Passive dynamics in animal locomotion

Author:
Te-yuan CHYOU
Department of Mathematics and Statistics
University of Otago

Supervisors:
Dr. Mike PAULIN
Department of Zoology
University of Otago

Dr. Gerrard LIDDELL
Department of Mathematics and Statistics
University of Otago

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Abstract

For decades, biologists believed that animals walk because the brain calculates motion trajectories for their limbs. However, an alternative hypothesis on animal locomotion suggests the opposite. The animal framework is built to walk “naturally” and the dynamics work without relying on controls. Instead, the walking gait is simply generated by the interaction of gravity and inertia, establishing a stable, naturally emerging limit-cycle known as passive dynamic walking. The feasibility of passive dynamic walking has been demonstrated for a biped system consisting of only a pair of legs. This thesis examines full-body passive dynamic walking models with simple and animal-like mechanical linkages that can generate walking gaits using only gravity that are able to recover from small perturbations without the need for controller input. The contribution of a torso to the stability and efficiency of passive biped walking is also addressed. When an upper-body is added, passive dynamic walking takes place on level ground but for energetic reasons, it is unstable. The second part of the thesis looks at how to stabilize the passive walking trajectory on level ground in a physically feasible and biologically relevant way. Findings suggest that the role of locomotion control is to provide stability, rather than drive the limb onto a pre-calculated trajectory.
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Chapter 1

Introduction

1.1 Animal locomotion at a glance

How legged animals move may seem to be a straightforward question, but it emerges from a complex interaction between the animal’s nervous system and its passive body dynamics [1, 2]. A full neuro-mechanical model of animal locomotion is an integration of the neuromuscular controls and the passive dynamics [3]. Central pattern generators can actively generate muscular activities and produce movements in the absence of feedbacks [4]. Feedback controls and reflexes are responsible for intra-limb coordination as a response to changes in the musculoskeletal states and the environment. Passive dynamics are motions due to the force of gravity, elasticity, inertia and interaction with the environment [1]. The passive dynamics of an animal’s body can change the musculoskeletal states of an animal, independent of the nervous system.

A musculoskeletal system consists of bones, muscles, tendons and ligaments comprise the structural framework of an animal. The nervous system consists of nerves and sensory organs that receive, transmit and process signals from the environment, and generate responses. These two systems are tightly coupled, and both contribute to animal locomotion. Motor neurons connect the central nervous system to the muscles, and are responsible for translating nervous responses into forces by activating muscles. At each joint, muscles join neighbouring bones together. Muscle activation results in muscle contraction, which causes the neighbouring limbs to rotate about the joint and a locomotion trajectory is formed.

1.2 Locomotion control mechanisms

The control processes involved in animal locomotion are summarized in Figure 1.1. The nervous system generates motor signals in two ways; through feedback or feed-forward. In the case of feedback control, the generation of force depends
on the state of the sensory organs and the environment [5]. In the case of feed-forward control, the generation of force is sensor-independent. The motor signals are translated into active muscle forces through molecular mechanisms that are not explicitly modelled in this research. The force of gravity and the inertial properties of the limbs also contribute to trajectory formation, and the interaction between them allows passive dynamic motions. The modern view is that neural locomotor control is about stabilizing and guiding the natural passive trajectories of the animal’s body, not about generating trajectories by applying forces.

Feed-forward controls are independent to the signals from sensory organs. Periodic body movements, such as leg swings and wing flapping, are often generated in a feed-forward manner [6, 7]. This force generation mechanism is commonly known as the central pattern generator (CPG) [1].

Passive dynamic motions refer to the limb movements that come from the interaction between the inertial properties of the limbs and the force of gravity [8, 2, 9], independent to active force generation and motor coordination. The passive dynamics of an animal’s body contribute to feed-forward control. Because the effects of passive dynamics can be sensed by the animal, it becomes possible for animals to use passive dynamics in the process of locomotion trajectory formation. Robots that move by using only passive mechanisms have been built in previous research and it is possible to generate locomotor patterns for legged walking entirely by passive mechanisms. While feed-forward controls can be achieved passively without any effort from the nervous system [10], the relative contribution between passive dynamics and the neural inputs requires investigation. This point will be addressed later in the chapter.

There are three types of feedback control mechanisms: guidance feedback, equilibrium feedback and rapid phasic feedback. In guidance feedback, animals use directional sensory organs (for example eyes and ears) to determine their orientation relative to the global environment, and to maintain the desired direction of movements [2, 9]. In equilibrium feedback, animals use specialized equilibrium organs such as inner ear, statocysts, halteres, and muscle and joint sensors to sense their current mechanical states, and to generate forces to prevent them from falling over [2]. Both guidance and equilibrium feedbacks use sensory organs and the central nervous system, so these feedbacks are fall under the category of sensory feedback [2]. In the case of rapid phasic feedback, animals sense their current mechanical state and the state of the surrounding environment using mechanosensors (for example, stress and pressure sensors) instead of directional sensory organs [11, 12, 13]. This process may involve the use of the brain, but a much quicker response can be generated via reflexive neural feedback. Reflexive neural feedback generates a force in response to environment
perturbations that change the state of the musculoskeletal system and the sensory organs [14, 15, 16, 17, 18, 19, 20]. This process does not involve brain function.

The feed-forward force generation can be coupled to feedback control mechanisms, so that the pattern of force generation can be modified. Research shows that some sensory neurons are also integral members of the pattern generating neural network. This coupling allows the feed-forward force generation to be driven by feedback mechanisms as the animal moves [21, 22]. The main difference, compared to a true feedback control, is that in this case the feedback mechanism modifies the state of the feed-forward control, and does not generate forces directly.

To sum up, the formation of the locomotion trajectory in animals involves both passive dynamics and active controls. The classical view of animal locomotion is that feed-forward and feedback controls dominate the trajectory formation. Inspired by passive walker models that walk stably without relying on controls, the modern view of animal locomotion is that passive dynamics dominate the trajectory formation, and the role of feedback and feed-forward controls is to ensure stability and balance during the course of motion.

1.3 Muscular force generation

Control signals must be translated into mechanical forces in order to affect limb motions. Muscles generate forces actively by converting chemical energy, and passively by storing and releasing elastic energy and dissipating energy in viscous drag. The chemically generated muscular forces come from muscle activations; a biochemical process that generates a contractile force. Readers are directed to the paper by Holmes et al. (2006) [1] for details on the chemical reactions that are involved in muscular force generation.

Because muscles only shorten on activation, they can only pull and cannot push. This means that muscles must be organized into synergistic groups at each joint, to enable the joint to move with all degrees of freedom. In the case of a single pair of muscles, one member is called an agonist or a flexor, and the other is called an antagonist or an extensor, which flex or extend a joint respectively. Contracting the agonist draws the limb together, but contracting the antagonist pulls the limb from the other side of the joint, generating a movement that is equivalent to a push. With a pair of muscles working as an agonist-antagonist couple, a two-sided force generation around a joint is achieved. (See Figure 1.2 for clarification.)
Muscles can do positive and negative mechanical work [23]. Forces generated by the muscles may either increase or decrease the total mechanical energy of the system. Regardless of whether a muscular force is doing positive or negative work, chemical energy is consumed in both cases. For example, muscles can generate resistive forces to stop limb motion and consume energy with this effort, but the total work done by the resistive forces is negative.

The chemical energy stored in muscles can be transformed into kinetic energy, but the kinetic energy of the limbs cannot be transformed back into chemical energy. However, because muscles are elastic, a proportion of the kinetic energy can be stored as elastic potential energy, which can then be transformed back into kinetic energy. Therefore, strictly speaking, we cannot simply model muscles as springs. However, for this research, we are demonstrating the principle of passive dynamic walking, so instead of building a realistic model of an animal, as in previous research we are treating the muscles as springs.

1.4 The mechanics of a walking gait

In the context of terrestrial locomotion, ultimately the objective is to produce a stable and efficient gait cycle. In this section, we discuss the mechanical aspects of a gait cycle.

In a bipedal walking gait, one leg pivots freely on the ground, while the other leg is free to swing about the hip. The leg motions are driven by the force of gravity, inertia and elasticity, as well as active forces. The pivoting leg is a stance leg, the swinging leg is a swing leg. When the swing foot reaches the ground, the gait cycle comes to an end. The ground brings the swing foot to a stop, and in most situations an inelastic collision takes place at this instant.

A new step starts after the swing foot strikes. The stance leg and the swing leg in the previous step swap roles and this action is called “support-transfer”. During a bipedal gait cycle, the stance leg carries the swing leg and the upper-body components, and rotates about the foot-ground contact. The entire system vaults upward in a way that is analogous to an inverted pendulum swing [2, 24, 25, 26]. Over the first half of the gait, the kinetic energy is transformed into gravitational potential energy. In the second half of the gait, the gravitational potential energy is transformed into kinetic energy again. The exchange between kinetic energy and the gravitational potential energy accounts for about 70% of all the energy exchanged. The remaining 30% of the energy exchanged is between the kinetic and elastic potential energies, and the one-way energy transfer from chemical energy to kinetic energy [24].
1.5. THE SIGNIFICANCE OF PASSIVE DYNAMICS IN WALKING GAITS

In a biped running gait, the stance leg does not always stay on the ground. There is a flight phase, during which both legs are in air and the support-transfer process does not simply involve an inelastic collision and a foot role-swapping. Instead, the legs behave like springy pogo sticks. After the swing foot strikes [27], an inelastic collision will still take place, but a proportion of the kinetic energy will be stored as elastic potential energy. The stored elastic potential energy can be released as kinetic energy when the new step starts. This reduces the amount of energy loss during the support transfer, and thus a faster motion can be achieved without increasing power consumption [27].

In the case of quadruped and poly-pod, the concepts of stance and swing legs still apply. In poly-pod walking gaits, a subset of legs function as stance legs whereas the rest function as swing legs. When several legs function as the stance or the swing legs, locomotion control involves controlling the phase and the relative timing of leg swings. Poly-pods also have running gaits and there is also a period of time in the running gait during which all legs are not in contact with the ground [1]. Quadruped galloping is a good example.

As a generalization, Cavanaugh [28] proposed that all animal motions can be modelled as motions of spring-coupled inverted pendulums. In a running gait, the legs bounce like pogo sticks as described previously. A walking gait can be considered as a special case of the running gait, in which the flight phase has an insignificant duration, and negligible kinetic energy is transformed into and stored as elastic potential energy at foot fall. Poly-pods can be considered as a set of coupled bipeds. Therefore, early mechanical models of animal locomotion are mainly in the form of spring-mass system, and the spring represents the legs. A spring loaded inverted pendulum (SLIP) model is used to model rapid motions such as running, in which the body parts of an animal stay close to the sagittal plane during the course of motion [28, 29]. A lateral leg spring (LLS) model is used to model the crawling of multi-legged arthropods (slow motion), in which body parts of an animal stay close to the horizontal plane during the course of motion [30, 31].

1.5 The significance of passive dynamics in walking gaits

Remarkably, walking gaits can be generated totally passively without controls. McGeer [10] demonstrated, using computer simulations and physical models, that a pair of legs in the form of a double pendulum, is capable of walking autonomously down a shallow slope. During each step, the mechanical system gains gravitational potential energy, but loses kinetic energy due to the inelastic
collision between the swing foot and the ground. When the gravitational potential energy gain equals the inelastic collision loss, a passive periodic gait cycle can emerge. Such passive, gravity-powered periodic gaits can be dynamically stable, such that when the system is perturbed, within certain limits the passive gait cycle can be restored without controls. The double-pendulum walker proposed by McGeer is known as the compass-gait walker and is the simplest known passive walker. McGeer further demonstrated the passive dynamic locomotion in a passive walker with knees [32], and a biped runner [27].

As we have discussed in Section 1.2, the neuromuscular system also plays a role in locomotion. People may have seen, or have experienced how robotic arms in an assembly line can be controlled to perform the required tasks. A computer system or an operator drives the robotic arms to do activities very accurately, so by the same analogy, people might intuit that animals too rely on the brain to calculate a locomotion trajectory for the limbs, and then use muscles to drive the limbs onto that trajectory. In this way, the muscular control dominates and the brain takes full control. However, this intuition overlooks the contribution of the neuromuscular system to the generation of walking gaits. Below we justify why this is so.

As described previously, 70% of all energy exchanged in a biped gait cycle comes from the natural energy exchange between the kinetic energy and the gravitational energy, and this excludes the possibility that actuations from muscles dominate the gait formation. Moreover, research shows that the metabolic cost of leg swing agrees reasonably well with the mechanical energy required to swing a pendulum, indicating that the passive dynamics of animal’s body play an important role in a gait cycle.

Research also indicates that significant muscle activations occur near toe-off, and near foot strike, but the activation is low during mid-flight [23, 33]. It was believed that the muscle activation near toe-off is to compensate the loss in kinetic energy due to the inelastic collision during the support transfer, and the muscle activation near foot strike was believed to have a guiding role, in the way that the muscles are activated to generate forces to slow down the swing leg to minimize the collision loss, and to redirect the limbs to avoid obstacles [23, 33].

Furthermore, it is evident that the mechanical design of animals has evolved for efficiency, agility and stability in particular environments, while allowing manoeuvrability and mechanical stability [34]. The stream-lined body design of aquatic animals is a good example. A fish would do poorly as a terrestrial predator, even if it had lungs. Thus, in retrospect the observation that mechanical design is at least as important as neural control for agile legged locomotion, is trivial. If all
1.6. WHY IS LOCOMOTION CONTROL STILL NECESSARY?

that is needed is a stable locomotion trajectory and environmental constraints are irrelevant, then aquatic animals would not need to possess the stream-lined body design to minimize hydraulic drag.

1.6 Why is locomotion control still necessary?

One might argue, if a walking gait can be achieved without control, then why do animals use active neuromuscular control in locomotion? The quick answer is that passive dynamic walking is undirected and not very stable, even under ideal conditions. Generally, some work is needed to overcome energy dissipated by friction, drag and impulses. Work is also needed to maintain, direct and stabilize passive trajectories. However, such instability control can be achieved by making extensive use of passive dynamics without trajectory planning and continuous force application like robotic servo control.

McGeer’s compass-gait walker can walk passively and withstand perturbation without any controls. However, the dynamic stability of the passive gait is only possible on shallow slopes. With slopes steeper than 0.015 radians, passive periodic walking is still possible, but becomes dynamically unstable. This means that once perturbed, the perturbation error cannot be repaired without control, and the periodicity of the passive gait is destroyed [35, 36]. On slopes that allow passive periodic walking to be dynamically stable, only small perturbation errors can be repaired passively [35, 36]. Similar limitations apply to other passive bipeds. The 3D straight-leg passive biped proposed by Coleman (i.e., the “Tinkertoys walker”) [37] is a good example. Mathematical analysis shows that the Tinkertoys walker has a dynamically stable passive periodic gait on downhill slopes, but experiments using a physical model with the same mechanical parameters reveal that in reality, a small ground step can make it stumble.

The other limitation is that McGeer’s compass-gait walker cannot walk with finite walking speed on level ground [38]. As the slope angle approaches zero, the step size and the walking speed of the passive periodic gait both approach zero. On a downhill slope, passive periodic walking gait is achieved by using the gravitational potential energy gained during support-transfer to compensate for the loss of energy due to the inelastic collision in the process of support-transfer. On level ground, the biped cannot gain gravitational potential energy during support-transfer, so passive periodic walking requires the swing foot to reach the ground with zero velocity, to avoid the collision loss. However, it is possible to design a biped that can walk passively on level ground. This will be the subject of Chapter 4.
Chaterjee [38] and Garcia [39] conjectured that by adding a torso to a compass-gait walker, it may be possible to construct an “ideal walker” that, by avoiding footfall collisions, uses no energy at all to walk on level ground. Gomes and Ruina succeeded in modelling such a device [40] by adding a torso to McGeer’s compass-gait walker, and using springs to bring the swing foot to a halt the moment it reaches the ground. This mechanism shows that, remarkably, horizontal legged locomotion can be accomplished with no actuation, no control, and no energy consumption.

The ideal walker proposed by Gomes and Ruina uses no energy, but simple energetic considerations make it clear that the gait cannot be dynamically stable. A slight deviation from the zero-energy trajectory will lead to impulsive collisions between the feet and the ground. The energy dissipated in these collisions cannot be recovered on level-ground, and therefore some source of external work is required to maintain the periodic walking.

Because of the inherent instability of passive dynamic walking, external controls are needed. Spong and Bullo [41] showed that a stable, gravity-powered periodic gait on a downhill slope can be reproduced exactly on any slopes and level ground by using controls. The control forces “simulate” the force of gravity experienced by the passive biped on the downhill slope. In this way, the exchange between the kinetic energy and the gravitational energy during the entire journey happens in the exact way as on the original slope. Spong and Bullo’s method resolves the issues of the steeper slope instability and passive walker’s inability to walk on level ground. This means that if a biped can walk stably and passively on a downhill slope, we can find the control force required for walking stably on level ground.

McGeer [42, 43] and Kuo [44, 45], and many other researchers (for example, [46]) showed that on the level ground, the missing gravitational energy gain required for stable periodic walking can be compensated by applying an impulsive force at the start of each step and then allowing the limbs to move passively. Where and how the impulsive force is applied does not matter. These examples demonstrated that while controls are necessary in practice for “passive” walking gaits, control forces can be applied to maintain, rather than override, passive trajectories. Gomes and Ruina’s zero-cost biped [40] can walk on level ground, but for energetic reasons it cannot be dynamically stable. Therefore, instead of applying control forces at the start of every step, we apply control forces only when a step is perturbed, to push the passive walker back to the zero-energy trajectory for stability. In this way, control forces ensure stability. Chapters 8 and 9 investigate how this can be done efficiently using impulses or force pulses.
1.7 Open questions in passive dynamic locomotion research

1.7.1 Overview

Passive dynamic mechanisms of legged locomotion has a long history in scientific literature and applications, but important questions remain. These include the role of the upper-body, passive walking with multiple legs, and the existence of passive but unstable gait cycles. Below, we discuss each of them in detail.

1.7.2 The role of the upper-body

Since McGeer proposed the compass-gait walker, there has been little progress on passive bipedal locomotion in the presence of an upper-body. Dynamically stable and periodic bipedal walking with a torso was achieved, but only by using state feedback control [47, 42, 48, 49], or kinematic constraints [50]. Gomes and Ruina showed that by adding a torso to a compass-gait walker, it can walk passively on level ground, but the gait is not stable. Furthermore, the gait involves exaggerated joint movements that seem unnatural. Whether this level-ground capable biped can walk stably down a slope using gravity, and if so, how its stability compares with that of a compass-gait walker, has not been reported. This raises the question of whether the upper-body is part of the locomotor mechanism, or just a payload.

1.7.3 Passive dynamic walking with multiple legs

McGeer showed that biped walkers can be designed to allow the existence of a dynamically stable, passive, gravity-powered gait. However, the analysis has not been extended to poly-pod walkers.

Passive quadruped toys that can walk passively downhill are available commercially (for example McMahon’s design [51], see [43] for what it looks like). These quadrupeds are essentially mechanically coupled bipeds in the form of two connected straight-leg biped subsystems.

Of course, we are not talking about replicating these quadruped toys computationally; we are interested in “why” they walk. In particular, is it possible to build a passive quadruped walker, such that with some idealizations, it can walk stably, passively and periodically, like the compass-gait walker, with an additional pair of legs? If this is possible, we can understand how people succeeded in designing quadruped toys that walk without controls.
1.7.4 Re-stabilizing passive but unstable gait cycles

Inspired by McGeer’s insights, for a long period of time, research on passive dynamic locomotion focused on dynamically stable periodic gaits that make good use of the principle of passive dynamic walking. Gomes and Ruina’s biped shows that unstable passive periodic gaits can exist, and be exploited for efficient locomotion using minimal control. This raises the question of whether unstable periodic gaits may exist in other circumstances and could be used in a similar way.

There are situations where passivity and dynamic stability cannot simultaneously be achieved at all. As explained previously, passive walking gaits cannot be dynamically stable on level ground, but it is possible to generate efficient walking gait on level ground, by using minimal control to stabilize an unstable passive trajectory. Animals do not generally have the luxury of walking continuously down an inclined plane, and so this may be a useful strategy for efficient locomotor control.

From an engineering stance, sometimes the dynamic stability of passive walking and the design requirements set by the client are incompatible. The 3D biped proposed by Michael Coleman has a dynamically stable period gait on downhill slopes, but has a rather unusual and space-inefficient leg mass distribution [37]. A similar 3D passive biped that has a more biologically-inspired and space-efficient leg mass distribution was studied by McGeer, and this research found only dynamically unstable passive periodic gaits [52].

1.8 Thesis outline

1.8.1 Overview

The goal of this thesis is to address the issues discussed in the previous section. The first half of the thesis (Chapters 3-7) is about full-body passive dynamic walking models. In these chapters, we extended McGeer’s leg-only passive walking models to account for the effect of adding a torso and arms. We also examine the passive walking models with multiple legs. For each passive walking model, we calculate the gait trajectory and find the design parameters that allow the passive gait to be dynamically stable. The aim is to model the efficiency and agility of animal locomotion, and to understand the principles that may be applied to the design of agile robots.

The second half of the thesis (Chapters 8-10) looks at how the instability of passive dynamic walking can be handled by biologically-inspired control mecha-
anisms. In particular, we are interested in stabilizing the unstable collision-free periodic gait on level ground. In reality, a walking gait is most likely to be perturbed by environmental obstacles. A perturbation is picked up by sensory mechanisms and causes a reflex response, which in turn results in a rapid adjustment to the state of the neuromuscular system, causing rapid change in the patterns of force generation [9]. A simple way to model such biologically-inspired control is by parameter triggering. The detection of a perturbation leads to changes in the adjustable parameters of control [53, 54]. In our case, we use a mathematical function of time, with coefficients parameterized by the landscape and mechanical states of the system to model the control.

1.8.2 Chapters 2 and 3

In Chapter 2, we provide a detailed description about the mathematical techniques used for modelling passive dynamic motions, and we use the 2D rimless wheel to illustrate the methodologies. In Chapter 3, we review the leg-only passive walkers proposed by McGeer, and explain in terms of energy conservation, why these passive walkers cannot walk periodically and passively on level ground with a finite walking speed.

1.8.3 Chapter 4

This chapter examines a straight-leg biped with a torso. Gomes and Ruina showed that with the right model parameters, this biped can be an “ideal walker” that walks passively on level ground with a finite walking speed using no energy. This is like a rolling wheel, a familiar example of a device that can move on level ground without consuming energy. However, a wheel is not stable on downhill slopes. McGeer’s torso-less bipeds are non-ideal, but with the correct body design they can walk stably on downhill slopes. Whether we can build an ideal walker that walks stably and passively on downhill slopes, and walks passively with finite walking speed on level ground so that it behaves like a wheel as well as a legged walker, is still an open question.

We restricted the ideal walker design such that it must have a torso, with centre of mass above the hip during the gait because we are also studying the contribution of the torso to passive-legged locomotion.

1.8.4 Chapter 5

In Chapter 5, we increase the complexity of the passive walker by introducing arms and knees. Although Gomes and Ruina had succeeded in achieving zero-cost level-ground walking, their solution involves an exaggerated torso swing and leg extension which is unnatural. We show that with arms added to the torso,
zero-cost level-ground walking can be achieved with natural-looking arm and leg movements. The torso and arms together do not have to swing as wildly as a torso alone.

We also study the passive gaits of the armed walker on downhill slopes. We developed a 2D passive walking model based on a human-like template that accounts for the major masses of a human (thighs, shanks, torso and arms), and showed that this complex model has a stable and passive walking gait. This result gives constructive evidence passive dynamics may be important in human.

1.8.5 Chapter 6

In Chapter 6, we look at passive walking gaits of quadrupeds. In this chapter, quadrupeds are modelled as a pair of passive bipeds, linked together by springs. When the passive bipeds at the front and the back are identical, the leg motions can be kept in phase passively by coupling the legs using damped springs. This is because when the springs are in their rest lengths, each biped takes care if its own passive gait, and the biped subsystems have identical passive gaits. Perturbing one of the biped subsystems will, in general, add energy into the spring, and transfer energy to the other biped subsystem. We show that as the energy stored in the springs dissipates, the two biped subsystems move back into the correct phase relationship. The stability of the gait can be optimized by adjusting the stiffness and damping of the coupling springs.

If the bipeds are not identical, they have different passive gaits. In this case, the passive phase control needs is accomplished by using the coupling spring to alter the leg dynamics, so that the two bipeds walk with the same walking speed. This phase of the leg movements will then lock at a constant value, which is usually non-zero. Again, the stability of the gait can be optimized by adjusting the stiffness and the damping of the coupling springs.

1.8.6 Chapter 7

In Chapter 7, we modify the “Tinkertoy walker” proposed by Michael Coleman [37] to add an upper-body. Coleman’s original design has a dynamically stable gait trajectory on downhill slopes, but the leg mass distribution is very unnatural, and it doesn’t have an upper-body. Our version has an upper-body and a more space-efficient leg mass distribution compared to Coleman’s design, but it still doesn’t fully resolve the problem of unnatural leg mass distribution.
1.8.7 Chapter 8

In Chapter 8, we approach the problem of designing a biologically-inspired control mechanism that stabilizes a dynamically unstable collision-free periodic gait on level ground. As stated, a perturbation results in a fast and brief reflexive response, which in turn results in a rapid adjustment in the state of the neuromuscular system, so we assume that the control forces are approximately impulsive, and we model them mathematically using Dirac Delta function. We then work out the ideal impulse application times to minimize the total unsigned mechanical work done by the control impulses.

We consider two scenarios. In the first scenario, control impulses are generated reactively to correct for the perturbations after they occur. In the second, control impulses are generated pre-emptively to minimize or prevent perturbations. We show that the predictive/pre-emptive control mechanism is more energetically efficient than the reactive control mechanism.

1.8.8 Chapter 9

In Chapter 9, we repeat the same study as in Chapter 8, but with a more realistic model of control force generation. We assume that a perturbation from the environment results in a reflexive response, which in turn changes the level of muscle activation and causes a muscle contraction. Mathematically, the contractile force is modelled as the effect of the adjustments of spring stiffness. Spring stiffness deviates from the base value by a finite amount over a short but non-infinitesimal period of time. As in Chapter 8, we then work out the impulse application times that minimize the control costs. For practical reasons, the control cost is calculated based on the amount of change in stiffness required for the stabilization, rather than the total unsigned mechanical work done by the control forces generated as a result of spring stiffness adjustments.

We also consider the scenarios discussed in Chapter 8; the reactive and the predictive control mechanisms with basically the same conclusion: the cost of recovery can be significantly reduced if the deviation of the ground height can be predicted in advance and the model pre-adjusted to account for this.

1.8.9 Chapter 10

In order for a passive periodic gait to occur on level ground, the swing foot must reach the ground with zero velocity. However, the acceleration of the swing foot is not subjected to any constraint. Therefore, when the swing foot reaches the ground with zero velocity, generally its next move will be swinging upwards due to the positive swing foot acceleration along the vertical axis of the global
CHAPTER 1. INTRODUCTION

reference frame. This implies that an infinitesimal downward ground step will result in a non-infinitesimal delay in foot strike time, and hence a non-infinitesimal deviation from the collision-free trajectory and recovery cost.

This explains why, if the ground height deviation is downward and a reactive control strategy is used (i.e., the ground step remains unknown until it is encountered), the cost of recovery is much larger than in the case of predictive control, and remains non-infinitesimal when the size of the ground height deviation goes to zero. When using the non-impulsive muscular control (Chapter 9), we found no control solutions that can handle an infinitesimal downward ground step. These results suggest that without prediction, the ideal passive walker cannot be stabilized efficiently.

We show in this chapter that it is possible to adjust the spring stiffness, so that the biped has a collision-free periodic gait on level ground, and in this gait the swing foot reaches the ground with zero velocity, as well as zero acceleration along the world vertical. In this way, regardless whether the ground height deviation is upward or downward, infinitesimal ground height deviation implies infinitesimal delay in foot strike time.

When the cost of recovery calculations are repeated with this tuned walker, small ground height deviation implies small cost of recovery, regardless of the control strategy, the force generation mechanism (impulses or non-impulsive muscular forces), and whether the ground step is upward or downward.
Figure 1.1: A diagram summarizing the control mechanisms involved during locomotion. Feed-forward force generation process is indicated by the blue arrows, which include passive dynamic motions. The sensory feedbacks and reflex controls are indicated by the green and red arrows respectively.
Figure 1.2: A diagram showing how two-sided muscle force generation can be achieved under the limitation that muscle can only contract (pull). Muscles wrap around and cover both sides of a joint. On one side, the muscles function as a flexor, and on the other side the muscles function as an extensor. With this arrangement, a two-sided force generation is possible, and the direction of joint movement depends on if the flexor or extensor is contracting.
Chapter 2

Modelling legged locomotion

2.1 Commonly used modelling approaches

2.1.1 Overview

In bipedal walking, one leg pivots freely on the ground, and the other leg pivots freely about the hip-joint. The leg that pivots freely on the ground is called a “stance leg”, and the leg that swings freely about the hip is called a “swing leg”. A step begins when a foot leaves the ground and ends when it touches the ground again. At the end of the step, the stance leg and the swing leg swap roles.

Due to the presence of bones, the legs, as well as other body parts, have fixed lengths and do not deform easily. So in most research, each body part is modelled as a rigid-body, joined to other body parts by revolute joints. Muscles and tendons are modelled as damped (linear or nonlinear) springs. This follows the standard conventions in neuromuscular modelling.

The dynamics of the walking gait described above can be computationally modelled in two different ways: the discrete momentum-impulse gait model and the continuous contact-force model. Below, we describe both methods and their underlying assumptions in detail.

2.1.2 The discrete momentum-impulse gait model

In the discrete momentum-impulse gait model, a biped has a stance leg and a swing leg. During a step, a “sticky-foot constraint” is enforced so that the stance leg pivots freely on the ground. The swing leg swings freely about the hip. At foot fall, the sticky foot constraint is removed from the stance leg and imposed on the swing leg. The swing foot becomes the new stance foot, and the stance leg simultaneously loses ground contact and becomes the new wing leg. This process is known as “support-transfer”. Because the swing foot is brought to a
sudden stop, the support-transfer is therefore an inelastic collision, during the support-transfer, the conservation of momentum is still satisfied, but there is a loss in kinetic energy.

Computationally, the motion is modelled as a system of ordinary differential equations (ODE), and solved numerically until the swing foot touches the ground. Then we compute the initial state for the new step by applying the law of momentum conservation. This calculation is known as the “support-transfer transformation” of the pre-impact mechanical state. We then restart the numerical differential equation solving from the post-impact mechanical state (i.e., the mechanical state of the animal after the support-transfer). To simulate a multi-step journey, we iterate the process several times, and this is why we call this method “discrete”.

2.1.3 The continuous contact-force gait model

In the continuous contact-force gait model, the biped is still treated as a rigid-body linkage; however during a step the sticky foot constraint is not enforced. Instead, we calculate the contact force between the stance leg and the ground, and when the swing foot strikes, we calculate the contact force between the swing foot and the ground. A foot will pivot freely on the ground if the contact force is positive in the direction of the upward normal off the ground, otherwise the foot will be released.

Furthermore, the contact force is not assumed to be an impulsive force that suddenly halts the swing foot. Instead, the contact force is assumed to be a one-sided elastic force between the stance leg and the ground. This confirms to physical intuition that ground contact may not be perfectly rigid and solid deformation could occur at the contact point and the stance foot might slip. While the solid deformation that occurs at the contact point is difficult to model, it is approximated as an elastic tension from a damped non-linear spring that joins the stance leg and the ground. The deformation of the spring is assumed to be one-sided, so it does not extend above the ground height.

The continuous contact force gait model has no discontinuity introduced, so that a trajectory that takes several steps can be modelled as a single smooth trajectory. This computational method, although more realistic than the discrete momentum-impulse model, has the major disadvantage that even with simplified assumptions, the resulting equation of motion can be stiff, causing the numerical ODE solver to fail. Mathematical programming packages such as MATLAB often call this “stiff system error”, and with this error, it becomes technically difficult to obtain meaningful results. To model the near rigidity of the ground, the
2.1. COMMONLY USED MODELING APPROACHES

2.1.4 Side notes on the sticky-foot constraint

When a biped vaults over the contact, it is under the influence of the force of gravity and the centripetal forces. The force of gravity acts through the stance leg and pushes it against the ground, but the centripetal force is act to lift the leg. At a slow walking speed, the centripetal force exerted on the stance leg is smaller than the force of gravity in magnitude, so the stance leg can be pushed against the ground. In this case, as long as the ground is not slippery, the sticky-foot constraint exists naturally and requires no active controls.

When walking fast, the centripetal force exerted on the stance leg can be greater in magnitude than the force of gravity. The net force will be in the direction of pulling the stance leg off the ground. Moreover, when the ground is slippery, the stance foot can slip. In both cases, the sticky foot constraint will only exist if the stance foot can grasp the ground. Biewener [55] pointed out that surface grasping can be accomplished by animals in two ways. One way is to lock onto the surface, as observed in animals with claws that can penetrate into the ground [55]. The other mechanism is to grasp with adhesive or suction forces between the foot and the surface, as observed in animals with specialized foot pads [55, 56]. The limbs of apes with specialized foot pads are very capable of grasping [57, 58] and reptile and amphibian feet use capillary adhesion to adhere to steep surfaces [55].

Animals with clawless feet, or feet with a limited grasping capability, are still capable of maintaining a positive contact force when walking [28, 59, 2]. Force platform analysis shows that in human walking, the force exerted by the stance foot on the ground increases in magnitude when walking at high speed [60, 59]. Therefore, when a biped has to walk fast, even if the feet are not well-adapted to surface grasping, the foot-ground contact can still be maintained by doing some muscle work.

The sticky-foot constraint mentioned previously is similar to the “workless grasping assumption” considered by Gomes [61]. When it is not possible to passively pivot the stance leg, actuation is needed to enforce the sticky-foot constraint during a step. Gomes [61] argues that if a limb is well-adapted to grasping, active
surface grasping can be achieved by consuming only a small amount of energy. In this way, when the swing foot touches the ground, the ground-grasping can be modelled as a passive inelastic collision, and the contact between the stance leg and the ground can be enforced over the step without taking the energetic cost of grasping into account.

Gomes [61] also argues that enforcing a limb-surface contact does not defeat the principle of passive dynamics. Suddenly imposing a surface contact constraint can change the total energy, but enforcing a surface contact during a step is workless, because the contact force that allows the stance to pivot freely on the ground is a constraint force, which does no mechanical work. So regardless of whether the sticky-foot constraint requires controls, the dynamics of the mechanical system remains energetically passive. Furthermore, the contact force holds the stance leg but does not drive the mechanical system, regardless of whether or not it comes from active grasping.

In computer simulations, the sticky-foot constraint can be implemented by treating the leg-ground contact as a hinge joint. When the sticky foot constraint is used, it is assumed that the leg has a specialized foot pad that is capable of surface-grasping. How a foot pad that is capable of surface-grasping can be designed beyond the scope of this research.

In the context of realizing a totally passive biped in a mechanical workshop, the implementation of active surface grasping is difficult. In this case, we need to make sure that the use of a sticky-foot constraint is realistic. We may eliminate gait trajectories that involve the unrealistic use of sticky-foot constraint.

### 2.1.5 Our choice

For technical simplicity, we are considering only the discrete momentum-impulse gait model with a sticky-foot constraint imposed on the stance foot. In this model, the two major parts involved are the equation of motion, which is a system of differential equations that describe the motions of the limbs, and the support-transfer transformation that involves the collision impulse calculation. For both parts, the calculations apply Lagrangian mechanics.

### 2.2 Assumptions

In this research, we are replicating the passive dynamic walking models described in Chapter 1, and extending them to account for the effects of an added torso and arms, or to solve locomotion control problems which have not previously been addressed in the literature. The focus is on simple mathematical models to
investigate the principles of passive dynamic locomotion, but the ultimate goal is the detailed engineering of realistic robots. Therefore we adopt the simplifying assumptions used in previous research on passive dynamic locomotion. These assumptions are used solely for technical simplicity in some chapters, but are removed in other chapters to provide more realism to the models.

1. **Foot scuffing:** We ignore foot scuffing. In straight-leg walkers, due to the geometrical properties of the design, there will be times during the middle of a step when the swing foot goes slightly below the ground. This “foot scuffing” is normally ignored in research on passive dynamics, with the understanding that there are modifications to the model that avoid foot scuffing with a negligible consumption of energy, but leave the dynamics essentially unchanged. (for example for example [10, 35, 39, 47, 48]). This idealization will be relaxed in Chapter 5 and Chapter 7.

2. **Perfect 2D limb trajectory in 3D environment:** We assume that the constraint force that keeps the biped gait in the 2D sagittal plane has a negligible consumption of energy. This is a common assumption [36, 54, 35, 39, 47, 40, 48, 49]. In Chapter 7, this simplifying assumption will be relaxed, as we investigate 3D bipeds with motion in both the lateral and the sagittal planes.

3. **Perfect-plastic foot collision and stance leg pivoting:** We assume that the foot strike is a perfectly plastic collision. At foot strike, the swing foot grasps firmly onto the ground, and after foot strike, the swing leg pivots freely about the ground contact and becomes the stance leg of the new step. This simplified assumption is widely used in previous research (for example [36, 54, 35, 39, 47, 40, 48, 49]).

4. **Infinitesimal support-transfer period:** Continuing from the previous assumption, once the swing foot strikes the ground, we assume that the stance foot is released from ground contact simultaneously, so that the support-transfer has an infinitesimal duration. This essentially assumes that the walker is a rigid-body linkage, such that the footfall impulses are instantaneously transmitted through the structure. This assumption, like the sticky-foot constraint, is widely adopted in previous research (for example [36, 54, 35, 39, 47, 40, 48, 49]).

5. **McGeer’s knee model:** For bipeds with knees, we use McGeer’s knee model to model the dynamics of the knee. This is a set of simplifying assumptions based on the behaviour of the kneed biped designed by McGeer [32]. Briefly, the stance leg knee is locked until the end of the step and after the support-transfer. The swing leg knee is free until the swing leg straightens, and then the swing leg knee becomes locked during the rest of
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the step and the support-transfer. Then, the knee-locked swing leg becomes the knee-locked stance leg in the new step. It is also assumed that knee locking happens before foot strike. More details on McGeer's knee model will be discussed in Chapter 5. The McGeer's knee model had been considered in research on the dynamics and control of kneed bipeds (for example [32, 35, 39, 38, 62]).

6. Unrealistic controller installation: When investigating stabilization controls, we start with the assumption that control forces are defined in an external reference frame. In this way, the controls can be applied to each limb segment independently. This simplification has been considered in some locomotion research (For example, [63]) to be unrealistic. In reality the controller will need to be installed between two limbs, and can only apply equal forces on neighbouring limbs but in opposite directions. In this way, the limbs cannot be controlled independently, and this renders the problem of control harder. This assumption will be used in Chapters 4 and 5, but will be relaxed in Chapters 8, 9 and 10 to achieve more realistic models.

7. The use of impulsive control force: Impulsive control forces have been considered in many previous studies on controlled passive locomotion (for example [42, 43, 64, 65]). If we can assume that the control forces are pulse-like, the effects are often modelled as the effects of applying impulsive forces, and the cost of control is often defined as the total unsigned mechanical work done by the impulses [42, 43].

8. Control cost as mechanical work done by impulse: From the previous assumption, in general, the energy consumed by the controller during the process of control does not equal to the unsigned mechanical work done by the control force generated. Ideally, we should use the energy consumed by the controller as the control cost. However, it is reasonable to assume that controllers are designed to have a positive correlation between the energy consumed by the controller and the unsigned mechanical work done, in order to guarantee that small mechanical work implies small power consumption. With this assumption, the unsigned mechanical work done by the controller is a good indication of the cost of the control, although usually it is not the actual power consumed by the controller.

Continuing from Assumption 7, the true impulsive force is not physically feasible so in reality the velocity adjustments required for the re-stabilization will need to be achieved by applying non-impulsive control forces over finite periods of time. Exactly how to generate the required non-impulsive control forces, whether the force generations are realistic, and if so, how much power will be consumed, are questions taken into consideration when specific controllers are designed, and
2.3. EQUATION OF MOTION

this is beyond the scope of this thesis. Impulsive control forces are considered in Chapters 4, 5 and 8. However, in Chapter 9 we attempted to address the efficacy and efficiency of non-impulsive control strategies for completeness.

To calculate the amount of power that will be required for doing a certain amount of mechanical work, the model will require a description the engineering of the controller. This is beyond the scope of this research. To calculate the mechanical work done by the control force, all that is needed is the force-time curve and the controlled trajectory, and the engineering of the controller is unnecessary. So basing the cost calculation on the unsigned mechanical work done by a control force simplifies the cost calculation significantly. The mechanical work done by a control force places a lower bound on the control cost, since the work done by a controller cannot exceed the energy it uses. In this way, we can still determine the expense of control force generation. In Chapters 4, 5 and 8, the cost of the control is taken as the total unsigned mechanical work done by the control forces. However, in Chapter 9 we made an initial attempt on addressing the question of how to measure the control cost in a realistic way.

2.3 Equation of motion

2.3.1 Deriving the equation of motion using Lagrangian mechanics

In this section, we describe in detail how Lagrangian mechanics can be applied to derive the equation of motion. We use the simplest walking device, a rimless wheel (Figure 9.1), to illustrate how the equation of motion and the various other components of a walking gait model can be derived by applying Lagrangian mechanics. A rimless wheel has a very simple mechanical structure, so all of the dynamic quantities we are interested in can be easily derived in closed form.

As stated, an animal can be modelled mechanically as a system of linked rigid-bodies, and the motion of each limb can be described by an Euler-Lagrange equation. During the course of motion, without the effects of damping and drag, the total energy of the system is conserved. Therefore, the mechanical state of each limb must comply with the law of energy conservation. As time goes by, the state of the system must follow a contour curve on the energy surface. The energy surface is a surface defined by a function that gives the total energy of the system, based on the system’s state variables, known as the “Hamiltonian” function. The contour curve that the system follows is a solution to the Euler-Lagrange equation which is a set of second-order non-linear ODEs also known as an “equation of motion”. Below, we describe how an equation of motion can be derived by applying Lagrangian mechanics.
CHAPTER 2. MODELLING LEGGED LOCOMOTION

Figure 2.1: A schematic diagram showing the design parameters and the configuration variables of a rimless wheel.

In the case of a passive bipedal walker made of rigid-body linkages, the configuration variables $q$ include the orientation angles of the limbs relative to the vertical axis of the reference frame of the global environment $(\theta_1, \theta_2, \cdots, \theta_n)$, and the position of the stance foot $(x, y)$ relative to the origin of the reference frame of the global environment. To make the subsequent discussions simpler and clearer, it is helpful to define $\theta$ as $(\theta_1, \theta_2, \cdots, \theta_n)$ where $\theta_i$ is the orientation angle of the $i$th limb, that $x = (x, y)$, and that $q = (x, \theta)$.

Let $T$ be the total kinetic energy, and $V$ be the total gravitational potential energy of the system, and the Lagrangian expression of the system $L = T - V$ is

$$L(q, \dot{q}) = \sum_{i=1}^{n} \frac{1}{2} m_i c_i^T \dot{c}_i + \frac{1}{2} I_{cm} \dot{\theta}_i^2 - m_i c_i^T g,$$  \hspace{1cm} (2.1)

where $c_i$ is the coordinate of the centre of mass of the $i$th rigid-body link, $I_{cm}$ is the rotation inertia of the $i$th rigid-body link with respect to the centre of mass about an appropriate axis, and $m_i$ is the mass of the $i$th rigid-body link.

In the special case of a serial rigid-body linkage with the limbs’ motions constrained to the sagittal plane (for example, as in the compass-gait walker proposed by McGeer), each link has only one child. Then, the parent of the $i$th link is the $(i - 1)$ link. The centre of mass of the $i$th link, $c_i$ is given by
2.3. EQUATION OF MOTION

\[ c_i = L_i^{COM} \left( \frac{\sin(\theta_i)}{\cos(\theta_i)} \right) + (L_{i-1} - L_{i-1}^{COM}) \left( \frac{\sin(\theta_{i-1})}{\cos(\theta_{i-1})} \right) + c_{i-1}, \quad (2.2) \]

\[ c_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \]

where \( L^{COM}_j \) is the distance between the centre of mass of the \( j \)th link and the joint that joins the \( j \)th link and its parent, \( L_j \) is the length of the \( j \)th link, and \( (x_f, y_f)^T \) is the location of the stance foot relative to the origin of the global reference frame.

Real-world bipeds are often do not take the form of serial rigid-body linkages. For example, a human is made of a pair of leg and a torso, so the stance leg has two child links, which are the torso and the swing leg. Therefore, a branched rigid-body linkage is a better model of a biped, and in a branched rigid-body linkage the parent can have more than one child link. Assuming that the motions of the limbs are constrained to the sagittal plane, the centre of masses of the links can be calculated using the recurrent relation

\[ c_i = L_i^{COM} \left( \frac{\sin(\theta_i)}{\cos(\theta_i)} \right) + (L_{\text{parent}(i)} - L_{\text{parent}(i)}^{COM}) \left( \frac{\sin(\theta_{\text{parent}(i)})}{\cos(\theta_{\text{parent}(i)})} \right) + c_{\text{parent}(i)}, \]

\[ c_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (2.3) \]

where the subscript \( i - 1 \) is replaced by \( \text{parent}(i) \), that says: “Find the index of the parental link of \( i \)th link”.

Let \( U \) be the total elastic potential energy in the system. Ignoring the effects of spring damping, the dynamics of each body part over time are given by the Euler-Lagrangian equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i}, \quad (2.4) \]

This gives a system of non-linear, second-order ordinary differential equations that describe the dynamics of the entire mechanical system over time, and this set of ordinary differential equations is called the equation of motion. The effect of spring damping and drag can be derived separately and added to the differential equations. The effect of controls and constraint forces (see later) can be treated in the same way.
After considering the effects of spring damping, the equation of motion can be written in terms of matrices and vectors as

\[ M(q) \ddot{q} + C(q, \dot{q}) \dot{q} - G(q) = F_{spring}(\theta, \dot{\theta}), \]  

(2.5)

\( M \) is the mass matrix, \( C \) is the centrifugal matrix, and the vector \( G \) is the conservative force, which in this case describes the effects of the force of gravity.

By assumption, the stance foot pivots freely on the ground, and this constraint can be introduced in by adding a constraint force term to the right-hand side of the equation of motion. The constraint force is a force that acts in the direction perpendicular to the constraint surface that forces the mechanical system to comply a particular kinematic constraint. A constraint surface is a surface defined by the constraint function \( f_c \), which describes how the state variables of the system are constrained over time.

In general, when the configuration variables \( q \) are under the constraint of \( f_c(q) = 0 \), the dynamics of the system are given by the equation of motion

\[ M(q) \ddot{q} + C(q, \dot{q}) \dot{q} - G(q) = F_{spring}(\theta, \dot{\theta}) + J_{f_c}^T F_{f_c}(q, \dot{q}), \]  

(2.6)

where

\[ J_{f_c,ij} = \frac{\partial f_{c,i}}{\partial q_j} \]

the constraint normal, which is the Jacobian of the constraint function \( f_c(q) \).

When the expression of \( f_c(q) \) is reasonably simple, instead of constructing the constraint force term, it is more convenient to simply write the constrained configuration variables as functions of unconstrained configuration variables, and then construct the equation of motion using the subset of unconstrained configuration variables.

In particular, for the sticky-foot constraint, the constraint expression only needs to describe the fact that the stance foot remains at a fixed location during a step \( (x = c) \). When deriving the equation of motion we can treat the stance foot position as a constant, and this treatment gives an equation of motion that assumes fixed stance foot

\[ M_g(\theta) \ddot{\theta} + C_g(\theta, \dot{\theta}) \dot{\theta} - G_g(\theta) = F_{spring}(\theta, \dot{\theta}). \]  

(2.7)
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The dynamics of the system can equivalently be described by this equation of motion.

The equation of motion now depends only on $\theta$ and there is no constraint force term. The subscript “g” is used to indicate that a dynamic quantity is derived under the assumption that the stance leg pivots freely on the ground. Alternatively, we can derive the equation of motion in $q$ and work out the constraint force that allows the stance leg to pivot freely on the ground.

### 2.3.2 Example: the equation of motion for the rimless wheel

In this section, we demonstrate how the equation of motion of a rimless wheel under the sticky-foot constraint can be derived by applying Lagrangian mechanics. A rimless wheel has only one mass, the stance foot remains in contact with the ground without slipping until a foot next to it strikes. At the instant of the foot strike, the stance foot is released and the striking foot becomes the new stance foot. The new stance foot remains in contact with the ground without slipping. In this way, the two energies are involved: the rotational kinetic energy and the gravitational potential energy. The Lagrangian expression of a rimless wheel is

$$L = \frac{1}{2} I \dot{\theta}^2 - mgL \cos(\theta), \quad (2.8)$$

where $I$ is the rotational inertia of the rimless wheel relative to the swing foot contact.

Using Equation 2.4, the dynamics of a rimless wheel are given by the equation of motion

$$\ddot{\theta} = \frac{mgL}{I} \sin(\theta), \quad (2.9)$$

and when the rimless wheel is made of a point mass and massless legs, the dynamics of a rimless wheel are given by the equation of motion

$$\ddot{\theta} = \frac{g}{L} \sin(\theta). \quad (2.10)$$
2.4 Collision model

2.4.1 Deriving the momentum balancing equation with Lagrangian mechanics

In general, an inelastic collision occurs when additional configuration constraints are suddenly imposed. Recall that when the motion of a mechanical system is under the configuration constraint $f_c(q) = 0$, the dynamics of the mechanical system are given by the equation of motion

$$M(q)\ddot{q} + C(q, \dot{q}) \dot{q} - G(q) = F_{\text{spring}}(\theta, \dot{\theta}) + J_f^T F_{f_c}(q, \dot{q}).$$

The constraint force $F_{f_c}$ makes the state of the system comply with the constraint $f_c(q) = 0$.

If the constraint $f_c(q) = 0$ is suddenly imposed on the system, the state of the system will be forced to comply with the constraint immediately. This will generally result in a collision (i.e., a discontinuity in the motion trajectory). Therefore, we can treat the constraint force $F_{f_c}$ as an impulsive force, which can be modelled using a Dirac Delta function. If the constraint $f_c(q) = 0$ is imposed during the infinitesimal time interval $[t^-, t^+]$, the effects of the constraint force $F_{f_c}$ are given by the integral expression

$$\int_{t^-}^{t^+} M(q) \ddot{q} + C(q, \dot{q}) \dot{q} - G(q) \, dt = \int_{t^-}^{t^+} F_{\text{spring}} + J_f^T F_{f_c} \, dt.$$  \hspace{1cm} (2.11)

A generalized acceleration due to the force of gravity or a conservative force is finite, and hence it takes a finite period of time to produce a finite change in the state of a mechanical system. Therefore, we can assume that all quantities that depend on the state of a mechanical system, such as the mass matrix, the impact constraint matrix, the generalized centrifugal force and the conservative force, will remain unchanged during an infinitesimal time interval. With this assumption, the integral expression that describes the effects of imposing the constraint $f_c(q) = 0$ (Equation 2.11) can be simplified to

$$M\dot{q}^+ - M\dot{q}^- = \lim_{t^- \to t^+} \int_{t^-}^{t^+} J_f^T F_{f_c} \, dt = J_f^T \rho.$$  \hspace{1cm} (2.12)
Differentiating the constraint expression \( f_c(q) = 0 \) with respect to time, we can show that when the constraint \( f_c(q) = 0 \) is imposed, the generalized velocity of the system \( \dot{q} \) is constrained by

\[
J_{f_c} \dot{q} = 0.
\]

As mentioned previously, imposing the constraint \( f_c(q) = 0 \) will in general result in a collision, so the post-impact velocity of the mechanical system is constrained by

\[
J_{f_c} \dot{q}^+ = 0.
\]

At the end, we can show that the post-impact velocity \( \dot{q}^+ \) and the pre-impact velocity \( \dot{q}^- \) are related through

\[
\dot{q}^+ = \left( I - M^{-1} J_{f_c}^T (J_{f_c} M^{-1} J_{f_c}^T)^{-1} J_{f_c} \right) \dot{q}^-.
\tag{2.13}
\]

Suddenly imposing a constraint will usually generate an impulsive effect, but suddenly removing a constraint will not. The moment before the removal of a constraint, the mechanical states satisfy all of the constraints that have previously been enforced. Therefore, at the moment after the removal of a constraint, the mechanical state still satisfies the remaining constraints, and the mechanical system can continue from the state at the instant of the constraint removal without discontinuity.

In the case of support-transfer, once the swing foot reaches the ground, the stance foot is released. The rigidity of the ground brings the swing foot to a full stop and changing it to the stance foot in the new step. Then, the support-transfer takes place, and afterwards, the swing leg pivots freely on the ground and becomes the new stance leg. In the special case of the rimless wheel, the leg next to the stance leg of the current step becomes the stance leg of the new step. In general, the coordinate of the pending stance foot \( x_{pst} \) relative to the global reference frame can be written as a function of the configuration variables \( q \).

\[
x_{pst} = \xi (q).
\]

A “swing-foot locking constraint” is the configuration constraint that describes the swing leg pivoting during the support-transfer. In terms of \( x_{pst} \), the expression of this constraint is

\[
\xi (q) - c = 0.
\]
The constraint normal of the swing-foot locking constraint $J_g$ is

$$J_{g,ij} = \frac{\partial g_i}{\partial q_j},$$

where

$$g(q) = \xi(q) - c.$$

We can view the stance foot releasing as a sudden removal of a configuration constraint. So it does not generate any impulsive effect and we do not need to take any action about it.

At the end, we can work out the post-strike velocity $\dot{q}^+$ from the pre-strike velocity $\dot{q}^-$ by using $J_g$ and 2.13.

### 2.4.2 Example: The momentum balancing equation of the rimless wheel

In this section, we demonstrate how the momentum balancing equation of a 2D rimless wheel can be derived by applying Lagrangian mechanics. We choose the centre of mass (i.e., the hip position) of the rimless wheel to be the reference coordinate. Let $(x, y)$ be the coordinates of the hip position, and the configuration variables $(q)$ that will be used when deriving the momentum balancing equation of a rimless wheel will include $(x, y)$ and the orientation angle of the rimless wheel relative to the world vertical,

$$q = (x, y, \theta). \quad (2.14)$$

Without the sticky-foot constraint, the rimless wheel is a rigid-body in free-fall, and the mass matrix is therefore given by

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix}, \quad (2.15)$$

where the rotational inertia ($I$) is defined at the centre of mass.

The rimless wheel rotates over the stance foot until the next foot strikes the ground. Then the reaction force brings the new stance foot to a sudden stop. Therefore, the expression of $J_g$ is
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\[ J_g = \begin{pmatrix} 1 & 0 & -L \cos \left( \theta - \frac{\alpha}{2} \right) \\ 0 & 1 & L \sin \left( \theta - \frac{\alpha}{2} \right) \end{pmatrix}. \quad (2.16) \]

At the moment the next foot reaches the ground, the generalized velocity of the system is given by

\[ \dot{q}^- = \begin{pmatrix} L \cos \left( \frac{\alpha}{2} + \gamma \right) \\ L \sin \left( \frac{\alpha}{2} + \gamma \right) \end{pmatrix} \dot{\theta}. \quad (2.17) \]

The orientation angle of the system relative to the world vertical is

\[ \theta^- = -\frac{\alpha}{2} - \gamma, \quad (2.18) \]

where \( \gamma \) is the slope angle.

Using Equation 2.13, we can show that the angular velocity after the foot strike \( \theta^+ \), and the angular velocity before the foot strike \( \theta^- \), are related through

\[ \dot{\theta}^+ = \frac{I + L^2 m \cos (\alpha)}{I + L^2 m} \dot{\theta}^- . \quad (2.19) \]

This is the momentum balancing equation of a rimless wheel. For a point mass model, the momentum balancing equation can be simplified to

\[ \dot{\theta}^+ = \cos (\alpha) \dot{\theta}^- . \quad (2.20) \]

### 2.5 Step-transition Poincare-map

#### 2.5.1 Definition of step-transition Poincare-map

When walking, a biped starts from an initial state, and over time, the limbs follow the trajectories specified by the equation of motion until the swing foot reaches the ground. When the swing foot reaches the ground, a support-transfer occurs. The stance foot and the swing foot swap roles, and an inelastic collision takes place. The support-transfer brings the biped into a new state, which is the initial state of the biped at the start of the next step.
Therefore, the initial state at the start of the next step $\eta_{n+1}$, and the initial state of the current step $\eta_n$ are related through a first-order non-linear recurrence

$$\eta_{n+1} = S(\eta_n),$$

(2.21)

where $\eta = (x_0, y_0, \theta_{1,0}, \ldots, \theta_{n,0}, \dot{x}_0, \dot{y}_0, \dot{\theta}_{1,0}, \ldots, \dot{\theta}_{n,0})$ is the initial state of a step that assumes a free stance foot.

This difference equation is called the “step-transition Poincare map”. (Figure 3.2), Next, we describe in detail how the function $S$ can be obtained.

To start with, we define $\Gamma(\eta, t)$ as the solution to the equation of motion. We can split it into the configuration and the velocity components.

$$\Gamma(\eta, t) = \left( q(\eta, t) \quad \dot{q}(\eta, t) \right).$$

(2.22)

Define $Q_{s/tf}$ as the mapping between the pre-transfer and the post-transfer states of the biped that account for the foot role swapping, and also the velocity change due to the inelastic ground collision. Let $\tau_{\text{strike}}$ be the time that the swing foot takes to reach the ground, the function $S$ is defined by

$$S(\eta) = Q_{s/tf}(\Gamma(\eta, \tau_{\text{strike}}(\eta, \gamma))) \circ \Gamma(\eta, \tau_{\text{strike}}(\eta, \gamma)).$$

(2.23)

In the case of passive walking on an inclined plane, the foot-strike time is a function of the initial state $\eta$ and the slope angle $\gamma$.

Because of the sticky-foot constraint, the position of the stance foot is at a fixed location during a step. The location of the stance foot will not affect the trajectories of the limb. Therefore, whenever a sticky-foot constraint is used, the stance foot position can be re-zeroed at the start of each step. With the use of the sticky-foot constraint, the expression of the step-transition Poincare map is the same as for the case of free stance foot (Equation 2.22),

$$\eta_{g,n+1} = S(\eta_{g,n}),$$

(2.24)

except that the initial state that assumes a fixed stance foot $\eta_g$ replaces the initial state that assumes a free stance foot $\eta$, and

$$\eta_g = (\theta_{1,0}, \ldots, \theta_{n,0}, \dot{\theta}_{1,0}, \ldots, \dot{\theta}_{n,0}).$$
2.5. STEP-TRANSITION POINCARE-MAP

Collision impulse

Equation of motion

\[ q_0 \quad \text{Start} \quad q_1 \quad q_2 \quad q_3 \quad \ldots q_n \]

Poincare map (red curve)

Foot-strike condition

Figure 2.2: A conceptual diagram illustrating the definition of step-transition Poincare-map.

2.5.2 Foot-strike time and foot scuffing

In this section, we provide more detail on the definition of a foot-strike time \( \tau_{\text{strike}} \). The vertical distance between the swing foot and the slope \( h_{\text{slope}} \) over time is a transformation of the solution to the equation of motion, so it is a function of the initial state of the system, the slope angle and time, \( h_{\text{slope}} (\eta, \gamma, t) \). The foot-strike time is a non-trivial zero root of the equation \( h_{\text{slope}} (\eta, \gamma, t) = 0 \), so \( \tau_{\text{strike}} \) is a function of the slope angle and the time,

\[
\tau_{\text{strike}} (\eta, \gamma) = \min \{ t \mid h_{\text{slope}} (\eta, \gamma, t) = 0, t > 0 \}.
\] (2.25)

While mathematically the foot trajectory can cross the slope surface many times, so the equation \( h_{\text{slope}} (\eta, \gamma, t) = 0 \) may have multiple non-trivial zero roots. Since the swing foot cannot penetrate into the ground (with the exception of foot scuffing, which we will discuss later), the only physically realistic foot-strike time is the instant that the swing foot crosses the slope surface for the first time, and this justifies the use of the minimum (min) operator.
For a straight-leg 2D biped, as a result of its geometrical properties there will be a brief period of time near the middle of the step, during which the swing foot is slightly below the ground surface. This imperfection is technically known as foot scuffing. Foot scuffing can be remedied by having the biped walker walk on a surface with evenly spaced pads that elevate the stance foot slightly and prevent the swing foot from scuffing the surface. So, foot scuffing can be treated without any actuation. For this reason, foot scuffing is ignored in most locomotion research that involves the use of a straight-leg 2D biped.

If an anti-scuffing foot path is unavailable, foot scuffing can be avoided by knee bending. Side-to-side rocking and ankle flexing also resolve the problem of foot scuffing, as all these actions will shorten the effective length of the swing leg and hence prevent it from scuffing the ground during the middle of a step. Fortunately, the depth of foot scuffing is typically much shorter than the length of the leg, so a passive trajectory with scuffing is a good approximation of a lightly controlled trajectory without scuffing. While a walking trajectory without scuffing may not be passive, if it closely resembles a passive trajectory with controllable scuffing, foot scuffing may be treated by consuming only a small amount of energy. For this reason, it is still reasonable to ignore foot scuffing.

In this research, scuffing is treated in the conventional way that if it happens, it is simply ignored. In order to skip off the foot-slope crossing due to foot scuffing, we ignore the when the foot strikes near the middle of a step, by introducing a cut-off time, $t_0$, which can be any time after the stance leg is sufficiently past the vertical, and we take the smallest foot-strike time above this cut-off as the foot-strike time. To ignore foot scuffing, the foot-strike time needs to be calculated through

$$\tau_{\text{strike}}(\eta, \gamma) = \min \{t| h_{\text{slope}}(\eta, \gamma, t) = 0, t > t_0\}.$$  \hspace{1cm} (2.26)

2.5.3 Example: the foot-strike time of rimless wheel

In this section, we demonstrate how the foot-strike time of a rimless wheel can be calculated. In general situations, because the solution to the equation of motion cannot be written in closed form, the foot-strike time will have to be obtained using numerical methods. It can be obtained by using an event locator that is available with most numerical ODE solvers. Alternatively, one can find the foot-strike time by using a root-finding algorithm like the Newton’s search algorithm. Because the mechanical structure of a rimless wheel is simple, the foot-strike time can be written as an integral of a closed form expression.
Let $\mathbf{\Gamma}(\theta_0, \dot{\theta}_0, t)$ be the solution to the equation of motion of a rimless wheel which can be split into the configuration and the velocity components:

$$\mathbf{\Gamma}(\eta, t) = \left( \begin{array}{c} \theta(\theta_0, \dot{\theta}_0, t) \\ \dot{\theta}(\theta_0, \dot{\theta}_0, t) \end{array} \right).$$ (2.27)

The foot-strike time is given by

$$\tau_{\text{strike}}(\theta_0, \dot{\theta}_0, \gamma) = \min \left\{ t \mid h_{\text{slope}}(\theta_0, \dot{\theta}_0, t) = 0, t > 0 \right\}. \tag{2.28}$$

Due to the rigidity of the structure, the foot-strike time can also be defined as the time the leg next to the stance leg hits the ground, so in the case of a rimless wheel, the foot-strike time can alternatively be calculated via,

$$\tau_{\text{strike}}(\theta_0, \dot{\theta}_0, \gamma) = \min \left\{ t \mid \theta(\theta_0, \dot{\theta}_0, t) = -\frac{\alpha}{2} - \gamma, t > 0 \right\}, \tag{2.29}$$

where $\gamma$ is the slope angle and $\alpha$ is the angle between the spikes.

Because the energy is conserved during a step, it is true that

$$\frac{1}{2} I \dot{\theta}^2(0) + mgL \cos(\theta(0)) = \frac{1}{2} I \dot{\theta}^2(t) + mgL \cos(\theta(t)). \tag{2.30}$$

Rearranging Equation 2.30 and make $\dot{\theta}$ the subject, we get

$$\dot{\theta}(t) = \sqrt{\dot{\theta}^2(0) + \frac{mgL}{I} (\cos(\theta(0)) - \cos(\theta(t))).} \tag{2.31}$$

Recall that

$$\dot{\theta}(t) = \frac{d\theta(t)}{dt}, \tag{2.32}$$

We can rearrange Equation 3.21 and make $dt$ the subject. At the end we get

$$dt = \frac{d\theta(t)}{\sqrt{\dot{\theta}^2(0) + \frac{mgL}{I} (\cos(\theta(0)) - \cos(\theta(t)))}}. \tag{2.33}$$
The current step starts when the foot of the leg next to the stance leg of the previous step strikes, and then the stance foot of the previous step is released from the ground contact. Because the stance foot of the current step is on the ground at the instant that the foot of the leg next to the stance leg of the previous step strikes, at the start of the current step the orientation angle $\theta$ must be $\frac{\alpha}{2} - \gamma$. As such, at the end of the current step, simple geometrical reasoning makes it clear that $\theta = -\frac{\alpha}{2} - \gamma$.

Therefore the foot-strike time is given by

$$\tau_{\text{strike}} \left( \theta_0, \dot{\theta}_0, \gamma \right) = \int_{\frac{\alpha}{2} - \gamma}^{\frac{\alpha}{2} - \gamma} \frac{d\theta(t)}{\sqrt{\dot{\theta}_0^2 + \frac{mgL}{I} \left( \cos(\theta_0) - \cos(\theta(t)) \right)}}.$$  (2.34)

This integral can be evaluated in closed form in terms of the Jacobi amplitude function, which is a standard transcendental function like sine and cosine, but less well-known.

### 2.5.4 Example: Step-transition Poincare-map time of rimless wheel

In this section, we demonstrate how the step-transition Poincare map of a rimless wheel is derived. In the simple case of a rimless-wheel, the step-transition Poincare map can be written in closed form. However, for other passive walkers, the step-transition Poincare map cannot be written in closed form in any length, and can only be defined computationally as modules of computer codes. By conservation of energy, we can show that the angular velocity at foot fall but before the support-transfer, is given by

$$\dot{\theta} \left( \tau_{\text{strike}} \left( \theta_0, \dot{\theta}_0, \gamma \right) \right) = \sqrt{\dot{\theta}^2(0) + \frac{2mgL}{I} \left( \cos\left(\frac{\alpha}{2} - \gamma \right) - \cos\left(\frac{\alpha}{2} + \gamma \right) \right)}.$$  (2.35)

By definition, the pre-strike velocity is the velocity at foot fall but before the support-transfer, so we have

$$\dot{\theta} \left( \tau_{\text{strike}} \left( \theta_0, \dot{\theta}_0, \gamma \right) \right) = \dot{\theta}^-.$$  (2.36)

We know that the pre-strike and the post-strike velocities of the rimless wheel are related through

$$\dot{\theta}^+ = \frac{I + L^2m \cos(\alpha)}{I + L^2m} \dot{\theta}^-.$$  (2.37)
so the post-strike velocity can be written in terms of the pre-strike velocity as

\[ \dot{\theta}^+ = \frac{I + L^2 m \cos(\alpha)}{I + L^2 m} \sqrt{\dot{\theta}^2(0) + \frac{2mgL}{I} \left( \cos\left(\frac{\alpha}{2} - \gamma\right) - \cos\left(\frac{\alpha}{2} + \gamma\right) \right)}. \] (2.38)

While the post-transfer velocity is the initial velocity of the new step, the step-transition Poincare map is therefore

\[ \begin{pmatrix} \theta_{n+1} \\ \dot{\theta}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} - \gamma \\ \frac{I + L^2 m \cos(\alpha)}{I + L^2 m} \sqrt{\dot{\theta}_n^2 + \frac{2mgL}{I} \left( \cos\left(\frac{\alpha}{2} - \gamma\right) - \cos\left(\frac{\alpha}{2} + \gamma\right) \right)} \end{pmatrix}. \] (2.39)

Because of the rigidity of the structure, the rimless wheel always starts a step with the same orientation, so the step-transition Poincare map is actually one-dimensional. With a passive biped, because it is not possible to pivot the stance foot in the air, a new step can only start when both feet are on the ground. Therefore, for a passive biped it is also true that the dimensionality of the step-transition Poincare map is one less than the dimensionality of the equation of motion.

### 2.5.5 Issues with the event locator

When finding the foot-strike time numerically, one needs to bear in mind that an event locator can only handle the foot-strike events in which the swing foot approaches the slope with a non-zero velocity relative to the slope normal, so that once the swing foot reaches the slope, without applying the support-transfer transformation to the system's state, the swing foot's next move is to penetrate the slope. If the swing foot approaches the slope with zero velocity relative to the slope normal, the foot-strike time is in the form of a root at the tangent point, and due to current technical limitations, an event locator will fail to pick up foot-strike events of this type. A solution is to use a standard root-finding algorithm that does not require the presence of a crossing point, such as the Newton search algorithm.

### 2.6 Periodic walking gait and fixed point

#### 2.6.1 Definition of fixed point

If a biped starts out the gait with a particular initial state, the swing foot hits the ground and after the support-transfer, the initial state of the new step happens to be the same as the initial state of the current step, and a periodic walking gait emerges. When this happens, the initial state is called a "fixed
point” on a step-transition Poincare map. A pictorial illustration of the definition of step-transition Poincare map fixed point is shown in Figure 3.3.

The fixed point $\eta_g^*$ gives an identity on step-transition Poincare-map. Therefore $\eta_g^*$ satisfies

$$\eta_g^* = S(\eta_g^*).$$

(2.40)

In this way, to find the fixed point, we want to find an initial state $\eta_g$ such that

$$S(\eta_g) - \eta_g = 0.$$  

(2.41)

Figure 2.3: A conceptual diagram illustrating the definition of a fixed point on a step-transition Poincare-map. When the biped starts with a fixed point, the step-transition Poincare-map is an identity. The step-transition Poincare-map curve collapsed down to a point.
2.6. PERIODIC WALKING GAiT AND FIXED POINT

2.6.2 Example: Fixed point expression for rimless wheel

Here we demonstrate how the fixed point of a rimless wheel can be derived. Given that we know the closed-form expression for the rimless wheel step-transition Poincare map, the fixed point that gives a periodic gait satisfies

\[
\begin{pmatrix}
\dot{\theta}^* \\
\theta^*
\end{pmatrix} = \left( \frac{I + L^2 m \cos(\alpha)}{I + L^2 m} \sqrt{\left( \frac{\theta^*}{2} - \gamma \right)^2 + \frac{2mgL}{I} \left( \cos \left( \frac{\theta^*}{2} - \gamma \right) - \cos \left( \frac{\theta^*}{2} + \gamma \right) \right)} \right).
\] (2.42)

Rearranging for \( \theta^* \) and \( \dot{\theta}^* \), the fixed point of a rimless wheel that allows it to walk passively and periodically is given by

\[
\theta^* = \frac{\alpha}{2} - \gamma
\] (2.43)

\[
\dot{\theta}^* = \sqrt{\frac{2mgL}{I} \left( \frac{I + L^2 m \cos(\alpha)}{I + L^2 m} \right)^2 \left( \cos \left( \frac{\theta^*}{2} - \gamma \right) - \cos \left( \frac{\theta^*}{2} + \gamma \right) \right)} \left( \cos \left( \frac{\theta^*}{2} - \gamma \right) - \cos \left( \frac{\theta^*}{2} + \gamma \right) \right) \left( 1 - \left( \frac{I + L^2 m \cos(\alpha)}{I + L^2 m} \right)^2 \right)}
\] (2.44)

2.6.3 Finding fixed-point for a general passive walker

In general, finding a fixed point that gives a periodic gait requires the use of numerical root-finding algorithms. For solving a scalar equation \( f(x) = 0 \), we use the Newton search algorithm that uses the recurrence

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\] (2.45)

For solving a vector function \( f(x) = 0 \), we use the Newton search algorithm that uses the recurrence

\[
x_{n+1} = x_n - J(x_n)^{-1} f(x_n),
\] (2.46)

where

\[
J_{ij}(x_n) = \left[ \frac{\partial f_i}{\partial x_j} \right]_{x=x_n}.
\] (2.47)
For a general passive walker, the solution to the equation of motion and the step-transition Poincare map cannot be written in closed-form and therefore we cannot always obtain the required derivatives in the form of closed-form expressions. In these situations, we approximate the required derivatives by first-order finite differences. A standard Newton search algorithm with the required derivatives approximated by first-order finite differences is known as the “Quasi-Newton search algorithm”.

2.7 Stability of the periodic walking gait

2.7.1 Gait stability analysis

The existence of a fixed-point implies the existence of a periodic gait. The next question is how stable is the periodic gait? Certainly, without perturbations, a biped can walk periodically given that there is a fixed point on the step-transition Poincare map. If it is perturbed, however, will it return to the periodic gait, or fall over after a few steps?

The stability of the gait can be determined by looking at the dynamic behavior of the step-transition Poincare map near the fixed point $\eta^*_g$. By linearizing the step-transition Poincare map at the fixed point, we get a system of linear difference equations

$$\eta_{g,n+1} = \eta^*_g + A(\eta^*_g)(\eta_{g,n} - \eta^*_g),$$

(2.48)

where $A(\eta^*_g)$ is the Jacobian matrix of $\eta_g - S(\eta_g)$ at the fixed point. It is a constant matrix describing the local behaviour of the step-transition Poincare map near the fixed-point. This matrix can be obtained numerically by using first-order finite differences.

Once we find $A$, we can obtain its eigenvalues. If all eigenvalues fall within a unit circle in the complex plane, then the local behaviour of the step-transition Poincare map is stable. This implies that as long as the perturbation does not disturb the system out of the attractive basin, the gait can be self-stabilized without the need for control.

If at least one eigenvalue falls outside a unit circle, then the local behaviour of the step-transition Poincare map is unstable, and a perturbation of any size will destroy the periodic gait. However, the gait can still be dynamically stable if it re-stabilized as a multi-period gait. The multi-periodic gait occurs when there is a Poincare-map limit-cycle surrounding the unstable fixed point. Perturbations will then cause the biped to miss the single-period gait, but as time goes on,
the state at the start of each step moves around the limit cycle, and so the
perturbation error will remain bounded. Alternatively, the perturbation error
will grow large as time goes on, and the passive walker will either fall over or
stop.

If at least one eigenvalue falls on a unit circle, the gait is neutrally stable. Error
due to a small perturbation cannot be damped out as time goes on, but does not
grow either. The gait is still self-sustaining.

From the rimless example, it is easy to recognize that the eigenvalue of the fixed
point can be expressed in terms of the mechanical parameters. This demonstrates
the basic principle of passive dynamic walking, and an understanding that loco-
motor agility in animals entails mechanical analysis and design, and is not simply
neuromuscular control mechanisms.

2.8 Passive dynamics with control

When controllers are installed between neighbouring limbs, the dynamics of
the system are given by the controlled equation of motion

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + G(q) = J^T_{\theta \rightarrow \phi} \left( F_{\text{spring}} + F_{\text{control}}^{n}(p_n, t) \right),
\]

where \( p_n \) is a collection of controllable parameters used in the nth step, that
determines how control forces are generated over time.

Because a controller can only be installed around a joint, neighbouring limbs
that are connected by a controller experience torques of the same size but opposite
direction. When constructing the equation of motion, we need to express the
control torques in terms of the joint angles, \( \phi \), to ensure that the control torques
are physically realistic. However, it is mathematically convenient for us to use the
orientation angles of the limbs relative to the world-vertical, \( \theta \), when constructing
the Lagrangian expression. So, in the above expression, the joint angles and the
orientation angles of the limbs are related through coordination transformation
\( J_{\theta \rightarrow \phi} \). The transformation matrix \( J_{\theta \rightarrow \phi} \) is a constant matrix that ensures that
neighbouring limbs connected by the controller experience torques of the same
size but opposite direction. So the control force terms, \( F_{\text{control}} \), should be written as

\[
F_{\text{control}} = J^T_{\theta \rightarrow \phi} F_{\phi \rightarrow \theta}^{\text{control}}(p_n, t).
\]
Similar treatments are applied to the passive spring forces as well, because springs can also only be installed around a joint. Therefore, the spring force term, $F^{spring}$, should be written as

$$F^{spring} = J_{\theta \rightarrow \phi}^{T} F^{spring}_{\phi}.$$

With controls, the gait trajectory is parameterized by controllable parameters $p_n$, and thus the step-transition Poincare map is a function of the initial state and the controllable parameters. So a controlled step-transition Poincare map is given by

$$\eta_{g,n+1} = S(\eta_{g,n}, p_n).$$

Later, we will investigate how to re-stabilize a passive but unstable walking gait. To do this, we need to find the control force required to bring the passive walker from a perturbed initial state to an unstable fixed point. Given that the perturbed initial state is $\eta_p^g$, and the unstable fixed point is $\eta^*_g$, we can find the control force such that we can bring the biped from $\eta_p^g$ to $\eta^*_g$ by solving for the control parameter $p^d$ such that

$$\eta^*_g = S(\eta_p^g, p^d).$$

Depending on the capability of the controller, not all $p^d$ calculated using this approach will in reality be feasible because the solution may require applying unfeasibly large or rapid forces. However, at this point we are only concerned with theoretical results and we are ignoring practical design considerations.
Chapter 3

Leg-only passive dynamic walkers

3.1 Overview

Strictly speaking, the idea of using passive dynamics as the primary means of robotic control is dated. The idea of passive dynamic walking was discovered by McGeer almost 20 years ago [42, 43, 10, 32]. McGeer demonstrated the possibility of building a walking device that can walk, and self-correct small perturbative errors without intelligence or actuation, by just using the force of gravity. The passive walking models proposed by McGeer had only legs. Examples include a rimless wheel, a straight-legged bipedal walker known as the compass-gait walker, and also a passive bipedal walker with knees. In this chapter, we will revisit and work through important published results related to McGeer’s passive dynamic walking models.

3.2 Rimless wheel

3.2.1 Background

The rimless wheel, as described in Chapter 2, is made of a mass, with equally-spaced spikes extending outwards, like a wagon wheel without the outer rim (Figure 3.1). It has a stable periodic walking gait on a downhill slope. An example of the gait trajectory of a rimless wheel that has 6 equally-spaced spikes is shown in Figure 3.2. The trajectory follows the hyperbolic Hamiltonian contour of an inverted simple pendulum until reaching the foot-strike position (Figure 3.2). As time goes on, a stable limit-cycle emerges.
3.2.2 The non-existence of a passive periodic level-ground walking gait

A rimless wheel cannot walk passively on level ground. In this section, we prove why this is the case.

**Theorem 1.** A rimless wheel cannot walk passively on level ground at a finite walking speed.

*Proof*

From Chapter 2, we know that the angular velocities of the rimless wheel before and after the foot strike are related through

\[
\dot{\theta}^+ = \frac{I + L^2 m \cos (\alpha)}{I + L^2 m} \dot{\theta}^-.
\]

Therefore, a non-zero pre-impact angular velocity must imply a loss in kinetic energy. On level ground, support-transfer does not change the gravitational potential energy, so on level ground, non-zero pre-impact angular velocity must imply a loss in total energy. The total loss energy is zero only if the pre-impact angular velocity is zero. If the pre-impact angular velocity is zero, there is no inelastic collision, so the new step will start with the same angular velocity, which is also zero. However, given the mechanical structure of a rimless wheel, if the system starts with zero angular velocity, it will be in a static equilibrium.

\[ \Box \]
3.3 Compass-gait walker

3.3.1 Background

The compass-gait walker is essentially an inverted double-pendulum, with two linkages representing the two legs of a biped (Figure 3.3). The mass at the joint represents the body mass. The equation of motion can be derived using the methods outlined in Chapter 2. With the use of the sticky-foot constraint, the dynamics of the mechanical system are given by the equation of motion

\[
M_g(\theta) \ddot{\theta} + C_g(\theta, \dot{\theta}) \dot{\theta} - G_g(\theta) = 0, \tag{3.1}
\]

where

\[
M_g(\theta_1, \theta_2) = \begin{pmatrix}
\frac{1}{4} L^2 (5m_l + 4m_h) & -\frac{1}{2} L^2 \cos (\theta_1 - \theta_2) \\
-\frac{1}{2} L^2 \cos (\theta_1 - \theta_2) & \frac{L^2 m_l}{4}
\end{pmatrix}, \tag{3.2}
\]
\[
C_g(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{pmatrix}
0 & -\frac{1}{2}L^2m_1 \sin(\theta_1 - \theta_2) \dot{\theta}_1 \\
\frac{1}{2}L^2m_1 \sin(\theta_1 - \theta_2) \dot{\theta}_1 & 0
\end{pmatrix}, \quad (3.3)
\]

and

\[
G_g(\theta_1, \theta_2) = \begin{pmatrix}
\frac{1}{2}L \left( 2m_1g \sin(\theta_1) + 3m_2g \sin(\theta_1) \right) \\
-\frac{1}{2}Lm_1g \sin(\theta_2)
\end{pmatrix}. \quad (3.4)
\]

In the above expressions, \( M_g \) is the mass matrix, \( C_g \) as the centrifugal matrix and \( G_g \) is the conservative force under the sticky-foot constraint. These are written in terms of the limb orientation angles \( \theta = (\theta_1, \theta_2) \).

![Figure 3.3: A compass-gait walker.](image)

Conservation of momentum during the support-transfer gives the momentum balancing equation

\[
\dot{\mathbf{q}}^+ = \left( \mathbf{I} - M^{-1} J_g^T \left( J_g M^{-1} J_g^T \right)^{-1} J_g \right) \dot{\mathbf{q}}^-, \quad (3.5)
\]
which is an equation of velocity change. In this momentum balancing equation,

$$q = (\theta_1, \theta_2, x_1, x_2)^T$$  \hspace{1cm} (3.6)

is the vector of the free stance foot in terms of the orientation angles of the legs ($\theta_1, \theta_2$) and the stance foot coordinates ($x_1, x_2$). The matrix

$$M (\theta_1, \theta_2, x_1, x_2) = \begin{pmatrix} M_g (\theta_1, \theta_2) & Q (\theta_1, \theta_2) \\ Q (\theta_1, \theta_2)^T & (2m_l + m_h) I_{22} \end{pmatrix}$$  \hspace{1cm} (3.7)

is the mass matrix without the sticky-foot constraint. The matrix

$$J_g (\theta_1, \theta_2, x_1, x_2) = \begin{pmatrix} L \cos (\theta_1) & -L \cos (\theta_2) & 1 & 0 \\ -L \sin (\theta_1) & L \sin (\theta_2) & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (3.8)

is the constraint normal of the swing foot locking constraint as described in Chapter 2. The matrix $Q$ is given by

$$Q (\theta_1, \theta_2) = \begin{pmatrix} \frac{1}{2} L (3m_l + 2m_h) \cos (\theta_1) & -\frac{1}{2} L (3m_l + 2m_h) \sin (\theta_1) \\ -\frac{1}{2} L m_l \cos (\theta_2) & \frac{1}{2} L m_l \sin (\theta_2) \end{pmatrix}.$$  \hspace{1cm} (3.9)

A typical trajectory of a compass-gait walker is shown in Figure 3.4. The stance leg moves with a small angular acceleration, and the trajectory is approximately a straight line. This is known as compass-gait behaviour.

The equation of motion of a compass-gait walker can be simplified when the leg masses are replaced by foot masses (Figure 3.5), which is possible when we assume that the hip mass is much greater than the foot mass. The mass matrix $M_g$, the centrifugal matrix $C_g$ and the conservative force $G_g$ can be simplified to

$$M_g (\theta, \phi) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$  \hspace{1cm} (3.10)

$$C_g (\theta, \phi, \dot{\theta}, \dot{\phi}) = \begin{pmatrix} 0 & \dot{\theta} \\ \sin (\phi) \dot{\theta} & 0 \end{pmatrix},$$  \hspace{1cm} (3.11)

and

$$G_g (\theta, \phi) = \begin{pmatrix} \sin (\theta) \\ \cos (\theta) \sin (\phi) \end{pmatrix}$$  \hspace{1cm} (3.12)

respectively.
Figure 3.4: A typical stable downhill trajectory of a compass-gait walker. The stance leg trajectory is shown in dark blue, and the swing leg trajectory is shown in purple. The trajectories are the orientation angles of the limbs as functions of time.

The angle $\phi$ is the angle between the stance and the swing legs, $\phi = \theta_1 - \theta_2$, and the angle $\theta$ is the orientation angle of the stance leg relative to the vertical, which is the same as $\theta_1$. This simplified compass-gait walker equation of motion is expressed in the coordinates $(\theta, \phi)$.

The momentum equation for updating the leg velocity at support-transfer can be simplified into

$$
\begin{pmatrix}
\dot{\theta} \\
\dot{\phi}
\end{pmatrix}
= \begin{pmatrix}
\cos (2\theta^-) & 0 \\
\cos (\theta^-) (1 - \cos (\theta^-)) & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}^- \\
\dot{\phi}^-
\end{pmatrix}.
$$

(3.13)

The trajectories of the general compass-gait walker are similar to the examples of the trajectories of the compass-gait walker with the relatively small leg mass as shown here. The simplifications make a system with no free design parameters. The initial state and the slope are the only variables to consider, and thus the modelling and the analysis becomes much easier.

### 3.3.2 The non-existence of a passive periodic level-ground walking gait

Here we revisit Chatterjee’s proof [38] that the compass-gait walker proposed by McGeer cannot walk passively on level ground with a finite speed. In Chatterjee’s work, the proofs are based on the time-reversal property of the swing-foot
Figure 3.5: A compass-gait walker with a massive hip and small foot mass trajectory. We carry out the same proof, but use conservation of energy, which is perhaps clearer. For the case of a compass-gait walker, Chatterjee’s proof is based on the following assumptions, and these assumptions will also be used here.

1. After support-transfer, the swing foot pivots freely on the ground and becomes the new stance leg.

2. Each step starts with both feet on the ground (i.e. a double-stance phase).

3. The double-stance phase has infinitesimal duration, and once the swing foot reaches the ground, it pivots freely on the ground, and the stance foot is released immediately.

4. There are no springs in the system.

**Theorem 2.** For a compass-gait walker, zero foot velocity implies that the angular velocity of the legs is zero.

**Proof**
The swing-foot velocity of a compass-gait walker is given by

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{y}_f
\end{pmatrix} = \begin{pmatrix}
L \cos (\theta_1) & -L \cos (\theta_2) \\
-L \sin (\theta_1) & L \sin (\theta_2)
\end{pmatrix} \begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{pmatrix}.
\] (3.14)

The determinant of the matrix on the right-hand side of Equation 3.14 is

\[
\det \left( \begin{pmatrix}
L \cos (\theta_1) & -L \cos (\theta_2) \\
-L \sin (\theta_1) & L \sin (\theta_2)
\end{pmatrix} \right) = L^2 \sin (\theta_1 - \theta_2).
\] (3.15)

Therefore, when \( \theta_1 = -\theta_2 \) and \( \theta_1 > 0 \), the transformation is non-singular. Therefore, under the ground-contact constraint, zero foot velocity implies zero leg angular velocities.

\[\square\]

**Theorem 3.** A compass-gait walker cannot walk passively on level ground with finite speed.

**Proof**

To begin with, we define \( E^{CGW}(\theta, \dot{\theta}) \) as the total energy of the compass-gait walker. This is a function of the orientation angles of the limbs relative to the world vertical, and their angular velocities relative to the world vertical.

Because the legs are straight, at a fixed point on the step-transition Poincare map both stance and swing feet are on the ground, and the fixed point is collision-free, therefore, the angular velocities of the stance leg and the swing leg must both be zero (Theorem 2). Since McGeer’s compass-gait walker has no springs, the system is in a static equilibrium, standing stably on the ground.

We can also base our arguments on the conservation of energy. From Theorem 2 and Chatterjee’s assumptions, we can easily see that in order to walk passively and periodically on level ground with a finite walking speed, the system must start with zero kinetic energy. We can easily show that under Assumptions 2 and 4 (if the step size is non-zero and the periodic gait starts without kinetic energy), then the total energy of the system at the start of the step must be less than \( E^{CGW}(0, 0) \). In order to complete a gait, the orientation angle of the stance leg starts from a positive value, and finishes with a negative value. This can only be achieved by starting the gait with a total energy not less than \( E^{CGW}(0, 0) \). Therefore, a passive periodic gait with a finite walking speed is not possible by contradiction.
3.4 Passive walker with knees

3.4.1 Background

McGeer had also proposed a kneed passive walker [32], which is an inverted four-link pendulum, where the links representing the thighs and the shanks of the stance and swing legs. At the top of each shank, there is a “kneecap” that prevents knee hyper-extension of the knee (Figure 3.6). A typical gait cycle of the kneed walker is shown in Figure 3.7. If the shank swings toward the thigh, when it reaches 180 the knee locks up and the shank and the thigh move together as long as the forces are towards extending the knee. In general, an inelastic collision will take place during knee-locking. Knee-locking is collision-free if the thigh approaches the kneecap with zero velocity.

![Figure 3.6: McGeer’s passive biped with knees.](image)

Because of the way McGeer designed the kneed walker, the inertial properties of the legs allow the thigh and shank on the stance leg to be knee-locked after the start of the step, while the shank on the swing leg is free to swing about the
knee joint until knee-locking happens. Because the stance leg remains straight, the kneed passive walker behaves as a three-link inverted pendulum until knee-locking occurs. After knee-locking, both the stance and the swing legs remain straight, so the system behaves as a compass-gait walker until the foot strikes.

After foot strike, the knee-locked swing leg becomes the new stance leg, so in the new step this leg will remain knee-locked. The impulse of inelastic collision during the support-transfer unlocks the knee of the previously knee-locked stance leg and makes it a knee-free swing leg in the new step.

Figure 3.7: A diagram showing the gait cycle of a McGeer-type kneed walker. In this figure, knee-locking happens after the stance leg reaches vertical. It is also possible that knee-locking takes place before the stance leg reaches the vertical, or after foot strike. These possibilities are not considered in McGeer’s works.
3.4.2 The non-existence of a passive periodic level-ground walking gait with a straight stance leg and infinitesimal support-transfer

Here we revisit the proof by Chatterjee [38] whereby the McGee r-type kneed walker proposed by McGeer cannot walk passively on level ground without allowing the stance leg to flex with a non-infinitesimal support-transfer. As for the case of the compass-gait walker, in Chatterjee’s work, the proofs are based on the time-reversal property of the swing foot trajectory. We carry out the same proof but use the conservation of energy and Chatterjee’s assumptions:

1. After support-transfer, the swing leg pivots freely on the ground and becomes the new stance leg.

2. Each step starts with both stance and swing legs straight and both feet on the ground.

3. The stance leg is strictly knee-locked and cannot flex at all until the new step starts and the stance and swing legs swap roles.

4. Knee-locking happens before foot strike, so that the swing leg strikes with a straight configuration, and after support-transfer the new step will start with a straight stance leg.

5. The double-stance phase has an infinitesimal duration, and once the swing foot reaches the ground, it pivots freely there, and the stance foot is released immediately.

6. There are no springs in the system.

Ideally, we should also account for the case in which knee-locking happens after the foot strike so that after support-transfer the new step will start with a flexed stance leg. We should also consider the case where the double-stance phase has a non-infinitesimal duration. As this is a revision chapter, these alternatives are not investigated.

Theorem 4. A kneed walker cannot passively walk on level ground at a finite speed without allowing the stance leg to flex and without allowing a non-infinitesimal support-transfer.

Proof
If an inelastic collision occurs when the knee locks, there is a loss in energy and on the level ground the gravitational potential energy is not readily available. Hence the periodic walking condition cannot be realized without active controls. Therefore we focus on the case in which the knee-locking is collision-free and does not change the total energy of the system.

The velocity of the swing foot in a kneeed biped with a knee-locked stance leg can be written as

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{y}_f
\end{pmatrix} =
\begin{pmatrix}
L_1 \cos(\theta_1) & -L_2 \cos(\theta_2) & -L_3 \cos(\theta_3) \\
-L_1 \sin(\theta_1) & L_2 \sin(\theta_2) & L_3 \sin(\theta_3)
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{pmatrix}.
\]

In order to walk on level ground, the swing foot must reach the ground with zero velocity. In order to achieve collision-free walking, the swing-foot velocity at footfall must be constrained by

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{y}_f
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

When the swing foot knee is locked, the orientation angles of the knee-locked stance leg, and the swing leg thigh and shank are constrained by

\[\theta_2 = \theta_3\]

These constraints can be combined as

\[
\begin{pmatrix}
L_1 \cos(\theta_1) & -L_2 \cos(\theta_2) & -L_3 \cos(\theta_3) \\
-L_1 \sin(\theta_1) & L_2 \sin(\theta_2) & L_3 \sin(\theta_3)
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

The determinant of the matrix on the left hand side is

\[
\Delta = \det
\begin{pmatrix}
L_1 \cos(\theta_1) & -L_2 \cos(\theta_2) & -L_3 \cos(\theta_3) \\
-L_1 \sin(\theta_1) & L_2 \sin(\theta_2) & L_3 \sin(\theta_3)
\end{pmatrix}
\]

\[
= -L_1 L_2 \sin(\theta_1 - \theta_2) - L_1 L_3 \sin(\theta_1 - \theta_3)
\]

Applying Chatterjee’s assumptions, the expression for the determinant becomes
3.5. **THE NEXT STEP FROM McGEER’S BASIC INSIGHT**

\[
\Delta = - \sin (2\theta_1) (L_1 L_2 + L_1 L_3).
\]

We can easily realize from this expression, that as long as \(\theta_1 \neq 0\), a zero foot-strike velocity implies zero angular velocities of all thighs and shanks, and the total kinetic energy is zero at foot strike. This implies that in order for the kneed biped to walk passively and periodically with a finite walking speed, the gait must start with zero kinetic energy.

To make the remaining part of the proof clearer and shorter, we define \(E^{KW}(\theta, \dot{\theta})\) as the total energy of the kneed walker, which is a function of the orientation angles and the angular velocities of the limbs, relative to the world vertical.

We have just showed that under Chatterjee’s assumptions, in order for the kneed walker to walk passively and periodically on level ground with a finite walking speed, the system must start with zero kinetic energy. As for the compass-gait walker, we can show that under Assumptions 2 and 6, if the periodic gait starts without kinetic energy and the step size of the periodic gait is non-zero then the total energy of the system at the start of the step must be less than \(E^{KW}(0, 0)\). (see Figure 3.8 for clarification). Again, as for the compass-gait walker, in order to complete a gait, the orientation angle of the stance leg starts from a positive value, and finishes with a negative value, and this can only be achieved by starting the gait with a total energy not less than \(E^{KW}(0, 0)\) (see Figure 3.9 for clarification). Therefore, a passive periodic gait with a finite walking speed is not possible by contradiction.

### 3.5 The next step from McGeer’s basic insight

We know that McGeer’s passive walkers cannot walk passively on level ground at a finite speed because without actuation, because an insufficient energy is available for the stance leg to vault over the vertical. One can easily imagine that if we mount a torsion spring around the hip of McGeer’s passive walker, then the elastic potential energy stored in the spring raises the system’s total energy and thus collision-free level-ground walking may be feasible energetically.

Therefore, the use of springs is an important consideration. However, in order to achieve collision-free level-ground walking, at the moment the swing foot reaches the ground, the kinetic energy of the legs must be fully be captured by the springs, and stored as elastic potential energy. It might be possible that the legs will have to be over-stretched in order to store the required amount of energy.
A typical pose of the kneeed walker at the start of a step, under the assumption that the step starts with legs straight.

The starting pose of the kneeed walker that gives the maximum total energy, when the total kinetic energy is zero.

Figure 3.8: A diagram illustrating that when all angular velocities are zero and both feet are on the ground (Assumption 2), the total energy is at the maximum when both legs are vertical.

This would prohibit the existence of a realistic collision-free walking gait. For this reason, we will consider adding a torso. With a torso, the kinetic energy of the legs can be stored as a combination of elastic potential energy and as energy in the torso. In this way, only part of the leg’s kinetic energy needs to be stored in the springs, and thus a realistic collision-free walking gait is more likely to emerge.

The next section will consider extensions on McGeer’s basic insight, involving springs and an upper-body. Animals have muscles, and we model muscles (with their tendons) as biological springs when they are not actively generating forces. Bipedal animals have an upper-body too. In order to gain further insights into passive dynamics, a mechanical model should account for the major body masses that are present in a real animal.
A typical pose of the kneed walker when stance leg is vertical

The pose of the kneed walker that gives the minimum GPE when stance leg is vertical

Figure 3.9: When the stance leg is vertical, the gravitational potential energy (GPE) is at the minimum when the swing leg is vertical, because as the swing leg moves away from vertical, the masses will be lifted.
Chapter 4
Simplest ideal walker

4.1 Introduction

This chapter is a reproduction of a paper by Te-yuan Chyou, Gerrard Liddell and Mike Paulin titled “An upper-body can improve the stability and efficiency of passive dynamic walking”, with additional information on the strategies used to calculate the fixed points, and a review of a similar biped that uses a bisection mechanism to keep the torso upright passively, which are not included in the paper due to the space limitations. The paper was published in the Journal of Theoretical Biology 285(2011) 126-135.

4.1.1 Gravity-powered stable walking on downhill slopes

The simplest bipedal passive walker is the compass-gait walker [36, 10]. It has a massive hip and two straight legs of equal length. Each leg behaves as a simple pendulum during the swing phase, but the legs are light and have only minor effects on the dynamics of the hip, so the hip acts as an inverted pendulum over the stance leg. The compass-gait walker can walk passively, periodically and stably on shallow slopes with a gait period that is dependent on the slope angle. Under the right conditions, small perturbation errors can passively be repaired, so the periodic walking can be maintained without controls.

However, the uncontrolled compass-gait walker has two major limitations. One limitation is that stable walking is only possible on shallow slopes [39, 36]. The other limitation is that as the slope angle approaches zero, the step size falls to zero. The compass-gait walker, as well as other bipedal walkers without an upper-body, cannot walk passively and periodically at all on a horizontal plane with a finite walking speed [38].
4.1.2 Passive walking on level ground using torso

Chatterjee [38] conjectured that by adding a torso to a leg-only bipedal walker, it is possible to construct an “ideal walker” that uses no power to walk on level ground, by avoiding footfall collisions. Gomes and Ruina [40] recently demonstrated that a compass-gait walker with a torso can walk passively on level ground with a finite walking speed, by requiring the swing foot to land on the ground with exactly zero velocity. This design shows that the essential work required for level-ground transportation is zero, and this is expected because the force of gravity and the supportive forces are orthogonal to the direction of motion.

An ideal walker cannot be stable on level ground, because small perturbations will cause the walker to leave the collision-free trajectory, which will, in general result in inelastic collisions and a loss in total energy. The collision loss is the main cause of energy consumption in passive dynamic walking [38, 10, 43]. Garcia [35] conjectured that the work required for stabilizing a passive but unstable periodic gait approaches zero when the size of the perturbation approaches zero.

Whether it is possible to build an ideal walker that can walk passively and stably down a slope using gravity, and if so how its stability and efficiency compare with that of a compass-gait walker, has not previously been reported.

4.1.3 Stable bipedal walking with a torso

Prior to the research of Gomes and Ruina [40], very little progress had been made on the passive bipedal locomotion with an upper-body. A stable bipedal walking gait with a torso had previously been achieved, but only by using either state-feedback controls [47, 42, 48] or kinematic constraints [50].

McGeer [42] showed that a control input supplied from the muscle that joins the stance leg and torso, can be used to hold the torso motionless at a particular orientation, and correct torso orientations allow the bipedal walker to walk passively, periodically and stably. If the control does not hold the torso perfectly motionless, the torso jerks forward during the support-transfer, and this will, in most cases, reduce the step size of the next step. McGeer therefore conjectured that torso swaying is undesirable.

In other research, the proposed feedback control schemes or kinematic constraints also aim to reduce the amount of torso swaying [47, 48], or to force the torso to follow a particular trajectory so that it remains upright during the course of motion [50]. In general, stable walking with a torso can be achieved by eliminating the contribution of the torso to the passive dynamics of the lower-body.
4.2. THE CHALLENGE OF ADDING A TORSO

As McGeer put it, “given a good pair of legs, it is hard to go wrong” [42], the implication of this and subsequent studies is that while legs may take care of themselves via their beautiful mechanical design, actuators and controllers are required to balance an upper-body on top of them.

Our goal in this chapter is to show that, it is actually possible to integrate the torso into a mechanical design that improves the stability and efficiency of passive dynamic walking.

4.2 The challenge of adding a torso

4.2.1 What makes torso incorporation a challenge and an unsolved problem?

It is clear that if a torso is to be included, provision must be made for keeping it upright. The torso can be stabilized on the two legs by springs and this will introduce an extra degree of freedom. The step-transition Poincare map becomes 5D and cannot be visualized easily. This makes the search for a fixed point on the step-transition Poincare map that allows a stable periodic gait rather difficult. Furthermore, because the springs must keep the torso upright throughout the entire journey, and the torso must be heavy enough to be realistic, the springs must be reasonably stiff. Stiff springs will always exert recoil forces on the stance and the swing legs, therefore, the periodic gaits of a compass-gait walker will not provide any useful clues about the periodic gaits of a bipedal walker with a torso.

In the case of the simplest biped, the fixed points can be found by using a 2-dimensional grid search. Because the simplest biped has no free design parameters, in order to find a passive periodic walking gait, we only need to find and start the stance leg with the correct velocity and the orientation angle (Chapter 3). After adding the torso, six more free parameters are introduced. They are the orientation angle and velocity of the torso, the mass and the length of the torso (relative to hip mass and leg length respectively), the leg mass (relative to the hip mass), and the stiffness of the springs connecting the torso and the legs. We need to find the correct values for all six parameters so that the biped can walk passively, periodically and stably. If we try to find the working parameters using a 6-dimensional grid search, simple counting arguments make it clear that the grid search will become very computationally expensive. This might explain why research in the field of passive dynamic walking has not gone beyond the complexity of a leg-only biped.
4.2.2 Incorporating the torso in a realistic way

A mistake that people might make is that from biological intuition the torso can be viewed as an inverted pendulum connected to the hip by springs. By making the springs stiff, the dynamics of a bipedal walker with a torso closely resemble the dynamics of a compass-gait walker, and hence the fixed points of a compass-gait walker are good approximations of the fixed points of a bipedal walker with a torso.

This argument is correct mathematically, but it is unrealistic. Firstly, if the upper-body has only one degree of freedom and the hip is a point mass, we can treat the hip part of the torso without affecting the dynamics. Then, connecting the spring between the torso and the hip essentially means connecting the spring between any two points on the torso, so the spring cannot generate the force required for holding the torso upright. If the hip is a rigid-body, it is physically possible to hold the torso upright by stabilizing the torso on the hip using springs. However, this is equivalent to having a 2-DOF torso, and thus brings us back to the same problem outlined in Section 4.2.1.

Wisse [50] suggests that instead of using springs, it is possible to build a mechanical framework that keeps the torso at the bisector of the hip during the course of motion without controls. For brevity, we call it "Wisse’s bisection mechanism". With Wisse’s bisection mechanism, the straight-leg biped with a torso has only two degrees of freedom, the step-transition Poincare map can be visualized in 3D, and we can see the stable fixed points on the step-transition Poincare map. Thus, when finding the stable fixed points by using the Newton search algorithm, it is easy to have good initial guesses.

To keep the torso aligned with the hip bisector passively, we consider the mechanical framework as outlined in Figure 4.1. The torso is extended below the hip, passing through a massless cylinder linked to the legs by massless rods and revolute joints. The entire mechanical framework is a parallelogram linkage. The major disadvantage of Wisse’s bisection mechanism is that a bipedal walker cannot walk passively on level ground if we use it to stabilize the torso. Later in the chapter, we will prove this argument mathematically.

4.2.3 Wisse’s bisection mechanism and level-ground inefficiency

In this section, we prove that if we keep the torso upright by using Wisse’s bisection mechanism (Figure 4.1), the biped cannot walk passively on level ground.
4.2. THE CHALLENGE OF ADDING A TORSO

Figure 4.1: The design of Wisse’s bisection mechanism that keeps the torso aligned with the hip bisector during a step.
CHAPTER 4. SIMPLEST IDEAL WALKER

**Theorem 5.** If we keep the torso upright by using Wisse’s bisection mechanism, the biped cannot walk passively and periodically on level ground at a finite walking speed.

**Proof**

During the course of motion, the torso is constrained at the bisector of the hip. Therefore the velocity of the torso is constrained by

\[ \dot{\theta}_3 = \frac{1}{2} (\dot{\theta}_1 - \dot{\theta}_2). \]

The swing-foot velocity is given by

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{y}_f
\end{pmatrix} =
\begin{pmatrix}
L \cos(\theta_1) & -L \cos(\theta_2) & 0 \\
-L \sin(\theta_1) & L \sin(\theta_2) & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{pmatrix}.
\]

In order to walk on level ground, the swing foot must reach the ground with zero velocity, so at foot strike, the velocity of the swing foot must comply with the constraint

\[
\begin{pmatrix}
\dot{x}_f \\
\dot{y}_f
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

These constraints can be combined as

\[
\begin{pmatrix}
L \cos(\theta_1) & -L \cos(\theta_2) & 0 \\
-L \sin(\theta_1) & L \sin(\theta_2) & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

The determinant of the matrix on the left hand side is

\[
\det
\begin{pmatrix}
L \cos(\theta_1) & -L \cos(\theta_2) & 0 \\
-L \sin(\theta_1) & L \sin(\theta_2) & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} \\
-\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{pmatrix} = -L^2 \sin(\theta_1 - \theta_2).
\]

When the swing foot reaches the ground, the orientation angles of the stance leg and the swing leg are related by

\[ \theta_1 = -\theta_2. \]
At this point, we can easily recognize that if the step size is non-infinitesimal, zero swing-foot velocity at foot strike implies that all the limbs will have zero angular velocities at foot strike. This, in turn, implies a static equilibrium.

\[ \square \]

If the torso dynamics are independent of the leg dynamics, to achieve collision-free walking, the biped can have non-zero kinetic energy at foot strike. Furthermore, a rapidly swaying torso places a reaction torque on the stance leg that counter-balances the force of gravity near foot strike and slows down the lower-body, and thus allows a collision-free foot strike.

The simplest ideal walker design would therefore be a compass-gait walker with a torso. In practice, provision for keeping the torso upright can be achieved using springs. The ideal walker design that we will be working on is shown in Figure 4.2.

### 4.3 The Simplest Ideal Walker

#### 4.3.1 The design and the equation of motion

An ideal walker, as we have defined it, means a walker that is able to maintain a steady legged locomotion while doing no work. On a downhill slope, the walker dissipates energy in footfall impacts and regains the energy from the force of gravity. Steady walking occurs when the energy gained equals the energy dissipated on each step. On level ground, steady walking can only occur if the feet touch the ground with exactly zero impact velocity.

The configuration of the simplest ideal walker is detailed in Figure 4.2. It has two straight and equal length legs each made of a massless rod with a point mass in the middle. The torso is a massless rod with a point mass at the top. The torso is stabilized around the hip with a matching pair of un-damped linear torsion springs. Each spring joins the torso to the leg.

With the use of sticky-foot constraint, the dynamics of the mechanical system are given by the equation of motion

\[
M_g(\theta)\ddot{\theta} + C_g(\theta, \dot{\theta})\dot{\theta} - G_g(\theta) = F_{\text{spring}}(\theta),
\]

where \( M_g \) is the mass matrix, \( C_g \) is the centrifugal matrix, \( G_g \) is the gravitational force, \( F_{\text{spring}} \) is the spring force, and \( \theta = (\theta_1, \theta_2, \theta_3)^T \) represents the configuration variables of the system. They are the orientation angles of the
stance leg, the swing leg and the torso respectively. The orientation angles are defined relative to the world vertical.

The mathematical expressions of $M_{g}$, $C_{g}$, $G_{g}$ and $F_{\text{spring}}$ are

$$M_{g}(\theta) = \begin{pmatrix}
\frac{1}{4}L^2 (5m_l + 4m_T + 4m_h) & -\frac{1}{2}L^2 m_l \cos (\theta_1 - \theta_2) & LRm_T \cos (\theta_1 - \theta_3) \\
-\frac{1}{2}L^2 m_l \cos (\theta_1 - \theta_2) & \frac{L^2 m_l}{4} & 0 \\
LRm_T \cos (\theta_1 - \theta_3) & 0 & R^2 m_T
\end{pmatrix},$$

(4.2)

$$C_{g}(\theta, \dot{\theta}) = \begin{pmatrix}
0 & -\frac{1}{2}L^2 m_l \sin (\theta_1 - \theta_2) \dot{\theta}_2 & LRm_T \sin (\theta_1 - \theta_3) \dot{\theta}_3 \\
-\frac{1}{2}L^2 m_l \sin (\theta_1 - \theta_2) \dot{\theta}_1 & 0 & 0 \\
-LRm_T \sin (\theta_1 - \theta_3) \dot{\theta}_1 & 0 & 0
\end{pmatrix},$$

(4.3)

$$G_{g}(\theta) = \begin{pmatrix}
gL \left(\frac{3}{2}m_l + m_T + m_h\right) \sin (\theta_1) \\
-\frac{1}{2}gLm_l \sin (\theta_2) \\
gRm_T \sin (\theta_3)
\end{pmatrix},$$

(4.4)

and

$$F_{\text{spring}}(\theta) = -k \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
1 & -1 & 0
\end{pmatrix} \begin{pmatrix}
\theta_3 - \theta_1 \\
\theta_2 - \theta_3
\end{pmatrix},$$

(4.5)

respectively, where $m_l$, $m_T$ and $m_h$ are the masses of the legs, torso, and hip respectively; $L$ and $R$ are the lengths of the legs and torso respectively; $k$ is the stiffness of the torso-leg spring, and $g$ is the gravitational acceleration. The numerical values of $m_l$, $m_T$, $m_h$, $L$, and $R$ considered in this chapter are shown in Table 4.1.

If the orientation angles are relative to the slope normal, the mass matrix and the centrifugal term remain unchanged because kinetic energy is invariant under rotation. The spring force term also remains unchanged too because spring forces depend on joint angles, which are invariant under rotation. However, the gravitational force term $G_{g}$ changes. It becomes

$$G_{g}(\theta) = \begin{pmatrix}
gL \left(\frac{3}{2}m_l + m_T + m_h\right) \sin (\theta_1 + \gamma) \\
-\frac{1}{2}gLm_l \sin (\theta_2 + \gamma) \\
gRm_T \sin (\theta_3 + \gamma)
\end{pmatrix},$$

(4.6)
4.3. **THE SIMPLEST IDEAL WALKER**

![Diagram](image)

**Figure 4.2**: A schematic diagram that details the design parameters and configuration variables of the ideal walker, which is made of two straight legs and a torso.

When the orientation angles are defined relative to the slope normal instead of the world vertical.

The collision impulse is calculated using the calculations outlined in Chapter 2. This involves the use of the mass matrix without the sticky-foot constraint $\mathbf{M}$ and the constraint normal of the sticky-foot constraint $\mathbf{J}_g$. The mass matrix without the sticky-foot constraint is

$$
\mathbf{M} (\theta, \mathbf{x}) = \begin{pmatrix}
\mathbf{M}_g (\theta)
\mathbf{X} (\theta)
\Lambda
\end{pmatrix},
$$

where $\mathbf{x} = (x, y)^T$ is the position of the stance feet.

---

1Non-dimensional units
Table 4.1: Design parameters of the ideal walker

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Length</td>
<td>1</td>
</tr>
<tr>
<td>Distance between the leg COM and the hip</td>
<td>0.5</td>
</tr>
<tr>
<td>Leg mass</td>
<td>0.5</td>
</tr>
<tr>
<td>Torso length</td>
<td>0.5</td>
</tr>
<tr>
<td>Distance between the torso COM and the hip</td>
<td>0.5</td>
</tr>
<tr>
<td>Torso mass</td>
<td>0.5</td>
</tr>
<tr>
<td>Hip Mass</td>
<td>1</td>
</tr>
<tr>
<td>Torso-leg spring stiffness (both)</td>
<td>$375.05$</td>
</tr>
<tr>
<td>Torso-leg spring equilibrium angle (both)</td>
<td>$\frac{98.1}{\pi}$</td>
</tr>
</tbody>
</table>

The constraint normal of the sticky-foot constraint ($J_g$) is

$$J_g(\theta, x) = \begin{pmatrix} L \cos (\theta_1) & -L \cos (\theta_2) & 0 & 1 & 0 \\ -L \sin (\theta_1) & L \sin (\theta_2) & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (4.10)

By conservation of generalized momentum, the pre-strike velocity ($\dot{\mathbf{q}}^-$), and post-strike velocity ($\dot{\mathbf{q}}^+$) are related by

$$\dot{\mathbf{q}}^+ = \left( \mathbf{I} - \mathbf{M}^{-1} J_g^T (J_g M^{-1} J_g^T)^{-1} J_g \right) \dot{\mathbf{q}}^-,$$  \hspace{1cm} (4.11)

where $\dot{\mathbf{q}} = \left( \dot{\mathbf{\theta}}, \dot{\mathbf{x}} \right)^T$, and $\mathbf{M}$ is the mass matrix without the sticky-foot constraint (Equation 4.7).

After foot strike, support-transfer takes place and the stance leg and the swing leg swap roles, so the pre-impact ($\mathbf{\theta}^-$), and post-impact ($\mathbf{\theta}^+$) joint angles are related by
4.3. THE SIMPLEST IDEAL WALKER

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \theta^+ = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \theta^- = W \theta^-.
\]

\[\text{(4.12)}\]

4.3.2 A non-exhaustive searching routine for periodic walking gaits on downhill slopes

Equation of motion estimator

When the orientation angles (with respect to slope normal), the angular velocities and accelerations of the limbs are all small, we can linearize the equation of motion to approximate the dynamics of the system. The linearized equation of motion of the ideal walker \( \Theta(t) \) is,

\[
\ddot{\Theta} = Q \Theta,
\]

where

\[
Q = \hat{M}_g^{-1} K,
\]

\[\text{(4.13)}\]

\[\text{(4.14)}\]

\[
\hat{M}_g = \begin{pmatrix}
\frac{1}{4} L^2 (5m_l + 4m_T + 4m_h) & -\frac{1}{2} L^2 m_l & LRm_T \\
-\frac{1}{2} L^2 m_l & \frac{L^2 m_l}{4} & 0 \\
LRm_T & 0 & R^2 m_T \\
\end{pmatrix},
\]

\[\text{(4.15)}\]

\[
K = \begin{pmatrix}
gL \left( \frac{3}{2} m_l + m_T + m_h \right) - k & 0 & k \\
0 & -\frac{1}{2} gLm_l - k & k \\
k & k & gRm_T - 2k \\
\end{pmatrix}.
\]

\[\text{(4.16)}\]

The solution to the linearized equation of motion is

\[
\hat{s}(t) = e^{\hat{A}t} \eta_g,
\]

where:

\[
\hat{s}(t) = \left( \Theta(t)^T, \dot{\Theta}(t)^T \right)^T
\]

\[\text{(4.17)}\]
and

\[ \hat{A} = \begin{pmatrix} 0 & I_{33} \\ Q & 0 \end{pmatrix}. \]

**Gait period estimator**

Foot strike occurs when \( \theta_1 = -\theta_2 \) and \( \theta_1 < 0 \). Hence we can obtain the approximated gait period, \( \hat{\tau} \), by solving for \( t \) so that \( \Theta_1(t) = -\Theta_2(t) \). This equation contains exponential and trig terms, and in order to obtain a closed-form expression for the gait period estimator \( \hat{\tau} \), we need to make some assumptions.

In the limiting case that the legs are planted firmly on the ground (\( \theta_1 = \theta_2 = 0, \dot{\theta}_1 = \dot{\theta}_2 = 0 \)), the motion of the torso is equivalent to the motion of a spring-mounted inverted pendulum. Therefore, if the legs move slowly and the step size is small, we can expect that the dynamics of the torso closely resembles the dynamics of a spring-mounted inverted pendulum.

Since periodic walking requires the torso to return to its initial orientation at the end of a step, we can estimate the gait period using the period of torso oscillation. The dynamics of the torso are given by the ordinary differential equation (ODE)

\[ m_T R^2 \ddot{\Theta}_3 = -2k\theta_3 + m_T g R \sin(\Theta_3). \] 

(4.18)

By small angle approximation, the ODE above can be simplified to

\[ \ddot{\Theta}_3 = - \left( \frac{2k - m_T g R}{m_T R^2} \right) \Theta_3. \]

(4.19)

The solution to the simplified ODE is

\[ \Theta_3(t) = c \sin \left( \sqrt{\frac{2k - m_T g R}{m_T R^2}} t \right). \]

(4.20)

Hence the approximated gait period \( \hat{\tau} \) is given by

\[ \hat{\tau} = 2\pi \sqrt{\frac{m_T R^2}{2k - m_T g R}}. \]

(4.21)

This approximation is accurate only when the step size is small and the legs move slowly.
4.3. THE SIMPLEST IDEAL WALKER

Momentum equation estimator

By combining Equation 4.11 and Equation 4.12, we can show that the pre-strike state $\eta_\text{g}^-$ and post-strike state $\eta_\text{g}^+$ are related by

$$\eta_\text{g}^+ = H (\eta_\text{g}^-) \eta_\text{g}^-.$$  \hspace{1cm} (4.22)

The matrix $H$ is the support-transfer transformation matrix that defines the mapping between the mechanical states of the system before and after the support-transfer. We can obtain the full expression for $H$ by using the matrix terms in Equation 4.11 and Equation 4.12 and show that $H$ depends only on the joint angles $\theta^-$. However, we describe it a function of state (angles and velocities) for the purposes of presentation.

By combining together the gait period estimator (Equation 4.21) and the equation of motion estimator (Equation 4.17), we can show that the approximated pre-impact state is given by

$$\hat{\eta}_\text{g}^- = e^{A\tau} \eta_\text{g},$$  \hspace{1cm} (4.23)

where $\eta_\text{g}$ is the state of system at the start of the step.

Using Equation 4.22, we can show that the step-transition Poincare map can be approximated by the difference equation

$$\eta_{\text{g},n+1} = H \left(e^{A\tau} \eta_n\right) e^{A\tau} \eta_{\text{g},n},$$  \hspace{1cm} (4.24)

and from this difference equation we can show that the fixed-point condition is given by

$$\eta_\text{g} = H \left(e^{A\tau} \eta_\text{g}\right) e^{A\tau} \eta_\text{g}.$$  \hspace{1cm} (4.25)

Assuming that the step size is small, we can approximate the transcendental terms in $H$ using second-order Taylor series expansions. By doing so, we get a system of polynomial equations that can be written as

$$\eta_\text{g} = \hat{H} \left(e^{A\tau} \eta_\text{g}\right) e^{A\tau} \eta_\text{g},$$  \hspace{1cm} (4.26)

where $\hat{H}$ is the support-transfer transformation matrix $H$ with all transcendental functions approximated by their second-order Taylor series expansions.
Fixed point estimator

Because Equation 4.26 is a system of polynomial equations, it can be solved numerically using the built-in “NDSolve” function in Mathematica. The function NDSolve does not require the user to provide an initial root guess for, and it gives more than one root, including the trivial solution 0 and solutions with complex numbers.

The real, non-zero roots from NDSolve can be approximated fixed points. We can obtain the true fixed point by using the Newton search algorithm to correct the approximation errors. In this case, the Newton search algorithm is used as an “error-corrector” rather than as a searching tool.

In this way, the fixed point finding requires only one Newton’s search in the best case. If convergence fails during the correction step, or NDSolve does not yield any realistic root, we can plot $\Theta_1(t) - \Theta_2(t)$ to find a better $\hat{\tau}$ by using the Newton search algorithm.

Programmatically, we can use Equation 4.21 as an “estimator of the estimated gait period”, $\hat{\tau}$, to obtain $\hat{\tau}$. This can be done by solving the equation $\Theta_1(t) - \Theta_2(t) = 0$ using Newton’s search, and taking $\hat{\tau}$ as the initial guess.

In either case, we reduce the number of initial guesses in Newton’s search from $O(n^k)$ down to just $O(1)$.

4.3.3 Searching for the collision-free periodic gaits

The searching routine

In order to show that the bipedal walker in Figure 4.2 is an ideal walker with the parameter setting from Table 4.1, we need to show that a passive periodic gait exists, by finding a fixed point that has zero swing-foot velocity. On level ground, the gravitational potential energy cannot be used to compensate for the loss in kinetic energy during the support transfer, when the stance leg and the swing leg swap roles. When the moving swing foot becomes the stationary stance foot, an inelastic collision occurs and hence a loss in kinetic energy. This loss in kinetic energy can be avoided if the swing foot approaches the ground with zero velocity. Our model closely resembles the one proposed by Gomes and Ruina [40]. Therefore, to find a collision-free fixed point, we use their procedure.

The collision-free fixed point searching procedure proposed by Gomes and Ruina is based on mirror-reflection and time-reversal symmetries of the equation of motion when there is no dissipation of energy. We can find a gait trajectory
such that the second half of the gait cycle is the first half of the gait cycle under simultaneous spatial reflection and time-reversal, so half a step fully characterizes the full gait trajectory. More specifically, we solve for \((\alpha, \beta)\) such that

\[
F(\alpha, \beta) = f((\alpha, -\alpha, 0, 0, 0, \beta)^T) = (0, 0)^T.
\]

(4.27)

The function \(f\) has no closed form, it evaluates and reports \((\theta_2, \dot{\theta}_3)_{t=T}\), and \(T\) is a time at which \(\theta_1 = 0\). Starting with the initial condition \((\alpha, -\alpha, 0, 0, 0, \beta)^T\), the trajectory described by the equation of motion over \([0, 2T]\) is a one-cycle trajectory of the symmetric collision-free level-ground gait. We need to use the Newton search algorithm when solving for \((\alpha, \beta)\).

One might ask, apart from simplicity, is there a better reason why we limit the collision-free fixed point search to those with a vertically upright torso posture. From Chapter 3, we learned that a straight-leg bipedal collision-free foot strike implies that both legs are at rest at foot strike. So, at the moment the collision-free foot strike occurs, the total angular momentum of the biped relative to the contact point equals the angular momentum of the torso relative to the contact point. The moment arm that defines the angular momentum of the torso relative to the contact point changes after the support-transfer, as does the angle between the moment arm and the ground. (The moment arm is the vector from the contact point to the centre of mass of the torso. See Figure 4.3 for clarification.) This implies that the support-transfer is still impulsive although the swing foot reaches the ground with zero velocity. If the torso is vertically upright at the end of the step, the length of the moment arm remains the same before and after the support-transfer, and so does the angle between the moment arm and the ground. This implies a non-impulsive support-transfer, and a collision-free periodic gait can exist.

4.3.4 Existence of a passive gait on level ground and downhill

Our ideal walker (Figure 4.2) has a passive gait on level ground (Figure 4.4) with the design parameters shown in Table 4.1. The initial conditions that give the collision-free periodic gait (with 13 figure accuracy) are \(\theta_1^* = 0.961990712677\), \(\dot{\theta}_3^* = -13.19669791285\), \(\theta_2^* = -\theta_1^*\) and \(\dot{\theta}_1^* = \dot{\theta}_2^* = \theta_3^* = 0\). This suggests that with a torso, it is possible to overcome the limitation that a torso-less bipedal walker cannot walk on level ground passively, even under idealized conditions.

Our ideal walker has a stable downhill gait on a slope of 0.05 radians (Figure 4.5). The torso sways back and forth over a small range, and the maximum angle
The torso is not vertically upright:

The torso is vertically upright:

Figure 4.3: The total momentum of the system relative to the ground contact (large dot) is the momentum of the torso when the foot strike is collision-free. If the torso is not vertically upright at the collision-free foot strike, the momentum of the torso changes after the support-transfer, because the moment arm before the support-transfer \((r^-)\) and the moment arm after the support-transfer \((r^+)\) are different. However, this is not the case if the torso is vertically upright at the collision-free foot strike.
4.3. **THE SIMPLEST IDEAL WALKER**

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**Figure 4.4:**
(a) The level-ground collision-free gait trajectory of the ideal walker over one step cycle. Time is non-dimensional \( \tau = t/\sqrt{L/g} \) where \( L \) is the leg length. (b) Level-ground collision-free gait pattern of the ideal walker over one step. At the end of the step the torso returns to the initial orientation. After swapping labels on the stance and swing legs they are in the initial configuration. The generalized velocities of the body components also return to the initial value at the end of the step. Because the foot strikes with zero velocity, there are no impulses, the velocities at the end of the step are carried over, and a periodic gait occurs.
between the two legs is approximately 0.2 radians.

The stability of the passive downhill gait is studied by estimating the eigenvalues of the fixed point using finite-difference, and confirmed by using a 200-step simulation from a perturbed initial condition (Figure 4.5). If all the estimated eigenvalues are inside the unit disc, the system is stable. If at least one of the eigenvalues is outside the unit disk, the system may have a stable periodic multistep gait. In this case, the 200-step simulation is used to check stability.

The 200-step simulation (Figure 4.6) shows that the fixed point is an attractor on the step-transition Poincare map. The error in each state due to the perturbation reduced to near zero after about 20 steps. This agrees with the approximated maximum eigenvalue modulus of 0.7834.

4.3.5 The stability and the walking speed of the ideal walker downhill gait in comparisons to the compass-gait walker

There are many stable downhill gaits of the ideal walker exist over a wide range of spring constants (Figure 4.7). The walking speed increases as the spring constant increases, but the stability reduces. When the slope angle is between 0.01 and 0.05 radians, both the stability and the walking speed of the biped increase as the slope angle increases (Figure 4.7).

We compared the non-dimensional walking speed and the dynamic stability of the passive downhill gait between the ideal walker and the massive-hip compass-gait walker. The massive-hip compass-gait walker is the simplest biped proposed by Garcia [39]. It has a hip mass significantly larger than the foot mass. After non-dimensionalization, the foot-to-hip mass ratio can be assumed as infinitesimal, thus leaving no free design parameters. Therefore the passive dynamic walking gait of the simplest biped is a very good benchmark for studying how modifications in the passive biped design affect the various aspects of passive dynamics walking.

The non-dimensional walking speed is defined as $v/\sqrt{gL}$, where $L$ is the leg length, $v$ is the walking speed defined as the steady-state step length divided by the steady-state gait period and $g$ is the gravitational acceleration (9.81). Non-dimensional walking speeds of the massive-hip compass-gait walker on slopes ranging from 0.005 to 0.05 radians are much slower than speeds of the ideal walker (Fig. 6). The ideal walker is more stable and much faster than the compass-gait walker on slopes greater than 0.014 radians.
4.3. THE SIMPLEST IDEAL WALKER

Figure 4.5: (a) The stable gravity-powered gait trajectory of the ideal walker over one step cycle on a 0.05 radians slope. (b) The gait pattern of the ideal walker over one step on a 0.05 radians slope.
Below 0.014 radians, both walkers are stable and the ideal walker is still much faster. The maximum eigenvalue moduli show that the compass-gait walker is slightly more stable (Figure 4.8). Once the slope reaches 0.014 radians, the compass-gait walkers stability drops dramatically. The gait becomes unstable after 0.015 radians, and the instability increases as the slope angle increases. As for the ideal walker, when the slope angle increases beyond 0.015 radians, the eigenvalues of the fixed point remain in the stable range. So overall, the ideal walker is a more stable design, in that it has stable gaits over a wider range of conditions.

A massive-hip compass-gait walker can only tolerate slopes below 0.015 radians. The observation that the ideal walker possesses a stable gait on a 0.05 radians slope means that the slope tolerance can be significantly improved when a torso is included. A bifurcation analysis (Figure 4.9) together with additional stability analysis (Figure 4.10) shows that stable period-one gait persists until the slope reaches 0.72 radians, which is steeper than the world’s steepest street (Baldwin Street in Dunedin, New Zealand). Beyond 0.72 radians, the period-one gait becomes unstable, but a stable period-two gait can be found until 0.77 radians.
4.3. **THE SIMPLEST IDEAL WALKER**

Figure 4.7: (a) Non-dimensional walking speed of the periodic gait and (b) log of maximum eigenvalue magnitude of the fixed point of the same periodic gait as functions of slopes for the ideal walker and the massive-hip compass-gait walker. The torso mass and length are 0.5 and 0.6 (dimension less) respectively.
Figure 4.8: (a) Non-dimensional walking speed and (b) maximum eigenvalue magnitude as functions of non-dimensional spring constant on different slopes. The dimension less torso mass and length are 0.5 and 0.6, respectively. The design parameters are as detailed in Table 4.1.
Figure 4.9: Bifurcation diagram on slope size (0.65 to 0.77 radians) and $\theta_1$, demonstrating the effect of slope size on the stability of period-one passive downhill gait.
For slopes between 0.05 and 0.65 radians, the non-dimensional walking speed increases as the slope becomes steeper and the increase in walking speed does not cause the walker to lose stability (Figure 4.10). The stability reaches a maximum on a 0.14 radians slope.

4.3.6 Zero-limiting work to maintain steady walking on level ground

Because the collision-free gait on level ground is dynamically unstable, if the gait does not start at the exact collision-free fixed point, the system cannot return to the collision-free trajectory without control, regardless of how small the perturbation is. The instability of the level-ground collision-free gait is mainly because that once the system deviates from the collision-free trajectory; an inelastic collision takes place, but on level ground gravitational potential energy is not available to compensate for the collision losses.

If the gait starts with an initial state that deviates from the collision-free fixed point by a small amount, then, depending on the controllability of the collision-free gait, it may be possible to achieve a controlled periodic gait that closely resembles the passive collision-free gait by applying impulsive controller forces to the biped at the start of each step, so that periodic walking can be restored.

Although it seems intuitive that a small perturbation error implies that the controller has to do a small amount of work to maintain steady walking, this intuition is misleading. For a collision-free periodic gait, the velocity of the foot must decrease to zero at the contact point and therefore, assuming that the foot reaches the ground from above, there is a local minimum or an inflection in the trajectory at that point. As a consequence, in the level-ground model, it is possible that an arbitrarily small perturbation will cause the foot to swing past the contact point and subsequently collide with a large impact.

Therefore, we need to determine the control impulses required to achieve periodic walking after perturbation and calculate the work done by the impulses. Below we describe in detail how the impulse calculation is done.

The biped system we are using is the ideal walker detailed in Figure 4.2 and Table 4.1. We assume that in the current step, at the beginning the biped walker is on its collision-free periodic gait trajectory, then perturbation occurs at some time during the step. We assume that the perturbation is known, and that no further perturbations occur during the step, nor during the rest of the journey.
4.3. THE SIMPLEST IDEAL WALKER

Figure 4.10: Plots showing (a) the non-dimensional walking speed of the periodic gait and (b) the maximum eigenvalue modulus of the fixed point of the same periodic gait of the ideal walker on steeper slopes. The design parameters are as detailed in Table 4.1.
The initial state at the start of the next step, $\eta_{g,n+1} = (\theta_{n+1}, \dot{\theta}_{n+1})$, and the initial state at the start of the current step, $\eta_{g,n} = (\theta_n, \dot{\theta}_n)$, are related by

$$\eta_{g,n+1} = S^c(v_n, \eta_{g,n}).$$  \hspace{1cm} (4.28)

Here, $S^c$ is the controlled step-transition Poincare-map, with an impulsive control signal, $v_n$, applied at the start of the $n$th step. In terms of the standard definition of the step-transition Poincare-map $S$, it can be written as

$$S^c(v_n, \eta_{g,n}) = S(\eta_n + (0, M^{-1}_g(\theta_n) v_n)).$$  \hspace{1cm} (4.29)

After being perturbed away from the unstable collision-free trajectory, control impulses need to be generated at the start of each step to ensure periodic walking. For the controlled step-transition Poincare-map, the periodic walking condition satisfies

$$\eta_g = S^c(v, \eta_g).$$  \hspace{1cm} (4.30)

Suppose that the current step (step 0) started with the collision-free fixed point, $(\eta_{g,0} = \eta^*_g)$, and a perturbation happened some time during this step. We can assume this perturbation is known. Supposing also that as a result of this perturbation, at the start of the next step, the biped will be in the state $\eta_{1,g} = (\theta_1, \dot{\theta}_1)$, we can find an impulse $v_1$ such that

$$S^c\left((\theta_1, \dot{\theta}_1), v_1\right) = \left(\theta_1, \omega(\theta_1, \dot{\theta}_1, v_1)\right).$$  \hspace{1cm} (4.31)

We want to find a control impulse such that, if it is applied at the start of the first step, then at the start of the second step, the pose of the biped returns to the pose at the start of the first step ($\theta_2 = \theta_1$). The velocity at the start of the second step will be dependent on the initial state and the control impulse calculated ($\dot{\theta}_2 = \omega(\theta_1, \dot{\theta}_1, v_1)$).

The control impulse $v_1$ can be found using Newton’s search algorithm. After finding the control impulse $v_1$, the velocity at the start of the second step $\omega(\theta_1, \dot{\theta}_1, v_1)$ can be found by integrating the equations of motion from the initial state $\left(\theta_1, \dot{\theta}_1 + M^{-1}_g(\theta_1) v_1\right)$, until the swing foot reaches the ground, and then we apply the standard support-transfer transformation on the terminal state.
4.3. THE SIMPLEST IDEAL WALKER

At the start of the second step and onwards we want to find a control impulse and apply it consistently at the start of every step such that the periodic walking condition (Equation 4.32) can be satisfied.

\[
S^c \left( \left( \theta_1, \omega \left( \theta_1, \dot{\theta}_1, v_1 \right) \right), v_n \right) = \left( \theta_1, \omega \left( \theta_1, \dot{\theta}_1, v_1 \right) \right), (n \geq 2). \tag{4.32}
\]

Given that the perturbation is known, we know \( \left( \theta_1, \dot{\theta}_1 \right) \), and can calculate the first control impulse \( v_1 \) so that \( \theta_2 = \theta_1 \). Therefore the required control impulse \( v_n \) can be calculated as

\[
v_n = M_g(\theta_1) \left( \dot{\theta}_1 - \omega \left( \theta_1, \dot{\theta}_1, v_1 \right) \right). \tag{4.33}
\]

In the above expressions, the impulse \( v \), the mass matrix \( M_g \), the configuration variables and the generalized velocity are functions of \( \theta = (\theta_1, \theta_2, \theta_3) \). The sticky-foot constraint is enforced.

Under periodic walking conditions, the average size of the impulse \( \|v\|_{av} \) per step in the long run satisfies

\[
\|v\|_{av} \propto \sqrt{v^T v}. \tag{4.34}
\]

This quantity reflects the amount of work required to retain periodic walking. We want to show that \( \|v\|_{av} \) goes down to zero as the size of perturbation approaches zero. If a perturbation affects the initial velocity but not the initial configuration, then apply an impulse of the same size but in the opposite direction to the perturbation. In this case \( \|v\|_{av} \) will certainly go down to zero as the size of the perturbation approaches zero.

As for perturbations that affect the initial configuration, simulation results show that when the error \( \Delta \phi \) in the angle between the two legs decreases, the average size \( \|v\|_{av} \) of the impulse required to maintain periodic walking goes to zero (Figure 4.11). This implies that the actual amount of work required for maintaining periodic walking also goes to zero as the perturbation decreases.

A similar observation is made if the error is in the torso orientation \( \Delta \theta_3 \). However, in practice, the actual cost for maintaining periodic walking, and the effectiveness of the control scheme may also depend on certain factors not included in this simple model.
4.4 Discussion

The massive-hip compass-gait walker proposed by McGeer [36, 10] has no torso, but it can walk downhill without actuation, and passively correct for small perturbations without intelligence. With the torso added, the walker can still walk downhill stably, and does not lose the ability to handle perturbations without intelligence.

Compared with the torso-less compass-gait walker, the addition of the torso to a straight-leg biped walker can provide the biped with the following mechanical advantages:

1. **The ability to walk passively on level ground with non-infinitesimal speed**: A compass-gait walker cannot walk passively on level ground with non-infinitesimal speed at all, even as an unstable gait. On level ground, the zero-cost gait is unstable, but the work required for maintaining periodic walking after being perturbed goes to zero as the size of perturbation goes to zero.

2. **The ability to walk stably on steeper slopes**: The stability of the passive walking gait is no longer sensitive to the slope angle after adding...
3. **The ability to walk faster with the same cost of transport without losing stability:** On each slope, after adding the torso, the walking speed increases, and the gait is still dynamically stable.

To clarify the third point, the mass-specific cost of transport can be defined as the energy consumed per unit distance per unit mass. For a walking robot walking passively and periodically by consuming the gravitational potential energy, the cost of transport $C_T$ is given by $C_T = \sin(\gamma)$, where $\gamma$ is the slope angle. It is universal for all designs and independent of the mass and the step length of the periodic gait. If the slope is shallow, then $C_T \approx \gamma$. Therefore, from Figure 4.8 there is clear evidence that with a torso, the walker can walk faster with the same cost of transport.

The motion of the ideal walker on the level ground resembles a wheel. An ideal 2D wheel rolling on level ground has the following properties of motion:

1. The essential cost of transport is zero, because the direction of motion is perpendicular to the direction of gravity. Therefore, like an ideal wheel on perfectly-level ground, the ideal biped can walk steadily without using energy and without requiring control.

2. In reality, an ideal wheel cannot exist, and a perfect environment is impossible, therefore a wheel cannot keep up with the steady motion due to the environmental imperfections that cause the total energy to be lost through inelastic collisions, friction, etc..

3. If we can add energy to compensate for the energy loss due to environmental imperfection, we can keep the motion of the wheel steady.

4. The amount of work required to maintain the steady motion approaches zero as the environment becomes ideal.

The ideal walker has a passive but unstable gait on level ground, which implies that the ideal walker satisfies Properties 1 and 2. The fact that the ideal walker possesses a stable, gravity-powered gait on downhill slopes implies that it satisfies Property 3. On level ground, it is always possible to reproduce a stable downhill walking gait by using the control mechanism proposed by Spong and Bullo [41], or by using potential energy shaping [66, 62]. Figure 4.11 implies that the ideal walker has Property 4 as well. Therefore, the motion of the ideal walker on the level ground resembles a rolling wheel.
The only exception is that in 2D, a wheel cannot fall over, but a biped can. In the case of a wheel, the energy can be added in many ways, but with a biped walker, energy can only be added in limited ways, subjected to the constraint that the walker must not fall over. In our research, this was achieved by finding a toe-off impulse at the start of each step, such that periodic walking can be maintained. With the use of this one-per-step impulsive control, the trajectory is no longer collision-free, but at the start of each step the control impulse adds energy to the system, and compensates for the energy loss due to the inelastic ground collision. Steady walking is achieved in a similar way on a downhill slope.

In conclusion, a good integration of the torso and the lower-body can improve the stability and the efficiency of walking gaits. For the case of a pair of straight legs, the addition of the torso makes the bipedal walker behave as an enhanced version of McGeer’s compass-gait walker on downhill slopes, and behave like a rolling wheel on level ground. The findings in this research disagree with McGeer’s suggestion that the torso may need to be perfectly stabilized using controls. The findings also suggest that the feedback controls or the kinematic constraints that deal with the undesired effects of torso swaying are in fact unnecessary. However, they may still be useful if one wants to improve the gait stability further by trading off transport costs, or to meet design requirements set by the client that may not be compatible with passive stability.
Chapter 5

Human-like passive bipedal walker

5.1 Passive biped with arms

5.1.1 Modelling the effect of arms

The first section of this chapter (Section 5.1) is published as part of the paper by Te-yuan Chyou, Gerrard Liddell and Mike Paulin titled “An upper-body can improve the stability and efficiency of passive dynamic walking”. The paper was published in Journal of Theoretical Biology 285(2011) 126-135.

Can we add weighted hands to an ideal walker? Previously we have assumed that the arms are much lighter than the torso, so they have negligible inertial effects on the ideal walker. For hands with significant weights, the inertial effects of the weights will change the dynamics of the ideal walker. To investigate the effect of arms, we model each arm as a pendulum attached to the torso. The arms swing freely, so there will be four more states in the equation of motion and the fixed points. They are the orientation angles and the angular velocities of the arms. In this chapter, the design parameters of the arms are shown in Table 5.1.

In order to walk passively and periodically on level ground, the support-transfer must be non-impulsive. So, when finding the fixed points that give passive collision-free periodic walking gaits, we require the total momentum of the system relative to the stance foot to be conserved before and after the support-transfer. We extend the method proposed by Gomes and Ruina to find the collision-free fixed points for bipeds with torso and arms. We want to find a symmetric periodic gait where the second half of the gait cycle is the first half of the gait trajectory under simultaneous spatial reflection and time reversal. The calculations involved are the same except that we solve for \((\alpha, \beta, \zeta, \varphi)\) so that
Table 5.1: The design parameters of the arms

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between the torso-arm joint and the hip (both arms)</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>Arm length (both)</td>
<td>0.5</td>
</tr>
<tr>
<td>Distance between the arm COM and the torso-arm joint (both arms)</td>
<td>0.5</td>
</tr>
<tr>
<td>Arm mass (both)</td>
<td>1</td>
</tr>
<tr>
<td>Torso-arm spring stiffness (both springs)</td>
<td>39</td>
</tr>
<tr>
<td>Torso-arm spring equilibrium angle (both springs)</td>
<td>$\frac{981}{100}$</td>
</tr>
<tr>
<td>Torso-arm spring damping used for level-ground walking (both)</td>
<td>0</td>
</tr>
<tr>
<td>Torso-arm spring damping used for downhill walking (both)</td>
<td>$\frac{0.125}{10\sqrt{0.81}}$</td>
</tr>
</tbody>
</table>

$$F(\alpha, \beta, \zeta, \varphi) = f\left((\alpha, -\alpha, 0, 0, 0, 0, \beta, \zeta, \varphi)^T\right) = (0, 0, 0, 0)^T.$$ \hspace{1cm} (5.1)

The function $f$ is the same as the one described in Chapter 4, except that it evaluates and returns $(\theta_2, \theta_3, \theta_4, \theta_5)_{t=T}$, where $\theta_4$ and $\theta_5$ are the configurations of the arms.

In Chapter 4 we showed that a straight-leg biped with a torso but no arms can walk passively and stably down a slope. When the arms are connected to the torso by stiff springs with high damping, the dynamics of the biped approximately match the dynamics of the same biped without arms. Assuming that the fixed point is smoothly parameterized by the spring parameters, we can find a passive periodic gait for the armed biped by using the “variational method”.

### 5.1.2 Fixed-point searching using the variational method

The variational method relies on known fixed points for a special case of the design. The fixed point $\eta^*$ is a function of the design parameters $u$:

$$\eta^* = f\left(u\right).$$

We assume that $f$ is a continuous function of $u$, and we know that the design parameters $u_0$ give a fixed point $\eta_0^*$. When the design parameters are changed slightly, we can simply use $\eta_0^*$ as a search seed to find the new fixed point by using the Newton search algorithm. Sometimes the fixed point can be sensitive to changes in design parameters, and in this case we can use the first-order Taylor series
\[ \tilde{\eta}^* = \eta_0^* + \nabla f(u_0) (u - u_0) \]

to get a better approximation of the new fixed point \( \tilde{\eta}^* \).

For the case of the armed biped, we apply variational method to the arm mass, and the parameters of the torso-arm springs.

### 5.1.3 Existence of level-ground collision-free gait

Previously, we have demonstrated that when the arms are massless, the collision-free periodic gait looks rather unnatural (Chapter 4). After adding arms, collision-free periodic walking is still possible, but a torso with arms does not have to sway over an enormous angle, it only has to sway over a 10-degree range. The legs only need to be opened up to a maximum of 10 degrees (Figure 5.1). Overall, the collision-free periodic gait looks more natural than the collision-free periodic gait of Gomes-Ruina’s ideal biped (Figure 5.2).

The arms and torso set up an internal oscillation that prevents collision impulses at foot-fall. The torso starts out with a high velocity to allow the swing leg to lift up, and then the arm swing provides opposing torque on the torso that reduces the amplitude of the torso oscillation and at the same time provides the required balance. The arms swing over a wide angle, about 30 degrees in each direction. At the end, the collision-free periodic gait becomes more biologically inspired, and it would be possible for a person to mimic it, by swinging both arms straight and in unison, and goose-stepping the straight legs. This observation might explain why people swing their arms when walking, because arm swings make walking more efficient.

We then investigate how much work is required to maintain periodic walking when a small perturbation interrupts the dynamically unstable collision-free periodic walking. Similar to Chapter 4, we assume that we know when the perturbation will occur. In particular, we assume that at the start of journey the hip angle deviates away from the collision-free fixed point value slightly, but there are no further perturbations after the journey starts.

Because periodic walking gaits on level ground cannot dynamically be stable, maintaining periodic walking requires controls. To maintain periodic walking, we apply a control impulse at the start of each step. We assume that the perturbation only affects the angle between the legs \( \phi \).

---

1Non-dimensional units
The results show that when the error in the angle between the two legs ($\Delta \phi$) decreases, the average size of the control impulses required for maintaining periodic walking ($\| \nu \|_{av}$) goes to zero (Figure 5.3). This implies that (like the case of the Gomes-Ruina ideal biped; Chapter 4), the amount of work required for maintaining periodic walking also goes to zero as the amount of perturbation decreases.

As discussed in Chapter 4, if a perturbation affects the initial velocity but not the orientation angles of the limbs at the start of a step, then applying an impulse that counter-balances the velocity change is all that is needed to maintain periodic walking. In this case, $\| \nu \|_{av}$ will certainly go down to zero as the perturbation approaches zero.
5.1.4 Stability of passive downhill gaits

While arms can substantially reduce the “silliness” of collision-free walking, they also reduce the stability of passive downhill walking. A passive periodic gait is found on a downhill slope of 0.01 radians (Figure 5.4), but eigenvalue analysis shows that the maximum eigenvalue modulus is 1.0000, so the armed walker is only neutrally stable.

Stability is easy to achieve with a straight-leg biped with arms by adding damped springs to the arms. With a small amount of damping in the torso-arm springs (Table 5.1), we find a periodic gait on a downhill slope of 0.01 radians (Figure 5.5), and the trajectory looks similar to the trajectory of the same biped without damping. The maximum eigenvalue modulus is now 0.9367, indicating stability. A 200-step simulation confirms the stability because the step-transition Poincare-map converges to a single value (Figure 5.6).

5.1.5 The in-phase arm swing of the 2D armed model

In real life, humans swing arms in opposing directions. This may be due to the lateral plane motions of the body. With this simplified two-dimensional model,
the motions of the limbs are constrained to the sagittal plane, and this may explain why in the simulation, the arms swing together in phase. Modelling and analyzing a 3D legged locomotion model with a torso and two arms is computationally expensive because of the number of state variables necessary. However, we intend to study similar models in 3D in the future.

The observation that the arms swing together in phase with identical trajectories indicates that when the lateral plane balancing is provided, one arm is sufficient for passive walking on level ground or downhill. Balancing in the lateral plane can be achieved passively by using the double-leg support described in [52].

A broad and polygonal foot can also eliminate at least some of the lateral plane motion passively, and thus allow the lateral plane balancing to be achieved with reduced use of energy input and motor coordination.
Figure 5.4: The neutrally stable downhill gait trajectory of the armed ideal walker over a one-step cycle on a downhill of slope of 0.01 radians. The time is taken as the actual time. (a) The trajectories of the legs and torso, and (b) the trajectories of the arms are shown. The arms happen to swing in phase following identical trajectories.

5.2 Passive biped with knees

5.2.1 McGeer-type kneed walker with a torso

Two-dimensional passive bipedal walking with McGeer-type knees had been demonstrated both in mathematical simulations and physical models. Like the case of the straight-leg compass-gait walker, these kneed bipeds can walk passively and stably down a shallow slope, and with the right model parameters, stable and passive walking can be achieved without foot scuffing. The gait pattern closely resembles a typical bipedal walking gait.

A major drawback of McGeer’s kneed walker [32] is the lack of an upper-body. The model has a stable and passive gait that looks natural and biologically inspiring. It is probably just a lucky coincidence because it is impossible in reality to incorporate an upper-body into a leg-only biped without affecting the dynamics
CHAPTER 5. HUMAN-LIKE PASSIVE BIPEDAL WALKER

Figure 5.5: The stable downhill gait trajectory of the armed walker with arm-damping over a one-step cycle on a downhill of slope of 0.01 radians. The time is taken as the actual time. (a) The trajectories of the legs and torso and (b) the trajectories of the arms are shown. The arms happen to swing in phase following identical trajectories.

of the legs (Chapter 4). There is certainly a possibility that after adding an upper-body to McGeer’s kneed walker, the recoil forces from the torso-leg springs is too strong to allow the existence of a passive and stable periodic walking gait.

We want to explore whether by choosing the design parameters carefully, a biped with McGeer-type knees and a torso can walk stably and passively downhill.

5.2.2 McGeer’s knee model

Because the kneed model is quite complex, we use McGeer’s knee model to simplify the modelling. McGeer’s knee model is a set of simplifying assumptions based on the dynamics of the kneed biped designed by McGeer [32]. In McGeer’s kneed walker [32], each leg has a thigh and a shank, with a kneecap that prevents the knee from hyper-extending. With experimental evidence [32, 39], the dynamics of McGeer’s kneed biped can be described as follows:
Figure 5.6: A 200-step simulation of the armed walker with arm damping walking down a 0.02 radian slope, starting with a perturbation of size 0.001 on each state. Initial conditions at the start of each step are plotted. The configuration variables are shown in the left column, and the velocities are shown in the right column.
1. The biped starts with both legs straight. The knee of the stance leg is locked, and remains locked during the entire step.

2. Then, the swing foot lifts off and the knee in the swing leg flexes. The dynamics of the lower-body is equivalent to the dynamics of a passive inverted 3-link pendulum, where the links represents the stance leg, the swing-leg thigh and the swing-leg shank. The stance leg pivots freely on the ground. The thigh swings freely about the hip and the shank swings freely about the knee.

3. In mid-step, the swing leg straightens. The kneecap prevents the knee from hyper-extending, so the swing leg knee is locked up. This event is called a knee strike. It is assumed that the knee strike has infinitesimal duration and thus can be modelled as an inelastic collision. It is also assumed that knee strike happens before foot strike.

4. The post-knee-strike dynamics of the lower-body are equivalent to the dynamics of an inverted double pendulum. The dynamics of the system are similar to the dynamics of a straight-leg biped.

5. Support-transfer occurs when the swing foot strikes. At support-transfer, the swing foot pivots freely on the ground. The inelastic collision impulse changes the angular velocities of the limbs. The stance foot lifts off and the knee in the stance leg is unlocked. It is assumed that during the support-transfer, the knee-locked swing leg remains knee-locked, and that the support-transfer has infinitesimal duration.

6. After the support-transfer, the previously knee-locked swing leg becomes the new stance leg, and the knee-unlocked stance leg becomes the swing leg, and a new gait cycle starts.

McGeer’s knee model assumes that the walking gait of a kneed biped has the same dynamics.

If the knee-locking torque is positive, the knee remains locked because a positive knee-locking torque will move the limbs in the direction of hyper-extension, but this movement is prevented by the kneecap. With McGeer’s kneed biped, the mass distribution gives the correct dynamics so that the knee-locking torque around the stance leg knee remains positive during the entire step.

If the knee-locking torque is negative, then knee-locking requires controls. In animals, this can be achieved by contracting both the flexor and extensor muscles simultaneously [53, 54]. In some research (for example [62]), controllable knee joints are assumed so that we can lock or unlock the knees at any time. This
assumption will not defeat the principle of passive dynamic walking, because the role of the controller is to lock or unlock the knees, and the biped can still walk by letting the passive dynamics drive the limbs rather than the controller.

During the support-transfer, the knee-locking torque around the stance-leg knee must be negative in order to unlock the stance-leg knee. At the same time, the knee-locking torque around the swing-leg knee must be positive so that the knee-locked swing leg will remain locked during the support-transfer, and becomes the knee-locked stance leg after the support-transfer. Garcia [35] argues that passive bipeds with a McGeer-type knee may have mildly negative knee-locking torque at lift-off. This can be considered a minor imperfection in the passive walking trajectory, like foot scuffing and can be ignored provided that the time period during which the knee-locking torque has negligible length and the magnitude of the negative knee-locking torque is close to zero.

The impulse due to the inelastic ground collision at footfall can, in principle, change the knee-bending velocity of the knee-locked swing leg. In the case of McGeer’s kneed biped, because of its mechanical properties, the knee-bending velocity of the knee-locked swing leg is in the direction of hyper-extension during the support-transfer. The kneecap inhibits this motion, and hence the observation that the knee-locked swing leg remains knee-locked during the support-transfer.

To sum up, when the knee-locking torque is negative, the engineering of the knee control and the energetic cost required to lock or unlock the knee should ideally be explicitly modelled. In McGeer’s knee model the swing-leg knee-locking is simply modelled as a passive inelastic collision, and the knee-locked legs are treated as straight legs without a knee. These treatments are the same for positive knee-locking torques. For simplicity, in this chapter, we assume that the knee control is energetically cheap and can be achieved easily, so McGeer’s knee model can always be applied.

The use of McGeer’s knee model allows the following simplifications:

1. The knee-locked leg can be treated as a single rigid-body linkage; this simplifies the equation of motion and the momentum balancing equation. Furthermore, when searching for the fixed points, the dimension of the search space is reduced.

2. There is only one collision sequence to be dealt with: swing-leg knee-locking and subsequently the foot strike.

3. The assumption that the knee-locked swing leg remains knee-locked during the support-transfer and in the new step means that we do not need to take the stance-leg flexing into account.
4. The assumption that the support-transfer has infinitesimal duration implies that after foot strike we do not need to consider the dynamics over the double-stance phase and/or ground-grasping. They can simply be modelled as an inelastic ground collision.

Next, we describe how the walking gaits of a kneed biped with or without an upper-body can be modelled using the simplifying assumptions of McGeer’s knee model in greater mathematical detail.

### 5.2.3 Model formulation with the use of McGeer’s knee model

Consider a human-like biped: under the assumption that the motion is constrained to the sagittal plane, the configuration variables \( q \) include the stance foot position \((x, y)\), the orientation angles of the thigh and shank of the stance leg \((\theta_1, \theta_2)\), defined relative to the vertical axis of the reference frame of the global environment), the orientation angles of the thigh and shank of the swing leg \((\theta_3, \theta_4)\), the orientation angle of the torso \((\theta_5)\), and the orientation angles of the arms \((\theta_6, \theta_7)\). Let \( q = (x, y, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7) \). The dynamics of the kneed walker are given by the equation of motion

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} - G(q) = F_{\text{spring}}(\theta, \dot{\theta}) + \sum J_{f_{c,i}}^T F_{f_{c,i}}(q, \dot{q}),
\]

(5.2)

where \( \theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7) \).

The system may simultaneously be under more than one configuration constraint \((f_{c,i}(q) = 0)\) at one time, In particular, over the second half of the gait cycle the biped is simultaneously under knee-locking and sticky-foot constraints. So there may be more than one constraint force \((F_{f_{c,i}})\) acting on the system at any time and hence the summation. Matrix \( J_{f_{c,i}} \) is the constraint normal of the configuration constraint \((f_{c,i}(q) = 0)\).

To model the walking gait using McGeer’s knee model, we need to consider the sticky-foot constraint \((x, y) = c\) (where \( c \) is a constant vector), the knee-locking constraint \((\theta_1 - \theta_2 = 0)\) for stance-leg knee-locking, and \((\theta_3 - \theta_4 = 0)\) for swing-leg knee-locking, and the swing-foot locking constraint (the expression is lengthy, but has the general form \( \xi(q) = 0 \)) so that the sticky-foot constraint can be introduced and enforced in the new step.
5.3 Results

5.3.1 A stable periodic downhill gait of the McGeer-type kneed walker with a torso

We start by considering just the knees and the torso. We use McGeer’s knee model to model the dynamics of the knee. During the model development process, we were able to achieve a stable and passive periodic walk without a torso when the leg parameters are chosen to be the values shown in Table 5.2. This justifies the leg parameter choices. To achieve stable and passive periodic walking with a torso, we apply the variational method.

With a 0.5 m torso, we are unable to find a passive and stable periodic gait when the mass of the torso is greater than 0.0001 kg, nor an unstable one (data not shown). So with our next attempt we add damped springs around the knee.
Figure 5.7: A typical trajectory of a kneeed walker. During the step, there is an additional inelastic collision at the stance and swing leg alignments, and the anatomical structure of the knee prevents hyper-extension of the joint so the stance and swing legs move together after this point. The planes indicate the collision events (knee-locking and foot strike), the green lines indicate the impulses due to inelastic collisions, and the curves represent the gait trajectories between collision events. The step-transition Poincare map is taken on the heel strike condition hyperplane.
5.3. RESULTS

Table 5.2: The design parameters of the kneed biped with a torso

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shank length (both)</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Thigh length (both)</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Torso length</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Shank mass (both)</td>
<td>2.5</td>
<td>kg</td>
</tr>
<tr>
<td>Thigh mass (both)</td>
<td>5.0</td>
<td>kg</td>
</tr>
<tr>
<td>Torso mass</td>
<td>4.0</td>
<td>kg</td>
</tr>
<tr>
<td>Hip mass</td>
<td>5.0</td>
<td>kg</td>
</tr>
<tr>
<td>Shank rotational inertia (both)</td>
<td>0</td>
<td>kg.m²</td>
</tr>
<tr>
<td>Thigh rotational inertia (both)</td>
<td>0</td>
<td>kg.m²</td>
</tr>
<tr>
<td>Torso rotational inertia</td>
<td>0</td>
<td>kg.m²</td>
</tr>
<tr>
<td>Torso-leg spring stiffness (both)</td>
<td>41.0</td>
<td>N.rad⁻¹</td>
</tr>
<tr>
<td>Torso-leg spring equilibrium angle (both)</td>
<td>π</td>
<td>rad</td>
</tr>
<tr>
<td>Torso-leg spring damping</td>
<td>0</td>
<td>N.rad⁻¹.s</td>
</tr>
<tr>
<td>Knee-spring stiffness</td>
<td>1.5</td>
<td>N.rad⁻¹</td>
</tr>
<tr>
<td>Knee-spring equilibrium angle</td>
<td>π</td>
<td>rad</td>
</tr>
<tr>
<td>Knee-spring damping</td>
<td>4.5</td>
<td>N.rad⁻¹.s</td>
</tr>
</tbody>
</table>

joints and use the variational method to find a stable and passive periodic walking gait with a torso. When the parameters of the knee springs are the values shown in Table 5.2, we can achieve stable, passive and periodic kneed walking with a torso heavier than 0.0001 kg. Finally, by setting other model parameters to the values shown in Table 5.2, we find a passive and stable periodic gait on a downhill slope with a 4.0 kg torso. The trajectory of this periodic gait is shown in Figure 5.3 and the fixed point of this periodic gait is shown in Table 5.3. The maximum eigenvalue modulus of this fixed point is 0.8867, indicating stability. A 150-step simulation demonstrates the convergence of the step-transition Poincare map (Figure 5.8) and confirms the stability (Figure 5.9).

The knee flexes during the start of the gait are just like McGee’s kneed biped, but because there is damping, the swing-leg knee cannot flex by an amount large enough to allow a successful foot clearance. The amount of knee flexing increases as the stiffness and the damping reduces, but during the process of model development, we noticed that a 10% change in any knee spring parameters is sufficient to prevent the kneed biped from walking passively, stably and periodically. The choices for spring parameters are highly restricted, and this prevents us from finding a passive periodic gait that is stable and does not have foot scuffing. Nonetheless, the result suggests that stable periodic walking is still possible at this level of complexity.
CHAPTER 5. HUMAN-LIKE PASSIVE BIPEDAL WALKER

Figure 5.8: The stable downhill gait trajectory of the torso walker with McGeer-type knees over one step cycle on a slope of 0.02 radians. The time is taken as the actual time. A damped spring is mounted around each knee (a) The trajectories of the legs and torso and (b) the trajectories of the arms are shown. The arms happen to swing in phase following identical trajectories.

Table 5.3: The stable fixed point of the kneed biped with a torso.

<table>
<thead>
<tr>
<th>Initial state</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stance-leg orientation angle $\theta_1$</td>
<td>0.1826</td>
</tr>
<tr>
<td>Swing-leg shank orientation angle $\theta_2$</td>
<td>-0.2226</td>
</tr>
<tr>
<td>Swing-leg thigh orientation angle $\theta_3$</td>
<td>-0.2226</td>
</tr>
<tr>
<td>Torso orientation angle $\theta_4$</td>
<td>0.0008</td>
</tr>
<tr>
<td>Stance-leg angular velocity $\dot{\theta}_1$</td>
<td>-0.7338</td>
</tr>
<tr>
<td>Swing-leg shank angular velocity $\dot{\theta}_2$</td>
<td>-0.0809</td>
</tr>
<tr>
<td>Swing-leg thigh angular velocity $\dot{\theta}_3$</td>
<td>-1.9331</td>
</tr>
<tr>
<td>Torso angular velocity $\dot{\theta}_4$</td>
<td>-0.6858</td>
</tr>
</tbody>
</table>
Figure 5.9: A 200-step simulation of the torso walker with McGeer-type knees walking down a 0.02 radian slope, starting with a perturbation of size 0.001 on each state. Initial conditions at the start of each step are plotted. The configuration variables are shown in the left column, and the velocities are shown in the right column.
Table 5.4: The design parameters of the kneed biped with a torso and two arms

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shank length (both)</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Thigh length (both)</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Torso length</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Arm length (both)</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Torso-arm joint distance from hip (both)</td>
<td>( \frac{2}{3} )</td>
<td></td>
</tr>
<tr>
<td>Shank mass (both)</td>
<td>2.5</td>
<td>kg</td>
</tr>
<tr>
<td>Thigh mass (both)</td>
<td>5.0</td>
<td>kg</td>
</tr>
<tr>
<td>Torso mass</td>
<td>4.0</td>
<td>kg</td>
</tr>
<tr>
<td>Arm mass (both)</td>
<td>0.25</td>
<td>kg</td>
</tr>
<tr>
<td>Hip mass</td>
<td>5.0</td>
<td>kg</td>
</tr>
<tr>
<td>Shank rotational inertia (both)</td>
<td>0.1</td>
<td>kg.m(^2)</td>
</tr>
<tr>
<td>Thigh rotational inertia (both)</td>
<td>0.072</td>
<td>kg.m(^2)</td>
</tr>
<tr>
<td>Torso rotational inertia</td>
<td>0</td>
<td>kg.m(^2)</td>
</tr>
<tr>
<td>Arm rotational inertia (both)</td>
<td>0</td>
<td>kg.m(^2)</td>
</tr>
<tr>
<td>Torso-leg spring stiffness (both)</td>
<td>60.0</td>
<td>N.rad(^{-1})</td>
</tr>
<tr>
<td>Torso-leg spring equilibrium angle (both)</td>
<td>( \pi )</td>
<td>rad</td>
</tr>
<tr>
<td>Torso-leg spring damping (both)</td>
<td>0</td>
<td>N.rad(^{-1}).s</td>
</tr>
<tr>
<td>Torso-arm spring stiffness (both)</td>
<td>42.0</td>
<td>N.rad(^{-1})</td>
</tr>
<tr>
<td>Torso-arm spring equilibrium angle (both)</td>
<td>0</td>
<td>rad</td>
</tr>
<tr>
<td>Torso-arm spring damping (both)</td>
<td>0.15</td>
<td>N.rad(^{-1}).s</td>
</tr>
<tr>
<td>Knee-spring stiffness</td>
<td>20</td>
<td>N.rad(^{-1})</td>
</tr>
<tr>
<td>Knee-spring equilibrium angle</td>
<td>( \pi )</td>
<td>rad</td>
</tr>
<tr>
<td>Knee-spring damping</td>
<td>3.25</td>
<td>N.rad(^{-1}).s</td>
</tr>
</tbody>
</table>

5.3.2 Uncontrolled two-dimensional human model

As a further development, a 2D model of a human is also built. The model includes knees in the legs (modelled using McGeer’s knee model), an upper-body and two arms. Muscles are modelled as damped springs, wrapping around knees, between the torso and each leg, and between the torso and each arm.

With the design parameters shown in Table 5.4, there is stable and passive periodic walking on a downhill slope of 0.05 radians. The trajectory shows knee bending in the early phase of the gait cycle, and arms swinging in phase (Figure 5.10). The fixed point is shown in Table 5.5. The 200-step simulation confirms the dynamic stability of the passive gait (Figure 5.11). Still, the amount of knee flexing is too small to avoid foot scuffing.

Two dimensional models always suggest that the two arms are swinging in phase following the identical trajectories, rather than alternating as one might
Table 5.5: The stable fixed point of the kneed biped with a torso and two arms.

<table>
<thead>
<tr>
<th>Initial state</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stance-leg orientation angle $\theta_1$</td>
<td>0.2258</td>
</tr>
<tr>
<td>Swing-leg shank orientation angle $\theta_2$</td>
<td>-0.3258</td>
</tr>
<tr>
<td>Swing-leg thigh orientation angle $\theta_3$</td>
<td>-0.3258</td>
</tr>
<tr>
<td>Torso orientation angle $\theta_4$</td>
<td>0.0020</td>
</tr>
<tr>
<td>Arm orientation angle (Left) $\theta_5$</td>
<td>-0.0334</td>
</tr>
<tr>
<td>Arm orientation angle (Right) $\theta_6$</td>
<td>-0.0334</td>
</tr>
<tr>
<td>Stance-leg angular velocity $\dot{\theta}_1$</td>
<td>-0.8705</td>
</tr>
<tr>
<td>Swing-leg shank angular velocity $\dot{\theta}_2$</td>
<td>-0.2146</td>
</tr>
<tr>
<td>Swing-leg thigh angular velocity $\dot{\theta}_3$</td>
<td>-1.8423</td>
</tr>
<tr>
<td>Torso angular velocity $\dot{\theta}_4$</td>
<td>-1.2434</td>
</tr>
<tr>
<td>Arm angular velocity (Left) $\dot{\theta}_5$</td>
<td>-1.0639</td>
</tr>
<tr>
<td>Arm angular velocity (Right) $\dot{\theta}_6$</td>
<td>-1.0639</td>
</tr>
</tbody>
</table>

Figure 5.10: A stable downhill gait trajectory of the 2D mechanical model of a human with McGeer-type knees over one step cycle on a downhill of slope of 0.05 radians. Time is the actual time. A damped spring is mounted around each knee. (a) The trajectories of the legs and torso and (b) the trajectories of the arms are shown. The arms happen to swing in phase following identical trajectories.
Figure 5.11: A 200-step simulation of the 2D mechanical model of a human with McGeer-type knees walking down a 0.05 radian slope, starting with a perturbation of size 0.001 on each state. Initial conditions at the start of each step are plotted. The configuration variables are shown in the left column, and the velocities are shown in the right column.
expected. This implies that one of the arms is redundant if a perfect lateral stabilization is achieved, and the alternating arm swing might be due to the lateral plane dynamics of the body. We hope a 3D model will capture this feature.

The choices of spring parameters are still highly restricted in this model. A stable gait is highly sensitive to changes in the design parameters. In particular, a 1% change in any spring parameter is sufficient to prevent the passive gaits from existing (data not shown).

5.3.3 Scuff-free solution in 2D with upper-body: An unsolved problem

Passive, stable and scuff-free periodic walking gaits with a torso were not found, although we checked through a large set of design parameters. The difficulty we have is that once knees are introduced, the existence of a stable gait is highly sensitive to changes in design parameters, so we cannot increase the amount of knee flexing by adjusting the parameters of the knee springs. If it is true that the existence of passive, scuff-free and stable periodic walking gait is incompatible with the presence of a torso, then we can rationalize the walking gaits with scuffing as follows.

While foot scuffing is a problem, even with straight legs, the depth of the scuffing is usually insignificant compared to the length of the leg. With kneed legs, foot scuffing can be avoided by flexing the swing leg knee slightly. We do not expect that a controller will need to invest a great deal of effort to slightly flex a knee. Given that an armed biped with a torso and a pair of straight legs can walk passively, the amount of energy required for handling foot scuffing can be minimized, by producing a controlled but scuff-free gait that closely resembles the passive periodic gait with scuffing.

Because the existence of a stable gait is extremely sensitive to variations in the mechanical parameters, even if we can avoid knee-scuffing passively, controls may still be needed to handle the natural uncertainties, to ensure the stability of the walking gaits under most situations.

By rocking side-to-side a 3D model can avoid foot scuffing without knee flexing, although using both knee flexing and side-to-side rocking avoids foot scuffing more effectively. This suggestion will be investigated in Chapter 7.
5.4 Discussion

Arm swings make the collision-free periodic gait more natural, but they do not help with the level-ground instability. There are real-life examples where humans achieve a high-efficiency gait by carrying weights. For example, Maloiy [67] shows that it is plausible that some African women have developed a technique to achieve higher than normal efficiency in walking when carrying a load, and we surmise that the weight carrying resembles this armed model where the motion is coordinated by the internal oscillation set up by the torso and arms, that reduces the collision-loss, and thereby reduce the metabolic effort.

Without damping, the arms have a negative contribution to the stability of the walking gaits, although a neutral stability can still be achieved. Passive and stable downhill walking can be achieved by damping the arms slightly. While damping prevents the biped from walking passively on level ground, the biped cannot simultaneously have the ability to walk passively and stably downhill, and the ability to walk on without energy consumption. This probably indicates that if we want to build an ideal biped that can walk passively and stably on a downhill slope, a pair of straight legs with a single-DOF upper-body is the maximum complexity we can have. This makes sense biologically because in humans, the arms are significantly smaller and lighter than the torso.

A torso can improve the stability and efficiency in comparison to a compass-gait walker (Chapter 4). Once arms are added, damping is needed in order to walk passively and stably downhill. While damping prevents the biped from walking passively on level ground, controls are necessary so that the arm damping can be adjusted at the transition between level ground and downhill. When knees are added, damping in the arms and the knees are required in order to walk passively and stably downhill. Furthermore, the existence of a stable periodic gait is highly sensitive to changes in the spring parameters. This trend indicates that while adding in additional body components can improve the stability of the gait, it is not a false intuition that there may be an upper limit to the complexity of the design, over which the biped can no longer walk passively. However, it appears that this limit is not at the complexity of a 2D biped that closely resembles a human body.

Alexandra [59] pointed out that athletes in walking race walk fast by keeping the legs fairly straight, and bending the lower part of the back. So, although we have not found a scuff-free passive periodic gait with knees and torso, it is plausible that a kneed biped with a torso do not have any passive periodic walking gaits that are stable and scuff-free. So avoiding foot scuffing by knee flexing requires controls and to walk efficiently, bipedal animals tend to avoid
knee flexing as suggested [59].

However, it is also likely that by describing the dynamics of the knee and the dynamics of models that use support-transfer that are more complete than the oversimplified models we are using, passive gaits with larger knee flexing may start to appear. So far, we have extended McGeer’s leg-only bipeds to bipeds with an upper-body. The effect of the upper-body on the dynamics of the lower-body had been investigated, but the lower-body is still modelled under the conventional assumptions considered by McGeer [32]: knee-locked stance leg, infinitesimal double-support period, and that the swing-leg knee locks before footfall. These assumptions are widely used in passive dynamic walking research because they simplify the periodic gait searching in many ways. However they rule out other types of periodic gait that are physically realistic and more biologically inspiring. For example, in the presence of an upper-body, or multiple-link legs, it is possible for a gait with extended double-support phase to exist. Perhaps there are passive gaits in which the benefits of the upper-body and the knee can both be incorporated, but here they are ruled out due to the simplifying assumptions.

We have very limited computing resources during model development, and the simplifying assumptions is largely due to this limitation. The kneed model with upper-body components was developed on a computer with dated software and hardware: single 1-GHz CPU, 512Mb RAM without GPU or any high-performance computing engines. Mathematica 7.0 is used for the model construction and simulations. With the given hardware and software, simulation-based biped models with a pair of straight legs, a torso and two arms (or models with the same degrees of freedom) run well, but the speed and memory usage start to become problematic once knees are introduced. With the simplifying assumption that the stance leg straight is knee-locked, the swing leg knee locks before footfall and remains knee-locked during support-transfer, and arms have no elbows. These constraints allow the simulations to run normally.
Chapter 6
Passive quadruped walking

6.1 Introduction

6.1.1 Previous research on quadruped locomotion

Overview

This chapter will be published as a co-authored paper by Te-yuan Chyou, Gerrard Liddell and Mike Paulin titled “Passive dynamics in quadruped locomotion” in the near future. The paper will be published in Journal of Theoretical Biology, and the manuscript is currently in preparation.

Passive dynamics can be applied to walking machines, so that on a downhill slope they can walk passively and stably without relying on controls. This idea has been demonstrated for leg-only bipedal walkers of various types [10, 32, 37].

Current research on passive locomotion is restricted to bipeds, however, the effect of upper-body components and the dynamics of poly-pod locomotion have not yet been investigated in detail. In this chapter, how poly-pod locomotion was modelled previously and the passive dynamics of quadruped locomotion are described.

Spring-mass templates

Full and Koditschek [25] modelled multi-legged locomotion by treating the legs as a single spring and the body as a rigid-body. The lateral leg spring (LLS) template proposed by Schmitt and Holmes [31], and the spring loaded inverted pendulum (SLIP) template [68, 69], are modelling approaches based on the same idea. The LLS model is normally used to model walking gaits of sprawl-postured animals, and the SLIP model is used to model the walking gaits of standing-postured animals [1].
Poulakakis [70] modelled a running quadruped by using a variant of the SLIP template. In Poulakakis’s work, the quadruped was modelled as a 2D mechanical system made up of a rigid-body and two springy legs. with motion restricted in the sagittal plane. This idea came from the quadruped robot “Scout II”. Their work demonstrated that a quadruped system can move itself forward using passive dynamics. However, the legs need to be kinematically adjusted to the right pose in order to achieve stable walking.

**Ballistic walking with kinematic constraints**

Near-passive periodic locomotion can be achieved by applying impulsive forces during each step [71, 51, 10, 72, 73]. Formalsky et. al. [65] demonstrated that a variety of quadruped gaits can be attained by applying an impulsive force at the start of every step, and letting the system move passively during the step. However, how stable the walking gait is, and what will happen if the controller does not generate the exact forces, was not addressed. Furthermore, the authors assume that legs with matching roles (i.e. stance legs and swing legs) are always moving in phase but in reality this is not always the case.

**Central pattern generator**

With the poly-pod walking models that we have discussed so far, the legs are represented by a spring. To model the dynamics of each leg, we need to consider that in poly-pod walking gaits, a subset of legs function as stance legs and the rest function as swing legs. When more than one legs function as the stance legs or the swing legs, locomotion control involves controlling the phase and the relative timing of leg swings.

In order to keep leg motions in phase, previous researchers suggest that animals use a central pattern generator (CPG), [74, 13, 75, 76] which is a neural circuit in the central nervous system. The CPG signals can be modelled as van der Pol oscillator [77, 78]. Tokiwa [79] demonstrates that it is possible to generate various types of quadruped gaits by using a set of linearly-coupled van der Pol oscillators to generate CPG signals to keep different pairs of legs in phase during the course of motion. However, Tokiwa’s work did not account for the effects of passive dynamics.

It is possible to keep the dynamics of the legs in-phase without using controls by using the mechanical properties of a parallelogram linkage. Animals do not use such parallelogram mechanisms in their limbs, so this model implies that a controller is generating the required constraint forces.
6.1.2 Our goal

McGeer [10] showed that bipedal walkers can be designed such that with some idealizations, it can walk stably and passively. Whether a passive quadruped can be designed based on the same idea remains an open question.

In this research we look at the passive dynamic behaviour of the simplest quadruped. It is made of two 2D straight-leg leg-only bipeds, connected together by massless springs. We will then extend our investigation to a quadruped model made of a straight-leg passive biped with a torso, and a compass-gait walker, coupled together with springs.

We want to demonstrate that with some idealizations, these quadrupeds can walk passively, stably and periodically on downhill slopes, like McGeer’s compass-gait walker [10]. In particular we want to demonstrate that leg motions can be passively phase-locked, and the walking speeds of the biped subsystems can be passively regulated, so that the entire system walks periodically as a four-legged system.

Successful outcomes demonstrate the feasibility of using the inertial property of the mechanical design as part of the locomotor pattern generator for quadrupeds.

6.2 Modelling approach

6.2.1 The modelling framework and assumptions

Our quadruped model is made of two biped subsystems, connected together by massless springs. In this way, the gait dynamics of the entire quadruped can be modelled as the gait dynamics of two compass-gait walkers in tandem, each under the influence of some externally applied forces which are equivalent to forces from springs. See Figure 6.1 for clarification.

The discrete momentum-impulse gait model described in Chapter 2 is used to model the dynamics of each biped subsystems. Each biped subsystem has a stance leg and a swing leg. The sticky-foot constraint is enforced during a step, so that the stance leg of each biped subsystem pivots freely on the ground until the swing foot strikes. The swing leg of each biped subsystem swings freely about the hip. Infinitesimal double-support period is assumed, so that at foot strike the swing leg of a biped subsystem pivots freely on the ground, and the stance foot of the same biped subsystem simultaneously loses ground contact and the stance leg becomes the swing leg.
Figure 6.1: When a quadruped is made of two biped subsystems connected by a massless spring, because the spring is massless, we can view the mechanical design as two unconnected bipeds that are each under the influence of an applied force, and the force comes from the spring.

### 6.2.2 Definition of a quadruped step

A “quadruped step” starts when a swing foot lifts off, and ends when a swing leg strikes (see Figure 6.2 for further details). A step-transition Poincare map is taken when a new quadruped step starts. Because we assumed that the stance leg of each biped subsystem pivots freely on the ground, and that the foot strike is an inelastic collision, over a quadruped step there will always be two legs that are pivoting freely on the ground, and two legs that are swinging. In this way, we only have three foot-strike patterns to consider:

1. The swing foot of biped subsystem at the front strikes the ground and this biped subsystem enters a support-transfer phase. Sometime later the swing foot of the biped subsystem at the back strikes the ground and this biped subsystem enters a support-transfer phase.

2. The swing foot of biped subsystem at the back strikes the ground and this biped subsystem enters a support-transfer phase. Sometime later the swing
foot of the biped subsystem at the front strikes the ground and this biped subsystem enters a support-transfer phase.

3. Both swing feet strike at the same time, and both biped subsystem enter the support-transfer phase simultaneously.
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The start of a quadruped step: A biped subsystem (the rear one in this example) is in the lift-off (post-strike state) state.

Both swing legs are in the flight.

The swing foot (rear) lifts off.

Some time later a biped subsystem (the front one in this example) strikes the ground.

The biped subsystem that strikes the ground enters the process of support-transfer (the front one in this example).

The other biped subsystem strikes the ground, and after the support-transfer, the system returns to the state with a biped subsystem in the lift-off state.

Both swing legs are in the flight again.

Figure 6.2: A schematic diagram illustrating the definition of a quadruped step. Although it is possible that a biped subsystem can complete two or more steps before the other biped strikes, for simplicity, this possibility is not investigated.
Other gaits may be ruled out by the simplifying assumptions used in this chapter. For example, we can have a quadruped gait in which one of the biped subsystems has both feet in flight, while the other one has both feet on the ground over a non-infinitesimal period of time. We can also have a quadruped gait in which one of the biped subsystems takes two or more steps before the other biped subsystem finishes a step. However, these assumptions for simplification do not over-simplify the model, because they do not rule out the possibility that a perturbation can make the leg motions out of phase over a finite number of steps, and they do not rule out the existence of a stable periodic quadruped gait in which legs do not move exactly in phase either.

6.3 The simplest quadruped

6.3.1 Mechanical design

The simplest quadruped model considered in this chapter is made of two compass-gait walker subsystems and three damped linear springs, connecting the hips, the stance legs and the swing legs shown in Figure 6.3. There are two stance legs and two swing legs. The stance legs pivot freely on the ground without slipping or detaching, and the swing legs swing freely about the hips. The hip of the biped subsystem at the front is called a “shoulder”.

Figure 6.3: The design parameters and configuration variable of the simplest passive quadruped walker.
The design parameters are non-dimensionalized and each mass is normalized by the mass of the hip. The length of each leg is normalized by its own length so the value after non-dimensionalization is 1. The gravitational acceleration is normalized by the gravitational acceleration on Earth. In this way, each compass-gait walker subsystem has only one free parameter, which is the ratio between the mass of the leg and the mass of the hip. For the entire system, the free parameters are the ratio between the leg mass and the hip mass of each biped subsystem, the spring anchor points on the legs, and the spring parameters (i.e. stiffness and damping).

6.3.2 The equation of motion

Considering the simplest quadruped, made of two identical compass-gait walkers, under the assumption that the motion is constrained in the sagittal plane, the configuration variables \( q \) include the stance-foot positions of the biped subsystems at the front \((x_1, x_2)\) and at the back \((x_3, x_4)\), and the orientation angles of the stance and swing legs of the biped subsystems at the front \((\theta_1, \theta_2)\) and at the back \((\theta_3, \theta_4)\). Orientation angles of the legs are defined with respect to the world vertical. Let \( q = (x_1, x_2, x_3, x_4, \theta_1, \theta_2, \theta_3, \theta_4) \), the dynamics of the mechanical system are given by the equation of motion

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} - G(q) = F_{\text{spring}}(q, \dot{q}) + \sum J_{f_{c,i}}^T F_{f_{c,i}}(q, \dot{q}). \tag{6.1}
\]

As for the kneed biped mentioned in Chapter 5, the quadruped system may be under more than one configuration constraint \((f_{c,i}(q) = 0)\) at one time. In particular, during a step we enforce a sticky-foot constraint on the stance foot of the biped subsystem at the front, and another sticky-foot constraint on the stance foot of the biped subsystem at the back. So there may be more than one constraint force \((F_{f_{c,i}})\) acting on the system at one time and hence the summation. The matrix \( J_{f_{c,i}} \) is the constraint normal of the configuration constraint \( f_{c,i}(q) = 0 \).

With the sticky-foot constraint, we also need to consider the swing foot locking constraint. It is used when a swing foot strikes, so that the sticky-foot constraint can be enforced after a biped subsystem completes the support-transfer. The sticky-foot constraint is enforced by treating the stance-foot positions as constants. However, the introduction of the swing-foot locking constraint is accomplished by applying the constraint force, so that the effect of the swing-foot locking constraint on the system dynamics can be calculated.

Exactly which constraints are to be enforced or imposed, and on which biped subsystems, depends on which biped subsystem is striking the ground. When the
swing foot of a biped subsystem strikes, we release this biped subsystem from the sticky-foot constraint, and impose the swing-foot locking constraint on the swing foot of this biped subsystem.

The derivation of the spring force term \( F_{spring} (q, \dot{q}) \) requires some explanation. First we look at the conservative force \( F^{IW S} (q) \) and the damping \( F^{IW D} (q, \dot{q}) \) from the inter-biped springs. The potential energy stored in the inter-biped springs connecting the anchoring points \( p_i \) and \( p_j \), \( V_{ij} \), is defined as

\[
V_{ij} = -\frac{1}{2} K (\|w_{ij}\| - L_0)^2 ,
\tag{6.2}
\]

where

\[
w_{ij} = p_i - p_j .
\tag{6.3}
\]

The anchoring points of the springs can be expressed as functions of \( \theta \), \( p_k = f_k (\theta) \). In this way, the \( k \)th component of the inter-biped spring force is given by

\[
F_{k,ij}^{IW S} (q) = \frac{\partial V_{ij}}{\partial \theta_k} ,
\tag{6.4}
\]

or equivalently

\[
F_{k,ij}^{IW S} (q) = -K (\|w_{ij}\| - L_0) \frac{(\frac{\partial w_{ij}}{\partial \theta_k} \cdot w_{ij})}{\|w_{ij}\|} .
\tag{6.5}
\]

The energy dissipated due to viscous damping in the inter-biped springs connecting anchoring points \( p_i \) and \( p_j \) (\( D_{ij} \)) is given by

\[
D_{ij} = -\frac{1}{2} \lambda \left( \frac{d\|w_{ij}\|}{dt} \right)^2
\tag{6.6}
\]

Hence the inter-biped damping force is given by

\[
F_{k,ij}^{IW D} (q, \dot{q}) = \frac{\partial D_{ij}}{\partial \theta_k} .
\tag{6.7}
\]

Using Equation 6.6 we can show that

\[
F_{k,ij}^{IW D} (q, \dot{q}) = -\lambda \left( \frac{(\frac{\partial w_{ij}}{\partial \theta_k} \cdot w_{ij})}{\|w_{ij}\|} \right) \frac{(\frac{\partial w_{ij}}{\partial \theta_k} \cdot w_{ij})}{\|w_{ij}\|} .
\tag{6.8}
\]
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The expressions of the conservative and damping forces of the torsion springs around the hip and the shoulder, $\mathbf{F}_{TS}(\mathbf{q})$ and $\mathbf{F}_{TD}(\dot{\mathbf{q}})$, are

$$\mathbf{F}_{TS}(\mathbf{q}) = -\kappa \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \theta_1 - \theta_2 \\ \theta_3 - \theta_4 \end{pmatrix}$$

(6.9)

and

$$\mathbf{F}_{TD}(\dot{\mathbf{q}}) = -u \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 - \dot{\theta}_2 \\ \dot{\theta}_3 - \dot{\theta}_4 \end{pmatrix}$$

(6.10)

respectively.

Defining $\mathcal{S}$ as a set containing all inter-biped spring connections, denoted by $(i, j)$, the total spring force is given by

$$\mathbf{F}_{spring}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{(i,j) \in \mathcal{S}} \left( \mathbf{F}_{iWS}^{iWS}(\mathbf{q}) + \mathbf{F}_{ij}^{iWD}(\mathbf{q}, \dot{\mathbf{q}}) \right) + \mathbf{F}_{TS}(\mathbf{q}) + \mathbf{F}_{TD}(\dot{\mathbf{q}}).$$

(6.11)

When calculating $\mathcal{S}$, one needs to be careful that the indexing depends on the role of the legs (i.e. as a stance or swing leg). When foot strike does not happen simultaneously on the two biped subsystems, the role of the legs are swapped in one of the subsystem after it completes the support-transfer, but not in the other one, so the indexing changes and $\mathcal{S}$ will need to be re-calculated.

In particular, at the start of a quadruped step, there will be a period of time during which the inter-biped springs are linking stance leg to stance leg, swing leg to swing leg, and hip to shoulder. When one of the compass-gait walker subsystems encounters a foot strike, the legs swap roles. The roles of the stance and the swings legs of the other biped subsystem remain the same. So the inter-biped springs link stance leg to swing leg, and hip to hip.

### 6.3.3 Step-transition Poincare map

A step-transition Poincare map is taken every time a new quadruped step starts. If the swing leg of the biped subsystem at the front strikes first, then we take the Poincare map immediately after the biped subsystem at the back completes the support-transfer. If the swing leg of the biped subsystem at the
6.3. THE SIMPLEST QUADRUPED

back strikes first, then we take the Poincare sections immediately after the front biped subsystem completes the support-transfer. If the swing legs of both bipeds strikes simultaneously, then we take the Poincare map immediately after both biped subsystems complete the support-transfers.

In this way, given that the gait starts with state $\eta_{g,n}$, after the first biped subsystem strikes and the support-transfer is completed, the post-strike state $(\sigma_g)$ can be written as a function of $\eta_{g,n}$. After the second biped subsystem strikes and the support-transfer is completed, the post-strike state $(\zeta_g)$, can be written as a function of $\sigma (\eta_{g,n})$, that is, $\zeta (\sigma (\eta_{g,n}))$.

The two biped subsystem must function together as a quadruped, rather than as two independent bipeds. So we want to find a periodic quadruped gait in which the two biped subsystems walk at the same speed, rather than two bipeds walking at different speeds. Therefore, the distance between the two biped subsystems is important, and we want to track it to ensure that the quadruped has the dynamics we want. Therefore, we add the distance between the two stance feet to the state $\eta_{g,n}$.

The step-transition Poincare map of a quadruped takes the same form as that of a biped. Given that the quadruped step starts with state $\eta_{g,n}$, the state at the start of the next quadruped step is given by

$$\eta_{g,n+1} = S (\eta_{g,n}),$$  \hspace{1cm} (6.12)

where

$$S (q) = \zeta (\sigma (q)).$$  \hspace{1cm} (6.13)

A fixed point on the step-transition Poincare map $\eta^*_g$ satisfies $\eta^*_g = S (\eta^*_g)$, and can be found using the Newton search algorithm.

The behaviour of the step-transition Poincare map near the fixed point $\eta^*_g$ is observed by linearizing the step-transition Poincare map near the fixed point. After the linearization we get a linear system of difference equations

$$\eta_{g,n+1} = \eta^*_g + A (\eta^*_g) (\eta_{g,n} - \eta^*_g),$$  \hspace{1cm} (6.14)

where $A (\eta^*_g)$ is the Jacobian matrix of $\eta_g - S (\eta_g)$ at the fixed point. This constant matrix describes the linearly approximated local behaviour on the step-transition Poincare map near the fixed point, and can be approximated using first-order finite difference.
If the eigenvalues of $A(\eta^*_g)$ all lie within a unit circle on the complex plane, then the fixed point is stable. If at least one eigenvalue lies on or close to the unit circle, then the stability of the fixed point is uncertain, and requires verification using a multi-step simulation with perturbation at the start of the journey, and determine if the error due to the perturbation is growing or bounded in the long run.

If at least one eigenvalues of $A(\eta^*_g)$ lies outside a unit circle, then the fixed point is unstable, but the system may still possess an attractor in the form of a limit cycle on the step-transition Poincare map. This can also be verified using a simulation of at least 100 steps with perturbation at the start of the journey, to see if the error due to the perturbation is growing, or bounded.

6.3.4 The existence of passive periodic gait

When the biped subsystems at the front and back are identical, springs are connected as shown in Figure 6.3 and they have the same rest length $L_0$, the existence of a passive quadruped periodic gait is guaranteed, if each biped subsystem has a passive periodic gait characterized by a fixed point $\eta^*_g$. In this case one of the fixed point of the quadruped must take the form $(\eta^*_g, \eta^*_g, L_0)$. The location of this “biped-equivalent” fixed point on the step-transition Poincare map does not change if the stiffness and the dampings of the inter-biped springs changes.

If the quadruped starts the step with the biped-equivalent fixed point, each biped subsystem moves with its own passive periodic trajectory. Legs swing in parallel and in phase, the springs remain at the rest length and no forces are generated. The quadruped gait trajectory is therefore equivalent to a pair of rigidly-coupled bipeds.

For simplicity, we restricted the fixed point search to the biped-equivalent fixed points. We are interested in fixed points that take the form $(\eta^*_g, \eta^*_g, L_0)$. With this simplification, the search for a periodic quadruped gait is the same as the search for a periodic bipedal gait, and the dimension of the search space is significantly reduced. Then we adjust the spring stiffness to make the to maximize the stability of the fixed point.

When the biped subsystems are no longer identical, or when the inter-biped springs have different rest lengths, the quadruped can no longer have the biped equivalent of a periodic gait. In these cases, this simplified method of finding the periodic gait cannot be applied.
6.4 Simulations on the simplest quadruped

6.4.1 An example of passive stable trajectory

Our simplest quadruped is made of a pair of identical, spring-coupled compass-gait walker. Each leg is 1 unit long (dimensionless) \((L = 1)\) with a point mass of 0.5 units and is located in the middle \((m_l = 0.5)\) of each leg. The hip and the shoulder (i.e. the hip of the other compass-gait walker subsystem in the front) are point masses of 0.5 unit \((m_h = 0.5)\). Springs are massless, each with a spring constant of 0.1 (dimensionless), a damping constant of 1.5 and a rest length of 1. All springs are linear. Springs connecting the legs are anchored on the legs, at a distance of \(\frac{1}{8}\) of the leg length from the hip (or the shoulder) \((L_r = \frac{1}{8})\).

With the design parameter listed in Section 6.1, there is a passive and stable periodic. The fixed point of this periodic gait is \((\theta^*_1, \theta^*_2, \dot{\theta}^*_1, \dot{\theta}^*_2, \theta^*_3, \theta^*_4, L^*_0) = (0.1296, -0.1496, -0.2159, -0.1787, 0.1296, -0.1496, -0.2159, -0.1787, 1.0000)\), where \(L^*_0\) is the separation distance between the stance feet in the periodic gait. When there are no perturbations, the biped subsystems walk with identical periodic trajectories, with legs swing in phase. The swing legs strike simultaneously and the body springs remain at rest lengths and, and do not generate tension or damping over the entire step.

By starting the journey with an initial state that deviates slightly from the fixed point, early on in the journey, due to the perturbation there is a significant phase difference between the gaits of the two compass-gait walker subsystems, but the phase difference between the dynamics of the biped subsystems decreases (Figure 6.4a) as time passes on. By looking at the step-transition Poincare map, there is a clear evidence that the quadruped gait is converging towards a dynamically stable passive periodic quadruped gait. (Figure 6.4b and 6.4c). Figure 6.5 confirms that this quadruped gait is dynamically stable. In the steady-state, the trajectories of the two compass-gait walker subsystems are in phase (Figure 6.5a).

Furthermore, as the journey goes on, the distance between the stance feet converges to a length equals to the rest length of inter-biped springs. This implies that in the steady state the body experiences no stretch or compression (Figure 6.5b).

These results show that in the case of quadruped, stable locomotion can be achieved without controls, and furthermore the phase of leg motions can passively be regulated. A good mechanical design is all that is needed.
6.4.2 Stiffness of the body spring and the dynamics of the biped-equivalent periodic gait

Next, we investigate how the dynamic properties of the biped-equivalent periodic gait and the parameters of the inter-biped springs are related. The fixed point search only covers the biped-equivalent fixed points. A location of the biped-equivalent fixed point on the step-transition Poincare map does not change when the stiffness and the dampings of the inter-biped springs change, so the steady-state walking speed of the simplest quadruped will not change when the parameters of the inter-biped springs change. However, when the parameters of the inter-biped springs changes, the stability of the biped-equivalent fixed point will vary.

Figure 6.6 shows that the steady-state walking speed remains unchanged when the parameters of the inter-biped springs change, as expected. The stability of the quadruped can be maximized by adjusting the spring parameters. Most combinations of stiffness and damping give unstable gaits, but a small set of combinations give stable gaits and the optimal combination gives a stable period gait. The biped-equivalent quadruped gait is stable when the stiffness of the inter-biped springs is significantly lower than their damping. (Figure 6.6c). This suggests that in order to use the bipedal passive dynamics to power up a quadruped without losing stability, the coupling springs need to be damped.

6.5 Passive quadruped model with head and neck

6.5.1 Overview

So far, we have looked at the simplest quadruped model that is made up of two identical compass-gait walkers. With this model, we have only managed to find the biped-equivalent trajectory, that at the steady state the quadruped gait is essentially a pair of compass-gait walkers in tandem, and the body spring has no dynamic effect. In order for the gait to be stable, springs need to have a very low stiffness, but a very large damping. A large-scaled numerical search was undertaken to find whether there exists some body spring parameters which give stable quadruped gaits that are not biped equivalent, but no parameters were found so the results are not shown here.

However, for a real walking quadruped animal, the phase difference between the front and hind legs is not always zero. Perhaps the simplest quadruped walking model is too simple to capture this feature. Also, it seems that in the simplest model the stability is achieved in an inefficient way because of the use of damping.
6.5. PASSIVE QUADRUPED MODEL WITH HEAD AND NECK

As a further development, we investigate the effect of including the 5th degree of freedom, a head and a neck as one rigid-body, on the simplest quadruped model. We expect that as discussed in Chapter 4, this modification will introduce more mechanical advantages than disadvantages, and will offer a flexible range of dynamics.

6.5.2 The model

Mechanical design

A realistic quadruped also has a neck, a head and possibly a tail, although some quadruped animals have short or light tails, which have very little effect on the dynamic properties of the quadruped. Therefore, a more realistic quadruped model includes a compass-gait walker and a 3-link torso biped connected together by a spring joining hip to hip (Figure 6.7).

With the biped subsystem on the back, each leg is 1 unit long (dimensionless) \((L = 1)\) and a point mass of 0.5 unit is located in the middle \((m_l = 0.5)\) of each leg. The hip is a point mass of 0.5 unit \((m_h = 0.5)\). A spring is mounted around the hip and the stiffness and the damping are variable. The biped subsystem in the front is the 3-link biped as described in Chapter 4.

Support transfer rule and Poincare map

The support-transfer is modelled in the same way as that of the simplest quadruped. Each biped subsystem undergoes the conventional bipedal support-transfer process. The support-transfers may not necessarily be happening simultaneously.

As for the simplest quadruped, we take a Poincare section immediately after both biped subsystems completed the bipedal support-transfer.

6.5.3 Existence of passive and stable downhill gait

We are looking for a fixed point that gives a dynamically stable periodic gait. To find this fixed point, random initial conditions are generated and a 200-step simulation is performed for each initial condition. (Figure 6.8). When the initial condition allows the system to walk for 200 steps without falling over, the entire journey is recorded as a step-transition Poincare map. Stable periodic gaits and their fixed-points can be inferred from any Poincare maps that show signs of convergence.
We start by setting the spring constant of the inter-biped spring to 28, and use no damping. The stiffness of the torso-leg springs of the biped subsystem in the front are both set at $138/98.1$, and the stiffness of the hip spring of the biped subsystem on the back is $138/98.1$. There is no damping of the torso-leg and hip springs. With a set of 1500 random initial conditions on a slope of 0.01 radians, a stable periodic gait is revealed. When the journey starts from an initial state that is close to the fixed point of the periodic gait, the step-transition Poincare map is converging (Figure 6.8). The trajectory of this periodic gait is shown in Figure 6.9. Note that the motion of the biped subsystems at the front and at the back are slightly out of phase while the body oscillates with the movements of the head and neck.

The change in gravitational potential energy for each hip (1 unit of weight) is approximately 0.0018 and the maximum elastic potential attained by the spring is 0.0045, based on Figure 6.9a. The maximum amount of elastic potential energy stored in the mechanical system is 250 percent of the gravitational potential energy gained by each hip, or 55.6 percent of the gravitational potential energy gained by the entire system (4.5 unit of weight in total). Therefore the body spring makes a significant contribution to the formation of this periodic gait.

6.5.4 Partial basin of attraction analysis

The partial basin of attraction can still provide some indication on the size of allowable perturbation, and which body parts are more sensitive to perturbation. The size of an attractive basin provides information on how large, and in what direction the perturbation can occur so that the perturbation error can be self-corrected without the need for control. The basin of attraction has a dimension of 11, so it cannot be visualized easily and is computationally expensive to generate. So we calculate a partial basin of attraction instead (Figure 6.10). We take each body part in turn, perturb its initial state from the fixed-point value, while keeping the initial states of other body parts at the fixed-point values.

The partial basin of attraction (Figure 6.10) shows that at the initial state of each body part can be significantly perturbed without destroying the stable walking gait. The stance legs and the separation between stance feet are more sensitive to perturbation, perhaps perturbations on these states cause the body spring (i.e. the inter-biped spring) to deform. When the body spring is very stiff, perturbations may cause a great increase in the spring recoil force that disrupts the gait.

The torso part of the quadruped, which is the head and neck of the front biped subsystem, is the body part that is least sensitive to perturbations. For the
swing legs, the orientation angles are sensitive to perturbations, but the angular velocities are not (Figure 6.10).

### 6.5.5 Gait with extreme phase differences

Quadrupeds possess a variety of gaits, ranging from the pacing gaits when the movements of the two biped subsystems are approximately in phase, to galloping-like gaits when the movements of the two biped subsystems are largely out of phase. Here we demonstrate that by adjusting the properties of the body spring, gaits with extreme phase difference can be generated.

#### In-phase, one-beat gait

By setting the stiffness of the body spring to 10, and keeping the stiffness of both the torso-leg springs and the hip spring at 138/98.1, and the damping of all springs at zero, in the steady-state the matching legs are nearly in phase (Figure 6.11a) and the oscillations of the body spring and torso are approximately in phase during each step (Figure 6.11b).

To verify that this gait is dynamically stable, we run a 200-step simulation starting with an initial state that deviates slightly from the fixed point, and take the step-transition Poincare map. The step-transition Poincare map is converging, indicating that the pacing gait shown in Figure 6.11 is dynamically stable (Figure 6.12).

#### Two-beat gait with large phase difference

A “galloping” gait was obtained by setting the stiffness of the body spring to a low value, 0.0808, and the damping of the body spring to to 0.8200. We keep the stiffness of the torso-leg the hip springs at 138/98.1, and the damping of all springs at zero. The body becomes soft and critically damped. In the steady-state the matching legs are almost half a step out of phase (Figure 6.13a) and the body and the torso oscillate in different ways. (Figure 6.13b). The torso oscillates once per step while the body oscillates twice with a different amplitude. Again, a 200-step simulation starting with an initial state that deviates slightly from the fixed point shows that the step-transition Poincare map is converging, indicating that the galloping-like gait shown in Figure 6.13 is stable (Figure 6.14).

This out-of-phase gait partially resembles a galloping gait, in that the leg movements are made of two sets of scissoring gaits differed by a constant phase. However, with a real galloping gait each of the biped subsystem also has a flight phase, and sometimes a double-stance phase as well.
CHAPTER 6. PASSIVE QUADRUPED WALKING

6.5.6 Stiffness of the body spring and the dynamics of the periodic gait

Overview

In this section, we investigate how the stiffness of the body spring and the dynamics of the periodic gait are related. We use a range of spring stiffnesses, and for each stiffness we do a 200-step simulation, take the mechanical states at the start of each quadruped step and calculate the separation distance between the biped subsystems, the step speed (i.e. the speed of quadruped over a quadruped step), and the phase difference of limb movements between the biped subsystems. These parameters are related to the dynamic properties of a quadruped gait, including the dynamic stability.

Changing the stiffness also changes the location of the fixed point on the step-transition Poincare map. However, we find that the location of the fixed point and its stability are smoothly parameterized by the body spring stiffnesses. After a slight change in the spring stiffness, the fixed point relocates, but in almost all cases we can expect that the original fixed point will stay inside the attractive basin of the relocated fixed point. In this way we can simply use the old fixed point as the starting point for finding the new fixed point.

For calculating the separation between the biped subsystems, we simply calculate the distance between the stance feet during a quadruped step. For calculating the step speed, we calculate the distance travelled by the front hip over a quadruped step, and divide this value by the duration of the quadruped step. For calculating the phase difference of limb movements, we take the difference between the time the biped subsystem at the front enters the support-transfer phase, and the time the biped subsystem at the back enters support-transfer phase. A negative difference means that the biped subsystem at the back enters the support-transfer phase before the biped subsystem at the front.

A multiple-step simulation can reveal a dynamically stable periodic quadruped gait, but cannot reveal dynamically unstable ones. To resolve this problem, we also use the Newton search algorithm to find fixed points on the step-transition Poincare map. In this way, a periodic quadruped gait can always be found regardless of whether it is dynamically stable or not. The step speed, and the phase difference of limb movements of the periodic quadruped gait can still be inferred from the fixed point.
Periodic gait properties versus body stiffness

Varying the stiffness gives a variety of stable periodic quadruped gaits (Figure 6.15a to 6.15d). In the steady state, the postures of the lower-body of the biped subsystems at the front and the back are not identical at the start of each step (Figure 6.15a to 6.15d).

The steady-state walking speed is not sensitive to changes in the stiffness of the body spring (Figure 6.15e), but changes in the stiffness of the body spring have a significant effect on the phase difference between the dynamics of both the front and back biped subsystems (Figure 6.15f). When the stiffness of the body spring is approximately 10.0 without damping, this produces a quadruped gait that is a perfect one-beat, pacing gait (Figure 6.16).

Dynamically stable quadruped gait exists over a wide range of body spring stiffness. When the stiffness of the body spring is between 5.0 and 28, dynamically stable quadruped gaits exists quite generally; except that when the body spring stiffness is between 11 and 14 (approximately), the periodic quadruped gait is dynamically unstable (Figure 6.17).

6.6 Discussion

It is known that agile animal locomotion is a result of body and brain coordination. In earlier research, it was believed that the brain sends control signals for the desired trajectory, with the cerebellum playing a key role. Our work shows how gait trajectories can emerge from the mechanical design of the body, which may require researchers to rethink the role of brain in the context of locomotion.

Our passive quadruped models demonstrate that two of the important attributes of multi-legged locomotion, perturbation handling and phase regulation, can be achieved totally passively, without the need of a CPG or any other control. Instead, the inertial property of the limbs under the influence of the force of gravity, and the recoil force from the springs takes on the role of a CPG.
Figure 6.4: The stability of a biped-equivalent periodic gait of the simplest quadruped. (a) The trajectory at the start of the journey. The journey starts with a state that deviates slightly from the fixed point. The step-transition Poincare map (b to d) converges, indicating stability. In sub-figure (a), (b) and (c), the red curve represents the dynamics of the stance leg of the front biped subsystem, the green curve represents the dynamics of the swing leg of the front biped subsystem, the magenta curve represents the dynamics of the stance leg of the back biped subsystem, and the black curve represents the dynamics of the swing leg of the back biped subsystem.
Figure 6.5: The trajectory of a biped-equivalent periodic gait of the simplest quadruped in a steady state. (a) The trajectory over two gait cycles. Once the system becomes stabilized, the dynamics of the two biped subsystems are in phase. (b) The length of the inter-biped spring that joins the hip and the shoulder during the steps. In sub-figure (a), the red curve is the trajectory of the front biped subsystem, the green curve is the trajectory of the swing leg of the front biped subsystem, the magenta curve is the trajectory of the stance leg of the back biped subsystem, and the black curve is the trajectory of the swing leg of the back biped subsystem. All trajectories are defined relative to the world vertical.
Figure 6.6: (a) The relationship between the maximum eigenvalue modulus and the spring parameters presented as a surface plot. (b) The relationship between the non-dimensionalized steady-state walking speed and the spring parameters as a surface plot. (c) The relationship between the maximum eigenvalue modulus and the spring parameters presented as a contour plot.
Figure 6.7: The mechanical design of a more realistic quadruped. The head and neck (the torso of the front biped subsystem) is anchored against both legs via identical torsion springs. Torsion springs are also used around the hip of the rear biped subsystem.
Figure 6.8: The convergence of the step-transition Poincare map shows a stable periodic gait for the quadruped model. A random search reveals a set of initial conditions that lies within its attractive basin. The fixed-point has a dimension of 11 so the Poincare map is presented (a) as the angle between the stance and the swing legs (i.e. the hip angle) of the two biped subsystems versus step number, and (b) as the orientation angle of the torso and the stance feet separation versus step number.
Figure 6.9: The stable gait trajectory of the fixed-point described in Section 6.5.3 over 2 step cycles. The trajectories of the legs and torso (a) and the body-spring deformation (b) were plotted. In (a), the red curve is the trajectory of the stance leg of the front biped subsystem, the green curve is the trajectory of the swing leg of the front biped subsystem, the magenta curve is the trajectory of the stance leg of the back biped subsystem, the black curve is the trajectory of the swing leg of the back biped subsystem, and the blue curve is the trajectory of the torso (i.e. the head and neck). All trajectories are defined relative to the world vertical.
CHAPTER 6. PASSIVE QUADRUPED WALKING

Figure 6.10: The partial basin of attraction of the fixed-point of the stable periodic gait as described in Section 6.5.3. Apart from the stance feet separation, perturbations are applied to one body part at a time. For each limb, the orientation angle and the angular velocity are perturbed (a to e). Because it is assumed that the stance feet hold on the ground firmly without slipping and detaching, for the stance feet separation (f), it is assumed that the velocities of the feet are fixed at zero, and the partial basin of attraction is one-dimensional. Red coloured regions indicate non-convergence. If a perturbation brings the mechanical state into one of these ranges the periodic walking will be destroyed. Blue coloured regions indicate convergence. If a perturbation brings the mechanical state into one of these ranges the periodic walking will be restored passively. Yellow coloured regions indicate that if a perturbation brings the mechanical state into one of these ranges either the periodic walking will be destroyed but very slowly, or the periodic walking will be restored but very slowly.
Figure 6.11: A two-cycle trajectory of a periodic pacing gait that is stable. The trajectories of the legs and torso (a) and the body-spring deformation (b) were plotted. In sub-figure (a), the red curve is the trajectory of the stance leg of the biped subsystem at the front, the green curve is the trajectory of the swing leg of the biped subsystem at the front, the magenta curve is the trajectory of the stance leg of the biped subsystem at the back, the black curve is the trajectory of the swing leg of the biped subsystem at the back, and the blue curve is the trajectory of the torso (i.e. the head and neck). All trajectories are defined relative to the world vertical.
Figure 6.12: The step-transition Poincare map from a 200-step simulation starting from a state that is perturbed slightly from the fixed point of the pacing gait. The step-transition map is converging, indicating stability. In sub-figure (a) and (b), the red curve is the orientation angle of the stance leg of the biped subsystem at the front on the Poincare map section, the green curve is the orientation angle of the swing leg of the biped subsystem at the front on the Poincare map section, the magenta curve is the orientation angle of the stance leg of the biped subsystem at the back on the Poincare map section, the black curve is the orientation angle of the swing leg of the biped subsystem at the back on the Poincare map section, and the blue curve is the torso orientation (i.e. the head and neck orientation). All orientation angles are defined relative to the world vertical.
Figure 6.13: A two-cycle trajectory of a galloping-like periodic gait that is stable. The trajectories of the legs and torso (a) and the body-spring deformation (b) were plotted. In sub-figure (a), the red curve is the trajectory of the stance leg of the biped subsystem at the front, the green curve is the trajectory of the swing leg of the biped subsystem at the front, the magenta curve is the trajectory of the stance leg of the biped subsystem at the back, the black curve is the trajectory of the swing leg of the biped subsystem at the back, and the blue curve is the trajectory of the torso (i.e. the head and neck). All trajectories are defined relative to the world vertical.
Figure 6.14: The step-transition Poincare map from a 200-step simulation starting from a state that is perturbed slightly from the fixed point of the galloping-like gait. The step-transition map is converging, indicating stability. In subfigure (a) and (b), the red curve is the orientation angle of the stance leg of the biped subsystem in the front at the Poincare map section, the green curve is the orientation angle of the swing leg of the biped subsystem in the front at the Poincare map section, the magenta curve is the orientation angle of the stance leg of the biped subsystem on the back at the Poincare map section, the black curve is the orientation angle of the swing leg of the biped subsystem on the back at the Poincare map section, and the blue curve is the torso orientation (i.e. the head and neck orientation). All orientation angles are defined relative to the world vertical.
6.6. DISCUSSION

Figure 6.15: A figure showing the effect of changing the stiffness of the body spring on the location of the fixed point of the periodic gaits (a to d). Other data related to the dynamics of quadruped gaits, in particular the speed of the step (e) and the phase difference between the dynamics of the two bipedal subsystems (f), are also shown. In sub-figures (a) and (b), the red curve is the orientation angle of the stance leg of the bipedal subsystem at the front on the Poincare map section, the green curve is the orientation angle of the swing leg of the bipedal subsystem at the front on the Poincare map section, the magenta curve is the orientation angle of the stance leg of the bipedal subsystem at the back on the Poincare map section, the black curve is the orientation angle of the swing leg of the bipedal subsystem at the back on the Poincare map section, and the blue curve is the torso orientation (i.e. the head and neck orientation). All orientation angles are defined relative to the world vertical.
Figure 6.16: Plots showing how the fixed point of a periodic quadruped gait varies as the stiffness of the body spring changes (a to d). It is assumed that the body spring has zero damping. The fixed points are calculated by using the Newton search algorithm instead of running 200-step simulations. In this way, fixed points can be obtained even if they are unstable. In sub-figures (a) and (b), the red curve is the orientation angle of the stance leg of the front biped subsystem on the Poincare map section, the green curve is the orientation angle of the swing leg of the front biped subsystem on the Poincare map section, the magenta curve is the orientation angle of the stance leg of the back biped subsystem on the Poincare map section, the black curve is the orientation angle of the swing leg of the back biped subsystem on the Poincare map section, and the blue curve is the torso orientation (i.e. the head and neck orientation). All orientation angles are defined relative to the world vertical.
Figure 6.17: The relationship between the maximum eigenvalue modulus of the fixed points and the body-spring stiffness.
Chapter 7

Passive 3D biped with an upper body

7.1 A passive 3D biped with an upper-body

This chapter concerns passive dynamic walking in 3D. McGeer [52] proposed the first 3D leg-only biped. This 3D biped is a straight-leg biped that can walk passively down a slope. The motion is stable in the sagittal plane, but is highly unstable in the lateral plane. Kuo [73] proposed a once-per-step impulsive control scheme that can resolve the instability issue of McGeer’s 3D biped by adjusting the impending foot strike position. Garcia [35] showed that by mounting a damped spring around the hip of McGeer’s 3D biped and adjusting the inertial properties of the legs, the stability of the biped can be largely improved. Garcia did not make the 3D periodic walking gait stable, however, Coleman [37] optimized the inertial properties of the legs and finally achieved stable 3D periodic walking.

To date, the effects of an upper-body on the dynamics of a 3D passive walker has not been studied in detail. From Coleman [37], it is possible to build a 3D leg-only biped that walks passively and stably down a slope. Whether or not a torso could improve the stability of a 3D passive biped, and if not how the instability issue could be resolved without resorting to servo-control, had not been investigated previously. These issues will be addressed in this chapter.

It is not possible for a 2D straight-leg biped to walk passively on a flat slope without scuffing the surface. In principle, the knees can resolve the scuffing problem by shortening the effective length of the swing leg. Resolving foot scuffing passively by flexing the knees is possible, if the passive biped does not have an upper-body [32]. In Chapter 5, we have investigated a 2D biped with a pair of kneed legs and an upper-body. Passive, periodic and stable downhill walking
was achieved, but we have not found a passive periodic gait that is stable and scuff-free. In this chapter, we also aim to resolve the foot-scuffing problem in passive walking by allowing the biped to rock side-to-side.

7.2 Model formulation

7.2.1 Gait simulations

The simulation code of the Tinkertoy walker without a torso (which is written in MATLAB) was kindly provided by Dr. Mariano Garcia. (The code can be found in the appendix section of Garcia [35].) The original MATLAB source code is modified to account for the effect of torso under the guidance of Dr. Garcia.

Garcia [35] did not use the Lagrangian-based approach (Chapter 2) to construct the equation of motion. Instead, Garcia constructed the equation of motion by balancing the forces applied to each linkage, so that all linkages remain in contact at the joints. Garcia’s calculations produce identical results to our Lagrangian-based calculations. Computationally, it is more cumbersome and one can easily make mistakes implementing the equation of motion using Garcia’s approach. However, Garcia’s approach does not require symbolic differentiations, and hence can be implemented in standard programming languages. Also, it runs faster.

7.2.2 Stride-to-stride Poincare map

In the 3D biped, the hip is not a simply a point, but a rigid link that sticks out of the sagittal plane, with the stance leg and the swing leg connected at each end. After the swing foot strikes, the whole system pivots at a contact point, and this contact point is not in line with the contact point of the foot in the previous step. Furthermore, because the stance leg and swing leg are not on the same plane, in order to walk straight and avoid foot scuffing, during a step, the biped must bank and roll in opposite directions as in the previous step. In this way, we would expect the biped to return to the starting state of the current step after a stride (i.e., two steps).

Therefore, when taking Poincare maps at foot strike, it would be convenient, in this case to consider the stride-to-stride Poincare map $\Xi$ instead of the step-transition Poincare map $S$. In this stride-to-stride Poincare map, the starting state of the next stride $\eta_{g,n+2}$ (i.e. the starting state of the step two steps after the current step) is a function of the starting state of the current stride $\eta_{g,n}$. The relationship between the two states can be described as a second-order recurrence

$$\eta_{g,n+2} = \Xi(\eta_{g,n}).$$
In terms of the step-transition Poincare map, the stride-to-stride Poincare map can be rewritten as

$$\Xi(\eta_g) = S(S(\eta_g)).$$

The fixed point on the stride-to-stride Poincare map and its eigenvalues can be calculated and analyzed in the same way as for the case of a fixed point on the step-transition Poincare map.

### 7.3 Physically realistic inertia tensor

#### 7.3.1 The masses-on-rods framework

When finding a stable periodic walking gait, very often, constant parameters in the model must be altered. The biggest problem lies with the inertia tensor. Inertia tensors are symmetric matrices, but not all symmetric matrices give a physically realistic inertia tensor. Whenever a stable gait is found, there is a major pitfall in that the optimization algorithm may have chosen a symmetric matrix that is not a physically realistic inertia tensor, and thus the stable gait does not physically exist.

One way to avoid constructing physically unrealistic inertia tensors is to use the “masses-on-rods” model proposed by Garcia [35] (Figure 7.1), with the inertia tensor realized by 6 point masses. The masses-on-rods framework involves 3 massless rods, intersecting with each other perpendicularly at the mid-points. Each rod has a point-mass on each end. The framework can be translated or rotated relative to the reference frame of the limb. Mathematically, it is possible to describe the inertial properties of rigid-bodies of arbitrary mass distributions and shapes using the masses-on-rods model.

When the three massless rods intersect at the origin of a reference frame, and the rods align with the axes of the same reference frame, the inertia tensor of the masses-on-rods framework is given by

$$\mathbf{I}_{cm} = \begin{pmatrix} 2m_2d_2^2 + 2m_3d_3^2 & 0 & 0 \\ 0 & 2m_1d_1^2 + 2m_3d_3^2 & 0 \\ 0 & 0 & 2m_1d_1^2 + 2m_2d_2^2 \end{pmatrix}. \quad (7.1)$$

The masses-on-rods framework can be rotated relative to a reference frame in 3D. Then, let $\{\alpha_1, \alpha_2, \alpha_3\}$ be the rotation angles about the $x$, $y$ and $z$ axes respectively,
CHAPTER 7. PASSIVE 3D BIPED WITH AN UPPER BODY

Figure 7.1: This diagram explains the masses-on-rods approach; a way to construct a physically realistic inertia tensor. The x, y and z axes are shown in red, green and blue respectively.

\[ I_{cm} = R(\alpha_1, \alpha_2, \alpha_3) \hat{I}_{cm} R^T(\alpha_1, \alpha_2, \alpha_3), \tag{7.2} \]

where \( R(\alpha_1, \alpha_2, \alpha_3) \) is the rotation matrix.

7.3.2 Reality check

When constructing the inertia tensor of the limbs using the masses-on-rods framework, we also need to ensure that the end product is able to walk in reality. In the simulation, the collision between the swing leg and the ground is handled by the event locator, but collisions between body parts are not handled. Therefore, it is necessary to ensure the following:

- The structure does not penetrate the ground during the course of motion.
- The structure does not clash with any other body parts during the course of motion.
The simplest way to do such a reality check is by direct visualization, by determining whether, during the course of motion body parts hinder each other.

7.4 Realistic passive downhill stable gaits

7.4.1 Overview

We have found two designs that can walk passively, stably and periodically downhill without violating the reality constraints. The design parameters were detailed in Table 7.1, and the visualizations of the resulting devices at their stable fixed points are shown in Figures 7.2, and 7.4. Their stable periodic gait trajectories are shown in Figure 7.3 and Figure 7.5.

7.4.2 Examples of stable 3D passive biped with a torso

In the case of Design 1 (Figure 7.2), the maximum eigenvalue modulus of the fixed point of the periodic gait is 0.6804 (Table 7.2). The periodic gait is stable but the torso is not quite human-like. With Design 2 (Figure 7.4), the leg extension is simplified into a pole with a point mass on each end, and the torso is re-orientated into a human-like posture. The fixed point of the periodic gait (Table 7.3) has a maximum eigenvalue modulus of 0.7983. For each design, a 200-step simulation starting from an initial state that deviates slightly from the fixed point shows that the step-transition Poincare is converging (Figure 7.3 and 7.5), indicating stability. In addition, for each design, the maximum eigenvalue modulus of the fixed point is significantly less than the maximum eigenvalue modulus of the Tinkertoy’s fixed point (0.8930, [37]).

7.4.3 Further reality check

As a final check, we plot the vertical distance between the lowest end of the leg extension and the slope when the gait stabilizes (Figure 7.6 for Design 1, and Figure 7.7 for Design 2). It turns out that during the step the legs are not hindered by the ground. Furthermore, from the design diagrams (Figure 7.2 and Figure 7.4), it is unlikely that the leg will hinder each other, or any other body parts, during the course of motion.

7.5 Parameter analysis on Design 2

7.5.1 Length of the leg extension

The leg extension parameter $d_2$ has a very strong influence on the dynamic properties of the gait. Increasing the value of $d_2$ reduces the walking speed and
When $d_2 = 0.45$, the dynamic stability of the gait reaches a maximum (Figure 7.9). As $d_2$ decreases, the eigenvalue modulus of the fixed point increases quite quickly, and eventually we cannot find any periodic gaits (data not shown). The stiffness of the torso-leg springs has little influence on the dynamic stability of the passive gait (Figure 7.9). The results suggest that we cannot reduce the lateral span of the legs extension by much without losing stability. In the case of a Tinkertoy walker, the maximum lateral span of each leg is 1.0 unit [37], but in the case of Design 2 it is only 0.5700 units.
7.5. PARAMETER ANALYSIS ON DESIGN 2

![Graphs showing dynamics of gait parameters](image)

Figure 7.3: The stable periodic gait of Design 1. In sub-figures (a), (b) and (d), the curves show the dynamics of the stance leg pitch, the swing leg pitch, the torso pitch, the stance leg bank and the stance leg yaw, and are coloured in red, green, blue, magenta, and black respectively.

### 7.5.2 Torso-leg spring stiffness and damping

Increasing the stiffness of the springs between the torso and legs increases the walking speed. Damping makes the biped walk slower (Figure 7.11), but when the damping is within a certain range, it helps with the dynamic stability of the gait. With a spring constant of 4.5 and a damping coefficient greater than 0.114, the gait starts to lose stability (data not shown).

### 7.5.3 Slope angle and foot radius

Like the 2D biped with a torso, the steady-state walking speed increases as the slope angle increases (Figure 7.12). From Figure 7.14, increasing the slope angle also increases the stability of the periodic gait, although we believe that there may be an upper limit beyond which the periodic gait becomes unstable. Increasing the foot radius increases the steady-state walking speed, but not by
In order to walk stably on downhill slopes, there seems to be a lower limit on
the slope angle (Figure 7.14). A similar observation, that stable gaits cannot
exist below a certain slope angle, was made on the dissipative (McGeer-type)
kneed walker [39]. The 2D torso-walker level-ground gait (Chapter 4) is unstable
because the collision loss caused by the perturbations cannot be compensated for
by the gravitational potential energy gain. If there are energy dissipations during
a step, such as an inelastic knee collision and damping, there must be a critical
slope angle greater than zero that will allow the energy dissipated during the
step to be compensated. This compensation is achieved by gaining just enough
gravitational potential energy, and on that slope, the foot must strike with zero
velocity. This gait, like the collision-free periodic gait discussed in Chapter 4, is
dynamically unstable. We would then expect that there will be a range of slope
angles within which the gait stability decreases as the slope angle decreases. Our
3D biped has damping in the spring, so as we can see in Figure 7.14, a stable
gait occurs over a range of slopes that are not near-zero slopes.
7.6. The partial basin of attraction of Design 2

The partial basin of attraction of the fixed point of Design 2 (Table 7.3) is shown in Figure 7.15. The roll angle (rotation about the slope normal) has the greatest tolerance for perturbation, compared with other states. The yaw angle (side-to-side rocking) has the least tolerance for perturbation, and the size of the partial basin of attraction is very small, so in reality controls may still be necessary to keep the walker stable.

The maximum eigenvalue modulus merely measures the local stability near the fixed point. When the maximum eigenvalue modulus of a fixed point is less than 1.0, the fixed point is stable. When the initial state is very close to the fixed point, it is expected that the step-transition Poincaré map will converge, and the perturbation error can be repaired passively. Because a step-transition Poincaré map is a non-linear dynamic system, a stable fixed point implies that

Figure 7.5: The stable periodic gait of Design 2. In sub-figures (a), (b) and (d), the curves show the dynamics of the stance leg pitch, the swing leg pitch, the torso pitch, the stance leg bank and the stance leg yaw, and are coloured in red, green, blue, magenta and black respectively.
Figure 7.6: The vertical distances between the lowest ends of the leg extensions of the stance leg (blue) and the swing leg (black) over time of Design 1, and the slope over time when the walker is on its stable limit cycle.

only some initial states allow the formation of a stable periodic gait, but others do not, so only some perturbations can be handled passively. Global stability is a measure of how large, and in which directions, perturbations can be in order to allow passive re-stabilization of a dynamically stable passive gait. We can work out the global stability by calculating the attractive basin around a fixed point. In general, when the maximum eigenvalue modulus suggests that a periodic gait is “highly stable”, it does not necessarily imply large a basin of attraction.

Although the 3D torso walker has an excellent stability measure according to the maximum eigenvalue modulus of the fixed point, by looking at the basin of attraction (Figure 7.15), it turns out that the yaw angle and velocity are both sensitive to perturbations.

Based on the basin of attraction analysis, the side-to-side rocking allows foot scuffing to be avoided passively, but the global stability of the passive periodic
7.6. THE PARTIAL BASIN OF ATTRACTION OF DESIGN 2

Figure 7.7: The vertical distances between the lowest ends of the leg extensions of the stance leg (blue) and the swing leg (black) over time of Design 2, and the slope over time when the walker is on its stable limit cycle.

gait is sensitive to the initial yaw angle and velocity (Figure 7.15). Knees allows foot scuffing to be avoided passively without having to rock side-to-side, but from Chapter 5, we see that when the torso is added, the knee bending in the passive periodic gait is too small to avoid foot scuffing without side to side rocking. It is plausible that bipeds use both side-to-side and knee bending to avoid foot scuffing, so that the scuffing problem can be resolved with a minimal use of yawing.
Figure 7.8: The relationship between steady-state walking speed, the leg extension, and the stiffness of the torso-leg spring.
Table 7.1: The design parameters of the two 3D straight-leg bipeds considered in this chapter.

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<tr>
<td>Torso inertia-tensor parameters $m_1, m_4$</td>
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<tr>
<td>Torso inertia-tensor parameters $m_3, m_6$</td>
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<td>Torso inertia-tensor parameter $\alpha_2$</td>
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<td>Torso inertia-tensor parameter $\alpha_3$</td>
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<td>Torso-leg spring rest angle</td>
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<tr>
<td>Hip mass</td>
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Table 7.2: The stable fixed point of Design 1 and its eigenvalues.

<table>
<thead>
<tr>
<th>State</th>
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<th>Eigenvalue</th>
</tr>
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<tbody>
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<td>$\theta_1$</td>
<td>0.1116</td>
<td>$-0.4687 - 0.4191i$</td>
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<td>$\theta_2$</td>
<td>1.5465</td>
<td>$-0.4187 + 0.4191i$</td>
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<td>$\theta_3$</td>
<td>-0.1580</td>
<td>$0.2155 - 0.6454i$</td>
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<td>$\theta_4$</td>
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<td>$0.2155 + 0.6454i$</td>
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<td>$\theta_5$</td>
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</tr>
<tr>
<td>$\dot{\theta}_1$</td>
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<td>$\dot{\theta}_4$</td>
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Table 7.3: The stable fixed point of Design 2 and its eigenvalues

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<th>Fixed point</th>
<th>Eigenvalue</th>
</tr>
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<tbody>
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<td>$\theta_4$</td>
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<td>$0.7315 + 0.3197i$</td>
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<td>$\theta_5$</td>
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<td>$\dot{\theta}_1$</td>
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<td>$\dot{\theta}_3$</td>
<td>0.00002</td>
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<tr>
<td>$\dot{\theta}_4$</td>
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<td>$\dot{\theta}_5$</td>
<td>0.4471</td>
<td>$0.0010$</td>
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7.6. THE PARTIAL BASIN OF ATTRACTION OF DESIGN 2

Figure 7.9: The relationship between stability (i.e. maximum eigenvalue modulus), the leg extension, and the stiffness of the torso-leg spring.
Figure 7.10: The relationship between the steady-state walking speed and the parameters of the torso-leg spring.
7.6. THE PARTIAL BASIN OF ATTRACTION OF DESIGN 2

Figure 7.11: The relationship between the stability of the periodic gait and the parameters of the torso-leg spring.
Figure 7.12: The relationship between the steady-state walking speed, the slope angle and the foot radius.
Figure 7.13: The relationship between the steady-state walking speed and the foot radius, on a slope of 0.0684 radians.
Figure 7.14: The relationship between the stability of the passive gait and the slope angle.
7.6. THE PARTIAL BASIN OF ATTRACTION OF DESIGN 2

Figure 7.15: The partial basin of attraction of Design 2, near the fixed point detailed in Table 7.3. Red regions mean non-convergence. If a perturbation brings the mechanical state into one of these regions the periodic walking will be destroyed. Blue regions mean convergence. If a perturbation brings the mechanical state into one of these regions the periodic walking will be restored passively. Yellow regions mean that if a perturbation brings the mechanical state into one of these regions either the periodic walking will be destroyed, but very slowly, or the periodic walking will be restored, but very slowly.
Chapter 8

Controlled level-ground walking

8.1 Collision-free gaits on level ground revisited

8.1.1 Overview

This chapter, will be published as a co-authored paper by Te-yuan Chyou, Gerard Liddell and Mike Paulin. The paper will be published in the Journal of Theoretical Biology, and the manuscript is currently in preparation.

Chapter 4 was concerned with the passive dynamics of a straight-leg biped with a torso stabilized with torsion springs. This passive walker can walk downhill stably, but unlike the torso-less compass-gait walker, it can walk stably on both shallow and steep downhill slopes, with the maximum slope tolerance being 45 degrees. The most interesting feature of this biped is that it can also walk passively on level ground, which cannot be achieved at all by a compass-gait walker.

The existence of the ideal gait demonstrates the possibility that 100% efficiency for level-ground transportation can be achieved by a legged device without the use of wheels. However, the ideal gait involves exaggerated torso and leg motions. By adding arms to the torso, 100% walking efficiency can be achieved with a more human-like gait trajectory, without the exaggerated torso swing and leg extension.

8.1.2 Handling perturbations from uneven ground

Ideal level-ground motion was highly unstable and once perturbed by the slightest unevenness of the ground, the walker collapsed. However, if a controller managed to place the biped back into the unstable collision-free gait again, then control forces would be unnecessary until further perturbations were encountered,
and hence allow an energetically efficient instability control mechanism. Here, we investigate whether this can lead to walking that is both efficient and stable.

One approach for handling perturbation and ensuring gait stability is to use servo-control. A dynamically unstable gait trajectory is used as a guide and the servo-controller turns this trajectory into an attractor, such that the controlled trajectory asymptotically converges into the collision-free gait trajectory and the control inputs diminish as time goes on. Gomes and Ruina had conjectured that the instability of the collision-free gait can be compensated using servo-control, by making the collision-free gait trajectory an attractor.

From experimental studies, muscles are activated at the start and the end of a step [51]. Shiavi [80] showed experimentally that muscles in the lower limbs exhibit a phasic activation pattern when walking. Van der Linden [54] argues that this observation may indicate that muscles only generate force when there is a need. This suggests that the assumption that the locomotion control mechanism drives the limbs onto a particular trajectory is inappropriate in the context of animal locomotion, and that control forces should be applied in short episodes, akin to impulses.

It is already known that stability on level ground does not require trajectory planning or continuous driving forces. Kuo [45] and McGeer [42, 43] showed that a compass-gait walker can walk stably on level ground by applying an impulse at the start of each step. They also showed it does not matter where the impulse is applied, and the choices of impulse magnitude and direction are flexible. In Chapters 4 and 5 we also used impulsive control to deal with perturbations. However, we used externally applied torques at the start of each step and that directed the ideal biped into a state close to the collision-free fixed point at the end of the step, after support-transfer. Although in our case, we calculated the impulse required to bring the ideal biped back into the target state after the support transfer to ensure periodic walking, the control system does not calculate the collision-free trajectory and drive the limbs onto the collision-free trajectory each time perturbation occurs.

The re-stabilization of the unstable collision-free passive gait on level ground has been investigated in Chapter 4, however this assumed that the control torques are externally applied. This is not physically realistic as in an animal, the control torques are generated between limbs, and there can be no moment around the stance foot. The other criticism is that the perturbation is pre-calculated due to the assumption that the perturbation is known, however, in reality, perturbations of gait trajectory are likely to be due to the unevenness of the ground. The perturbations may be foreseen, or they may remain unknown until the foot strikes
the uneven ground. In this chapter, the problem of re-stabilizing the ideal gait will be reconsidered, with these criticisms addressed.

8.2 Modelling approach

8.2.1 Biped model, control force and landscape

In this chapter, we revisit the problem of re-stabilizing the unstable collision-free level ground gait using impulsive control, which has been discussed in Chapter 4. We are still basing our study on the Gomes-Ruina ideal biped, and enforcing the sticky-foot constraint during a step, but this time the impulsive control torques are internal and applied at the joint so neighbouring limbs experience impulsive torques that are equal in size but opposite in direction. This is to address the practical consideration that the controller needs to be installed around a joint and cannot be installed between a limb and a non-existing vertical support. In addition, the objective of the control is to bring the biped back to the collision-free fixed point after perturbation, returning it to an unstable ideal gait. To achieve this, we only need to apply control impulses when the biped is perturbed, rather than at the start of every step. This is in contrast to a controlled periodic gait that closely resembles the collision-free gait requiring a small control impulse per step.

In the context of animal locomotion, control forces come from muscle contractions. Muscle forces are generated by a complicated biochemical process, and modelling this is beyond the scope of this research. From experimental studies, muscle activation is significant during the start and the end of a step [51]. Following muscle activation, muscle activity decays quite quickly, in milliseconds [1]. Based on these facts, in many previous studies [42, 64, 43, 65, 45], the muscular forces generated over a walking step are simply treated as pulse-like control forces and the internal mechanisms of muscle force generation are black-boxed. The pulse-like control forces are approximated as perfectly impulsive controls that make instantaneous changes in generalized velocities. The unsigned mechanical work of control forces (often approximated as the changes in the kinetic energy made by the control impulses) are used to measure the expense of the control process.

Of course, a perfectly impulsive control force is not physically feasible, so in reality control forces are applied over finite time periods using muscles (or more generally actuators). Exactly how perfectly impulsive controls (that instantaneous changes in generalized velocities) can be realized by muscles is a question for muscle modeling and we consider this a separate problem beyond the scope
of this research. For this chapter, we use the simple impulsive control model considered by previous researchers. However, for the sake of being thorough, in the next chapter we make an attempt to address the issues related to control force generation, and the physical impossibility of impulsive control force, by using a naive, non-impulsive model of a control force based on transient spring stiffness adjustments.

In this chapter, we assume that the perturbations come from the landscape and we consider a slightly idealized situation. We have an ideal biped that can walk passively on the level ground by avoiding collision-loss. Our landscape is a level ground with a small step that perturbs the biped. We define the direction (up or down) and the size of the ground step as “ground height deviation”. A ground height deviation perturbs the ideal biped away from the ideal trajectory. Because the ideal trajectory is unstable, controls will be necessary to restore collision-free walking.

Reactive control scenario

We do not always assume that the ground step is known in advance, so we consider two possible scenarios. The first scenario is a “reactive control scenario”, in which the ground step remains unknown until it perturbs the ideal biped, or it is known but we decide to take no action until it is encountered. The second scenario is a “predictive control scenario” in which the ground step is predicted in advance, and control actions are taken before it is encountered.

In the reactive control scenario, the animal does not know anything about the ground step until it is encountered and the ground step passively introduces a perturbation. A sequence of control impulses is then applied to bring the biped back to the collision-free fixed point and thus restore collision-free periodic walking. A diagram explaining how the reactive control strategy works is shown in Figure 8.1.

Predictive control scenario

In the predictive control scenario, the animal knows about the ground step; where it is, and its elevation. Because information about the ground step is provided by the observation mechanism, animals can choose a target state to be reached at the instant the ground step is encountered and plan a sequence of control impulses to achieve that goal. Afterwards, another sequence of control impulses is used to bring the system back to the collision-free fixed point to restore collision-free periodic walking. Figure 8.2 shows how the predictive control strategy works.
In the predictive scenario, the ground step is observed by a mechanism, such as vision. This information is used to decide what the target state should be. We can define a control sequence function, which takes as input the information about the ground step as observed, and gives as output the sequence of control impulses, with their times, that are to be used to bring the biped back to the collision-free periodic gait.

The target state over the ground step must be a double-contact state. The stance and swing feet must both be in contact with the ground, with the difference in the height of the feet equal to the ground height deviation. We choose the target state to be the rotated collision-free fixed point with the joint angles and angular velocities of the limbs at collision-free fixed points for the following reasons:

1. We do not want the target state to be too sensitive to ground height deviation, as this can make it technically difficult to find the control impulses required to bring the biped from the collision-free fixed point to the target state. By choosing the target state as a rotated collision-free fixed point, the limb orientation angles deviate from the collision-free fixed-point values by an amount approximately proportional to the size of the ground height deviation. So small ground steps imply a small perturbation from the collision-free fixed point on foot strike, because without perturbation at all the biped can make use of the ideal gait without controls, we would therefore expect that small perturbation implies small control impulses. When the ground height deviation is small, this allows us to use zero control as the initial guess for root-finding by the Newton search algorithm.

2. We want the target state over the ground step to approach the collision-free fixed point as the ground height deviation approaches zero, so that work done by the controller diminishes to zero as the ground step diminishes to zero.

3. We want the target state to remain collision-free, so that the controller will not do additional work to compensate for the collision loss, and the collision-free condition is ensured by having the angular velocities of the limbs in the target state equal to the angular velocities of the limbs in the collision-free fixed point.

Elaborating point 1, we have investigated matching just the swing-foot height and enforcing the collision-free constraint, to save the controller the work required to compensate for collision-losses. This gives a more comprehensive and rigorous way to set up the predictive control. However, so far we have not managed to get this method to work. We were unable to find an initial guess such that the quasi-Newton search converges. Changing control placement times does not resolve
the problem. Furthermore, for small ground steps, zero control is not a good initial guess and don’t allow the quasi-Newton search to converge. A possible explanation is that over the first phase we use control impulses to match only the swing foot velocity and the swing-foot height, but the deviation between the collision-free fixed point and the resulting post-strike state is very sensitive to the ground height deviation. Therefore, the size of the deviation between the post-strike state and the collision-free fixed point can be much greater than zero, and zero control is not a good initial guess. This makes finding the control impulses over the second phase difficult.

By requiring that both the joint angles and the angular velocities of the joints be those of the rotated collision free fixed point when the ground height deviation is small, and using zero control as the initial guess, control placement times can be found such that the Newton’s search converges and gives a control solution.

8.2.2 Achieving the state-matching required

In both the reactive and the predictive controls, re-stabilization of the ideal gait after a ground height perturbation is achieved by bringing the biped from a starting state to the target state that we want at the end of a step after the support-transfer. In fact, we can match the state before the support-transfer. However, because the starting state of the new step is also the state of the current step after the support-transfer, and we usually want the new step to start with a specific state in order to resolve the instability issue, for example the collision-free fixed point, matching the post-strike state (the state after the support-transfer) makes the calculation simpler. For brevity, we refer to finding a control strategy that brings an animal from an initial state to a desired final state as “state-matching”.

State-matching with impulsive controls, and its application to periodic walking gaits, has been investigated by other researchers. Formalsky et al. (2000) [65] used impulsive control to generate various types of a quadruped gaits, by bringing the quadruped from a chosen initial state to the same initial state at the end of the step. Multiple control impulses were used and the impulses were placed at the start and the end of the step. This decision was based on the suggestion the muscles are more active near the start and the end of the step. In the work of Blajer and Schielen [64], control impulses are used to bring a biped from an impact-less state into the same impact-less state, so that the biped can walk with a controlled collision-free periodic gait. The impact-less state is not a passive collision-free fixed point, therefore controls will be needed for state-matching but not for thrust.
8.2. MODELLING APPROACH

The biped starts the step with a starting state which has both feet on the ground and control impulses are calculated and applied in the middle of the step, to bring the biped into a target state which also has both feet on the ground. A new step starts immediately after the target state is reached, as outlined in Figure 8.3. We want each state-matching process to take one step, so between the time the biped starts the step, and the time the biped is brought to the target state, there are no foot-strikes. We assume also that the support-transfer has infinitesimal duration.

We base our simulation on the Gomes-Ruina ideal biped. The configuration space is the 3-dimensional space of the angles of the orientations of the stance leg, the swing leg and the torso. The dynamical state space (with the angular velocities) is 6-dimensional. So the state-matching for a general perturbation will need 6 degrees of freedom. Each control episode is allowed to apply impulses to the inter-joint angles: around the stance-torso joint and around the swing-torso joint. Hence, state-matching will need two control episodes to realize a 6 dimensional adjustment, because the times control impulses applied add control dimensions. Standard quasi-Newton root finding methods are used to find the control sequence for state-matching.

It is worth pointing out that we tried solving for the two control impulses and the two control placement times required for state-matching. However, we were not able to get this method to work. Newton searches either diverged, or encountered near-singular, ill-conditioned Jacobian matrices after a certain number of iterations. Changing the target state or changing the initial guess did not help with the problem. Exactly what was causing the problem was unknown, but this approach was not useful, although it is the most mathematically comprehensive way to set up the state-matching control.

Our next attempt was to choose 3 control placement times and apply a control impulse at each time. Because the control placement times were chosen, each control impulse application introduced only 2 free variables, because the control impulses were only applied to the stance-torso and the swing-torso joints. As before, we had six mechanical states to match, and with three control impulses applied at the chosen control placement times, we still had enough free parameters to find required three control impulses for the state-matching, using Newton’s algorithm.

With control placement times chosen near the start of the gait and near the collision-free gait period, Newton’s algorithm converged to a control solution. The control placement times where an initial guess would allow Newton’s algorithm to converge to a control solution were very restricted. The three impulses need to
be placed at times near the start of the gait, and when the time (relative to the start of the step) is close to the collision-free gait period. We then find control placement times within this range such that the total unsigned work done by the control impulses required for achieving the state-matching objective can be minimized.

The Gomes-Ruina ideal biped is a straight-leg biped, so we also had to account for foot scuffing during mid-swing. Foot scuffing is handled by postponing checking for foot strike until near the time when the step is expected to end. This is when the stance leg is sufficiently past the vertical.

Since the foot-strike condition is not checked until the stance leg is sufficiently past the vertical, we can keep all control impulses within a step by applying them to times before we start to check the foot-strike condition. If the control impulses cause the swing foot to go under the ground before the last impulse is applied, it is regarded as foot scuffing and is ignored, provided that the scuffing depth is small relative to the length of the leg.
In the next section, we describe in greater mathematical detail the calculations involved in the non-impulsive control model. The notations to be used are listed below. For all calculations described in this chapter, the sticky-foot constraint is applied to the stance foot during the step, and the support-transfer is assumed to be an inelastic collision. For presentation purposes, the subscript “g” is not always used in the notations.

1. $M_g$: Mass matrix with pivoting stance leg.
2. $C_g$: Centrifugal matrix with pivoting stance leg.
3. $G_g$: Conservative force vector with pivoting stance leg.
4. $\theta$: The configuration variables of the system. With the use of the sticky-foot constraint the configuration variables are the orientation angles of the limbs with respect to the vertical axis of the reference frame of the global environment.
5. $\phi$: The collection of all joint angles.
6. $F_{\phi}^{\text{spring}}$: The spring forces expressed in joint angles.
7. $F_{\phi}^{\text{control}}$: The forces from the controller expressed in joint angles.
8. $B$: The force-coupling matrix, which ensures a controller exerts one force on each neighbouring limb, equal in size but opposite in direction. This is also the transformation matrix that defines the coordinate mapping between $\theta$ and $\phi$.
9. $\upsilon_k$: The $k$th control impulse.
10. $\eta_n$: The initial state of the $n$th step.
11. $\eta^*$: The collision-free fixed point on level ground.
12. $\tau_k$: The time (relative to the start of the step) when the $k$th control impulse is applied
13. $\Delta \eta_k$: The change in the mechanical state due to the $k$th control impulse.
14. $\tau_k$: The times control impulses are applied (collected as a vector).
15. $\Psi_n$: The control impulses applied during the $n$th step (collected as a matrix; we call it the control matrix for brevity).
16. $\Gamma^c$: The controlled gait trajectory after toe-off but before support-transfer.
17. $W_{k,n}$: The mechanical work done by the $k$th control impulse in the $n$th step.

18. $W_{net,n}$: The net mechanical work done by all control impulses in the $n$th step.

19. $C_n$: The controller cost over the $n$th step, which is defined as the sum of the unsigned mechanical work done by all control impulses in the $n$th step.

20. $\Omega$: The sequence of control impulses required to bring the biped from a starting state to a target state at the end of a step after support-transfer.

21. $\eta^+$: The post-strike state of the ideal biped given that the starting state is the collision-free fixed point and the gait is passive. It is a function of the ground step height.

22. $\tau_{s/sf}$: The time that support-transfer occurs.

23. $\tau_{\text{strike}}$: The foot-strike time of the ideal biped given that the starting state is the collision-free fixed point and the gait is passive. It is a function of the ground step.

24. $\eta^d$: The target state we want the ideal biped to be in at the end of the first phase of the predictive control process. It is a function of the ground step.

25. $\Omega^{react}$: The sequence of control impulses required to bring the biped from the starting state back into the collision-free fixed point in a reactive control process.

26. $\Omega_{1}^{pred}$: The sequence of control impulses required to bring the biped from the collision-free fixed point into the target state in the first phase of a predictive control process.

27. $\Omega_{2}^{pred}$: The sequence of control impulses required for bringing the biped from the target state into the collision-free fixed point in the second phase of the predictive control process.

28. $C_{tot}$: The cost of recovery, defined as total controller cost over all steps in which controls are involved.

8.3 Controlled walking model

8.3.1 Passive dynamic walking with control

The passive dynamics of the biped can be modelled using rigid-body mechanics as in Chapter 2. With the use of the sticky-foot constraint, during the step, the
stance-foot position can be treated as a constant, so the configuration variables include only the orientation angles of the limbs \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \). When the passive springs around each joint are replaced by controllable springs, and each joint is coupled with a force generating mechanism, a controller exists at each joint where control forces can be applied. The dynamics of the controlled system are given by the equation of motion

\[
M_g(\theta) \ddot{\theta} + C_g(\theta, \dot{\theta}) \dot{\theta} + G_g(\theta) = B \left( F_{\phi}^{\text{spring}} + F_{\phi}^{\text{control}} \right). \tag{8.1}
\]

Because control forces can only be generated between limbs, the control force \( F_{\phi}^{\text{control}} \) must be expressed in terms of the joint angles \( \phi = (\theta_1 - \theta_2, \theta_2 - \theta_3) \) between neighbouring limbs. The effect of the control force, \( F_{\phi}^{\text{control}} \), can be described in terms of the coordinate transformation \( f_{\phi} : \theta \mapsto \phi \) with derivative matrix \( B = \partial_{\theta} f_{\phi}^T \) by the expression \( B F_{\phi}^{\text{control}} \). When the control force is impulsive, the force function is written in terms of Dirac Delta functions:

\[
F_{\phi}^{\text{control}} = \sum_i v_i \delta(t - \tau_i^c). \tag{8.2}
\]

8.3.2 Passive dynamic walking with impulsive control

Impulsive forces result in instantaneous changes in velocity. The effect of applying impulsive forces to a mechanical system at multiple instants is given by the integral expression

\[
\int_{\tau_i^c-k}^{\tau_i^c+\delta t} \dot{\theta} dt = \int_{\tau_i^c-k}^{\tau_i^c+\delta t} M_g^{-1}(\theta) \left( -C_g(\theta, \dot{\theta}) \dot{\theta} - G_g(\theta) \right) dt + \int_{\tau_i^c-k}^{\tau_i^c+\delta t} M_g^{-1}(\theta) B \left( F_{\phi}^{\text{spring}} + \sum_i v_i \delta(t - \tau_i^c) \right) dt.
\]

The values of \( \theta \) and \( \dot{\theta} \) are bounded, so the first term on the right-hand side converges to zero as \( \delta t \to 0 \). Similarly the term involving \( F_{\phi}^{\text{spring}} \) converges to zero. This gives us an expression that describes the effect of impulsive forces generated between limbs on the changes in the vertically-referenced angular velocities.

The generalized acceleration due to passive dynamics is bounded, and hence it takes a finite period of time to produce a change in the state of the mechanical system, so in this way the first term on the right-hand side diminishes to zero after taking the zero limit. The left-hand side is the instantaneous change in generalized velocity. At the end, we get the expression that describes the effect of the impulsive forces generated between limbs on the changes in the vertically-referencing angular velocities,
We will consider applying multiple control impulses during a step, within the period after toe-off and before footfall. It is convenient to parameterize the control force function as a sequence of impulse applications in the form of a matrix,

$$\Psi_n = (\nu_1, \nu_2, ..., \nu_N).$$

We call this a “control matrix”.

Because all ground crossings before the last control application are ignored and are treated as foot scuffing, the control placement time can be defined relative to the foot-strike time of the previous step,

$$\tau_c = (\tau_1, \tau_2, ..., \tau_N).$$

The gait trajectory will be a discontinuous curve that is parameterized by the initial state, the control matrix, and the control placement times,

$$\Gamma_c(\eta_n, \Psi_n, \tau_c, t) = \begin{pmatrix} \theta(\eta_n, \Psi_n, \tau_c, t) \\ \dot{\theta}(\eta_n, \Psi_n, \tau_c, t) \end{pmatrix}.$$

### 8.3.3 Work done by impulsive control forces and the cost of recovery

The change in the mechanical state $\Delta \eta_k$ due to the $k$th control impulse involves only the change in the velocities:

$$\Delta \eta_k = \begin{pmatrix} 0 \\ \Delta \dot{\theta}_k \end{pmatrix}.$$

The work done by each impulse equals the change in total energy before and after the application of the impulse. Defining $E_{\text{tot}}(\eta)$ as the total energy of state $\eta$, the work done by the $k$th control impulse is given by

$$W_{k,n}(\eta_n, \Psi_n, \tau_c) = E_{\text{tot}}(\Gamma_c(\eta_n, \Psi_n, \tau_c, \tau_k) + \Delta \eta_k) - E_{\text{tot}}(\Gamma_c(\eta_n, \Psi_n, \tau_c, \tau_k)).$$
The net work done by all impulses is the sum of the work done by all impulses. By conservation of energy, this must be equal to the difference between the energy of the initial state and the energy of the state at the end of the gait. So we can write

$$W_{\text{net},n} (\eta_n, \Psi_n, \tau_c) = \sum_{i=1}^{N} W_{i,n} (\eta_n, \Psi_n, \tau_c).$$  \hspace{1cm} (8.5)

In McGeer’s work, with impulsive controls the cost of control is based on the mechanical work done by the control impulses [42, 43]. A force can do positive or negative mechanical work to a mechanical system, and so too for an impulse. Regardless of whether the control force is doing positive or negative mechanical work, a positive power of consumption is required for generating the control force. In this sense, a control impulse that does negative control work has a positive cost. To calculate the cost of the entire impulse control process, we sum up the unsigned work done by the control impulses:

$$C_n (\eta_n, \Psi_n, \tau_c) = \sum_{i=0}^{N} |W_{i,n} (\eta_n, \Psi_n, \tau_c)|. \hspace{1cm} (8.6)$$

In other research (for example [65]), the control costs are taken as magnitudes of the control impulses, and with this calculation, a positive cost is always incurred when a non-zero impulse is applied. In McGeer’s work [42, 43], only one control impulse is used at the start of each step, with the objective of achieving cyclic walking so the control impulses do positive mechanical work to compensate for the collision losses during support-transfer. So there is no need to consider the sign of the work done by the impulses.

By the law of energy conservation, the net controller work must always be equal to the difference in total energy between the target and the starting states, minus the energy changed during support-transfer. We define $\eta_+^{\text{end}}$ as the target post-strike state that we want the biped to be in at the end of the step after support-transfer, $\eta_-^{\text{end}} (\eta_+^{\text{end}})$ as the pre-strike state as a function of the post-strike state $\eta_+^{\text{end}}$, and $\Omega (\eta_+^{\text{start}}, \eta_-^{\text{end}}, \tau_c)$ as the control matrix required to bring the biped from the starting state $\eta_+^{\text{start}}$ to the target post-strike state $\eta_+^{\text{end}}$. When the control placement times are chosen to be $\tau_c$, the net controller work ($W_{\text{net},n}$) will always satisfy

$$W_{\text{net},n} (\eta_+^{\text{start}}, \Omega (\eta_+^{\text{start}}, \eta_-^{\text{end}}, \tau_c), \tau_c) = E_{\text{tot}} (\eta_-^{\text{end}} (\eta_+^{\text{end}})) - E_{\text{tot}} (\eta_+^{\text{start}}). \hspace{1cm} (8.7)$$
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Calculating the required control matrix $\Omega (\eta^{start}, \eta^{end}, \tau_c)$ will be described in the next section.

Observe that the controller cost (Equation 8.6) also satisfies

$$C_n (\eta^{start}, \Omega (\eta^{start}, \eta^{end}, \tau_c), \tau_c) \geq \left| E^{tot} (\eta^{end}) - E^{tot} (\eta^{start}) \right| .$$

Therefore, it is always true that the cost of recovery cannot be lower than the absolute value of the net controller work required.

8.3.4 Step-transition Poincare map

Mathematically, the initial state of the next step can be written as a function of the initial state of the current step. This function is typically known as the "step-transition Poincare map". The "step-transition Poincare map" defines the relationship between the initial state of the current step and the initial state of the next step (ie. $\eta_{n+1} = S (\eta_n)$). In many cases, people are interested in the effects of variations in the model and the environmental parameters on the dynamical properties of a walking gait, and in these cases, the step-transition Poincare map is defined as a recurrence relation that involves not only the initial state of the next and the current step, but also the model and the environmental parameters, so the recurrence is written as $\eta_{n+1} = S (\eta_n, p_1, p_2, \ldots, p_n)$ [54, 35, 42]. In McGeer's work [42], the time at which support-transfer occurs is also involved in the definition of the step-transition Poincare map, on the consideration that variation of the environment, such as the slope angle, which affects the time foot strike occurs.

In this chapter, impulsive control is considered so the control impulses are parameters of the control function. The control force function is a vector of Dirac Delta functions, and its parameters will be varied to bring the biped from an initial state to a desired state. Therefore, we consider the step-transition Poincare map definition that involves model parameters. A ground step can change the time at which foot-strike occurs, so the support-transfer time is included in the definition of our step-transition Poincare map. This treatment had also been considered by McGeer. We consider that the initial state of the next step is given by the step-transition Poincare map function parameterized by the initial state of the current step $\eta_n$, the control matrix $\Psi_n$ (the sequence of control impulses), the control placement times $\tau_c$, and the time support-transfer occurs $\tau_{s/tf}$,

$$\eta_{n+1} = S (\eta_n, \Psi_n, \tau_c, \tau_{s/tf}) .$$
Support-transfer occurs when the swing foot strikes the ground, so the support-transfer time is the same as the foot-strike time. Under the influence of impulsive controls, the foot-strike time is a function of the initial state of the current step $\eta_n$, the control matrix $\Psi_n$, the control placement times $\tau_c$, and the ground height deviation $h$, $\tau_{\text{strike}}^{im}(\eta_n, \Psi_n, \tau_c, h)$. With the constraint that in reality, support-transfer can only occur when the swing foot strikes the ground, the initial state of the next step is now defined by the step-transition Poincaré map with the “realistic” support-transfer time, or the “foot-strike time” $\tau_{\text{strike}}^{im}$:

$$\eta_{n+1} = S(\eta_n, \Psi_n, \tau_c, \tau_{\text{strike}}^{im}(\eta_n, \Psi_n, \tau_c, h)).$$

In the case of the reactive control, the step before encountering the ground step starts with the collision-free fixed point without control. If the current step starts with the collision-free fixed point $\eta^*$ and no controls are applied, the time the swing foot strikes the ground depends on the ground height deviation only, so in terms of $\tau_{\text{strike}}^{im}$, this passively-determined foot-strike time $\tau_{\text{strike}}(h)$ is defined by

$$\tau_{\text{strike}}(h) = \tau_{\text{strike}}^{im}(\eta^*, 0, 0, h).$$

In this way, when the current step starts with the collision-free fixed point and no controls are applied, the initial state of the next step (ie. post-strike state of the current step) is a function of the ground height deviation only. In this case, the initial state of the next step can be written as

$$\eta^+(h) = S(\eta^*, 0, 0, \tau_{\text{strike}}(h)).$$

The superscript “im” indicates that the dynamics of the system is under the influence of impulsive control.

### 8.3.5 Calculating the required sequence of control impulses

Given that we know that the state of the biped is $\eta_1$ at the start of the step, and at the start of the next step we want the state of the biped to be $\eta_2$, the state-matching constraint can be written as

$$\eta_2 - S(\eta_1, \Psi^d, \tau_c, \tau_{\text{strike}}^{im}(\eta_1, \Psi^d, \tau_c, h)) = 0.$$  

We want to find the desired control matrix $\Psi^d$ such that the state-matching constraint is satisfied. In this way, the desired control matrix is the root of the equation that describes the state-matching constraint.
Alternatively, because the orientation angle of the torso will be carried over after the support-transfer, we can use an event locator function, \( f_{\text{event}} \) to match the orientation angle of the torso, and then work out the control matrix that gives the full state-matching. In this case, the support-transfer time is taken as the time the orientation angle of the torso matches the targeted value \( \tau_e \) instead of the foot-strike time. This is fine as long as the target state is chosen to be a state such that both feet on the ground. With this approach, we solve for \( \Psi^d \) so that

\[
\eta_2 - S (\eta_1, \Psi^d, \tau_c, \tau_e (\eta_1, \eta_2, \Psi^d, \tau_c)) = 0 \\
\tau_e (\eta_1, \eta_2, \Psi^d, \tau_c) = \min \{ \tau | f_{\text{event}} (\Gamma (\eta_1, \Psi^d, \tau_c, \tau)) = 0 \}\]

In both cases the desired control matrix \( \Psi^d \) is a function of the starting and the target states (\( \eta_1 \) and \( \eta_2 \) respectively), and the control placement times \( \tau_c \):

\[\Psi^d = \Omega (\eta_1, \eta_2, \tau_c).\]

The desired control matrix takes the form

\[\Psi^d = \Omega (\eta_1, \eta_2, \tau_c) = (\nu_1^d (\eta_1, \eta_2, \tau_c), \nu_2^d (\eta_1, \eta_2, \tau_c), ... \nu_N^d (\eta_1, \eta_2, \tau_c)).\]

The columns in the desired control matrix are the control impulses we want.

### 8.4 Control scenario

#### 8.4.1 Reactive control scenario

In the reactive scenario, the animal does not know anything about the ground step. The animal is only able to find out about the presence of a ground step after being perturbed by it. Before encountering the ground step, the step starts with a collision free fixed point \( \eta^* \), and the biped moves passively along the collision-free periodic gait trajectory. Because of the ground step, the step will not end at the collision-free strike time \( \tau^* \), and hence the post-transfer state will not be the collision-free fixed point \( \eta^* \). After encountering the ground step, the post-strike state is passively determined as a function of the ground step. For brevity, we call this function \( \eta^+ (h) \).

After encountering the ground step, the impulsive control will bring the biped from the passively determined post-strike state \( \eta^+ (h) \) to the collision free fixed point \( \eta^* \). The desired sequence of control impulses that bring the system back
onto the collision-free trajectory can be written as a function of ground step and the control placement times:

$$ \Omega^{\text{react}}(h, \tau_c) = \Omega(\eta^+(h), \eta^*, \tau_c). $$

Define $C_{\text{tot}}$ as the total unsigned mechanical work done by the control impulses that allow the required state-matching, and for brevity we call $C_{\text{tot}}$ the “cost of recovery”, the cost of recovery ($C_{\text{tot}}$) can be expressed as a function of the ground step and the control placement time,

$$ C_{\text{tot}}(h, \tau_c) = C_{2}(h, \tau_c) = \sum_{i=0}^{N} \left| W_{i,2}(\eta^+(h), \Omega^{\text{react}}(h, \tau_c), \tau_c) \right|. \quad (8.9) $$

To reduce the arbitrariness, we can choose the control placement times such that the cost of recovery can be minimized.

### 8.4.2 Predictive control scenario

In the predictive control scenario, the target state we want can be specified through a decision function $\eta^d(h)$. The decision function is an input-output function. The input is the information about the ground step captured by predictions, and the output is a decision on the target state to be used at the instant the biped reaches the ground step. Details about the mathematical definition of the decision function will be discussed in Section 8.4.3.

When the biped reaches the ground step, it is in the chosen target state, because with a non-zero ground height deviation, the target state has a non-zero deviation from the collision-free fixed point, therefore we have to bring the biped from the target state $\eta^d(h)$ to the collision-free fixed point, $\eta^*$, using another sequence of control impulses.

Over the first phase of the control, we want a sequence of control impulses that brings the biped from the collision-free fixed point, $\eta^*$, to a target state calculated from the decision function $\eta^d(h)$ in one step. Therefore the sequence of impulses over the first phase of the control is given by

$$ \Omega^{\text{pred}}_1(h, \tau_c) = \Omega(\eta^*, \eta^d(h), \tau_c). $$
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Over the second phase of control, we want a sequence of control impulses that brings the biped from the target state \( \eta^d(h) \) to the collision-free fixed point \( \eta^* \) in one step. Therefore the sequence of impulses over the second phase of the control is given by

\[
\Omega^{\text{pred}}_2(h, \tau_c) = \Omega \left( \eta^d(h), \eta^*, \tau_c \right).
\]

The cost of recovery can be expressed as a function of ground height deviation and the control placement times,

\[
C_{\text{tot}}(h, \tau_c) = C_1(h, \tau_c) + C_2(h, \tau_c)
= \sum_{i=0}^{N} \left( |W_{i,1} \left( \eta^*, \Omega^{\text{pred}}_1(h, \tau_c), \tau_c \right)| + |W_{i,2} \left( \eta^d(h), \Omega^{\text{pred}}_2(h, \tau_c), \tau_c \right)| \right),
\]

(8.10)

All of the work done by the control impulses applied during both phases of the predictive control process counts towards the cost of recovery.

Again, to reduce the arbitrariness, we can choose the control placement times such that the cost of recovery can be minimized.

8.4.3 Predictive strategy: Defining the decision function

To recap: In the predictive scenario, the animal knows everything about the ground step. Because information about the ground step is provided by the predictive mechanism, if the ground height deviation is predicted to be \( h \), one can use this information and plan ahead, through a decision function, for a desired ground-contacting state for the biped when the swing foot strikes the ground, and use impulsive control to achieve it. Over the next step, we bring the biped back to the collision-free gait using another impulsive control.

The decision function is therefore a function of the ground step that gives a desired post-strike state. Exactly how the decision function should be formulated is up to the designer. Ideally, the decision function should give a post-strike state that satisfies the ground contact constraint, and when the ground step is zero, it should give the collision-free fixed point.

In this study, the decision function is chosen such that the desired post-strike state is a state in which all joint angles matches exactly with the collision-free fixed-point values. Both the stance and the swing feet are in contact with the ground, but at different heights due to the ground step. The angular velocities of
the limbs (relative to the vertical) are exactly the same as that of the collision-free fixed point.

Therefore, in this chapter the decision function \( \eta^d(h) \) is chosen to be

\[
\eta^d(h) = \left( \begin{array}{c}
\theta^d(h) \\
\dot{\theta}^d(h)
\end{array} \right) = \left( \begin{array}{c}
\theta^* + \sin^{-1}\left( \frac{h}{L^*} \right) c^T
\end{array} \right), c = (1, 1, 1, \ldots, 1)
\]

(8.11)

where \( L^* \) is the step length of the collision-free periodic gait, and \( h \) is the ground height deviation.

8.5 Cost of recovery simulations

8.5.1 Overview

In this section, our main objective is to investigate the relationship between the cost of recovery and the ground height deviation. We consider both the reactive and the predictive control strategies. The simulations will be based on the Gomes-Ruina biped, and controls are in the form of impulses applied between the torso and the legs.

Because the collision-free trajectory is dynamically unstable, without control, a perturbation error cannot be repaired, even if the perturbation error is near zero. So the controller will still play a significant role when dealing with the instability due to a ground step that falls within the near-zero margin. Therefore, we base our investigation on small, near-zero ground steps.

8.5.2 Working demonstrations

Predictive control strategy

We start with a demonstration of the predictive control (Figure 8.4). The simulation is based on the Gomes-Ruina ideal biped. It has a collision-free fixed point. To six decimal places, the collision-free fixed point is:

\[ \eta^* = (0.716750, -0.716750, 0.000000, 0.000000, 0.000000, -7.43121) \]
Table 8.1: Control impulses used in the predictive-control demonstration (Figure 8.4)

<table>
<thead>
<tr>
<th>Application time</th>
<th>Phase-1 impulses(^1)</th>
<th>Phase-2 impulses(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0.00124120, −0.000948784)</td>
<td>(0.00182331, 0.00272785)</td>
</tr>
<tr>
<td>2.2</td>
<td>(0.000468137, −0.00210049)</td>
<td>(0.00177401, 0.00265674)</td>
</tr>
<tr>
<td>2.4</td>
<td>(0.00389599, 0.00389599)</td>
<td>(−0.00190527, −0.00190527)</td>
</tr>
</tbody>
</table>

The ground step chosen for the demonstration is 0.01, so when the swing foot reaches the ground step, we want the biped to be in the target state

\[ \eta^d(0.001) = (0.717511, −0.715989, 0.000761165, 0, 0, −7.43121). \]

The first three control impulses are used to bring the biped from the collision-free fixed point to the target state, and the control impulses bring the biped to the target state (six-figure accuracy). In the second phase, another three control impulses are applied to bring the biped from the target state to the same collision-free fixed point but at a different height. Finally, the control impulses successfully achieve this state-matching (six-figure accuracy).

In this demonstration, the control impulses are very small in size and have only minor effects on the dynamics. So the effects of these impulses cannot be visualized easily. This is expected because the target state is chosen to be close to the collision-free fixed point, and the mechanical work required for achieving the required state-matching in both phases of the predictive control are expected to be small. The numerical values of the control impulses are listed in Table 8.1.

**Reactive control strategy**

We show here a demonstration of the reactive control strategy based on the Gomes-Ruina ideal biped (Figure 8.5). We assume that the ground step remains unknown until it is encountered. In this demonstration, the ground height deviation is 10\(^{-3}\). The ground step interrupts the collision-free periodic walking. The control impulses bring the biped from a perturbed state to collision-free fixed point (six figure accuracy). This time, the control impulses are large in magnitude, and the discontinuities in the velocities of the limbs due to impulse applications can be visualized. At the end, the collision-free periodic walking is restored at a different height. The numerical values of the control impulses are listed in Table 8.2.

---

1Impulses are non-dimensional, and are defined in coordinate \( \phi \).

2The control application times are non-dimensional and relative to the start of a step.
8.5. COST OF RECOVERY SIMULATIONS

Table 8.2: Control impulses used in the reactive-control demonstration (Figure 8.5)

<table>
<thead>
<tr>
<th>Application time</th>
<th>Impulses¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(−0.136254, −0.149063)</td>
</tr>
<tr>
<td>2.2</td>
<td>(−0.100038, −0.115284)</td>
</tr>
<tr>
<td>2.4</td>
<td>(0.0651738, 0.0651738)</td>
</tr>
</tbody>
</table>

At this point, it is easy to recognize that although the predictive control involves twice as many control impulses as the number of control impulses necessary for reactive control, the magnitude of the control impulses is much smaller. In the extreme case, when the collision-free gait is perturbed by a downward ground step with a size much less than a millimeter (Figure 8.6), the control impulses need to produce a change in velocity in the magnitude of 1 in order to restore the collision-free periodic walking, which is more than 1000 times the size of the ground step! Therefore, it is plausible that prediction can reduce the energy required for handling environmental obstacles. In the next section, we will look further into this argument.

8.5.3 The relationship between the net controller work and the ground height deviation

Previously, our study only covered a set of arbitrarily chosen control placement times and only a few ground height deviations, in this section we extend our investigation. We investigate the relationship between the net controller work required for handling a small ground step, the ground height deviation and the control strategy (predictive or reactive). We plot mechanical work done against the ground height deviation as a log-log plot (Figure 8.7).

When the ground step is downward, with the predictive control strategy, it is evident from Figure 8.7 that the amount of controller work reduces to zero as the ground step approaches zero. The controller does negative work. With the reactive control strategy, it is evident from Figure 8.7 that comparing with the case of predictive control strategy, the reactive control strategy requires a larger net controller for handling each ground step, and the net controller work done does not go to zero as size of the ground step approaches zero.

As a reminder, when the log-log plot is approximately a straight line with a positive, non-zero slope, it suggests a power curve passing through the origin. When the log-log plot is approaching a horizontal line, it suggests either a power curve that does not pass through the origin, or an asymptotic behaviour.
When the ground step is upwards, it is evident from Figure 8.7 that the amount of controller work reduces to zero as the ground step approaches zero regardless of whether the control strategy is predictive or reactive. However, compared to the case of predictive control strategy, the reactive control strategy requires a larger net controller work for handling each ground step.

In this part of the work, we always fix the control placement times at \( \tau_c = (0, 2.2, 2.4) \). Because the control objective is to bring the system from a starting state to a target state. Given that the state-matching is achieved, the net controller work must always equal to the change in total energy over the step by energy conservation, regardless of how the control placement times are chosen, although the works done by individual control impulses will vary when the control placement times vary. Therefore, if we repeat the same calculations with a different set of control placement times the results will remain the same.

### 8.5.4 The relationship between the cost of recovery, the control placement time and other model parameters

In this section, we use computer simulations to investigate the relationship between the cost of recovery, ground step and the control placement times. The questions to be answered are: How important is the choice of control placement time in terms of reducing the cost of recovery? Can prediction really reduce the cost of recovery?

From experimental studies, muscle activation is significant near the start and the end of a step [51]. Following muscle activation, muscle activity decays quite quickly in the magnitude of milliseconds [1]. Therefore, a major proportion of the gait trajectory is expected to be nearly passive. For this reason, when studying the effect of changing control placement times on the cost of recovery, the control placement times are chosen such that the control impulses are located at the start and the end of a step.

While we are working with a straight-leg biped, we will need to handle foot scuffing. We also want to keep the end-step control impulses close to the end of the step but before foot strike. Therefore, we place the end-step impulses when the stance leg is sufficiently past the vertical, and we do not detect foot strike until the last control is applied. In this way, we can handle the foot scuffing and at the same time keep all control impulses within a step. If control impulses make the swing foot hit the ground before the last control impulse application, it is treated as foot scuffing. This is fine as long as controls do not make the swing foot dive too far into the ground.
8.5. **COST OF RECOVERY SIMULATIONS**

From a non-systematic study, not all choices of control placement times will allow control impulses required for the state-matching to be found. The existence of an impulsive control solution that allows the required state-matching is quite sensitive to the control placement times. Working demonstrations offer some clues about when the control impulses should be placed such that control impulses required for the state-matching can be found.

Therefore, we use Figures 8.4, 8.5 and 8.6 as guides to determine when control impulses should be placed so that an impulsive control solution can be found. We place the first control impulse at the toe-off ($\tau_1 = 0$), and the second and third impulse will be placed near the time we expect the step to end. In order to keep the second and third impulses close to the end of the step while keeping them well-separated, we introduce a “timing parameter”, $\tau$. In the case of the reactive control strategy, the controls are placed at times $\tau_c = (0, \tau^* - 2\tau, \tau^* - \tau)$. For the case of predictive control strategy, the control impulses are placed at $\tau_c = (0, \tau^* - 2\tau, \tau^* - \tau)$ for both phases of the control. The value $\tau^*$ is the period of the collision-free periodic gait which, in our case, is approximately 2.62. The timing parameter is constrained to be between 0.05 and 0.21. This keeps the last two controls within the late phase of a step cycle while keeping them separated.

In this way, the cost of recovery at each ground step can be minimized through a one-dimensional constrained optimization problem by varying the timing parameter $\tau$. The cost of recovery as a function of the ground step and the timing parameter, and the minimum cost of recovery observed at each ground step are presented in Figure 8.8.

By varying the timing parameter $\tau$, the cost of recovery can be minimized. For the case of predictive control strategy with small ground step, the minimum cost of recovery is a local minimum within the chosen limits of the timing parameter $\tau$, and the location of the minimum is unaffected by the size of ground step. As the ground step approaches zero, the minimum cost asymptotically approaches a fixed value. A similar trend is observed for the case of reactive control strategy, when the ground step is upward.

In the case of the reactive control strategy, when the ground step is upward and small, the minimum cost of recovery occurs at the upper boundary of the chosen limits of $\tau$. As the ground step approaches zero, the minimum cost asymptotically approaches a fixed value. It is easy to see from Figure 8.8 that the cost of recovery is approximately proportional to the square root of the ground height deviation when the ground step is small and upward. The change in control placement times has an insignificant effect on the relationship between the cost of recovery and the ground height deviation. When the ground step is downward, the cost of recovery
does not approach zero as the ground step approaches zero, and the change in control placement times has an insignificant effect on the relationship between the cost of recovery and the ground step for this case too. Mathematically, the observed relationship between the cost of recovery, \( C_{\text{tot}} \), the ground step \( h \), and the control placement time \( \tau_c \) can mathematically be described as

\[
C_{\text{tot}} (h, \tau) \approx \begin{cases} 
  f_1 (\tau) \sqrt{h} & h \geq 0 \\
  f_2 (\tau) + f_3 (\tau) |h| & h < 0 
\end{cases}
\]

In the case of the predictive control strategy, the simulations suggest that when the ground step is small, regardless of its direction of the ground step, the cost of recovery is approximately proportional to the size of the ground step. The change in control placement times has an insignificant effect on the relationship between the cost of recovery and the ground step. The observed trend can mathematically be described as:

\[
C_{\text{tot}} (h, \tau) = g (\tau) |h|.
\]

As for the minimum cost of recovery, results from simulations suggest that for the case of reactive control strategy it is proportional to the square root of the ground step when the ground height deviation is small. When the ground step is downward, it does not approach zero as the ground height deviation approaches zero. The observed trend can mathematically be described as

\[
C_{\text{min}}^{\text{tot}} (h) = \begin{cases} 
  k_1 \sqrt{h} & h \geq 0 \\
  k_2 + k_3 |h| & h < 0 
\end{cases}
\]

As for the predictive strategy, computer simulation suggests that when the ground step is small, the minimum cost of recovery is proportional to the size of ground step, regardless of whether it is up or down. The observed trend can mathematically be described as

\[
C_{\text{min}}^{\text{tot}} (h) = k |h|.
\]

The most important observation is that the predictive control strategy allows a much cheaper cost of recovery compared to the cost required in the case of reactive control strategy. Therefore, if the ground step can be predicted in advance and controls are applied to pre-adjust for the upcoming ground step, the minimal cost of recovery is much less compared to attempting to fix the perturbation error introduced by the ground step after it has happened.
8.6 Disadvantages of the reactive control strategy

8.6.1 Overview

From the simulations, with the Gomes-Ruina ideal biped we found the energy required for repairing the perturbation error due to a small ground step can be reduced if the ground step can be predicted in advance, and controls applied to pre-adjust for the upcoming ground step. Otherwise, the energy consumption for handling the ground step will be higher, and may not always diminish to zero as the ground step approaches zero. In this section, we discuss the general ideal biped and whether or not it is more energetically expensive to bring the ideal biped back to the unstable collision-free periodic gait without the ability to predict the ground step and take advance action than with the ability to predict the ground step and taking advance action.

8.6.2 An infinitesimal ground height deviation does not imply an infinitesimal change in foot-strike time

The problem to be investigated here is how changes in the ground height deviation affect the foot-strike time, given that the gait starts with the collision-free fixed point at zero height and the gait is passive. The problem can alternatively be described as how changes in the height of the stance foot affect the time the swing foot reaches zero height, given that the gait starts with the collision-free fixed point and the gait is passive. The effect of a ground step with a positive ground height deviation and zero stance-foot height is equivalent to the effect of placing the stance foot below zero height. Similarly, the effect of a ground step and a negative ground height deviation with zero stance-foot height is equivalent to the effect of placing the stance foot above zero height.

In order to walk passively and periodically at a finite walking speed on level ground, the swing foot must reach the ground with zero velocity. However, the swing foot can reach the ground with non-zero acceleration without making the foot strike impulsive. If the swing foot approaches the ground from above and reaches the ground with zero velocity, in general situations, the foot will accelerate upward in the next moment. Therefore, in a collision-free periodic gait, we expect the swing-foot height function $h_{vert}(t)$ (the swing-foot height as a function of time) to have a local minimum at $t = \tau^*$ (Figure 8.9a). In the special case when the swing foot reaches the ground with zero velocity and zero acceleration along the world vertical, we expect $h_{vert}(t)$ to have an inflection point at $t = \tau^*$ (Figure 8.9a).
In the limiting case when the step starts with the collision-free fixed point, the stance foot is at zero height and the motion is uncontrolled, the swing foot strikes impact-less at \( t = \tau^* \) (Figure 8.9b, the blue curve). If the step starts with the collision-free fixed point, the stance-foot height is below zero height and the motion is uncontrolled, the foot-strike time changes smoothly as the stance-foot height approaches zero from negative (Figure 8.9b, green curves). Therefore, if there is a small upward ground step, the walking step starts with the collision-free fixed point without controls, and the stance foot is at zero height, we expect that the foot-strike time will change smoothly as the ground height deviation changes.

If the step now starts with the collision-free fixed point and the motion is uncontrolled but the stance foot is lifted to a height slightly above zero, we can see that the time at which the swing foot reaches zero height suddenly deviates away from the collision-free gait period \( \tau^* \) (Figure 8.9b, red curves). Therefore, if there is a small downward ground step, the walking step starts with the collision-free fixed point without controls and the stance foot is at zero height, we expect that the foot-strike time will suddenly deviate away from the collision-free gait period when the ground height deviation deviates slightly from zero. Therefore, the foot-strike time function \( \tau_{\text{strike}} (h) \) has unequal left- and right-hand zero limits:

\[
\lim_{h \to 0^+} \tau_{\text{strike}} (h) = \tau^* \\
\lim_{h \to 0^-} \tau_{\text{strike}} (h) = \sigma, (\sigma \neq \tau^*).
\]

Therefore, an infinitesimal ground height deviation does not imply an infinitesimal change in foot-strike time.

If the collision-free zero-height contact is missed due to a downward ground height deviation, when the swing foot approaches the zero height again, it misses the collision-free strike time. So we do not expect the post-strike state to be the collision-free fixed point if the support-transfer happens at this instant. Rather, for the post-strike state, we would expect that

\[
\lim_{h \to 0^+} \eta^+ (h) = \eta^* \\
\lim_{h \to 0^-} \eta^+ (h) = q, (q \neq \tau^*).
\]
8.6.3 An infinitesimal ground step does not imply an infinitesimal essential controller work

Before reaching the ground step, the ideal walker is undergoing collision-free periodic walking. The periodic walking will be interrupted by the ground step, and the controller’s task is to bring the ideal walker back into the same collision-free period gait at a different height. The essential work of the controller, $W_{net}$, equals the kinetic energy loss, $\Delta K$, due to inelastic ground collision, plus the gravitational potential energy change due to support-transfer:

$$W_{net} = \Delta K + h \sum m_i g,$$

(8.12)

where $h$ is the ground height deviation, and the second term is the gravitational potential energy change due to support-transfer (see Figure 8.10 for clarification).

Figure 8.9a implies that if the ground step is not predicted and perturbation error can only be repaired after encountering it, an infinitesimal downward ground step will cause the swing foot to miss the contact and land with non-infinitesimal velocity and hence a non-infinitesimal loss in kinetic energy ($\Delta K$). Although a downward ground step will allow the gravitational potential energy to compensate the collision loss, the amount of gravitational potential energy gained (which is the second term in Equation 8.12) is infinitesimal when the ground step is infinitesimal. So the amount of gravitational potential energy gained is not enough to compensate for the loss in kinetic energy. Overall, the essential work that the controller must do will be positive and non-infinitesimal when the ground height deviation is infinitesimal and downward. This rules out the possibility that the cost of recovery will be infinitesimal when the ground step is infinitesimal and downward.

8.6.4 The case of zero vertical foot acceleration

For the case of zero vertical foot acceleration (Figure 8.9a), it is possible that by adjusting the design parameters (i.e., spring stiffness), such that the collision-free periodic gait has both zero foot-strike velocity and zero foot-strike acceleration in the direction of the world vertical, when the swing foot reaches zero-height, its next movement is downward. In this way, the foot-strike time will approach the collision-free gait period as the ground height deviation approaches zero from either positive or negative ground height deviation. In this case we expect the cost to diminish to zero as the ground height deviation approaches zero regardless of whether the ground height deviation is up or down. In Chapter 10, we will demonstrate this possibility.
However, the swing-foot height function has an inflection point behaviour near the collision-free strike time \( t \approx \tau^* \), and because of the inflection point behavior, the time at which the foot strikes the ground is still sensitive to the ground height deviation. Therefore, the controller cost can still be very sensitive to the ground height deviation.

### 8.6.5 How can predictions help?

If, on the other hand, the downward ground height deviation can be predicted in advance, and actions taken before encountering it, we can certainly try to set up the controller before reaching the ground step, so that the swing foot will land with zero velocity when reaching the ground step. We can only generate control forces before encountering the ground step unless we can see, or predict, the location of the ground step. If the control successfully avoids the collision, there will be no collision loss and the essential controller work will be equal to the gravitational potential energy required for bringing the ideal walker back into the same collision-free periodic gait at a different height:

\[
W_{\text{net}} = h \sum_i m_i g 
\]  

(See Figure 8.11 for clarification.)

Therefore, the essential controller work will be infinitesimal when the size of the ground step is infinitesimal. Therefore, regardless of whether the ground height deviation is upward or downward, with prediction it becomes possible that the cost of recovery is infinitesimal when the ground step is downward and infinitesimal in size.

In addition, for the case of an upward ground step, by comparing Equation 8.12 and 8.13, we easily realize that in order to bring the ideal walk back into the same collision-free trajectory at a higher level, if the ground height deviation can be predicted in advance, the essential controller work can be reduced by using controls to avoid the inelastic collision before encountering the ground step. Therefore, with the ability to predict, it becomes possible for the cost for handling an upward ground step to be reduced too.

It is also possible to make the post-transfer state of the biped at the ground step less sensitive to the ground height deviation by applying control forces before encountering the ground step. So when the ground height deviation is zero, the post-strike state at the ground step is the collision-free fixed point, and when the
ground height deviation is close to zero, the post-transfer state is the collision-
free fixed point plus a close-to-zero perturbation due to an infinitesimal ground
height deviation. In this way, with prediction it becomes possible that the cost
of recovery is infinitesimal if the size of the ground step is infinitesimal.

8.7 In greater mathematical detail

8.7.1 Overview

In this section, we continue in greater mathematical detail our discussion on
the energetic disadvantage of the reactive control strategy. It is clear already from
Figure 8.9b that when the ground height deviation is downward and the reactive
control strategy is used to handle it, an infinitesimal downward ground step can
cause the swing foot to swing past the contact and hit the ground with non-
infinitesimal velocity, and consequently a non-infinitesimal control power will be
necessary to compensate for the collision-loss. In this section, we focus our discus-
sion on perturbations due to a small, upward ground step, where the collision-free
fixed point has zero swing foot acceleration in the direction of the world vertical,
and zero swing foot velocity (Figure 8.9a).

Regardless of whether the control strategy is reactive or predictive, we can view
the entire control process as a two-step process. The first walking step starts with
the collision-free fixed point $\eta^*$ (Figure 8.12), and at the end of the second walking
step after support-transfer, we return to the same collision-free fixed point, but
at a different height.

In the limiting case, when the ground height deviation is zero and the first step
starts with the collision free fixed point $\eta^*$ at time $t = 0$, the post-strike state
is also $\eta^*$ and the gait is periodic, with gait period equals to $\tau^*$. In each step,
swing foot strikes at time $t = \tau^*$.

If the ground height deviation is non-zero, the post-strike state of the first step
must always be $\eta^* + \Delta \eta (h)$, where $\Delta \eta (h)$ represents a non-zero perturbation.
So if we want the biped to return to the collision-free fixed point again at the
end of the second step after support-transfer, controls will be necessary So, the
remaining question is, how sensitive $\Delta \eta (h)$ is to the ground height deviation $h$
when $h \approx 0$? and, hence, how sensitive the cost of recovery is to $\Delta \eta (h)$ when
$h \approx 0$?
8.7.2 The foot-strike time

When an ideal biped walks passively with the collision-free periodic gait at zero height, the swing foot reaches zero height at the collision-free strike time $\tau^*$ with zero velocity. Define $a_{f,y}(\eta^*, t)$ as the swing foot acceleration in the direction of the world-vertical at time $t$ given that the gait starts with the collision-free fixed point $\eta^*$. The height of the swing foot as a function of time near the collision-free strike time $\tau^*$ can be approximated as

$$h_{\text{vert}}(t) \approx \frac{1}{2} a_{f,y}(\eta^*, \tau^*) (t - \tau_{\text{strike}})^2.$$ (8.14)

The foot-strike time can be approximated as a function of ground height deviation $h$ by replacing the left hand side of Equation 8.14 with $h$ and rearranged to make $t$ the subject. So, given that the ground height deviation is positive and small, the foot-strike time follows approximately the trend

$$\tau_{\text{strike}}(h) \approx \tau^* + \alpha \sqrt{h}.$$ (8.15)

The approximation remains accurate when $h$ is small.

If the ideal biped is redesigned in the way that when it walks passively with the collision-free periodic gait at zero height, the swing foot reaches zero height at the collision-free strike time $\tau^*$ with zero velocity as well as zero swing foot acceleration in the direction of the world vertical, the height of the swing foot as a function of time near the collision-free strike time $\tau^*$ can be approximated as

$$h_{\text{vert}}(t) \approx \frac{1}{6} \dot{a}_{f,y}(\eta^*, \tau^*) (t - \tau_{\text{strike}})^3.$$ (8.16)

Hence in this case, when the ground height deviation is small in size, the foot-strike time follows approximately the trend

$$\tau_{\text{strike}}(h) \approx \tau^* + \beta h^{\frac{1}{3}}.$$ (8.17)

In the case of reactive control, the first step is passive, and controls are not generated until the second step starts. If we start the first step with the collision-free fixed point, leave the first step passive and let the swing foot strike the ground naturally, for a small change in ground height deviation ($h$), the foot-strike time deviates from the collision-free gait period by an amount proportional to $\sqrt{h}$ when $a_{f,y}(\eta^*, \tau^*)$ is positive and the ground step is upward. The foot-strike time deviates from the collision-free gait period by an amount proportional to $|h|^{\frac{1}{3}}$ when $a_{f,y}(\eta^*, \tau^*)$ is zero and the ground step is either upward or downward.
The square-root function $\sqrt{h}$ and the cube-root function $|h|^{\frac{1}{3}}$ have a slope of infinity at $h = 0$, which means in the case of reactive control, when the ground height deviation changes slightly from zero, the foot-strike time changes smoothly when the ground height deviation changes, but is sensitive to the ground height deviation.

8.7.3 The cost of recovery and the control strategies

Overview

We know that with the reactive control strategy, the foot-strike time is sensitive to change in ground height deviation. Here, we investigate how sensitive the recovery cost is to the ground height deviation. If we leave the first step passive and let the ideal biped be perturbed away from the collision-free trajectory naturally by the ground height deviation on footfall, we can then apply impulsive controls to recover the perturbation. We also ask whether the sensitivity can be reduced by allowing the ground height deviation to be predicted in advance and pre-adjust for it on the first step.

In this and subsequent sections, we assume that control forces are pulse-like so that impulsive force approximation can be applied. The controller cost is defined as the total unsigned work done by the control force pulse. The cost of recovery is defined as the total unsigned work done by all control force pulses.

In addition, in this and the subsequent sections, we base our discussion on Taylor series expansions of the various functions involved in cost calculation. For simplicity, we assume that except for the foot-strike time function $\tau_{\text{strike}}(h)$ at $h = 0$, all other functions are analytic with respect to the parameters we have established, at least within certain ranges of values (for example near the collision-free fixed point). We compare results derived from the Taylor series expansions with computer simulations.

The post-strike state

If support-transfer happens at time $T$, the post-transfer state is $S(\eta^*, 0, \tau_c, T)$. When the support-transfer time deviates slightly from $T$, the post-transfer state is perturbed, and the perturbed post-transfer state can be approximated as

$$S(\eta^*, 0, \tau_c, \tau_{s/tf}) \approx S(\eta^*, 0, \tau_c, T) + \left[ \frac{dS(\eta^*, 0, \tau_c, s)}{ds} \right]_{s=T}(\tau_{s/tf} - T). \quad (8.18)$$
Support-transfer happens when the swing foot strikes. We know that if we start at the collision-free fixed point, the foot strike happens exactly at the collision-free strike time $\tau^*$. Therefore, a small deviation in foot-strike time $\tau_{\text{strike}}(h)$ deviates post-strike state from the collision-free fixed point, and perturbed post-strike state can be approximated as:

$$S(\eta^*, 0, \tau_c, \tau_{\text{strike}}(h)) \approx \eta^* + \left[ \frac{dS(\eta^*, 0, \tau_c, s)}{ds} \right]_{s = \tau^*} (\tau_{\text{strike}}(h) - \tau^*). \quad (8.19)$$

When the ground height deviation is positive and $a_{f,y}(\eta^*, \tau^*) > 0$, we can replace the $\tau_{\text{strike}}(h)$ term in Equation 8.19 with the expression on the right hand side of Equation 8.15 to approximate $\tau_{\text{strike}}(h)$ when $h \approx 0$. Finally, we can see that a small positive ground height deviation $(h)$ deviates the post-strike state from the collision-free fixed point by an amount that is approximately proportional to $\sqrt{h}$,

$$S(\eta^*, 0, \tau_c, \tau_{\text{strike}}(h)) \approx \eta^* + \left[ \frac{dS(\eta^*, 0, \tau_c, s)}{ds} \right]_{s = \tau^*} (\alpha\sqrt{h}). \quad (8.20)$$

When $a_{f,y}(\eta^*, \tau^*) = 0$, we can replace the $\tau_{\text{strike}}(h)$ term in Equation 8.17 to approximate $\tau_{\text{strike}}(h)$ when $h \approx 0$. In this case, we can see that a small ground height deviation $(h)$ deviates the post-strike state from the collision-free fixed point by an amount that is approximately proportional to $|h|^{\frac{1}{3}}$,

$$S(\eta^*, 0, \tau_c, \tau_{\text{strike}}(h)) \approx \eta^* + \left[ \frac{dS(\eta^*, 0, \tau_c, s)}{ds} \right]_{s = \tau^*} (\beta|h|^{\frac{1}{3}}), \quad (8.21)$$

regardless of whether the ground height deviation is positive or negative.

**Impulsive control sequence (the control matrix)**

We know that if the gait starts with the collision-free fixed point $(\eta_1 = \eta^*)$, no control would be needed if the target state is the collision-free fixed point $(\eta_2 = \eta^*)$, so $\Omega(\eta^*, \eta^*, \tau_c) = 0$. If the starting and the target states deviate slightly from $\eta^*$, controls will be needed for the required state-matching. This implies that the control matrix is no longer zero, and the changed control matrix can be approximated as

$$\Omega_{ij}(\eta_1, \eta_2, \tau_c) \approx [\nabla_u (\Omega_{ij}(u, \eta^*, \tau_c))]_{u = \eta^*} \bullet (\eta_1 - \eta^*) + [\nabla_v (\Omega_{ij}(\eta^*, v, \tau_c))]_{v = \eta^*} \bullet (\eta_2 - \eta^*). \quad (8.22)$$
The target state is the collision-free fixed point ($\eta^*$), and in the case of the reactive control strategy, the starting state is the passive post-strike state $\eta^+ (h)$. Therefore, when the ground height deviation is small and positive, the control matrix required for the re-stabilization can be approximated as

$$
\Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \approx [\nabla_u (\Omega_{ij} (u, \eta^*, \tau_c))]_{u=\eta^*} \cdot (\eta^+ (h) - \eta^*). \quad (8.23)
$$

(We have defined previously that $\eta^+ (h) = S (\eta^*, 0, \tau_c, \tau_{\text{strike}} (h))$.)

When the ground height deviation is positive and $a_{f,y} (\eta^*, \tau^*) > 0$, we can use Equation 8.20 to approximate the $\eta^+ (h)$ term in Equation 8.24. Because the ground step perturbs the system, we need to change the elements in the control matrix to allow re-stabilization, and the required changes are approximately proportional to $\sqrt{h}$. So the new control matrix can be approximated as:

$$
\Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \approx \omega_{ij} (\tau_c) \sqrt{h}. \quad (8.24)
$$

When $a_{f,y} (\eta^*, \tau^*) = 0$, we can use Equation 8.21 to approximate the $\eta^+ (h)$ term in Equation 8.24. In this case, the new control matrix can be approximated as:

$$
\Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \approx \xi_{ij} (\tau_c) h^{\frac{1}{3}} \quad (8.25)
$$

regardless of whether it is positive or negative.

**Work done by control impulses and cost of recovery**

If in the $n$th step, the walking step starts with the collision-free fixed point ($\eta_n = \eta^*$) without controls (i.e. $\Psi_n = 0$), all control impulses are doing zero mechanical work because all of them are zero. When the starting state varies, we need to vary the elements in the control matrix to achieve the state-matching. If the variation is small, the work done by the $k$th impulse can be approximated by using the Taylor series

$$
W_{k,n} (\eta_n, \Psi_n, \tau_c) \approx [\nabla_u (W_{k,n} (u, \Phi, \tau_c))]_{u=\eta^*, \Phi=0} \cdot (\eta_n - \eta^*) \\
+ \sum_i \sum_j \left[ \frac{\partial W_{k,n} (u, \Phi, \tau_c)}{\partial \Phi_{ij}} \right]_{u=\eta^*, \Phi_{ij}=0} \Psi_{n,ij}. \quad (8.26)
$$
In the case of the reactive control strategy, we use impulsive in the second step to bring the biped from the passive post-strike state \( \eta^+ (h) \) to the collision-free fixed point. We know that \( \eta^+ (0) = \eta^* \), and \( \Omega (\eta^*, \eta^*, \tau_c) = 0 \), so when the passive post-strike state \( \eta^+ (h) \) deviates slightly from the collision-free fixed point \( \eta^* \), non-zero control impulses are needed to repair the perturbation error and the work done by the \( k \)th impulse can be approximated by using the Taylor series

\[
W_{k,2} (\eta^+ (h), \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c), \tau_c) \approx \left[ \nabla_u (W_{k,2} (u, \Phi, \tau_c)) \right]_{u=\eta^*, \Phi=0} \cdot (\eta^+ (h) - \eta^*) + \sum_i \sum_j \left[ \frac{\partial W_{k,2} (u, \Phi, \tau_c)}{\partial \Phi_{ij}} \right]_{u=\eta^*, \Phi_{ij}=0} \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \approx w_{k,2} (\tau_c) \sqrt{h}. \tag{8.28}
\]

When the ground height deviation \( (h) \) is positive and \( a_{f,g} (\eta^*, \tau^*) > 0 \), we can use Equation 8.20 to approximate the \( \eta^+ (h) \) term, and use Equation 8.24 to approximate the \( \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \) term, in Equation 8.27. Then, from Equation 8.27 we can show that the mechanical work done by the \( k \)th control impulse is approximately proportional to \( \sqrt{h} \),

\[
W_{k,2} (\eta^+ (h), \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c), \tau_c) \approx w_{k,2} (\tau_c) \sqrt{h}. \tag{8.28}
\]

When \( a_{f,g} (\eta^*, \tau^*) = 0 \), we can use Equation 8.21 to approximate the \( \eta^+ (h) \) term, and Equation 8.25 to approximate the \( \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c) \) term, in Equation 8.27. In this case, from Equation 8.27 we can show that the mechanical work done by the \( k \)th control impulse is approximately proportional to \( |h|^{1/3} \), regardless of whether the ground height deviation is positive or negative,

\[
W_{k,2} (\eta^+ (h), \Omega_{ij} (\eta^+ (h), \eta^*, \tau_c), \tau_c) \approx u_{k,2} (\tau_c) |h|^{1/3}. \tag{8.29}
\]

Recall that with the reactive control strategy, no controls are used before encountering the ground height deviation, and that the cost of recovery is defined as the sum of the unsigned mechanical work done by control impulses over all steps, the cost of recovery as a function of the ground height deviation \( h \) and the control placement time \( \tau_c \) can be written as

\[
C_{tot} (h, \tau_c) = C_2 (h, \tau_c) = \sum_{i=0}^{N} \left| W_{i,2} (\eta^+ (h), \Omega (\eta^+ (h), \eta^*, \tau_c), \tau_c) \right|. \tag{8.30}
\]
When the ground step is small and upward, and \( a_{f,g} (\eta^*, \tau^*) > 0 \), the biped will be perturbed and the re-stabilization requires a small but non-zero cost of recovery. We can see that the cost of recovery required for handling a small positive deviation in the ground height \( h \) is approximately proportional to \( \sqrt{h} \).

This result matches reasonably well with the result from computer simulation shown in (Figure 8.8), that

\[
C_{\text{tot}} (h, \tau_c) \approx c_i (\tau_c) \sqrt{h}. \tag{8.31}
\]

Similarly, when \( a_{f,g} (\eta^*, \tau^*) = 0 \), we can work out that the cost of recovery due to a small ground step is approximately proportional to \( \left| \frac{1}{3} h \right| \),

\[
C_{\text{tot}} (h, \tau_c) \approx c_i (\tau_c) \left| \frac{1}{3} h \right|. \tag{8.32}
\]

**How can the predictive mechanism help?**

With the predictive control strategy, we have the freedom to choose a target post-strike state we would like to be in at the end of the first step (Figure 8.2). In the case of the predictive control strategy, the target post-strike state is decided using the decision function \( \eta^d (h) \) given by Equation 8.11.

Recall that \( \sin^{-1} (x) \approx x \) when \( x \) is small, Equation 8.11 can be approximated using the Taylor series

\[
\eta^d (h) \approx \eta^* + c h. \tag{8.33}
\]

More generally, because we have the freedom to choose the decision function \( \eta^d (h) \), we can certainly make \( \eta^d (h) \) such that it is analytic at \( h = 0 \), giving collision-free fixed point when \( h = 0 \) and satisfying ground contacting constraint. In this way, when \( h \) is small, \( \eta^d (h) \) can be approximated using the Taylor series

\[
\eta^d (h) \approx \eta^* + c h. \tag{8.34}
\]

In the first phase of the predictive control process, the step starts with the collision-free fixed point \( \eta^* \), and we want to bring the biped into the target post-strike state \( \eta^d (h) \). When \( \eta^d (h) \) is close to the collision-free fixed point \( \eta^* \), the desired control matrix that would bring the biped from state \( \eta^* \) to \( \eta^d (h) \) can be approximated as

\[
\Omega_{ij} (\eta^*, \eta^d (h), \tau_c) \approx \left[ \nabla_v (\Omega_{ij} (\eta^*, v, \tau_c)) \right]_{v=\eta^*} \cdot (\eta^d (h) - \eta^*). \tag{8.35}
\]
Due to the need to handle a slight perturbation, the changes in the mechanical work done by the $k$th control impulse in the first phase of the predictive control process can be approximated as

$$ W_{k,1} (\eta^*(h), \eta^d(h), \tau_c) $$

approximated as

$$ \sum_i \sum_j \left[ \frac{\partial W_{k,1} (\eta^*, \Phi, \tau_c)}{\partial \phi_{ij}} \right] \phi_{ij} = 0 $$

which is equation (8.36).

In the second phase of the predictive control strategy, the step starts with the target post-strike state $\eta^d(h)$ we have chosen in the first step, and we want to bring the biped to the collision-free fixed-point $\eta^*$. When $\eta^d(h)$ is close to the collision-free fixed point $\eta^*$, the desired control matrix required for re-stabilizing the collision-free periodic gait can be approximated as

$$ \Omega_{ij} (\eta^d(h), \eta^*, \tau_c) \approx [\nabla_u (\Omega_{ij} (u, \eta^*, \tau_c))]_{u=\eta^*} \cdot (\eta^d(h) - \eta^*), $$

and the mechanical work done by the $k$th control impulse in the second phase of the control can be approximated as,

$$ W_{k,2} (\eta^d(h), \Omega_{ij} (\eta^d(h), \eta^*, \tau_c), \tau_c) $$

approximated as

$$ [\nabla_u (W_{k,2} (u, \Phi, \tau_c))]_{u=\eta^*, \Phi=0} \cdot (\eta^d(h) - \eta^*) $$

which is equation (8.38).

The cost of recovery in the case of predictive control is defined as the sum of the unsigned mechanical work done by control impulses over both phases of the process. In this case, the cost of recovery as a function of the ground height deviation and the control placement time can be written as

$$ C_{\text{tot}} (h, \tau_c) = C_2 (h, \tau_c) = \sum_{i=0}^{N} \left| W_{i,1} (\eta^*, \Omega (\eta^*, \eta^+ (h), \tau_c), \tau_c) \right| + $$

$$ \sum_{i=0}^{N} \left| W_{i,2} (\eta^d(h), \Omega (\eta^d(h), \eta^*, \tau_c), \tau_c) \right| $$

which is equation (8.39).

For small $h$, in Equation 8.39 we can use Equation 8.36 and Equation 8.38 to approximate $W_{i,1}$ and $W_{i,2}$ respectively; and for Equation 8.36 and Equation 8.38 we can approximate the $\eta^d(h)$ term using Equation 8.34. For Equation 8.36 we can approximate the $\Omega_{ij} (\eta^*, \eta^d(h), \tau_c)$ term using Equation 8.35, and the
\( \eta_d^i (h) \) term using Equation 8.34; and for Equation 8.38 we can approximate the \( \Omega_{ij} (\eta^i, \eta^j, \tau_c) \) term using Equation 8.37, and the \( \eta_d^d (h) \) term using Equation 8.34. At the end, we can see that the cost of recovery is proportional to \( h \) when the ground step is small,

\[
C_{tot} (h, \tau_c) \approx c_i (\tau_c) |h|.
\] (8.40)

This result matches reasonably well with the result from computer simulation shown in Figure 8.8.

Putting everything together, in the limiting case when ground height deviation is zero, regardless of whether the control strategy is reactive or predictive, all control impulses are zero in the absence of other forms of perturbation (Figure 8.12), and so the controller cost is zero. When the ground height deviation increases, with the predictive control strategy the controller cost is proportional to \( h \). However, with the reactive control strategy, controller cost impulse magnitudes is proportional to \( \sqrt{h} \) when \( a_{f,y} (\eta^*, \tau^*) > 0 \) and the ground height deviation is positive and small. When \( a_{f,y} (\eta^*, \tau^*) = 0 \), with the reactive control strategy the controller cost is proportional to \( |h|^{\frac{1}{3}} \). The square-root function \( \sqrt{h} \) and the cube-root function \( |h|^{\frac{1}{3}} \) grow faster than \( h \), when \( h \) is small. From this, we can see that the energetic disadvantage of the reactive control strategy is also due to the problem that the foot-strike time is sensitive to the ground height deviation, so a small ground height deviation can introduce in a perturbation error that is larger than expected, but in the case of reactive control strategy no action can be taken. With the use of the predictive control strategy, the issue of sensitivity is resolved by being able to choose a desirable post-strike state at the end of the first step.

8.8 Discussion

The results from the simulation suggests that although the collision-free gait is unstable, perturbation error can be fixed with physically realistic controls. The energetic cost for repairing perturbation can be reduced if perturbation can be predicted and adjusted for in advance. With prediction, we can ensure that the cost for re-stabilizing the unstable level-ground passive gait due to a change in ground height will always diminish as the ground height deviation approaches zero.

From the theoretical analysis, we showed by Taylor series expansions that the relationship between the cost of recovery and the ground height deviation follows power law, except for the case of reactive control strategy and downward ground
height deviation, in which the cost of recovery does not approach zero as the ground height deviation goes to zero. Although the case in which the use of Taylor series approximation is invalid was not investigated, the results derived based on Taylor series approximations match reasonably well with the results from computer simulations.

Due to the use of physically-unrealistic perfectly-impulsive control forces, our controlled walking model is too simple to be a biologically inspired, integrated neuro-mechanical locomotion model. However, in animal locomotion, the nervous system has a role in guidance [2, 9], which keeps the animal’s body upright, and able to avoid obstacles. It is also argued that animals attempt to slow down the centre of mass of the body before foot strike, using muscles to avoid collision loss [23, 33]. In addition, there are suggestions that animal locomotion has evolved in the direction of minimal energy use [60, 81, 82]. These points justify the predictive control strategy that directs the biped into a collision-free state close to the collision-free fixed point when there is a ground height deviation, and supports our finding that the predictive mechanism can contribute to the reduction of energetic cost when handling environmental obstacles.
8.8. DISCUSSION

Impulsive control brings the biped to the collision-free fixed point

Figure 8.1: A conceptual diagram illustrating how the reactive impulsive control strategy works. (GS, ground step)
Impulsive control brings the biped to the desired pose

Desired pose: Collision-free fixed point rotated to allow ground contact

Impulsive control brings the biped to the collision-free fixed point

Figure 8.2: A conceptual diagram illustrating how the predictive impulsive control strategy works. When landing over the ground step, the control impulses have ensured that the joint angles remain the same as that of the collision-free fixed point. In addition, not shown in the figure, the joint velocities are the same as that of the collision-free fixed point. Therefore, the desired landing state is the collision-free fixed point rotated to allow ground contact.
Figure 8.3: The conceptual diagram illustrating how the state-matching control works. The biped starts with a state where both feet are on the ground, and control impulses are applied in the middle of the step to bring the biped into a target state which also has both feet on the ground, and then a new step starts immediately.
Phase 1: Terminal state: $[0.717511, -0.715989, 0.000761165, 0, 0, -7.43121]$

Phase 2: Terminal state: $[0.71675, -0.71675, 0, 0, 0, -7.43121]$

Figure 8.4: A figure showing an example of the predictive control strategy. In this example, the ground height deviation is $10^{-3}$ . The angular velocities over both phases of the control process are plotted, and the times at which the control impulses are applied over both phases of the control process are indicated by the vertical bars. The numerical values of the control impulses are listed in Table 8.1. In both plots, the angular velocities of the stance leg $\dot{\theta}_1$, the swing leg $\dot{\theta}_2$ and the torso $\dot{\theta}_3$ are shown in red, green and blue respectively.
8.8. DISCUSSION

Figure 8.5: A figure showing an example of the reactive control strategy. In this example, the ground height deviation is $10^{-3}$. The angular velocities of the limbs are plotted and the times at which the control impulses are applied are indicated by vertical bars. The numerical values of the control impulses are listed in Table 8.2. The angular velocities of the stance leg $\dot{\theta}_1$, the swing leg $\dot{\theta}_2$ and the torso $\dot{\theta}_3$ are shown in red, green and blue respectively.

Terminal state: $(0.71675, -0.71675, 0, 0, 0, -7.43121)$
Figure 8.6: A figure showing another example of the reactive control strategy. In this example, the ground height deviation is $-10^{-4}$. The angular velocities are plotted and, the times at which impulses are applied are indicated by the vertical bars. The angular velocities of the stance leg $\dot{\theta}_1$, the swing leg $\dot{\theta}_2$ and the torso $\dot{\theta}_3$ are shown in red, green and blue respectively.
Figure 8.7: The relationship between the net controller work and the ground height deviation $h$. The control placement times are fixed at $\tau_c = (0, 2.2, 2.4)$. Both the predictive and the reactive control strategies are considered. Filled circle means positive work, and unfilled ellipse means negative work.
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Figure 8.8: The relationship between the cost of recovery, the ground height deviation $h$ and the timing parameter $\tau$. The relationship between the minimum cost of recovery and the ground height deviation is also shown. Both the predictive and the reactive control strategies are considered.
Figure 8.9: The relationship between the swing-foot height and the time near $t = \tau^*$. Various stance-foot heights are considered. The times at which swing foot reaches zero height are indicated by the dots on the time axis. The blue curve is the collision-free periodic gait with zero stance-foot height. The green curves are the same collision-free gaits with negative stance-foot height, and the red curves are the same collision-free gaits with positive stance-foot height. The effect of a ground step and a positive ground height deviation with zero stance-foot height is equivalent to the effect of placing the stance foot below zero height. Similarly, the effect of a ground step and a negative ground height deviation with zero stance-foot height is equivalent to the effect of placing the stance foot above zero height.
Figure 8.10: The gravitational energy of the system is defined relative to the height of the stance foot. After support-transfer, the swing foot becomes the stance foot, so if at the moment before the support-transfer the swing foot is at a different height relative to the stance foot, when the support is transferred, the gravitational potential energy of each limb (with mass \( m_i \)) will be changed by \( m_i g h \), where \( h \) is the ground height deviation. Therefore the total change in gravitational potential energy will be \( h \Sigma m_i g \).
Figure 8.11: The controller’s job is to bring the system back to the same collision-free periodic gait at a different height. If the ground height deviation is predicted in advance, and the inelastic ground collision is avoided by controls before encountering the ground step, then the net controller work will be equal to the gravitational potential energy change, which is the mechanical work required for bringing the ideal walker to its collision-free fixed point configuration at a different height.
Limiting case: Without a ground step, the walking gait is the collision-free periodic gait characterized by the fixed point $q^r$.
Chapter 9

Non-impulsive level-ground control

9.1 Overview

9.1.1 Issues with impulsive actuation model

In Chapter 8, we assumed that muscles generate pulse-like forces that quickly change the velocity of the limbs. In the controlled walking model, we used impulsive forces to approximate these force pulses. We black-boxed the force generation mechanism and took the control cost as the total mechanical work done by the impulses applied.

In general, the mechanical work done by a force does not equal the power consumed by the controller when generating the force. To determine the relationship between the power consumption and mechanical work output, it is important to know how actuators consume power and generate forces. This is an actuator modeling problem, which is beyond the scope of this research. However, in order to strengthen our argument on the energetic benefit of the predictive mechanism, we must fairly compare the expense of restoring the collision-free periodic walking after a ground step perturbation with or without prediction. So before we can talk about the cost of recovery, we cannot totally avoid looking into the engineering of actuators.

In this chapter, we make an initial attempt to develop a good model of muscular force generation that does not involve the use of impulsive forces, to allow a more realistic controller cost calculation.
9.1.2 Muscular force generation

To understand how muscles generate forces, we start by looking at the sequence of steps involved in the force generation process (Figure 9.1).

Muscle contraction is a chemical process, which (from [1]) can be summarized as follows: The nerve signals from motor-neurons are nearly impulsive, but they are in the form of shots of neural transmitters rather than forces. When a motor-neuronal action potential reaches the neuromuscular junction, another action potential is produced and propagates through the transverse tubuli. This second action potential causes the transverse tubuli to depolarize. When the transverse tubuli are depolarized, gates in the sarcoplasmatic reticulum are opened, and calcium ions are released through the gate. The calcium ions, together with the adenosine tri-phosphate (ATP) which serves as the source of chemical potential...
energy, cause the thick myofilaments to change the conformation, and the muscle is activated. The thick myofilaments contain an important protein called myosin. When the thick myofilaments change conformation, myosin molecules bind to thin myofilaments and pull them, and a contractile force is generated. In order to activate the muscle, a finite period of time is needed to allow the biochemical substances to reach the required concentration. As a result, the level of muscle activation as a function of time has a hill-like geometry. After activation, the muscle remains active for approximately 100 to 150 milliseconds [1]. Therefore, to increase the accuracy of the model, it is worthwhile to consider a non-impulsive model of force generation, such that the force generated has finite magnitude and lasts over a short but finite period of time.

9.1.3 Actuation model modifications and modelling approach

In this chapter, we assume that the control forces are generated by muscle contractions in the form of short pulses, but we do not approximate them as impulsive forces that instantaneously change the velocities of the limbs. Modelling muscle contraction in detail is a difficult problem. However, a simple modeling approach known as state triggering was proposed by van der Linn (1999) [54]. State triggering involves adjusting the muscle stiffness away from its base value for a short period of time. The underlying idea is that mathematically, we can always express a control force or torque in terms of stiffness adjustments and from the stiffness adjustments we can work out the changes in elastic potential energy, which can be used to represent the force generation cost. In this way we can account for both the force generation cost, and the mechanical work done by the control force that was generated.

A pair of muscles working as a synergistic group can generate two-sided forces but the stiffness of a spring cannot be negative. So the forces generated by adjusting the stiffness of springs are one-side limited. Therefore, when realizing two-sided muscular force generation, we tolerate negative stiffness adjustments that imply negative spring stiffness. This idea had also been considered by van der Linn (1999) [54]. Ideally, we should develop a proper model of muscle at the beginning to realize two-sided muscular force generation, but due to time limitations, we leave this suggestion for future research.

The problem to be investigated here is the same problem investigated in Chapter 8. We investigate control mechanisms that restore the collision-free periodic level-ground walking, after being perturbed by a ground step. The only difference is that instead of applying perfectly impulsive torques around the joints, we consider non-impulsive torques. The control torques, modelled by state trigger-
ing, come from muscle contractions and have finite sizes that last over small but non-infinitesimal time periods.

As for Chapter 8, we consider the reactive and predictive control strategies. For the case of predictive control strategy, we assume that the ground height deviation remains unknown until perturbation happens, and controls can only be generated to restore the collision-free periodic walking after this point. For the case of reactive control strategy, we assume that the ground height deviation is predicted in advance, and generated controls pre-adjust for it.

In the next section, we describe in greater mathematical detail about the calculations involved in the non-impulsive control model. Notations are listed below. As for Chapter 8, we are also enforcing the sticky-foot constraint on the stance foot during the step, and the support-transfer is assumed to be an inelastic collision. For the purpose of presentation, the subscript “g” is not always used in the notations.

1. $\theta$: The configuration variables of the system. With the use of the sticky-foot constraint, the configuration variables include only the orientation angles of the limbs with respect to the vertical axis of the reference frame of the global environment.

2. $\phi$: A collection of all joint angles.

3. $f^{\text{spring}}_i$: The spring force at the $i$th joint.

4. $f^{\text{control}}_i$: The controller force at the $i$th joint.

5. $\Delta k_i$: The change in spring stiffness of a spring at the $i$th, which is a function of time.

6. $p_i$: The spring stiffness adjustment parameter. When we need to apply a stiffness change to the spring at the $i$th joint, the stiffness of the spring is continuously adjusted to the peak value $p_i$ from the base stiffness, and the stiffness is subsequently adjusted continuously from the peak value back to the base stiffness. The entire process lasts over a short, but non-infinitesimal period of time.

7. $\Delta K$: The change in the stiffness matrix over time due to the changes in the stiffness of springs. The stiffness matrix is defined in the joint angle configuration.

8. $p_k$: A collection of all stiffness adjustment parameters used by all springs involved during the $k$th control episode. The stiffness adjustment parameters are collected as a vector.
9.1. OVERVIEW

9. \( \tau_k^c \): The time (relative to the start of the step) the \( k \)th stiffness adjustment is made.

10. \( \Psi^K_n \): A collection of all stiffness adjustment parameters used in all control episodes over the \( n \)th step.

11. \( \tau_c \): The times the stiffness adjustments are applied (collected as a vector).

12. \( F^\text{control}_\phi \): The control forces due to all stiffness adjustments made on all springs involved.

13. \( M_g \): Mass matrix with pivoting stance leg.

14. \( C_g \): Centrifugal matrix with pivoting stance leg.

15. \( G_g \): Conservative force vector with pivoting stance leg.

16. \( B \): The force coupling matrix, which ensures that a controller exerts a force on each neighbouring limb segments, equal in size but opposite in direction. It is also the coordination transformation matrix that defines the mapping between \( \theta \) and \( \phi \).

17. \( \Gamma_c \): The controlled gait trajectory after toe-off but before support-transfer.

18. \( \eta_n \): The initial state of the \( n \)th step.

19. \( \eta^* \): The collision-free fixed point on level ground.

20. \( W_{k,n} \): The mechanical work done by the \( k \)th stiffness adjustment in the \( n \)th step.

21. \( W_{\text{net},n} \): The net mechanical work done by the all stiffness adjustments in the \( n \)th step.

22. \( C_n \): The controller cost over the the \( n \)th step. In this chapter it is *not* defined as the total unsigned mechanical work. The definition of the controller cost will be provided in-text.

23. \( \Omega^K \): The sequence of impulses required for bringing the biped from a starting state to a target state at the end of a step after support-transfer.

24. \( \eta^+ \): The post-strike state of the ideal biped given that the starting state is the collision-free fixed point and the gait is passive. It is a function of the ground height deviation.

25. \( \tau_{s/sf} \): The time support-transfer occurs.
26. $\tau_{\text{strike}}$: The foot-strike time of the ideal biped given that the starting state is the collision-free fixed point and the gait is passive. It is a function of the ground height deviation.

27. $\eta^d$: The target state for the ideal biped at the end of the first phase of the predictive control process after the support-transfer. It is a function of the ground height deviation.

28. $\Omega_{\text{K}}^{\text{react}}$: The sequence of stiffness adjustments required for bringing the biped from a perturbed starting state back to the collision-free fixed point, given that the control process is reactive.

29. $\Omega_{\text{K},1}^{\text{pred}}$: The sequence of stiffness adjustments required for bringing the biped from the collision-free fixed point to the target state during the first phase of the predictive control process.

30. $\Omega_{\text{K},2}^{\text{pred}}$: The sequence of stiffness adjustments required for bringing the biped from the target state back to the collision-free fixed point during the second phase of the predictive control process.

31. $C_{\text{tot}}$: The cost of recovery, defined as the total controller cost from all steps that use controls

### 9.2 Passive dynamic walking revisited

#### 9.2.1 Passive dynamic walking with control in general

We start with revisiting the mathematical definitions of a controlled gait. The dynamics of a biped, made up of a system of linked rigid-bodies with a passive spring and a control force generator mounted around each joint, are given by the equation of motion

$$M_g(\theta) \ddot{\theta} + C_g(\theta, \dot{\theta}) \dot{\theta} - G_g(\theta) = B \left( F_{\phi}^{\text{spring}} + F_{\phi}^{\text{control}} \right). \quad (9.1)$$

Because controllers can only be installed between limbs, the control force $F_{\phi}^{\text{control}}$ control must be expressed in terms of joint angles $\phi$, and the effect of the control force can be described in terms of the vertical-referencing angular coordinates $\theta$ using the coordinate transformation matrix $B$.

In the context of animal locomotion, the control force $F_{\phi}^{\text{control}}$ comes from muscle contraction. In this case, instead of using a perfectly impulsive force that causes the limbs to instantaneously change their velocities, we assume that the contractile force has a finite magnitude and is applied over a small but non-infinitesimal period of time. Although modelling muscular force generation in
9.2. PASSIVE DYNAMIC WALKING REVISITED

detail is difficult, we model muscular force generation by adjusting the stiffness of springs, and the control cost is calculated based on the unsigned changes in the stiffness of the springs. We assume that the energy required for making a spring stiffness adjustment is positively related to the unsigned change in the spring stiffness.

During a step cycle, the spring stiffness stays at the base value until a control is needed. When control is necessary, it is changed to a new value transiently. Adjusting the stiffness of a spring generates a control force. The control force is defined by

\[
 f^\text{control}_{\phi_i} (t, \tau_c, p_i) = -\Delta k_i (p_i, t - \tau_c) (\phi_i - \phi_{i,0}). \tag{9.2}
\]

The stiffness adjustment function \( \Delta k_i \) is a function of time, which is defined as

\[
 \Delta k_i (p_i, t) = \begin{cases} 
 u_i (p_i, t) & 0 \leq t \leq \delta t \\
 0 & \text{otherwise}
\end{cases} \tag{9.3}
\]

The parameter \( p_i \) determines how the stiffness will vary as time goes on. We want to change the stiffness of the springs from their default values transiently, therefore, \( \Delta k_i \) can be chosen as a pulse-like function with the maximum amount of stiffness adjustment and the direction of adjustment (increasing/stiffening or decreasing/softening), determined by the parameter \( p_i \). So we let

\[
 \Delta k_i (p_i, t) = \begin{cases} 
 p_i \xi (t) & 0 \leq s \leq \delta t \\
 0 & \text{otherwise}
\end{cases}, \tag{9.4}
\]

where \( \xi (t) \) is a scaling function that varies between 0 and 1.

To describe the effect of all stiffness adjustments made during the control episode at time \( \tau_c \), we compute all control forces generated by the stiffness adjustments. In matrix form, the control forces are given by

\[
 F^\text{control}_{\phi} (t, \tau_c, p) = \Delta K (p, t - \tau_c) (\phi_c - \phi_0), \tag{9.5}
\]

where

\[
 \Delta K (p, s) = \begin{cases} 
 \text{Diag} (\Delta k_i (p_i, s)) & 0 \leq s \leq \delta t \\
 0 & \text{otherwise}
\end{cases}.
\tag{9.6}
\]

\[
p = \begin{pmatrix} 
 p_1 \\
 p_2 \\
 \vdots \\
 p_k
\end{pmatrix}
\]
The stiffness adjustment function is now a matrix. As for Chapter 8, when stiffness adjustments are made is relative to the starting time of the step. Each joint has a pair of muscles mounted around it, and we use springs with adjustable stiffness to model activated muscles. Muscular force generated can be modelled by dynamically changing the spring stiffness. The stiffness change at the \(i\)th joint over time is specified by the stiffness adjustment function \(\Delta k_i\), which is parameterized by stiffness adjustment parameters \(p_i\). The vector \(p\) represents the collection of stiffness adjustment parameters used by all springs in the control episode that takes place at time \(\tau_c\).

If stiffness control is used at multiple time instants, we describe each stiffness control as a separate control force term and index the parameters involved, and then add them up to describe the overall effect. We can also collect the control placement times relative to the start of the step \(\tau_j\) as a vector \(\tau_c = (\tau_1, \tau_2, ..., \tau_N)\). The control forces from the stiffness adjustments made during the entire course of control are given by

\[
F^\text{control}_\phi(t, \tau_c) = \sum_{j=1}^{N} F^\text{control}_{\phi,j}(t, \tau_j, p_j) \quad (9.7)
\]

where

\[
F^\text{control}_{\phi,i}(t, \tau_i, p_i) = \Delta K(p_i, t - \tau_i)(\phi_c - \phi_0). \quad (9.8)
\]

### 9.2.2 Passive dynamic walking under the control of a sequence of muscular forces

If multiple stiffness adjustments are made during a step, and the control process takes more than one step, it would be convenient to index each stiffness change parameter as the stiffness change parameter used at the \(i\)th joint, in the \(j\)th control episode in the \(n\)th step, \(p_{i,j,n}\), and in this way, we can collect the stiffness adjustments used over a step as a control matrix \(\Psi^K_n\) with elements defined by \(\Psi^K_{n,ij} = p_{i,j,n}\). The control placement time can still be kept as a vector as described previously \((\tau_c = (\tau_1, \tau_2, ..., \tau_N))\). In this way, the control forces are now parameterized by \(\Psi^K_n\) and \(\tau_c\). Defining \(p_{i,n}\) as the \(i\)th column of the control matrix \(\Psi^K_n\), the control forces from spring stiffness adjustments are given by

\[
F^\text{control}_\phi(\Psi^K_n, \tau_c, t) = \sum_{j=1}^{N} F^\text{control}_{\phi,j}(t, \tau_j, p_{j,n}). \quad (9.9)
\]
In this way, the equation of motion can be rewritten as

\[ M_g(\theta) \ddot{\theta} + C_g(\theta, \dot{\theta}) \dot{\theta} - G_g(\theta) = B \left( F^\text{spring} + F^\text{control}_\phi (\Psi^K_n, \tau_c, t) \right). \]  

(9.10)

The trajectory is now parameterized by the sequence of stiffness adjustments, so we have

\[ \Gamma^c (\eta_n, \Psi^K_n, \tau_c, t) = \left( \theta (\eta_n, \Psi^K_n, \tau_c, t), \dot{\theta} (\eta_n, \Psi^K_n, \tau_c, t) \right). \]  

(9.11)

### 9.2.3 Work done by controller in the context of muscular force

In general, the net work done, \( W_{\text{net}} \), by the control forces applied during a step is given by

\[ W_{\text{net}} = \int_0^{\tau_{\text{strike}}} BF^\text{control}_\phi \cdot \dot{\theta} dt. \]  

(9.12)

With our definition of \( F^\text{control}_\phi \), given a sequence of stiffness adjustments \( \Psi^K_n = (p_{1,n}, p_{2,n}, \ldots, p_{N,n}) \), that are made at time instances \( \tau_c = (\tau_1, \tau_2, \ldots, \tau_N) \), the net work done by the control forces can be calculated as

\[ W_{\text{net,n}} (\eta_n, \Psi^K_n, \tau_c) = \sum_{i=1}^{N} W_{i,n} (\eta_n, \Psi^K_n, \tau_c), \]  

(9.13)

where

\[ W_{i,n} (\eta_n, \Psi^K_n, \tau_c) = \int_{\tau_i}^{\tau_i + \delta t} BF^\text{control}_\phi (\Psi^K_n, \tau_c, t) \cdot \dot{\theta} dt. \]  

(9.14)

### 9.2.4 Redefining the cost of recovery

#### Force generation cost based on unsigned elastic potential energy change

The cost of recovery is not simply the total unsigned mechanical work done by the controllers as considered in Chapter 8 and McGeer’s works. Instead, we add a positive quantity representing the expensiveness of control force generation. We call it the “arming cost”.

\[ \]
While the control torque required for re-stabilization is expressed in terms of spring stiffness adjustments, the arming cost can be calculated as the change in elastic potential energy that occurs when changing the stiffness of springs from the base values to the desired values. If the biped has $M$ joints, and $N$ stiffness adjustments are made during a step, and we define $\phi_{j,0}$ as the rest angle of joint $j$, the arming cost $C_{i,n}^{\text{arm}}$ is given by

$$C_{i,n}^{\text{arm}} (\eta_n, \Psi_n^K, \tau_c) = \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{1}{2} p_{i,j,n} (\phi_j (\tau_i) - \phi_{j,0}).$$  \hspace{1cm} (9.15)$$

We assume that the arming is a sunk cost. Once the adjustments are made, the power consumed making the adjustments cannot be recycled, so controllers must consume power to revert the adjustments. We call this the “disarming cost”. The disarming cost $C_{i,n}^{\text{dar}}$ can be calculated as the change in the elastic potential energy when reverting the stiffness of the springs from the desired values back to the default values, as given by

$$C_{i,n}^{\text{dar}} (\eta_n, \Psi_n^K, \tau_c) = \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{1}{2} p_{i,j,n} (\phi_j (\tau_i + \delta t) - \phi_{j,0}).$$  \hspace{1cm} (9.16)$$

The cost of recovery is defined by the mechanical work of controllers plus the arming and disarming cost,

$$C_n (\eta_n, \Psi_n^K, \tau_c) = \sum_{i=0}^{N} \left| W_{i,n} (\eta_n, \Psi_n^K, \tau_c) \right| + C_{i,n}^{\text{arm}} (\eta_n, \Psi_n^K, \tau_c) + C_{i,n}^{\text{dar}} (\eta_n, \Psi_n^K, \tau_c).$$  \hspace{1cm} (9.17)$$

**Force generation cost based on unsigned stiffness adjustment**

It is not unreasonable to assume that the energy consumed on making a spring stiffness adjustment is positively related to the unsigned change in the stiffness of the spring. The stiffness adjustment parameters used during a step are collected in the control matrix, so an easy way to compute the cost is to compute the norm of each column of the control matrix, and then take the sum. In this way, the cost of recovery is given by

$$C_n (\eta_n, \Psi_n^K, \tau_c) = \sum_{i=0}^{N} \| p_{i,n} \|. \hspace{1cm} (9.18)$$
The arming and disarming cost calculations explicitly take into account the amount of stiffness adjustment made. Unlike cost calculations based on a mechanical work calculation, if the period of integration is small, costs calculated this way will not be squeezed down. However, if the stiffness adjustment is made when the springs are near their rest configuration, the calculation can still underestimate the cost, because when all springs are approximately at equilibrium configurations, stiffness adjustments produce little change in the elastic potential energy, even if they are large. The non-energetic cost of recovery calculation depends solely on the amount of stiffness changes that have been made instead of energy changes, so a large stiffness change that is intended to be costly will not be turned into a small cost.

9.2.5 Step-transition Poincare map, and the calculation of the desired sequence of stiffness adjustments

A desired sequence of stiffness adjustments, \( \Psi^K_d = \Omega_K (\eta_1, \eta_2, \tau_c) \) in this case, is one such sequence of stiffness adjustments that allows the resulting control force \( F^\text{control}_\phi (\Omega_K (\eta_1, \eta_2, \tau_c), \tau_c, t) \) to change the state of the mechanical system from \( \eta_1 \) to \( \eta_2 \). In this section, we describe how this sequence of stiffness adjustments \( \Omega_K (\eta_1, \eta_2, \tau_c) \) can be obtained.

The initial state of the next step \( \eta_{n+1} \) and the initial state of the current step \( \eta_n \) are related by the step-transition Poincare map. Due to the stiffness adjustment control, the step-transition Poincare map is parameterized by the stiffness adjustments \( \Psi^K_n \), the control placement times \( \tau_c \), and the time support-transfer occurs \( \tau_{s/\text{tf}} \),

\[
\eta_{n+1} = S (\eta_n, \Psi^K_n, \tau_c, \tau_{s/\text{tf}}).
\]

The foot-strike time is now under the influence of stiffness adjustments rather than impulsive controls, and therefore it is a function of the initial state of the current step \( \eta^* \), the “control matrix” that represents the sequence of stiffness adjustments \( \Psi^K_n \), the control placement times \( \tau_c \) and the ground height deviation \( h_c \).

\[
\tau^K_{\text{strike}} (\eta_n, \Psi^K_n, \tau_c, h_c).
\]

Given that support-transfer can only occur when the swing foot strikes the ground, the initial state of the next step is now given by

\[
\eta_{n+1} = S (\eta_n, \Psi^K_n, \tau_c, \tau^K_{\text{strike}} (\eta_n, \Psi^K_n, \tau_c, h_c)).
\]

If the current step starts with the collision-free fixed point \( \eta^* \) and no controls are applied, the foot-strike time will depend only on the ground height deviation. In terms of \( \tau^K_{\text{strike}} \) this passively determined foot-strike time is given by
\[
\tau_{\text{strike}}(h) = \tau_{\text{strike}}^K(\eta^*, 0, 0, h).
\]

The superscript “K” indicates that the dynamics of the system is under the influence of stiffness adjustment controls, as in mechanics K often represents the stiffness matrix.

In this way, if the current step starts with the collision-free fixed point and no controls are applied, the initial state of the next step, or the post-strike state of the current step, is a function of the ground height deviation only. In terms of the step-transition Poincare map definition, the initial state of the next step can be written as

\[
\eta^+(h) = S(\eta^*, 0, 0, \tau_{\text{strike}}(h)).
\]

Given that we know the state of the system at the start of the step \((\eta_1)\), and at the start of the next step we want to be in state \((\eta_2)\), the state-matching constraints can be written in terms of the step-transition Poincare map. In order to find the desired sequence of stiffness adjustments, we solve for \(\Psi_d^K\) so that

\[
\eta_2 - S(\eta_1, \Psi_d^K, \tau_c, \tau_{\text{strike}}^K(\eta_1, \Psi_d^K, \tau_c, h)) = 0.
\]

Alternatively, since the target-state is chosen to be a state with both feet in contact with the ground, we can use an event locator function \(f_{\text{event}}\) to achieve a partial state-match. We can then work out the control matrix so that the post-strike state matches the target state in full. In this case, as for the case of impulsive control the state-matching constraint can be written in terms of the step-transition Poincare map with unconstrained support-transfer time, and the unconstrained support-transfer time is taken as the time that the torso orientation matches the targeted value \(\tau_e\) instead of the foot-strike time. With this approach, we solve for \(\Psi_d^K\) so that

\[
\eta_2 - S(\eta_1, \Psi_d^K, \tau_e, \tau_{\text{strike}}^K(\eta_1, \eta_2, \Psi_d^K, \tau_e)) = 0
\]

\[
\tau_e(\eta_1, \eta_2, \Psi_d^K, \tau_e) = \min \{\tau | f_{\text{event}}(\Gamma(\eta_1, \eta_2, \Psi_d^K, \tau)) = 0\}.
\]

In both cases, the desired control matrix \(\Psi_d^K\) is a function of the starting and target state \((\eta_1\) and \(\eta_2\) respectively) and the control placement times \(\tau_e\):

\[
\Psi_d^K = \Omega^K(\eta_1, \eta_2, \tau_e),
\]
and furthermore

\[ \Psi^K_d = \Omega_K (\eta_1, \eta_2, \tau_c) = (p^d_1 (\eta_1, \eta_2, \tau_c), p^d_2 (\eta_1, \eta_2, \tau_c), \ldots, p^d_N (\eta_1, \eta_2, \tau_c)) . \]

9.3 Control strategies

We consider here the same control strategies that were used in Chapter 8, particularly the reactive and the predictive strategies. The definitions of the reactive and predictive strategies are the same as Chapter 8, so the details are omitted. Briefly, for the reactive strategy, we are not aware of the ground step until it is encountered, so the perturbation error can only be handled in the step after perturbation. For the predictive strategies, we know everything about the ground step in advance and we can prepare for it.

The perturbations are introduced by an upward or a downward ground step, and for both the predictive and reactive strategies, control actions are the control forces generated by transiently adjusting the stiffness of springs to new values, instead of impulses. With the predictive control, the predicted ground height deviation is used to plan for the desired ground-contacting state we want to be in when the swing foot strikes the ground via a decision function \( \eta^d (h) \) as defined in Chapter 8.

As for the impulsive controls, we optimize the cost of recovery by choosing the control placement times so that perturbation errors can be repaired in full and the cost of recovery can be minimized.

In the case of reactive control, after encountering the ground step, the stiffness adjustments will bring the biped from the passively determined poststrike state \( \eta^+ (h) \) to the collision free fixed point \( \eta^* \). The desired sequence of stiffness adjustments, \( \Omega^\text{react}_K \), that bring the system back from the perturbed state \( \eta^+ (h) \) to the collision-free trajectory can be written as a function of the ground height deviation and the control placement times as

\[ \Omega^\text{react}_K (h, \tau_c) = \Omega_K (\eta^+ (h), \eta^*, \tau_c) . \]

In the case of reactive control, perturbation errors can only be repaired after being perturbed by the ground height deviation, so no controls are generated in the step before encountering the ground step, and the cost of recovery accounts only for the stiffness adjustments made over the second step. For technical simplicity, we calculate the cost of recovery using Equation 9.18. Defining that \( p^\text{react}_{i,2} (\eta^+ (h), \eta^*, \tau_c) \) as the \( i \)th column of the control matrix \( \Omega^\text{react}_K \), the cost can
be written as a function of the ground height deviation and the control placement times as

$$C_{\text{tot}}(h, \tau_c) = C_2(h, \tau_c) = \sum_{i=0}^{N} \| p_{i,2}^{\text{react}}(\eta^+(h), \eta^*, \tau_c) \|.$$ 

We can vary the control placement times $\tau_c$ to optimize the cost of recovery.

With the predictive control strategy, we have the freedom to choose a target state when the biped lands over the ground step. The target state is calculated from a decision function $\eta^d(h)$. Over the first phase of the predictive control process, we want a sequence of control impulses that brings the system from the collision-free fixed point $\eta^*$ to the target state calculated from the decision function $\eta^d(h)$. The desired sequence of impulses for the first phase of the control can be written as a function of ground height deviation and the control placement times as

$$\Omega_{K1}^{\text{pred}}(h, \tau_c) = \Omega_K \left( \eta^*, \eta^d(h), \tau_c \right).$$

In the second phase of the control, we bring the biped from the target state $\eta^d(h)$ back to the collision-free fixed point $\eta^*$ using another sequence of control impulses,

$$\Omega_{K2}^{\text{pred}}(h, \tau_c) = \Omega_K \left( \eta^d(h), \eta^*, \tau_c \right).$$

Again, we can vary $\tau_c$ to optimize cost of recovery. However, in the case of predictive control, the controller pre-adjusts for the ground step before reaching it. After reaching the ground step, the controller brings the biped back to the collision-free gait. The cost of recovery accounts for the stiffness adjustments made over both steps. As for the case of reactive control, for technical simplicity we calculate the cost of recovery using Equation 9.18. Defining $p_{i,1}^{\text{pred}}(\eta^+(h), \eta^*, \tau_c)$ and $p_{i,2}^{\text{pred}}(\eta^+(h), \eta^*, \tau_c)$ as the $i$th columns of the control matrices $\Omega_{K1}^{\text{pred}}$ and $\Omega_{K2}^{\text{pred}}$ respectively, the cost can be written as a function of the ground height deviation and the control placement times as

$$C_{\text{tot}}(h, \tau_c) = C_1(h, \tau_c) + C_2(h, \tau_c) = \sum_{i=0}^{N} \| p_{i,1}^{\text{pred}}(\eta^*, \eta^d(h), \tau_c) \| + \sum_{i=0}^{N} \| p_{i,2}^{\text{pred}}(\eta^d(h), \eta^*, \tau_c) \|. $$
9.4. SIMULATIONS AND COST OF RECOVERY CALCULATIONS

We used the same decision function $\eta^d(h)$ considered in Chapter 8, that is

$$\eta^d(h) = \begin{pmatrix} \theta^d(h) \\ \dot{\theta}^d(h) \end{pmatrix} = \left( \theta^* + \sin^{-1}\left( \frac{h}{L^*} \right) \mathbf{c}^T \right), \mathbf{c} = (1, 1, 1, ..., 1).$$

9.4 Simulations and cost of recovery calculations

9.4.1 Overview

In this section, our main objective is to investigate the relationship between the cost of recovery and the ground height deviation, but we do not use perfectly-impulsive control forces. Instead, we adjust the stiffness of the springs over a short but non-infinitesimal period of time to generate pulse-like but non-impulsive control forces. As for Chapter 8, we consider both the reactive and the predictive control strategies.

As explained in Chapter 8, because the collision-free trajectory is dynamically unstable, a perturbation error cannot be repaired at all even if it is near zero, so a small, near-zero ground height deviation will not take out the role of the controller. Therefore, we can base the investigation on small ground steps.

9.4.2 Working demonstration

Predictive control strategy

Here we start with a demonstration of the predictive control strategy (Figure 9.2). The angular velocities of the limbs and the control torque around the stance-leg-torso and the swing-leg-torso joints are plotted over time. In this demonstration, the control placement times are chosen to be $\tau_c = (0, 0.2, 2.4)$. The durations of stiffness adjustments are fixed at $0.05\pi$. The simulation is based on the Gomes-Ruina ideal biped [40], which has a collision-free fixed point. To 6 decimal places, the collision-free fixed point is

$$\eta^* = (0.716750, -0.716750, 0.000000, 0.000000, 0.000000, -7.43121).$$

In this demonstration, the ground height deviation is 0.01. At the end of the first step, we want to bring the biped to the target state. To 6 decimal places the target state is

$$\eta^d(0.01) = (0.724361, -0.709139, 0.00761115, 0, 0, -7.43121).$$
The first three stiffness adjustments are to bring the biped from the collision-free fixed point to the target state and the stiffness adjustments accomplish this. This state-matching is achieved with 6-figure accuracy. In the second phase, another three stiffness adjustments are made to bring the biped from the target state to the same collision-free fixed point at a different height. At the end, this state-matching is also achieved with 6-figure accuracy.

**Reactive control strategy**

Here we show a working demonstration of the reactive control strategy (Figure 9.3). In this demonstration, the chosen control placement times are $\tau_c = (0, 2, 2.35)$. The duration of all stiffness adjustments are fixed at 0.02\pi. The angular velocity of the limbs and the torque around the stance-leg-torso and the swing-leg-torso joints are plotted. The demonstration is also based on the Gomes-Ruina ideal biped [40].

As for the predictive control, the torques around joints come from spring stiffness adjustments, but the stiffness adjustments are made after being perturbed by the ground height deviation. The ground height deviation is $10^{-5}$, but in order to restore the collision-free periodic walking, the control torques required for the state-matching can be in the magnitude of 10, which is $10^6$ times larger than the magnitude of the ground height deviation. Nonetheless, the required state-matching is achieved with 6-figure accuracy, and the collision-free periodic walking is restored.

From the reactive control strategy, we can see that with a ground step in the magnitude of micrometer (i.e. $10^{-5}$), control torques need to be in the magnitude of 10 in order to restore the collision-free periodic walking. However, from the demonstration of the predictive control strategy, we see that the size of the control torques in the magnitude of 1 are all that are needed to deal with ground steps in the magnitude of centimeter (i.e. $10^{-2}$). This result matches what we found in Chapter 8, that prediction can reduce the energetic cost to overcome environmental obstacles.

**9.4.3 The relationship between the cost of recovery, the control placement times and other model parameters**

In this section, through computer simulations we investigate the relationship between the ground height deviation, the control placement times and the cost of recovery. While we are working with a straight-leg biped, foot scuffing must be taken into account. We also want to keep the end-step control impulses close
to the end of the step but before foot strike. Therefore, we place the last stiffness adjustment when the stance leg is sufficiently past the vertical, and we do not detect foot strike until after the last stiffness adjustment. In this way, we can handle the foot scuffing and at the same time keep all stiffness adjustments within a step. If the stiffness adjustments make the swing foot hit the ground before the last stiffness adjustment, it is treated as foot scuffing and ignored. This treatment is fine as long as the controls do not make the swing foot dip too far into the ground.

From a non-systematic study, almost all choices of control placement times do not work. The existence of a stiffness adjustment solution that allows the required state-matching is highly sensitive to variations in control placement times. The working demonstrations (Figures 9.2 and 9.3) offer some clues about when stiffness adjustments should be applied so that periodic walking can be restored after being perturbed by a ground step. We use the working demonstrations as guides to determine the ranges of control placement times that would allow the existences of control solutions.

For the reactive control strategy, we use three stiffness adjustments. The adjustments are made at $\tau_c = (0, \tau^* - 2\tau, \tau^* - \tau)$, where $\tau^*$ is the gait period of the collision free gait, which is approximately 2.6, and $\tau$ is a timing parameter that will be varied. For the predictive control strategy, we choose $\tau_c = (0, \tau, \tau^* - \tau)$ for both phases of control. In animal walking, muscles are active near the start and at the end of a step. The level of muscle activation is insignificant during the middle of a step. A walking gait can make good use of the passive dynamics of the body by keeping the gait passive during the mid-flight. Therefore, we keep all stiffness adjustments near the start and the end of a step by constraining $\tau$ to be a small value. Three torso-leg spring stiffness adjustments are needed in order to expand the dimension of the control space so that the required state match can be achieved.

By keeping $\tau$ less than 0.25, there will be an uninterrupted time period with a length comparable to the collision-free gait period, during which the motions of the limbs are passive. We then vary $\tau$ to find the optimal control placement times such that after the perturbation, the collision-free periodic walking is stabilized with the minimal control cost. The durations of the stiffness adjustments are fixed at $0.05\pi$.

When the ground step is upward and prediction is allowed, within the chosen range of $\tau$, the optimal choice for $\tau$ is approximately 0.15. This means that the stiffness adjustments should be made at $\tau_c = (0, 0.15, 2.45)$. For smaller
upward ground steps, the optimal control placement times remain approximately constant as the ground height deviation changes (Figure 9.4).

When the ground step is downward and prediction is allowed, within the chosen range of $\tau$, the optimal choice for $\tau$ is approximately 0.125 when the ground height deviation is 0.01 based on the numerical data, although the minimum cannot be visualized clearly from the graph. As the size of ground height deviation gets close to zero, the optimal control placement times remain approximately constant as the ground height deviation changes (Figure 9.4).

If we can only generate controls after being perturbed by the ground step, for the case of an upward ground step with size 0.01, a much larger controller cost is required for the re-stabilization but an optimal solution still exists. The stabilization control requires the stiffness adjustments to be made somewhere nearer the middle of the step. Control solutions exist when $0.265 \leq \tau \leq 0.325$. With smaller ground height deviation, the optimal control placement times remain approximately constant as the ground height deviation changes (Figure 9.4).

We have not found any re-stabilization control solutions in the form of stiffness adjustments when the ground height deviation is downward and the control strategy is reactive. From Chapter 8, when the ground height deviation is downward, and the control strategy is reactive, re-stabilization control solutions exist in the form of impulsive forces. One possibility is that when the control strategy is reactive, we can only use perfectly impulsive control forces to handle a small downward ground step. If this is really the case, then with the new definition of cost of recovery, it will be infinitely large because a perfectly impulsive force is unrealistic in the first place.

Of course, the other possibility is that we have not managed to find a working initial guess for the Newton’s search algorithm, which is the numerical tool we use for finding the required stiffness adjustments. So, instead of following the numerical path, we infer from the swing-foot trajectory whether or not the cost of recovery will approach zero as the ground height deviation approaches zero. The results will be presented at the end of the chapter.

By plotting the minimum cost of recovery as a function of the ground height deviation, we found that the minimum cost of recovery approaches zero when the ground height deviation approaches zero if the ground height deviation is predicted and pre-adjusted for (Figure 9.5). In the case of reactive control, the minimum cost of recovery is higher than the case of predictive control, and does not necessarily approach zero as the ground height deviation goes to zero.
With the reactive control strategy, when the ground step is upward and small, the cost of recovery is approximately proportional to the square root of the ground height deviation. The changes in control placement times have an insignificant effect on the relationship between the cost of recovery and the ground height deviation when the ground height deviation is small. In this case, the relationship between the ground height deviation and the cost of recovery can mathematically be described as

\[ C_{\text{tot}} (h, \tau) = f (\tau) \sqrt{h}. \]

With the predictive control strategy, the cost of recovery is approximately proportional to the ground height deviation when the ground step is small, regardless of whether it is up or down. The changes in control placement times have an insignificant effect on the relationship between the cost of recovery and the ground height deviation for this case too, when the ground height deviation is small. In this case, the relationship between the ground height deviation and the cost of recovery can mathematically be described as

\[ C_{\text{tot}} (h, \tau) = g (\tau) |h|. \]

With the reactive control strategy, the minimum cost of recovery is approximately proportional to the square root of the ground height deviation when the ground height deviation is small and upward,

\[ C_{\text{tot}}^{\min} (h) = k_1 \sqrt{h}. \]

With the predictive strategy, when the ground height deviation is small the minimum cost of recovery is approximately proportional to the ground height deviation regardless of whether the ground height deviation is up or down,

\[ C_{\text{tot}}^{\min} (h) = k_2 |h|. \]

### 9.4.4 Resolving the ambiguity in the scenario of downward ground height deviation with reactive control strategy

We have not found a control solution for the case of downward ground height deviation with the reactive control strategy. However, this could be because we have not found a good search seed that will allow the quasi-Newton search algorithm to locate the root. In this section, instead of using numerical simulations, we infer from the foot-trajectory plot (Figure 9.6) how the cost of recovery should vary as the ground height deviation becomes infinitesimal.
CHAPTER 9. NON-IMPULSIVE LEVEL-GROUND CONTROL

From the foot-trajectory plot (Figure 9.6) of the collision-free periodic gait, it is clear that if the swing foot missed the zero-height contact due to an infinitesimal downward ground height deviation, the swing foot will land with non-infinitesimal velocity, pre-strike foot velocity.

We can also show this more clearly by plotting the striking speed of the swing foot at foot strike, $\|\mathbf{v}_{\text{strike}}\|$, against the ground height deviation $h$, and we can see that the striking speed is infinitesimal when the ground height deviation is infinitesimal and positive, although it is sensitive to the size of the ground height deviation. The striking speed is non-infinitesimal when the ground height deviation is infinitesimal and negative.

The net energy change over the support-transfer is the sum of the kinetic energy loss due to the inelastic foot collision $\Delta K$, and the change in gravitational potential energy due to the foot role-swapping, which is given by

$$\Delta E(h) = \Delta K + h \sum_i m_i g.$$  

The second term is the change in gravitational potential energy due to foot role-swapping.

It is easy to see that the net energy change during the support-transfer will not approach zero as the ground height deviation approaches zero from the negative. The change in gravitational potential energy due to support-transfer will be infinitesimal as the ground height deviation ($h$) becomes infinitesimal. The collision loss will not be infinitesimal when $h$ is negative and infinitesimal, because pre-strike swing foot velocity is not infinitesimal when the ground height deviation is negative and infinitesimal.

Therefore, energy conservation dictates that in order to return back to the collision-free trajectory again, the controller needs to do work to cover this impact loss and hence requires non-zero stiffness adjustments. Regardless of how the control placement times are chosen, the cost of recovery will not approach zero as the ground height deviation approaches zero from below. So if one attempts to minimize the cost of recovery by adjusting the control placement times, infinitesimal cost of recovery cannot be achieved if the ground step is infinitesimal in size but downward.

Simulation shows that when the control strategy is predictive, the cost of recovery will be infinitesimal when the ground height deviation is infinitesimal, regardless of whether it is up or down. Therefore we would expect that for the
9.5. DISCUSSION

In this chapter, the cost of recovery is calculated as the sum of the norms of the columns in the control matrix. Therefore the cost calculation is based on the amount of stiffness adjustment made, rather than the mechanical work done by the force generated by stiffness adjustment. It turns out that when this new definition is used, the cost calculated, based on the amount of stiffness adjustment made, is hundreds of times the cost calculated as the sum of the unsigned mechanical work done. It appears that the cost calculation in this chapter strengthens the conclusion in Chapter 8, that the predictive control strategy is energetically advantageous over the reactive control strategy, even when force generation is energetically expensive.

As a side note, the fact that the pre-strike velocity is non-infinitesimal when the ground height is infinitesimal and negative implies that, technically, a zero control is not a good search seed for finding stiffness adjustment solutions using the Newton search algorithm even when the downward ground step is infinitesimal in size. This suggests that it is technically difficult to find the stiffness adjustments required when the ground step is downward and the control strategy is reactive.
Figure 9.2: A figure showing an example of the predictive control strategy, when the ground height deviation is $10^{-2}$. The torques around the joints and the angular velocities are plotted on the same axis. The angular velocities of the stance leg $\dot{\theta}_1$, the swing leg $\dot{\theta}_2$ and the torso $\dot{\theta}_3$ are shown in red, green and blue, respectively. The torques around the torso-stance-leg and the torso-swing-leg joints are plotted as functions of time, shown in magenta and cyan respectively. The angular velocity and the joint torques are non-dimensionalized (N/D).
Figure 9.3: A figure showing an example of the reactive control strategy, when the ground height deviation is $10^{-5}$. The torques around the joints and the angular velocities are plotted on the same axis. The angular velocities of the stance leg $\dot{\theta}_1$, the swing leg $\dot{\theta}_2$ and the torso $\dot{\theta}_3$ are shown in red, green and blue, respectively. The torques around the torso-stance-leg and the torso-swing-leg joints are plotted as functions of time, shown in magenta and cyan respectively. The angular velocity and the joint torques are non-dimensionalized (N/D).
Figure 9.4: The relationship between the control placement times, the ground height deviation, the control strategy and the cost of recovery. For both phases of the predictive control process, the starting times of the three control episodes are 0, $\tau$ and $\tau^* - \tau$, where $\tau^*$ is the period of the collision-free gait, (approximately 2.6), and $\tau$ is a timing parameter. For the case of reactive control, the starting times of the three control episodes are 0, $\tau - 2\tau$ and $\tau^* - \tau$. The red dot represents the minimal cost. We did not find any stiffness adjustment solution for the case of reactive control strategy with a downward ground step, so with the new definition of the cost of recovery, the cost of recovery is infinite because in Chapter 8 we showed that for this case the stabilization can be achieved by using impulsive controls, but an impulsive force is unrealistic.
9.5. DISCUSSION

Figure 9.5: The relationship between the ground height deviation, the control strategy and the minimal cost of recovery, and it supplements Figure 9.4. Because we did not find any control solution in the form of stiffness adjustments when the control strategy is reactive and the ground step is downward, here, for comparison, we imported the results from the impulsive control study (Chapter 8).
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Figure 9.6: The swing-foot trajectory plot shows the height of the swing foot relative to a zero horizon in the passive collision-free periodic walking gait.

Figure 9.7: The relationship between the speed of the swing foot at foot strike and ground height deviation $h$. 

Chapter 10

Improving level-ground walking

10.1 Motivation

To walk passively on level ground, the velocity of the swing foot must be zero the moment it reaches the ground, so that the loss of kinetic energy can be avoided during the support-transfer, yet the acceleration of the swing foot cannot be subjected to any constraints. As the swing foot approaches zero height from above at non-zero acceleration, its next move must be upward. This will cause the foot strike to be delayed when there is a downward ground step, even if the ground height deviation is near-zero. Consequently, the swing foot strikes the ground at a non-infinitesimal velocity, and there will be a non-infinitesimal loss of energy during the support-transfer even if the ground height deviation is infinitesimal. In this situation, a non-infinitesimal energetic cost will be required to restore the collision-free periodic walking, even if the ground step is near zero in size.

This is why, in Chapters 8 and 9, we observed that the reactive control strategy works well for upward ground steps, but not so well for downward ground steps. In the case of impulsive control (Chapter 8), control solutions exist, but the cost of recovery does not diminish to zero as the ground height deviation approaches zero. In the case of stiffness control (Chapter 9), no stiffness adjustment solutions were found at all. We call this a “touching down” problem.

If the ground step is to be detected in advance and controls are generated to pre-adjust for the ground step by directing the biped into a target state, the touching down problem can be avoided by choosing a good target state (Chapter 8). Another solution would be to find a good set of mechanical parameters that would allow the biped to walk passively and periodically on level ground where, when the swing foot reaches zero height, its velocity and acceleration along the world vertical will be zero and the next movement would be downward. Then,
even if we choose to use the reactive control strategy to restore the collision-free periodic walking, after being perturbed by a downward ground step the cost of recovery is no longer sensitive to the ground height deviation.

In this chapter, we fine tune the spring parameters of the ideal biped described in Chapter 4, to achieve a better collision-free periodic gait, so that at the end of the step, the velocity and acceleration of the swing foot along the world vertical are both zero. After our fine tuning, we revisit the cost of recovery calculations and see whether or not the “touching down” problem can be resolved without resorting to the predictive control mechanism.

### 10.2 Searching for collision-free periodic gaits

#### 10.2.1 Overview

To find collision-free periodic gaits with the above-mentioned kinematic properties, the procedure is similar to the one described in Chapter 4, which is based on the work by Gomes and Ruina [40], except that we also want the acceleration of the swing foot to be zero in the direction of the world vertical.

#### 10.2.2 The searching algorithm

We start by using a set of spring stiffness values. For each value we try to find the passive collision-free periodic gait by applying the searching algorithm proposed by Gomes and Ruina [40]. If a collision-free periodic gait is found, we work out the acceleration of the swing foot along the world vertical at foot strike. The result shows that there exists a critical spring constant $k_c$ that gives a collision-free periodic gait in which the acceleration of the swing foot along the world vertical ($a_{CFP}^{CFP}$) is zero at foot strike (Figure 10.1). By looking at the swing-foot height trajectory near the collision-free strike times, we can see that as the spring stiffness varies, the behaviour of the swing-foot height trajectory near the collision-free strike time changes from a local minimum to a point of inflection, and then from a point of inflection to a local maximum (Figure 10.2).

The problem now is how to pinpoint the exact spring constant that allows the vertical component of the swing foot’s acceleration to be zero at footfall.

We start with an overview of the collision-free periodic-gait searching algorithm proposed by Gomes and Ruina [40]. The solution to the equation of motion of the Gomes-Ruina ideal biped can be written as a function of the initial state $\eta$, and the time $t$:
10.2. SEARCHING FOR COLLISION-FREE PERIODIC GAITS

The solution to the equation of motion also depends on the design parameters, and so we introduce in the stiffness of the torso-leg springs, $k$. The solution to the equation of motion can now be written as a function of the initial state $\eta$, the time $t$ and the torso-leg spring stiffness $k$:

$$
\mathbf{\Gamma} (\eta, k, t) = 
\begin{pmatrix}
\dot{\theta}_1 (\eta, k, t) \\
\dot{\theta}_2 (\eta, k, t) \\
\dot{\theta}_3 (\eta, k, t) \\
\dot{\theta}_1 (\eta, t) \\
\dot{\theta}_2 (\eta, t) \\
\dot{\theta}_3 (\eta, t)
\end{pmatrix}.
$$

In the algorithm proposed by Gomes and Ruina [40], the collision-free fixed point is constrained such that the initial orientation angle of the swing leg equals

![Figure 10.1: The relationship between the stiffness $k$ of the torso-leg spring and the acceleration of the swing foot along the world vertical at foot strike in a collision-free periodic gait](image)

$\theta_1 (\eta, k, t)$

$\theta_2 (\eta, k, t)$

$\theta_3 (\eta, k, t)$

$\dot{\theta}_1 (\eta, k, t)$

$\dot{\theta}_2 (\eta, k, t)$

$\dot{\theta}_3 (\eta, k, t)$
the negative of the initial orientation angle of the stance leg \( (\eta_2 = -\eta_1) \), the torso is vertically upright, \( (\eta_3 = 0) \), and the initial angular velocities of the stance and the swing legs are both zero \( (\eta_4 = \eta_5 = 0) \). So, there are only two free variables: the orientation angle of the stance leg at the start of the step \( (\alpha) \), and the angular velocity of the torso at the start of the step \( (\beta) \). Therefore, the fixed point that gives the collision-free periodic gait (i.e. the collision-free fixed point) can be parameterized as

\[
\eta^\ast (\alpha, \beta) = (\alpha, -\alpha, 0, 0, 0, \beta)^T.
\] \hspace{1cm} (10.3)

If the value of \( k \) is fixed, to find the collision-free fixed point, we solve simultaneously for \( \alpha \), \( \beta \) and \( \tau \) so that

\[
\begin{pmatrix}
\theta_1 (\eta (\alpha, \beta), k, \tau) \\
\theta_2 (\eta (\alpha, \beta), k, \tau) \\
\theta_3 (\eta (\alpha, \beta), k, \tau)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\] \hspace{1cm} (10.4)

Using this approach, the second half of the gait trajectory of the collision-free periodic gait is the first half of the gait trajectory under simultaneous mirror-reflection and time reversal. Therefore, if the acceleration of the swing foot is zero along the world vertical at the start of the step, at the end of the step the acceleration of the swing foot is also zero along the world vertical. Therefore, to find a fixed point that gives the collision-free periodic walking gait with zero
10.3 AN EXAMPLE OF A TUNED COLLISION-FREE PERIODIC GAIT

swing-foot acceleration along the world vertical at foot strike, we solve for $\alpha$, $\beta$, $\tau$ and $k$ such that

$$
\begin{bmatrix}
\theta_1(\eta(\alpha, \beta), k, \tau) \\
\theta_2(\eta(\alpha, \beta), k, \tau) \\
\theta_3(\eta(\alpha, \beta), k, \tau)
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
$$

(10.5)

$$
a_{f,y}(\alpha, \beta, k, 0) = 0
$$

where $a_{f,y}(\alpha, \beta, k, 0)$ is the acceleration of the swing foot along the world vertical at the start of the step. The solution to $k$ is the desired spring constant $k_c$.

10.3 An example of a tuned collision-free periodic gait

By setting the stiffness of each torso-leg spring to 1.24795, the collision-free periodic gait has a zero foot-strike acceleration (Figure 10.3) along the world vertical. The behaviour of the swing-foot trajectory is an inflection point near the collision-free strike time. To 6-figure accuracy, the new collision-free fixed point ($\eta^*$) is

$$
\eta^* = (0.601974, -0.601974, 0, 0, 0, -5.93761).
$$

The acceleration of the swing foot at foot strike is $-6.91507 \times 10^{-11}$, and so is zero with 10-figure accuracy.

10.4 Cost of recovery calculations revisited

In this section, we repeat the same sets of cost of recovery calculations described in Chapter 8 and Chapter 9, but this time we use the tuned ideal biped with zero foot-strike accelerations. We want to find out whether the cost of recovery is still sensitive to the ground height deviation after tuning. While we are only interested in the ground height deviation, for the impulsive control we fix the control placement times at $\tau_c = (0, 2.5, 2.7)$, and for the non-impulsive stiffness adjustment control, we fix the control placement time at $\tau_c = (0, 2.0, 2.25)$.

We can easily see that the cost of recovery goes to zero as the ground height deviation approaches zero after the stiffness tuning, regardless of whether the ground step is upward or downward (Figure 10.4). When the ground step is small and the control strategy is reactive, regardless of whether the control forces
are impulsive, or are coming from the non-impulsive spring stiffness adjustments, the cost of recovery asymptotically approaches the cubic scaling rule
\[ C_{\text{tot}} \approx k_1 |h^{\frac{3}{2}}|. \]

When the ground step is small and the control strategy is predictive, regardless of whether the control forces are impulsive or come from the non-impulsive spring stiffness adjustments, the cost of recovery asymptotically approaches the linear scaling rule
\[ C_{\text{tot}} \approx k_2 |h|. \]

These trends suggest that a good choice of spring stiffness can resolve the touching-down problem that we had been struggling with in Chapters 8 and 9. Still, a predictive mechanism allows a lower cost of recovery compared with the reactive case. However, unlike the version without the tuning, a small ground step now implies a small cost of recovery regardless of the control strategy. So, although a predictive mechanism can reduce the energy required for dealing with perturbations, a good mechanical design means that perturbation handling is not an energetically expensive exercise even if a prediction is unavailable.
10.4. COST OF RECOVERY CALCULATIONS REVISITED

Figure 10.4: The relationship between the cost of recovery, the control strategy and the ground height deviation. Both the impulsive control and the non-impulsive stiffness adjustment control are considered. In the case of impulsive control, the control placement times are fixed at $\tau_c = (0, 2.5, 2.7)$, and in the case of non-impulsive control, the control placement times are fixed at $\tau_c = (0, 2.0, 2.25)$.
Chapter 11

Conclusion

11.1 So, what have we learned?

11.1.1 An upper-body is a mechanical advantage for the stability and efficiency of locomotion

The massive-hip compass-gait walker proposed by McGeer [36, 10] can walk downhill without actuation or control. It is stable, meaning that it can passively correct for small perturbations. Gomes and Ruina [40] showed that a straight-leg walker with a torso can walk passively on a horizontal line. However, this ideal walker is highly unstable when it moves horizontally. We have shown that an ideal straight-leg walker with a torso is not only stable on a downhill slope, over a much wider range of slopes than a straight-leg walker without a torso, and it generally walks much faster on the same slope. Additionally, it is easy to stabilize the ideal walker during horizontal motion, by doing positive work on the linkage during the step cycle to match the energy lost in footfall impacts.

The Gomes-Ruina walker has a comically exaggerated gait that we have called a “silly walk”. Silly walking requires a range of joint motion that looks rather unnatural. We showed that by adding arms it is possible to greatly reduce the amplitude of leg and torso oscillations required for an ideal straight-leg passive dynamic walker. Our straight-leg walker with arms has a “slightly silly walk”, but it would be easy for a human to mimic it, with both arms straight and swinging in unison, and goose-stepping straight legs.

Collision-free walking can be achieved if the kinetic energy of the legs can be fully captured by the springs and the upper-body at footfall. In Gomes and Ruina’s case, the kinetic energy of the legs is captured as elastic potential energy as well as the kinetic energy of the torso. When arms are added to the torso, a significant proportion of the leg’s kinetic energy can be captured as the kinetic
energy of the arms at footfall. In this way, the amount of energy required to be stored as spring potential energy is reduced, which reduces the amount of leg opening. Similarly, the amount of energy required to be transformed into the torso’s kinetic energy is also reduced, and this resolves the problem of the exaggerated torso swaying. In addition, the arms can exert a passive torque on the torso during the course of motion to counter its swaying and provide a good passive torso-balancing mechanism.

While a two-dimensional rigid-body linkage with four degrees of freedom cannot have a gait that very closely resembles a natural human gait, it is surprising that it is not only possible for a passive mechanism to generate a gait that is as human-like as this is, but the mechanism dissipates no energy in steady horizontal locomotion.

Humans have long regarded the wheel as an emblem of our intellectual triumph over nature. In contrast, as noted by McGeer in his original paper on passive dynamic walking [10], legs at first sight seem to be an awkward kluge rather than an elegant solution to the problem of efficient locomotion. It now appears that the wheel may be regarded as a simple special case of a general class of mechanisms in which periodic motion in an internal configuration space with holonomic constraints couples the mechanism to its environment and results in motion through the environment [8]. Under ideal conditions, legs are as efficient as wheels, but they are more versatile under a range of conditions.

One design advantage of a 2D wheel over legs is that the 2D wheel cannot fall over, but a biped can stumble. A wheel remains on a neutrally stable orbit when it loses or gains kinetic energy, so its forward speed can be increased, decreased or regulated in arbitrary ways without worrying about stability. In contrast, an ideal walker has at best a narrow region of stability around any gait, and if a perturbation causes it to move out of that region then a controller must do some work on the structure in a specific way to return it to the periodic gait orbit. For example, we showed how actuator impulses could be used to correct for dissipation due to footfall impulses.

Passive dynamic walking mechanisms suggest that the classical view of animal and human brains generating locomotor movements by trajectory planning and servo-control may need to be revised, in favour of the view that locomotor patterns are essentially a consequence of mechanical design. The role of the brain and muscles then would be to do work to transfer the body between different gaits (including the “null” gait, i.e. stationary), to stabilize it in the face of perturbations, and to do external work to compensate for friction, drag and impact losses. Brains and muscles could also parameterize the mechanical design of the
11.1. SO, WHAT HAVE WE LEARNED?

body for different gaits by altering joint stiffness and set points [2]. For example, a controller for our slightly silly walker would damp arm swinging and do external work to compensate for the resulting dissipation, in order to maintain stability in horizontal locomotion. On small downhill slopes this damping and external work would not be required.

In conclusion, adding a torso to a straight-leg biped can be an advantage for faster and stable passive dynamic locomotion over a wider range of slopes. The addition of the torso makes the straight-leg passive biped walker behave as an enhanced version of McGeer’s compass gait walker on downhill slopes, and behave as a rolling wheel on level ground. These findings contrast with the expectation that adding a torso to a passive walker requires trajectory planning and servo-control.

11.1.2 The importance of vision for handling the gait instability on level-ground

While the passive level-ground periodic gait must be collision-free, so must the swing foot reach the ground with zero velocity. However, no restrictions are applied to the swing foot acceleration at foot-strike. Therefore, in general, when the swing foot reaches zero height from above but has missed ground contact due to a downward ground height deviation, the swing foot’s next move is to fly upward. This will delay the foot-strike time, and will, in general, cause the swing foot to land impulsively. So, a small downward ground step does not imply that once the biped is perturbed by it, the controller can bring it back to the unstable collision-free periodic gait by consuming a small amount of energy. For brevity, we call this issue the “touching down problem”.

If the ground step is predicted in advance, the touching down problem can be resolved easily by applying controls before reaching the ground step. We showed by computer simulations and Taylor series approximations that if the ground height deviation is known in advance and we pre-adjust for it by choosing a good target state and direct the biped into it, an inelastic collision is avoided and as a result the cost of recovery is significantly reduced, although more control actions are involved.

In animals or robots, an unexpected ground impact can be detected by using a touch sensor or a stress sensor to detect the amount of muscular compression and stretch at a joint. If the ground height deviation has to be known in advance, this may be achieved through sensory mechanisms, such as vision. We had considered an idealized situation that apart from the ground step, the landscape is perfectly smooth, and the prediction has an infinite precision so ground
height deviations can be detected even if they are in the sub-millimeter ranges. However, the results explain the observation that real animals, such as humans, use vision when walking. Most of us have the difficulty of walking in the dark. Furthermore, a blind person will use a cane to detect any incoming obstacles. Therefore, despite the idealization, our result suggests that the role of vision, or more generally a predictive mechanism, is required, at least, in part, to improve the energetic efficiency when walking on an uneven landscape while maintaining the gait stability.

The touching down problem also implies that there a trade-off between stability and the information processing cost, because it can be resolved by predicting obstacles. The brain consumes energy, but a better brain can lower the cost of control. For energetic reasons, we cannot avoid this trade-off, but the trade-off between stability and control costs is a simple issue. We showed that the touch down problem can be resolved by adjusting the spring stiffness gait to allow for the existence of a collision-free periodic gait that has a zero pre-strike foot velocity and foot acceleration along the world vertical, so that a small ground height deviation always implies a small cost of recovery, regardless of the direction of ground height deviation, the availability of a predictive sensory mechanism, and how controls are generated. The biped will be fine, even when walking in dark.

On level ground, it is not possible to achieve stable passive walking, and indeed, we can spend as much energy as possible to override the unstable passive dynamics in order to ensure stability. Dr. Paulin (personal communication) conjectured that there exists an essential cost of stability due to the minimal energetic expenditure, information plus mechanical work, required to keep an animal within the basin of its locomotor attractor(s) on a given terrain during the course of evolution, and the animal locomotion mechanism evolves toward this boundary to achieve optimal locomotion control in a given niche.

11.1.3 The complexity of a mechanical structure is not a barrier to passive walking

McGeer’s idea of passive dynamic walking is not restricted to one pair of legs. We showed that McGeer’s idea can be extended to bipeds with an upper-body, and to 2D quadrupeds. Passive and stable periodic gaits are also found for a 3D biped with a pair of straight legs and a torso. Therefore, the complexity of the mechanical structure is not a barrier to passive walking. It is plausible that in humans, as well as other animals, the body has evolved for efficient, agile locomotion with minimal control. The efficiency of biped walking on level ground
suggests that in humans, the mechanical design may have evolved under selection pressure for slow but highly efficient locomotion on hard, level surfaces.

11.2 Where to go next?

11.2.1 A concise model of muscular force generation

In this thesis, research is based on the biological fact that muscles are more active near the start and the end of a step, and in walking, muscular forces are pulse-like. We have not looked at the muscular force generation mechanism in detail. So a potential next step is to come up with a concise model of muscular force generation. To do this, we need to model the process of perturbation detection, the muscle activation and the process of muscle contraction mathematically, so that the force generation depends on the neuromuscular parameters and mechanical parameters and the environment. We then determine which parameters can be adjusted, and how to adjust them realistically.

A concise model of muscular force generation also helps determine the actual cost of stabilization on level ground. Given only the size and direction of the control force required for stabilization, we could work out the mechanical work done by the control force that is generated without knowing how the control force is generated, and because of this, in Chapter 8 and McGeer’s work, the controller cost is measured based on the mechanical work done by the controller. However, for realism, the cost of stabilization should be the power consumed by the actuator. This does not, in general, equal the mechanical work done by the control forces. A concise model of muscular force generation allows the real cost of stabilization to be measured objectively.

In Chapter 9, the control force is generated by changing the spring stiffness. However, with a pair of muscles acting as a synergistic couple, torque can be generated in either direction, but real spring stiffness adjustment only generates a one-side limited force because spring stiffness cannot be negative. So stiffness adjustment is not a good model of muscular force generation. We used this simple model because we want to address the question of “what” gives the control forces and “how”, so that the cost of force generation can objectively be calculated, and due to time limitation we have only made an initial attempt.

11.2.2 Dealing with controller errors

In the current work, we assume that the controller and the sensors are perfectly error-free. However, in reality, there will always be errors and we might expect that during each step, additional controls will be needed to handle the controller
errors. With controls, level-ground periodic walking can be dynamically stable, so there is an attractive basin within which periodic walking can be maintained without handling the controller errors.

Therefore, the investigation should be based on a controlled level-ground gait that is dynamically stable, and the cost of recovery should be the additional controller work required for bringing the system back to the attractive basin after being perturbed. We can minimize the cost required for stable periodic walking by optimization and preliminary studies have been done along this direction. We showed that we can use a one-per-step impulsive control to achieve stable bipedal periodic walking on level ground. We also showed that maintaining periodic walking after being perturbed by a small ground step requires additional controls, but we can reduce the control cost by detecting the ground step in advance without capturing its size accurately. For space reasons, we did not include this in these results.

11.2.3 Three-dimensional passive biped with multiple body parts

The 3D biped with the torso (Chapter 7) can walk passively, periodically and stably on downhill slopes, but the stability requires the legs to have a large lateral span. We conjecture that the lateral extensions of the legs can be replaced with arms connected to the torso through ball and socket joints. There are physical 3D passive walkers with swinging arms but no upper body that are able to walk passively and stably down a ramp with biologically inspired legs [83]. It was believed that the arm swing compensated the instability in the lateral plane dynamics, and we believe that we can achieve a similar result with arms attached to the torso.


