</tr>
<tr>
<td>Introduction 1</td>
<td>1</td>
<td>10</td>
<td>No</td>
</tr>
<tr>
<td>Tall bricks - Visible</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Introduction 2</td>
<td>1</td>
<td>10</td>
<td>No</td>
</tr>
<tr>
<td>Tall bricks - Broken</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Introduction 3</td>
<td>2</td>
<td>5</td>
<td>No</td>
</tr>
<tr>
<td>Long bricks - Visible</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Long bricks - Broken</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Introduction 4</td>
<td>1</td>
<td>10</td>
<td>No</td>
</tr>
<tr>
<td>Hidden bricks - Tall</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Hidden bricks - Long</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Totals</td>
<td>201</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX 5

Students’ qualitative feedback

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
</table>
| High achievers (N = 15)       | 4 (27%)                              | • I liked the game!!  
• It was very fun.  
• It was fun and easy to use. The instructions were easy to use and I got the hang of the game.  
• I thought that the fractions game was pretty good. I already knew heaps about fractions and most of it was easy. I liked how it was a sort of platform game, they are the ones I enjoy most. Thanks for letting me participate in your study. |
| Average students (N = 49)     | 4 (8%)                               | • I like this game but I think it should be longer.  
• It was cool in some parts! <(‘-‘)>  
• I enjoyed the game because we were learning while having fun at the same time.  
• I liked it. It was fun even though I found it hard to dodge the creatures :) oh well it was good oh and it was original and fun way to learn fractions. |
| Cause-for-concern students (N = 42) | 8 (19%)            | • I thought the game was cool.  
• It was great fun to do.  
• It was cool :)  
• I really liked the game I think it was COOL and I want to play it again.  
• I wish I had this game  
• I like the animals in the game but in a way didn’t because they kept hitting me. I think the game is a really good idea and I hope it works.  
• I like the game and it helps me learn fractions thanks now I know more about fractions.  
• I really liked this game because at the moment at school we are learning about fractions I think this will help me a lot. |
Table 2: Students’ negative feedback about the game

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedbacks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average (N = 49)</td>
<td>1 (2%)</td>
<td>• It was not fun.</td>
</tr>
<tr>
<td>Cause-for-concern (N = 42)</td>
<td>1 (2%)</td>
<td>• The game was fun but it gets boring really easily so yea.</td>
</tr>
</tbody>
</table>

Table 3: Students’ feedback about the game creatures

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
</table>
| High achievers (N = 15)       | 2 (13%)                             | • The Purple Squirrels (i.e., creatures) were cool =D, the poofy thing when you hit the Squirrels or birds was cool but an explosion would be better.  
• I found the creatures were moving too fast for my reaction time.  |
| Average students (N = 49)     | 4 (8%)                              | • I liked it. It was fun even though I found it hard to dodge the creatures :) oh well it was good oh and it was original and fun way to learn fractions.  
• I think that the game was pretty small but the creatures were cool. I think that the game should be 3D with you behind the main character and there should be weapons that you can use to defend yourself against the creatures.  
• The animals was too fast.^^  
• I think that the creatures in the game should become harder in harder levels. You should be able to see how many staircases you have made.  |
| Cause-for-concern students (N = 42) | 1 (2%)                           | • I like the animals in the game but in a way didn't because they kept hitting me. I think the game is a really good idea and I hope it works.  |
Table 4: Students’ feedback about the graphics and manipulation

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>High achievers (N = 15)</td>
<td>1 (7%)</td>
<td>• The boy shouldn't be able to jump higher than the three bricks put together.</td>
</tr>
<tr>
<td>Average students (N = 49)</td>
<td>1 (2%)</td>
<td>• The game had bad graphics!</td>
</tr>
<tr>
<td>Cause-for-concern students (N = 42)</td>
<td>4 (10%)</td>
<td>• The graphics could be a bit better.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• It was a bit boring and the graphics aren't very good, a bit too childish, and the story lines are boring and annoying.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• I think that if the game had better graphics, cooler characters, crazy aliens and weapons, I reckon it will attract kids to learn fractions and other mathematics. When the kid jumps he will take a long time to get down which I found annoying.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• I think when the character jumps he should jump so high &amp; so slowly.</td>
</tr>
</tbody>
</table>

Table 5: Students’ feedback about the game features

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average students (N = 49)</td>
<td>4 (%)</td>
<td>• I think that the creatures in the game should become harder in harder levels. You should be able to see how many staircases you have made.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• The game got a bit boring because it was the same level over and over so it got boring if you changed the level a bit each time it would be good thanx.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Why can't it be a girl with pretty clothes! For a girl it’s a girl and for a boy it’s a boy.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• I think there should be a timer when doing the fractions and if the time runs out a bear chases</td>
</tr>
</tbody>
</table>
you back to the start. There should be more different activities than just the stairs because it gets boring after a while. There could be a puzzle round and you have to create the puzzle with the fractions. There should be a girl character. You could design your own character. There should be more levels and every time you go to a new one it is different. There should be more colours so it’s brighter and looks more exciting. The person should find the ball at the top of the tower and the ball has got stuck inside a treasure chest and there is an evil creature next to the chest and you have to have a fraction showdown and you have to get more than 80% of the showdown (quiz) correct.

<table>
<thead>
<tr>
<th>Cause-for-concern students (N = 42)</th>
<th>1 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• “There should be a timer and if u don’t get the answer in like just say 30 seconds then u have to start again and three (sic) should be a little cat or a big bear chasing you up the tower and dodging those animals is quite hard and when you dodge the animals we should get like extra points, and if you hit the animals you should get a penalty. At the start of the game it should ask you if your male or female and if you choose male then a boy character should come up and if girl character is chosen then it should be a girl character and design what your character looks like, so it’s like the same but still not different questions each time and animals should come a little bit slower. and at the end there should be a table in the middle of the room and it has like a flying carpet and the flying carpet flies you back home and when you land a big message comes up on the screen saying congratulations and you should be able to print a little certificate off with your name on it. Example numbers up 1 and 2.”</td>
<td></td>
</tr>
</tbody>
</table>
### Table 6: Students’ feedback about learning fractions using the game

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average students (N = 49)</td>
<td>1 (2%)</td>
<td>• We don’t do fractions too much anymore in class so I learn some more</td>
</tr>
</tbody>
</table>
| Cause-for-concern students (N = 42) | 5 (12%)                           | • I think that I need to get better at fractions and try my very best at them otherwise I will not be able to know them at all.  
• Well…the game was a bit easy and hard in some areas of fractions that I didn’t realise.  
• I like the game and it helps me learn fractions thanks now I know more about fractions.  
• I think it is a great way to learn your fractions.  
• It was boring but it helped me learn little but I could learn that thing anyway. |

### Table 7: Students’ feedback about using the game as a teaching aid

<table>
<thead>
<tr>
<th>Students</th>
<th>Students who provided feedback n (%)</th>
<th>Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average students (N = 49)</td>
<td>1 (2%)</td>
<td>• I think the school should use this game for learning fractions.</td>
</tr>
</tbody>
</table>
| Cause-for-concern students (N = 42) | 3 (7%)                             | • I really liked this game because at the moment at school we are learning about fractions I think this will help me a lot.  
• It was better than learning fractions in school!  
• I think it would be a good game for schools to use to help those who are having trouble with fractions. |