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MACH'S PRINCIPLE

in

GENERAL RELATIVITY

and other Gravitational Theories.

by

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INTRODUCTION

An important problem associated with Einstein's general theory of relativity is that concerning the origin of inertia. In Newtonian theory one assumes the existence of an inertial frame of reference at each point of space, in which Newton's laws of motion are satisfied. But then how do we define this selection of inertial frames? In Newton's view these frames were determined by absolute space. However we cannot observe absolute space in any way, so why is this space (if it exists) so well hidden? We are unable to detect this space through velocity (and certainly not position), so we have to appeal to the second derivative, that is acceleration, before we can get any information.

There has been much criticism about the meaning of 'absolute' space, and it would be more satisfactory if we could correlate inertial frames with observable features of the universe. This last viewpoint was put forward by Berkeley and Mach (1893), and the following statement, which clearly negates the concept of absolute space is usually known as Mach's principle: "the local inertial frame is determined by some average of the motion of the distant astronomical objects".

Hence only relative accelerations exist in Mach's formalism, and inertial effects arise as a consequence of accelerations relative to
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distant galaxies. By Mach's principle, a reference frame which is locally inertial, will be one in which the distant astronomical objects are non-rotating. This has been verified experimentally, to a certain extent, by two independent methods of determining the speed of rotation of the earth. We will show that as a consequence of normal relativistic cosmology, but not general relativity, the reference frame relative to which inertial forces are observed is one in which the distant objects in the universe are non-rotating.

If the locally inertial frame is determined by the distribution of matter and energy in the universe we would expect the inertia of a body would be determined by the rest of the matter in the universe. As Einstein stated: "there can be no inertia of matter against space, only an inertia of matter against matter". However we know (within experimental accuracy) that local masses have no effect on the inertia of a body, hence it must be assumed that the main contribution to inertia comes from distant matter. This does not necessarily mean that inertia depends critically on distance, but since there is a far greater amount of matter at large distances the effect of this distant matter will be of predominant importance. Hence the inertial field in our astronomical neighbourhood is approximately constant which would seem to imply an 'absolute' space for local dynamics. This explains why Newton's theory describes the motion of planets in our solar system so accurately, and why local

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phenomena appear to be isolated from the rest of the universe.

Further since local inertia is determined by the masses of the universe, it would follow that local experiments would provide some information about the universe as a whole. Since the gravitational effect of a particle is apparently intrinsic, it follows that the ratio of gravitational and inertial masses, the constant of gravitation, is dependent on the masses in the universe and their distribution. Hence as far as Mach's principle is concerned, any theory of gravitation which deals with a time varying universe, must also consider a time dependent 'constant' of gravitation.

The field equations of general relativity relate the $g_{ik}$ (the inertial field) to the $T_{ik}$ (the matter tensor). We may consider these equations as second-order non-linear differential equations for the $g_{ik}$ in terms of the $T_{ik}$. However in order that the inertial field is completely determined by the distribution of matter we would have to solve these differential equations and therefore some boundary conditions on the metric tensor would be required. Also, in the absence of matter, that is $T_{ik} = 0$, we would require the field equations to have no solution for the $g_{ik}$, since by Mach's principle the inertia arises from the matter in the universe.

This thesis discusses the extent to which Mach's principle is incorporated into general relativity. In chapter 1, we consider a vector theory of inertia which attributes the origin of inertial forces
to an inductive effect of a moving universe, and which reduces to Newton's laws of gravitation in a reference frame at rest relative to the "smoothed-out" universe. We then show how general relativity provides a superior presentation of these ideas on the origin of inertia. Chapter 2 is chiefly concerned with boundary conditions on the metric tensor, brought about by a separation of local effects from the general cosmological structure due to the distribution of distant matter. In chapter 3, we consider the compatibility of Mach's principle with the steady-state theory, and Hoyle's C-field term is discussed in the light of Mach's principle. Some solutions of the field equations of general relativity which are incompatible with Mach's principle are discussed in chapter 4, and in chapter 5 we consider the inertial properties of a test-particle in a homogeneous isotropic closed model of the universe. An integral form of Einstein's equations is presented in chapter 6, involving retarded bi-tensor Green's functions. A direct-particle interaction theory of gravitation, which reduces to Einstein's theory in the smooth fluid approximation is also discussed.
Chapter 1.

1.1 A vector theory of inertia.

The principle of equivalence states that the effects of a gravitational field can be removed, locally at least, by employing an appropriately chosen accelerated coordinate system. In other words, gravitational and inertial effects are locally indistinguishable. This suggested to Einstein that inertia was due to the gravitational influence of a moving universe. We describe a theory of gravitation in which the motion of a body is influenced by the presence of other bodies in the universe, in accordance with Mach's principle.

The basic postulate of this theory (Sciama 1953) is that "in the rest frame of any body, the total gravitational field at the body arising from all the other matter in the universe is zero". The field due to the matter in the universe is characterised by a scalar potential $\phi$ and a vector potential $A$. Sciama uses Maxwell type field equations to relate the gravitational field to its sources.

The "gravelectric" part of the field is given by:

$$E = -\text{grad } \phi - \frac{1}{c} \frac{\partial A}{\partial t}$$

(1.1.1)

while the "gravomagnetic" field is

$$H = \text{curl } A$$

(1.1.2)
where
\[ \phi = - \int_V \frac{|\rho|}{r} \, dV \]  \hspace{1cm} (1.1.3)
and
\[ A = - \int_V \frac{|V\rho|}{r} \, dV \]  \hspace{1cm} (1.1.4)

In these expressions \( \rho \) is the mass density, \( V \) is the space velocity of the element of mass in volume \( dV \) at distance \( r \) from the point where these potentials are evaluated. Square brackets indicate retarded values corresponding to the propagation of the field with velocity \( c \). It must be emphasised that these equations are describing purely gravitational effects even though they are similar to Maxwell's electromagnetic equations describing the electromagnetic field of a charge distribution \( \rho \).

Since the inertia of a body arises as a result of interaction with the rest of the matter in the universe it will be necessary to specify the distribution of this matter in the universe. Since distant matter is of predominant importance for the origin of inertia, and since the universe is observed to be homogeneous and isotropic on the large scale, it will be convenient to take a 'smoothed-out' model of the universe as the source of the local inertial field. The 'smoothed-out' model is a homogeneous and isotropic distribution of matter of density \( \rho \), expanding (relative to any point as origin) according to the Hubble law \( \dot{r} = Hr \), where \( H \) is Hubble's constant.
Consider the case of a test-particle moving with small rectilinear velocity \( \mathbf{v}(t) \) relative to the 'smoothed-out' universe. Equivalently we can say that the universe moves rectilinearly with velocity \( \mathbf{u}(t) \) relative to the particle's rest frame. Hence taking into account the Hubble effect we must ascribe to every region in the universe the velocity \( \mathbf{v}(t) + H \mathbf{r} \) relative to the rest frame of the particle.

We make the further assumption that matter which goes beyond the effective radius \( c/H \) of the model makes no contribution to the field potentials. (The effective radius of a model is that distance to the horizon of the model where the velocity of matter, relative to the space origin, equals the velocity of light).

Neglecting terms of the order \( V^2/c^2 \), the integral (1.1.3) can now be taken over a spherical volume of radius \( c/H \). We then have

\[
\phi = - \int_{r=0}^{c/H} \frac{\rho}{r^2 + \pi r^2} dr
\]

that is

\[
\phi = - 2\pi \rho c^2 / H^2 \quad (1.1.5)
\]

where we have assumed that \( \rho \) does not vary appreciably with time.
For the vector potential $A$ we obtain

$$A = - \int V \frac{\partial}{\partial t} \, dV - \int H \frac{\partial}{\partial t} \, dV$$

and since the second integral vanishes by symmetry we obtain

$$A = \phi V(t)/c \quad (1.1.6)$$

since $V$ is a function of $t$ only.

Hence the field due to the inductive effect of a moving universe, relative to the rest frame of the particle is given by

$$E = - \phi \frac{dV}{dt} / c^2 \quad (1.1.7)$$

In order that we may take into account the effects of local masses we consider a body of gravitational mass $m$ superposed on our 'smoothed-out' universe and at rest relative to it. In the rest frame of the particle the potential of the body at the particle is $\phi = - m/r$ where $r$ is the distance of the body from the particle. Hence the field of this body in the rest frame of the particle is, using (1.1.1)

$$E = - \frac{m}{c^2} \frac{\dot{r}}{r} - \phi \frac{dV}{dt} / c^2 \quad (1.1.8)$$
We also have \( \mathbf{\hat{r}} \cdot \frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}}{dt} \) since \( V^2 = \mathbf{V} \cdot \mathbf{V} \).

We might picture our situation as that of a test particle in the gravitational field of the sun of mass \( m \). The test particle accelerates towards the sun, but we can equally as well suppose the particle to be at rest and the sun (and the rest of the matter in the universe) falling towards the particle with acceleration \( \mathbf{a} = \frac{d\mathbf{V}}{dt} \).

In the rest frame of the test particle the gravitational forces acting on the particle are:

(i) gravitational pull of the sun
(ii) gravitational wave, radiated by the distant accelerated matter of the universe.

The postulate of this theory was that the total gravitational field, in the rest frame of the particle, arising from all other matter in the universe is zero. Hence \( E_u + E_b = 0 \), and so from (1.1.7), (1.1.8) we have

\[
\frac{m}{r^2} \mathbf{\hat{r}} = - \left( \frac{\phi + \frac{c}{\mathbf{a}}}{c^2} \right) \frac{d\mathbf{V}}{dt}
\]

Taking the dot product of both sides with \( \mathbf{\hat{r}} \) we then obtain

\[
\frac{m}{r^2} = - \left( \frac{\phi + \frac{c}{\mathbf{a}}}{c^2} \right) \mathbf{a} \quad (1.1.9)
\]

where \( \mathbf{a} = \frac{d\mathbf{V}}{dt} \).
Suppose we now consider a reference frame in which the universe is at rest with the sun at the space origin. From (1.1.9) we see that the particle accelerates towards the sun with acceleration

\[ a = \frac{Gm}{r^2} \]  \hspace{1cm} (1.1.10)

where we have defined

\[ -\frac{1}{G} = \frac{\phi + \phi}{c^2} \]  \hspace{1cm} (1.1.11)

(G denotes the Newtonian gravitational constant).

It is clear that in this frame Newton's laws of motion and of gravitation hold, and it is important to note that this so-called inertial frame in Newtonian theory has been replaced by a reference frame which is at rest relative to the averaged-out motion of the matter in the universe. In other words Newton's theory holds in the inertial frame defined by Mach's principle. (It is important to note that a point of relative rest in the 'smoothed-out' universe is one at which the observed distribution of red-shifts of distant matter is isotropic).

If the main contribution to local inertia comes from distant matter, then \( \phi \ll \phi \) and so from (1.1.11) we have

\[ G\phi = -c^2 \]  \hspace{1cm} (1.1.12)

We see from this last equation that the gravitational constant at any point is determined by \( \phi \) and hence is determined by the distribution of matter in the universe. From (1.1.5) and (1.1.12) we obtain

\[ Gp \sim H^2 \]
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Hence a local determination of $G$ and an astronomical determination of the Hubble constant $H$, provides an estimate of the mean density of matter in the universe.

Suppose now the test particle moves in a circle with constant angular velocity under the attraction of a body of mass $m$ at the centre. In the rest frame of the particle we choose the body to be at the origin, and the system universe-plus-body rotates with constant angular velocity $w$ about the $z$-axis, the test particle being in the $x,y$-plane.

If the universe was not rotating relative to this frame, the potential near the origin would be $A = 0$ and $\phi = -1$ (we have chosen units so that $G = c = 1$). If the universe now rotates, the potential near the origin in the $x,y$ plane will be (Rosen 1947)

\[
A = (wy, -wx, 0) \\
\phi = -(1 + w^2 r^2)^{\frac{1}{2}}
\]

where $r^2 = x^2 + y^2$.

The field due to the rotating universe in the particle's rest frame will be, using (1.1.1)

\[
E_u = -\text{grad} \phi \\
= \frac{w^2 r}{(1 + w^2 r^2)^{\frac{3}{2}}}
\]

hence

\[
E_u \approx w^2 \frac{r}{c^2}
\]

since we are neglecting terms of the order $v^2/c^2$.

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The field of the body (neglecting its rotation) will be

$$\mathbf{E}_b = -\text{grad} \phi$$

$$= -\frac{m}{r^2} \hat{r}.$$

Hence the total field at the test particle is zero if

$$\frac{m}{r^2} = w^2 r$$  \hspace{1cm} (1.1.15)

From the Newtonian point of view, in the rest frame of the universe this is just the Newtonian law of gravitation for circular motion. The important point to note is that the "$w^2 r$" term has been attributed to the inductive effect of a moving universe, whereas in Newton's theory this centrifugal term is fictitious.

The Coriolis force term can also be attributed to the inductive effect of a moving universe. For a test particle moving with uniform velocity $\mathbf{v}$ relative to the first particle, the field of the universe in the rest frame of the second particle would be $w^2 r + \mathbf{v} \wedge \mathbf{H}$.

From (1.1.2) we have $\mathbf{H} = (0,0,-2w)$. Hence the field due to a moving universe will be

$$\mathbf{E}_u = (w^2 x - 2w y, w^2 y + 2w x, 0)$$

where we are restricted to the $x, y$ plane. Thus both the centrifugal and Coriolis force fields arise from an inductive effect of a moving universe.
13.

universe.

If $M$ stands for the finite mass of the visible universe, and $R$ is the radius of the boundary of the visible universe, then we would expect the potential $\phi$ in (1.1.3) to be of the order $-\frac{M}{R}$. Hence from (1.1.12) we obtain

$$\frac{GM}{Rc^2} \sim 1$$

(1.1.16)

The gravitational constant is expressed in terms of the mass distribution in the universe. For a universe in which $M/R$ is not constant one would expect a variable gravitational constant. We will discuss this further in section (2.7).

Finally, the principle of equivalence actually follows from this theory whereas in general relativity it is an initial axiom. Suppose we imagine an observer in a closed laboratory which is firstly placed in a gravitational field and then secondly, pulled by a rope in a gravitationally free field. In either case the motion of the universe relative to the laboratory will be the same and hence the inertial effects felt in the laboratory will be the same. The observer should not be able to tell the difference between these two motions.

1.2 Motion of a free particle in general relativity.

We consider the motion of a free particle in general relativity when the gravitational field is weak. That is we assume...
\( g_{ij} = \eta_{ij} + h_{ij} \)  \tag{1.2.1}

where the \( \eta_{ij} \) are the metric coefficients of special relativity 
\[(ds^2 = dt^2 - dx^2 - dy^2 - dz^2), \text{ and where products and squares of} \]
the \( h_{ij} \) and those of their derivatives are neglected.

We will show that the equation of motion can be described 
by a Maxwell type pondermotive equation. Since Einstein's derivation 
contains a slight error a corrected version was presented (Davidson 1957).

We let Latin letters denote space-time indices 1, 2, 3, 4 
while Greek letters denote space coordinates 1, 2, 3. From (1.2.1) 
we have

\[ g = \det (g_{ij}) = 1 + h_{11} + h_{22} + h_{33} - h_{44} \]
to the correct order of approximation. We then obtain

\[ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, g^{4\alpha} = h^{4\alpha}, g^{44} = 1 - h_{44} \]  \tag{1.2.2}

For the Christoffel symbols \( \Gamma^\mu_{4\nu} \) and \( \Gamma^\mu_{4\nu} \) we obtain

\[ \Gamma^\mu_{4\nu} = \frac{1}{2} \frac{\partial g_{\nu4}}{\partial x^\mu} - \frac{\partial g_{4\nu}}{\partial x^\mu} \]
\[ \Gamma^\mu_{4\nu} = -\frac{1}{2} \left( \frac{\partial g_{\mu4}}{\partial x^\nu} + \frac{\partial g_{\nu4}}{\partial x^\mu} - \frac{\partial g_{44}}{\partial x^\mu} \right) \]  \tag{1.2.3}
The world line of a free particle in general relativity is a geodesic having the equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$  \hspace{1cm} (1.2.4)$$

where

$$ds^2 = g_{\mu\nu} dt^2 + 2g_{\mu\alpha} dt \frac{dx^\alpha}{ds} + g_{\nu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$  \hspace{1cm} (1.2.5)$$

Neglecting squares and products of the spatial coordinate velocities $v^\alpha = \frac{dx^\alpha}{dt}$, for $i = \mu$ equations (1.2.4) become

$$\frac{d^2x^\mu}{ds^2} + 2\Gamma^\mu_{\alpha\alpha} v^\alpha \left(\frac{dt}{ds}\right)^2 + \Gamma^\mu_{\nu\nu} \left(\frac{dt}{ds}\right)^2 = 0$$  \hspace{1cm} (1.2.6)$$

or

$$\frac{dt}{ds} \frac{d}{dt} \left( v^\mu \frac{dt}{ds} \right) - \left( \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\mu\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \right) v^\alpha \left(\frac{dt}{ds}\right)^2 + \left( \frac{1}{2} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\mu} \right) \left(\frac{dt}{ds}\right)^2 = 0$$

we can now write this as

$$\frac{dt}{ds} \left\{ \frac{d}{dt} \left( v^\mu \frac{dt}{ds} \right) - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} v^\alpha \right\} - \frac{d}{dt} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} v^\alpha \left(\frac{dt}{ds}\right)^2 = \left\{ - \frac{1}{2} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\mu\nu}}{\partial x^\mu} + \left( \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\nu}}{\partial x^\mu} \right) \right\} \left(\frac{dt}{ds}\right)^2$$  \hspace{1cm} (1.2.7)$$

The solutions for the $h_{ij}$ in the weak field case (appendix 1) are

$$h_{ii} = -\frac{\kappa}{4\pi} \int \frac{1}{r} \rho \, dV, \quad h_{\alpha\alpha} = \frac{\kappa}{2\pi} \int \frac{1}{r} \rho u^\alpha \, dV, \quad h_{\alpha\beta} = 0 \quad \alpha \neq \beta$$

where $\rho$ is the mass density, $u^\alpha$ the space velocity of the element of
mass in volume $dV$ at distance $r$ from the point where the $h_{ij}$ are evaluated, all quantities being measured by observers at rest in the reference frame. The stress components of the energy momentum tensor have been neglected compared with the density and momentum components.

Hence $g_{\alpha\mu} = 0$ if $\alpha \neq \mu$, and since from (1.2.5) we have

$$\left( \frac{ds}{dt} \right)^2 = g_{\mu\mu} + 2g_{\mu\alpha} v^\alpha$$

equation (1.2.7) can now be written to the correct order of approximation as

$$\frac{d}{dt} \left( -g_{\mu\nu} \frac{\partial}{\partial s} \right) = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial t} + \left( \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \right) v^\alpha$$

or

$$\frac{d}{dt} \left( -g_{\mu\nu} \frac{\partial}{\partial s} \right) = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial t} + \left( \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \right) v^\alpha \quad (1.2.8)$$

Further using (1.2.1) the left hand side of (1.2.8) can be written as

$$\frac{d}{dt} \left\{ (1 - h_{\mu\nu} - \frac{1}{2} h_{\nu\nu}) v^\mu \right\}$$

We now define $\phi = -G \int \frac{1}{r} \, dV$ and $A^\mu = -4G \int \frac{\rho v^\mu}{r} \, dV$

then (1.2.8) can be written in vector form as

$$\frac{d}{dt} \left\{ (1 - 3\phi) v^\mu \right\} = -\text{grad} \phi - \frac{\partial A^\mu}{\partial t} + V \wedge \text{curl} A \quad (1.2.9)$$
If the source velocities of the field are also small then

\[ \frac{dV}{dt} = - \text{grad} \phi - \frac{\partial A}{\partial t} + V \wedge \text{curl} A \quad (1.2.10) \]

With reference to (1.2.9) we see that the inertial mass is apparently proportional to \( 1 - 3\phi \), and hence increases when ponderable masses are piled up in its neighbourhood. The term \(- \frac{\partial A}{\partial t}\) indicates an inductive action of local accelerated masses, on the test particle. Further, matter which is rotating relative to the compass of inertia at "infinity" generates a Coriolis field at the particle indicated by the term \( V \wedge \text{curl} A \). These effects are certainly favourable to Mach's principle, but it is clear that since the potentials \( \phi, A \) are determined only by local matter (otherwise the integrals would diverge) we have not considered how the entire cosmic distribution of matter influences inertia.

By writing out the geodesic equations to second order it was shown that the "Mach's principle" term \( \phi \) entering the left-hand side of (1.2.9) is really only a coordinate effect, and can be transformed away by a suitable choice of coordinate system. (Brans 1962). So it seems that Einstein may have been mistaken about this effect of local matter on the inertia of a body.

Another consequence of Mach's principle was that a rotating hollow body must generate inside of itself, Coriolis and centrifugal force fields which deflect moving bodies in the sense of the rotation.
General relativity does support this result, since Thirring (1918) showed that a slowly rotating mass shell drags along the inertial frames within it. Thirring calculated the gravitational field in the centre of the shell, neglecting the gravitational field of the shell itself, and showed that the shell near its centre produced forces analogous to the Coriolis and centrifugal force fields, working to second order in $\omega$ the rotation of the shell. This result was elaborated on by Bass and Pirani (1955) who added an extra term to the matter tensor to account for elastic stresses between the particles of the shell. Thirring had assumed that the shell was composed of incoherent matter, and as Bass and Pirani pointed out these particles would follow geodesics rather than circles, and further the conservation law, $T^k_{ik}=0$, would not be satisfied correct to the order Thirring was assuming.

Brill and Cohen (1966) further generalised this result by considering shells of arbitrarily large mass so that the gravitational field of the shell itself could not be neglected, however it was still assumed that the shell rotation was small. Instead of a flat space "base" metric as was used in Thirring's result, they assumed the Schwarzschild metric to be the "base" metric and considered a perturbation of this metric in the case of a rotating shell. By matching interior and exterior solutions for the shell, it was found that for a shell of small mass one obtained the Thirring result, but that for a large shell mass whose radius approaches the Schwarzschild
radius \( r^* = 2m \), the inertial frames were dragged along with the same rotation as that of the shell. Hence in this last case the inertial properties inside the shell are completely determined by the shell itself. A shell of matter of radius equal to the Schwarzschild radius has often been taken as a model universe. These results then show that there is no rotation of the large masses of the universe relative to a locally inertial frame, in accordance with Mach's principle.

Hence the treatment of a rotating mass shell according to general relativity, is certainly in agreement with the expectations of Mach's principle.

### 1.3 Inductive effect of the universe in general relativity.

In section (1.1) we described a theory of gravitation in which the fictitious forces, inherent in Newtonian theory, were attributed to the inductive effect of a moving universe. We will now show that general relativity is fully consistent with these ideas on inertia, and in particular we will give a superior presentation, due to general covariance, of the origin of fictitious Newtonian forces in a reference frame which is in general motion relative to a locally inertial frame (Davidson 1957).

We wish to obtain a Maxwell pondermotive type equation for the motion of a free particle in general relativity. The equation

\[ 20. \]
(1.2.9) shows only how local matter affects the motion of a particle since we had to assume the metric tensor is Galilean at infinity, in order that the potentials \( \phi \), \( A \) could be defined in terms of convergent integrals. Since distant matter plays a dominant role in the origin of local inertia, it will be necessary to generalize the quantities \( \phi \) and \( A \).

We generalize the assumptions made in the derivation in section (1.2):

(i) we assume the particle velocity \( V \) in the reference frame is small so that we may neglect terms of the order \( V^2/c^2 \) compared with \( V/c \).

(ii) the velocities of the sources of the field are assumed small in the region of space-time coordinates with which we shall be concerned, so that the same conditions as in (i) hold.

(iii) the \( h_{ij} \) as defined in (1.2.1) are small such that their squares and products and those of their derivatives can be neglected in our specified range of space-time coordinates, but in order to examine the cosmic influence on inertia, and not just the influence of local matter, we must assume that the \( h_{ij} \) do not necessarily remain small outside our
range of coordinates.

Referring to (1.2.8) we can write the equation of motion of a free particle under these conditions as

\[
\frac{dV^\mu}{dt} = \frac{1}{2} \frac{\partial g_{4\mu}}{\partial x^\nu} - \frac{\partial}{\partial t} (e_{4\mu}) + \left( \frac{\partial g_{4\mu}}{\partial x^\alpha} - \frac{\partial g_{4\alpha}}{\partial x^\mu} \right) V^\alpha
\]  

(1.3.1)

We now define the scalar and vector quantities \( \phi, A \) by

\[
\phi = \frac{1}{2} g_{4\mu} \quad , \quad A^\mu = -g_{4\mu}
\]  

(1.3.2)

Hence (1.3.1) can be written in vector form as

\[
\frac{dV}{dt} = -\text{grad} \phi - \frac{\partial A}{\partial t} + V \wedge \text{curl} A
\]  

(1.3.3)

In an effort to incorporate Mach's principle into general relativity we therefore define \( \phi, A \) in terms of the total \( g_{ij} \) and not just their deviations from the Galilean values. We require the whole inertial field, that is the total \( g_{ij} \), to be determined by the distribution of matter in the universe. Since the \( h_{ij} \) (the deviations of \( g_{ij} \) from Galilean values) are determined predominantly by local irregularities in the distribution of matter, we infer that the cosmic contribution to the \( g_{ij} \) potentials is present in the Galilean terms. We will discuss the validity of this inference in section (3.1).

Putting \( V = 0 \) in (1.3.3) we obtain

\[
-\text{grad} \phi - \frac{\partial A}{\partial t} = 0
\]  

(1.3.4)
This equation must hold in the particles rest frame. Comparing this with (1.1.1) we can say, in Sciama's language, that the 'gravel' field of the whole universe is zero in the particle's rest frame. Further we note that (1.3.4) is postulated by Sciama, whereas this equation is a consequence of general relativity.

For a reference frame at rest relative to the 'smoothed-out' universe we would expect that the derivatives of $\phi$ and $A$ would vanish near the origin due to the homogeneous and isotropic distribution of matter about it. Hence by (1.3.5) we have $V = \text{constant}$, which implies a locally inertial frame. General relativity predicts that such a reference frame will be locally inertial due to the spherical symmetry about the origin. For example a spherically symmetrical distribution of matter, characterized by an isotropic pressure and density has a metric of the form

$$ds^2 = c^2 \left[ A - B(1 - r^2/R^2)^{3/2} \right]^2 dt^2 - \frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$$

which clearly takes the form $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ near the space origin in a cartesian system.

Hence the static potential $\phi_0$ at the origin of such a locally inertial frame is given by

$$\phi_0 = \frac{1}{2} c^2$$

(1.3.5)

Consider now a reference frame at rest relative to the 'smoothed-out' universe with a massive body of gravitational mass $m$ at
the space origin. Assuming that deviations of the $g_{ij}$ from the
Galilean values are due only to the mass at the origin, we have in
this frame near the origin

$$h_{ii} = -2Gm/r , \quad h_{ij} = 0 \quad i \neq j$$

Hence in this region the metric will be approximately

$$ds^2 = \left(1 - \frac{2Gm}{r}\right) dt^2 - \left(1 + \frac{2Gm}{r}\right)(dx^2 + dy^2 + dz^2) \quad (1.3.6)$$

and so in this frame

$$A = 0 , \quad \Phi = \frac{1}{2}(1 - 2Gm/r) \quad (1.3.7)$$

Suppose we have a test-particle moving towards the mass
m along the x-axis. If the particle has spatial coordinates $(x_1, 0, 0)$
at the coordinate time $t$, then its coordinate speed is $dx_1/dt = -V$,
$(V > 0)$.

Transforming the metric (1.3.6) to the rest frame of
the particle by

$$x = X + x_1 , \quad y = Y , \quad z = Z , \quad t = T \quad (1.3.8)$$

we obtain, to the correct order of approximation

$$ds^2 = \left(1 - V^2 - \frac{2Gm}{r}\right) dt^2 - 2Vdxdt - \left(1 + \frac{2Gm}{r}\right)(dx^2 + dy^2 + dz^2) \quad (1.3.9)$$

So in the particle's rest frame we have

$$A = (-V, 0, 0) , \quad \Phi = \frac{1}{2} \left(1 - V^2 - \frac{2Gm}{r}\right) \quad (1.3.10)$$

Now applying (1.3.4) in this frame we obtain (due to general covariance)
24.

\[- \frac{3}{2X} \left\{ \frac{1}{2} (1 - V^2 - \frac{2Gm}{R^2}) \right\} - \frac{3}{2T} (- V) = 0 \quad (1.3.11)\]

which is equivalent to (in original coordinates)

\[\frac{Gm}{R^2} = \frac{dV}{dT} \quad (1.3.12)\]

For the case of a particle moving with uniform motion in a circle under the attraction of a mass \(m\) at the centre which is at rest relative to the smoothed out universe, we again transform to the rest frame of the particle by means of the relations

\[
\begin{align*}
  x &= X \cos \omega T - Y \sin \omega T \\
  y &= Y \cos \omega T + X \sin \omega T \\
  z &= Z, \quad t = T
\end{align*}
\]

(1.3.13)

The transformed metric becomes

\[ds^2 = (1 - \frac{2Gm}{R} - w^2 R^2) \, dt^2 - 2w(-YdXdT +XdYdT)\]

\[- (1 + \frac{2Gm}{R})(dX^2 + dY^2 + dZ^2) \quad (1.3.14)\]

where \(R^2 = X^2 + Y^2\).

Hence in the particle's rest frame we have

\[A = (-wY, wX, 0), \quad \phi = \frac{1}{2} \left(1 - \frac{2Gm}{R} - w^2 R^2\right) \quad (1.3.15)\]

and from equation (1.3.4) we have

\[\frac{Gm}{R^2} = w^2 R \quad (1.3.16)\]

25..
or in the rest frame of the universe

\[ \frac{Gm}{r^2} = w^2 r \]  (1.3.17)

We see that equations (1.3.12) and (1.3.17) are just the Newtonian equations that would be obtained in the rest frame of the universe. However the origin of the inertial terms \( -\frac{2}{3r} (-V) \) and \( w^2 R \) in (1.3.11) and (1.3.16) respectively lies in the motion of the universe relative to the particle's rest frame. As in Sciama's theory we attribute these inertial terms to the inductive effect of a moving universe.

It is important to note that we could have obtained (1.3.12) and (1.3.17), in the rest frame of the universe, directly by the use of equation (1.3.3). This is because of the generally covariant nature of (1.3.3), whereas in Sciama's theory his equations are coordinate dependent since we can only formulate them in the rest frame of the particle.

We now consider the most general motion of a reference frame relative to a locally inertial frame. For a particle near the origin of the locally inertial frame (1.3.3) gives \( \hat{r} = \) constant, where \( r \) is the position vector of the particle in this frame. We now transform to the second frame whose space origin has variable velocity \( V \) and variable spin \( W \) relative to the first frame. A result of Newtonian motion gives (appendix 2)
\[ \mathbf{\hat{r}} = \mathbf{v} + \mathbf{\hat{A}} + \mathbf{\hat{w}} \wedge \mathbf{R} \]  

(1.3.18)

or in differential form

\[
\frac{d\mathbf{r}}{dt} = (\mathbf{v} + \mathbf{\hat{w}} \wedge \mathbf{R}) \, dT + d\mathbf{R} \\
\frac{dt}{dt} = dT
\]  

(1.3.19)

where \( \mathbf{R} \) is the position vector of the particle in the second frame.

Near the origin of the first frame the metric takes the form

\[ ds^2 = dt^2 - dr^2 \]

By (1.3.19) the metric in the second frame becomes

\[
ds^2 = [1 - \frac{\mathbf{v}^2}{c^2} - 2\mathbf{v} \cdot (\mathbf{\hat{w}} \wedge \mathbf{R}) - (\mathbf{\hat{w}} \wedge \mathbf{R})^2] \, dT^2 - 2(\mathbf{v} + \mathbf{\hat{w}} \wedge \mathbf{R}) \cdot dR \, dT - dR^2
\]

(1.3.20)

so that in this frame

\[ \mathbf{A} = \mathbf{v} + \mathbf{\hat{w}} \wedge \mathbf{R}, \quad \phi = \frac{1}{2} [1 - \frac{\mathbf{v}^2}{c^2} - 2\mathbf{v} \cdot (\mathbf{\hat{w}} \wedge \mathbf{R}) - (\mathbf{\hat{w}} \wedge \mathbf{R})^2] \]

hence

\[ \text{grad} \, \phi = \mathbf{\hat{w}} \wedge \frac{\mathbf{v}}{c^2} + \mathbf{\hat{w}} \wedge (\mathbf{\hat{w}} \wedge \mathbf{R}) \]

\[ \frac{\partial \mathbf{A}}{\partial t} = \mathbf{\hat{v}} + \mathbf{\hat{w}} \wedge \mathbf{R} \quad \text{and} \quad \text{curl} \, \mathbf{A} = 2\mathbf{w} \]

Hence using (1.3.3) the equation of motion in the second frame becomes

\[ \mathbf{\hat{R}} = -\left[ \mathbf{w} \wedge \mathbf{v} + \mathbf{w} \wedge (\mathbf{\hat{w}} \wedge \mathbf{R}) + \mathbf{\hat{v}} + \mathbf{\hat{w}} \wedge \mathbf{R} + 2\mathbf{w} \wedge \mathbf{\hat{R}} \right] \]

in complete agreement with the Newtonian result (appendix 2).
The so-called fictitious forces of Newtonian theory, arise from the relative motion of the universe with respect to the second frame according to general relativity.
Chaprer 2.

2.1 Boundary conditions for the metric tensor in general relativity.

Mach's principle states that local inertia is determined by the masses of the universe and by their distribution. We know also that distant bodies have much greater influence on local inertia than local matter has. Hence it would be convenient if we could separate local effects from the general cosmological structure due to the distribution of distant bodies. Within the framework of general relativity we will impose boundary conditions on the metric tensor $g_{ij}$ at spatial infinity, which will provide such a separation of local and global effects. We will specify the boundary conditions in an inertial frame which is determined by the cosmological structure, and remembering that general relativity is a covariant theory we must be able to generalize the boundary conditions to a non-inertial frame (Gursey 1964). Firstly we will have to define these inertial frames in general relativity.

From the principles of normal relativistic cosmology, we obtain the Robertson-Walker line element describing the cosmological background,

$$ds^2 = c^2 dt^2 - \frac{R^2(t)}{(1 + kr^2/4)^2} (dx^2 + dy^2 + dz^2) \quad (2.1.1)$$
where $k = 0, \pm 1$, and the fundamental particles (galaxies) have constant spatial coordinates $x, y, z$ (appendix 3).

For most cosmological models it is possible to find a transformation (Infeld & Schild 1945), such that the line element (2.1.1) takes on the form

$$ds^2 = \lambda(t,r)(c^2 dt^2 - dx^2 - dy^2 - dz^2)$$

(2.1.2)

This implies that on the large scale the universe is conformally flat. It is necessary to note that an acceleration transformation has been found (Hill 1945) which does not destroy the isotropy of the cosmological background and hence the transformed metric will be of the same form as (2.1.2). Clearly one can see that if the cosmological background in an inertial frame is described by (2.1.2) then it is possible to find a non-inertial frame for which the cosmological background is described by the same type of metric. Hence the cosmological metric for an inertial frame will have to be more restrictive in form than (2.1.2). For this purpose we employ the perfect cosmological principle proposed by Bondi and Gold (1948), which states that the universe, on the large scale, is uniform not only in space but in time, and hence all points in space-time are equivalent. Our cosmological line element now takes the form

$$ds^2 = \phi(t^2)(c^2 dt^2 - dx^2 - dy^2 - dz^2)$$

$$= \phi(t^2)\eta_{ik}dx^i dx^k$$

(2.1.3)
where $\eta_{ik}$ are the metric coefficients of special relativity, and $\tau^2$ is the Lorentz invariant length defined by

$$\tau^2 = \eta_{ik} x^i x^k = c^2 t^2 - x^2$$  \hspace{1cm} (2.1.4)\]

The function $\phi$ characteristic of a homogeneous space time is given by

$$\phi(\tau^2) = \phi_0 / (1 - \tau^2/4R^2)$$  \hspace{1cm} (2.1.5)\]

where $R$ is the radius of curvature.

Since the universe is observed to be homogeneous and isotropic on the large scale we will assume that the metric is made up of two parts, one $c_{ik}$ referring to a geometry which is conformally flat and spatially homogeneous and another part which refers to deviations from this uniform cosmological structure. We now define the inertial frame as one in which the cosmological background takes the form (2.1.3), so that we have

$$c_{ik} = \phi^2(\tau^2)\eta_{ik}$$  \hspace{1cm} (2.1.6)\]

in an inertial frame. Our cosmological structure is conformally flat and so we have at spatial infinity, $g_{ik} + c_{ik}$ in a general frame and

$$g_{ik} + \phi^2(\tau^2)\eta_{ik}$$  \hspace{1cm} (2.1.7)\]

in an inertial system.

We now define

$$\phi = (-g)^{1/8}$$  \hspace{1cm} (2.1.8)\]
where \( g = \det(g_{ik}) \), then in an inertial system we have from (2.1.7),

\[ \phi + \phi \]  

(2.1.9)
at spatial infinity. Further, defining

\[ \gamma_{ik} = (-g)^{-\frac{1}{2}} g_{ik} \]  

(2.1.10)
we also have in the inertial system,

\[ \gamma_{ik} \rightarrow \eta_{ik} \]  

(2.1.11)

Hence we can see from the above relations that our metric tensor \( g_{ik} \) may be written in the form

\[ g_{ik} = \phi^2 \gamma_{ik} \]  

(2.1.12)
where \( \phi \) (or the determinant of the metric) contains all the information about the cosmological structure, and the \( \gamma_{ik} \) describe the local irregularities of the structure, provided we keep to the inertial system. We now have our separation of local and global effects, and it will be convenient to rewrite Einstein's field equations in terms of these new variables. Defining

\[ \xi = \phi - \phi, \quad b_{ik} = \gamma_{ik} - \eta_{ik} \]  

(2.1.13)
we have \( \xi \rightarrow 0 \) and \( b_{ik} \rightarrow 0 \) in the inertial frame of reference.
2.2 Reformulation of the field equations.

We now rewrite Einstein's field equations

$$R_{ij} - \frac{1}{2} g_{ij} R = -\kappa T_{ij} \quad (2.2.1)$$

in terms of the quantities $\phi$ and $\gamma_{ik}$ as defined in the previous section. We define $B_{ik}$ to be the Ricci tensor constructed out of the $\gamma_{ik}$, and $B$ as the quantity

$$B = \gamma^{ik} B_{ik} \quad (2.2.2)$$

where

$$\gamma^{ik} = (- g)^{ik} g^{jk} = \phi^2 g^{ik} \quad (2.2.3)$$

We can now rewrite the tensor $R_{ik} - \frac{1}{2} g_{ik} R$ in terms of these new quantities (appendix 4), by properties of conformal transformations, as

$$R_{ik} - \frac{1}{2} g_{ik} R = B_{ik} = \frac{1}{2} \gamma_{ik} B = 4 \phi^{-2} \left[ \phi_i \phi^i - \frac{1}{4} \gamma_{ik} \phi^i \phi^j \phi^k \right] + 2 \phi^{-1} \left[ \phi_i ; k - \gamma_{ik} \Box \phi \right] \quad (2.2.4)$$

where $\phi_i = \frac{\partial \phi}{\partial x^i}$ and the covariant derivatives on the right hand side of (2.2.4) refer to the tensor $\gamma_{ik}$.

$$\Box \phi = \gamma^{ik} \phi_{i;k} = (- \gamma)^{-\frac{3}{2}} \frac{\partial}{\partial x^i} \left( \sqrt{- \gamma} \gamma^{ik} \phi_k \right)$$

From (2.1.10) we have $\gamma = -1$ and so
Now (2.2.1) becomes

\[ B^i_k - \frac{1}{2} \delta^i_k B = - \kappa y^i j T_{jk} + M^i_k(\phi) \] (2.2.6)

Defining the tensor density \( j^i_k \) of weight 3/4 by (appendix 6)

\[ j^i_k = \phi^3 g^{ij} T_{jk} \] (2.2.7)

we finally obtain

\[ B^i_k - \frac{1}{2} \delta^i_k B = - \kappa \phi^{-1} j^i_k + M^i_k(\phi) \] (2.2.8)

where

\[ M_k^i(\phi) = 4 \phi^{-2} \left[ \phi_k^i \delta^i_k - \frac{1}{4} \delta^i_k \phi^d_k \delta^d_j - 2 \phi^{-1} \phi_\ell \delta^i_k \delta^\ell \right] \] (2.2.9)

So

\[ M(\phi) = 6 \phi^{-1} \square(\gamma)^\phi \] (2.2.10)

Hence from (2.2.8) we get

\[ - B = \phi^{-1} (- \kappa j + 6 \square(\gamma)^\phi) \] (2.2.11)

In the case of vanishing \( \xi \) and \( b^i_k \) as defined in (2.1.13) we have

\[ M^i_k(\phi) \sim M^i_k(\phi) = 3 \phi^2 \delta^i_k / \phi_0^2 R^2 \] (2.2.12)

and so from (2.2.10) we have

\[ \square(\eta)^\phi = 2 \phi^3 / \phi_0^2 R^2 \] (2.2.13)
Since $B$ corresponds to $n_{ik}$ in the zero approximation (2.2.11) gives

$$\frac{1}{6} J = 2\phi^3/\phi_0^2 \frac{R^2}{R}$$

(2.2.14)

and from (2.2.7)

$$J = \phi^3 T$$

(2.2.15)

hence

$$\frac{1}{6} T = 2/\phi_0^2 \frac{R^2}{R}$$

(2.2.16)

The geodesic equations in terms of the metric tensor $g_{ik}$ are

$$\frac{d^2 x^i}{ds^2} + \gamma_{ijk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

(2.2.17)

and in terms of $\phi$ and $\gamma_{ik}$ we have (appendix 4)

$$\frac{d}{dl} (\phi \frac{dx^i}{dl}) = \gamma_{ip} \phi_p - \phi \left\{ \gamma_{jk} \right\} \frac{dx^j}{dl} \frac{dx^k}{dl}$$

(2.2.18)

where the Christoffel symbols $\left\{ \gamma^i_{jk} \right\}$ are constructed out of the tensor density $\gamma_{ik}$, and the scalar density $dl$ is defined by

$$dl^2 = \gamma_{ik} \frac{dx^i}{dl} \frac{dx^k}{dl}$$

(2.2.19)

so that

$$ds = \phi dl$$

(2.2.20)

If

$$du^2 = n_{ik} \frac{dx^i}{dl} \frac{dx^k}{dl}$$

(2.2.21)

then in an inertial system we have $dl + du$ asymptotically.
2.3 The de Sitter cosmological background.

We first discuss the de Sitter universe, in which the functions $\xi$ and $b_{ik}$ vanish. It is a completely homogeneous universe with metric

$$ds^2 = \phi^2 \eta_{ik} dx^i dx^k$$

(2.3.1)

From (2.2.8) and (2.2.12) we have

$$\kappa J_{ik} = \left(3/\sqrt{\phi^2} R^2 \right) \phi^3 \eta_{ik}$$

(2.3.2)

this gives $J_{ik} = -J_{11} = -J_{22} = -J_{33}$ or an equation of state of the form $p + \rho = 0$. Hence the de Sitter universe cannot be composed of stable matter, nor can it represent a radiation filled universe since $J \neq 0$. Gursey has shown that this universe may be interpreted as being associate with a uniform distribution of mass scintillations, or unstable masses.

We now propose to define the total mass of this uniform universe, in the case of positive spatial curvature. From (2.2.5) and (2.2.11) we obtain

$$\Box(\phi) = \eta_{ik} \frac{\partial^2 \phi}{\partial x^i \partial x^k} = \left( \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \right) \phi$$

so

$$\left( \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \right) \phi = \frac{1}{\kappa} \kappa J$$

(2.3.3)

We consider the corresponding Newtonian situation for a
static system of mass points in flat space. Equation (2.3.3) corresponds to the Poisson equation, viz

\[ \nabla^2 V = 4\pi G \sum_i m_i \delta(r - r_i) \]

where \( V \) is the Newtonian potential. The total mass of the system is given by

\[ M = \sum m_i = \frac{1}{4\pi} \int \nabla^2 V \, d^3 \mathbf{r} \]

By analogy we define the total mass of the de Sitter universe at time \( t \) by

\[ M(t) = \frac{6}{\kappa c^2} \int \Phi(\eta) \, d^3 \mathbf{r} \quad (2.3.4) \]

Using (2.2.13) and (2.1.5) we have

\[
M(t) = \frac{12}{\kappa c^2 R^2} \int \Phi^3 \, d^3 \mathbf{r}
\]

\[
= \frac{12 \phi_0 4\pi}{\kappa c^2 R^2} \int_{r=0}^{\infty} r^2 \left(1 + \frac{r^2 - c^2 t^2}{4R^2}\right)^{-3} \, dr
\]

\[
\int_{r=0}^{\infty} r^2 \left(1 + \frac{r^2 - c^2 t^2}{4R^2}\right)^{-3} \, dr = R^2 \int_{r=0}^{\infty} \left(1 + \frac{r^2 - c^2 t^2}{4R^2}\right)^{-2} \, dr
\]

\[
= \frac{1}{2} \pi R^3 \left(1 - c^2 t^2 / 4R^2\right)^{-3/2}
\]

So we have

\[ M(t) = M(0) \left(1 - c^2 t^2 / 4R^2\right)^{-3/2} \quad (2.3.5) \]
where

\[
\frac{R}{M(O)} = \frac{\kappa c^2}{24\pi^2 \phi_0}
\]  

(2.3.6)

The equation of motion of a test particle in the de Sitter universe is, using (2.2.18)

\[
\frac{d}{du} \left( \phi \frac{dx}{du} \right) = \eta^i \frac{\phi^i}{\phi} = \frac{x^i}{2\phi_0 R^2} \phi^2
\]

(2.3.7)

or

\[
\frac{d}{du} \left( \frac{\phi}{\phi_0} \frac{dx}{du} \right) = \frac{x^i}{2R^2} \left( \frac{\phi}{\phi_0} \right)^2
\]

Neglecting terms of order $R^{-4}$ we then have in vector form

\[
\frac{d^2x}{du^2} = \frac{x}{2R^2}
\]

(2.3.8)

From (2.3.7) we see that $\phi$ and hence $\phi_0$ is proportional to the inertial mass of the test particle. From (2.3.6) it follows that $\phi_0$ is proportional to $M(O)/R$, and so the inertial mass will be dependent on the distribution of matter in this universe in accordance with Mach's principle.

2.4 A conformally flat universe.

In this section we will consider a space-time model which is conformally flat and satisfies the less restrictive cosmological principle, rather than the perfect cosmological principle as we assumed.
in the last section. We superimpose a stable mass point on our uniform de Sitter background, preserving the conformally flat character of the geometry. We have

\[ ds^2 = \phi^2 \eta_{ik} dx^i dx^k \]  

(2.4.1)

in an inertial system. If the stable mass point is at the origin of coordinates we have

\[ \phi = \phi(r^2) + \xi(r) \]  

(2.4.2)

where our Machian boundary conditions require

\[ \lim_{r \to \infty} \xi(r) = 0 \]  

(2.4.3)

In order to determine the form of the function \( \xi \) we consider equation (2.2.11). Since \( B = 0 \) we have

\[ \square (\eta) \phi = \square (\eta) (\phi + \xi) = \frac{1}{6} \kappa \mathcal{J} \]  

(2.4.4)

Now \( \mathcal{J} \) consists of two parts, one arising from the de Sitter structure and the other part arising from the stable mass at the space origin. That is

\[ \mathcal{J} = \mathcal{J}_s + mc^2 \xi(r) \]

where \( \mathcal{J}_s \) arises from the de Sitter cosmological background.

Since \[ \square (\eta) \phi = \frac{1}{6} \kappa \mathcal{J}_s \], we obtain from (2.4.4)

\[ \nabla^2 \xi = -\frac{1}{6} \kappa mc^2 \xi(r) \]  

(2.4.5)
(m is the mass of the inhomogeneity at the space origin).

The solution of (2.4.5), satisfying (2.4.3) is

\[ \xi(r) = \frac{\kappa c^2}{24\pi} \cdot \frac{m}{r} \quad (2.4.6) \]

So the metric of this conformally flat universe is

\[ ds^2 = \left( \phi + \frac{\kappa c^2}{24\pi\phi_0} \frac{m}{r} \right) \left( c^2 dt^2 - dx^2 - dy^2 - dz^2 \right) \quad (2.4.7) \]

The equation of a test particle in this universe is from (2.2.18)

\[ \frac{d}{du} \left[ \left( \frac{\phi}{\phi_0} + \frac{\kappa c^2}{24\pi\phi_0} \frac{m}{r} \right) \frac{dx^i}{du} \right] = \frac{i}{\phi_0} \frac{\phi}{\phi_0} + \frac{\kappa c^2}{24\pi\phi_0} \frac{i}{\phi_0} \frac{\phi}{\phi_0} \left( \frac{\phi}{\phi_0} \right) \left( \frac{2}{c^2} - \frac{\kappa c^2}{24\pi\phi_0} \text{grad} \left( \frac{m}{r} \right) \right) \quad (2.4.8) \]

or in vector form we have for \( i = 1, 2, 3 \)

\[ \frac{d}{du} \left[ \left( \frac{\phi}{\phi_0} + \frac{\kappa c^2}{24\pi\phi_0} \frac{m}{r} \right) \frac{dx^i}{du} \right] = \left( \frac{\phi}{\phi_0} \right)^2 \frac{2}{c^2} - \frac{\kappa c^2}{24\pi\phi_0} \text{grad} \left( \frac{m}{r} \right) \quad (2.4.8) \]

Neglecting the cosmical expansion (that is assuming \( \phi/\phi_0 \sim 1 \)) this last equation can be written

\[ \frac{d}{du} \left[ \left( 1 + \frac{R}{\kappa M(0)} \frac{m}{r} \right) \frac{dx}{du} \right] = - \frac{R}{\kappa M(0)} \text{grad} \left( \frac{m}{r} \right) \quad (2.4.9) \]

This last equation shows that the test particle will be subject to a repulsive force proportional to the mass \( m \) of the body at the origin.

If we consider a more general case in which we have \( N \) stable masses (galaxies) of mass \( m \) situated at the positions \( a_i \) at time \( t = 0 \),
then the total mass of the universe at time $t = 0$ will be

$$M_{u}(0) = \frac{6}{\kappa c^{2}} \int \left( \delta \phi \right)_{t=0} d^{3}r$$

$$M_{u}(0) = M(0) + Nm$$  \hspace{1cm} (2.4.10)

We have

$$v^{2} \xi = - \frac{1}{6} \kappa mc^{2} \sum \delta(r - a_{i})$$  \hspace{1cm} (2.4.11)

with the solution

$$\xi = \frac{\kappa c^{2}}{24\pi} \sum_{i=1}^{N} \frac{m}{|r - a_{i}|}$$  \hspace{1cm} (2.4.12)

Hence at time $t = 0$ we have

$$\phi_{0} = \frac{\kappa c^{2}}{24\pi} \left( \frac{M(0)}{r} \frac{1}{1 + r^{2}/4\pi^{2}} + \sum_{i=1}^{N} \frac{1}{|r - a_{i}|} \right)$$  \hspace{1cm} (2.4.13)

Introducing

$$V(r) = m \frac{1}{|r - a_{i}|}$$

and assuming that the galaxies are uniformly distributed within a sphere of radius $A$ we have (appendix 5)

$$V(r) = \begin{cases} \frac{3Nm}{2A} \left( 1 - \frac{r^{2}}{3A^{2}} \right) & r \leq A \\ \frac{Nm}{r} & r > A \end{cases}$$  \hspace{1cm} (2.4.14)

where $Nm$ is the total mass of the galaxies. Since we have shown the forces between galaxies to be repulsive (equation (2.4.9)), $A$ will be an increasing function of time.

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If we neglect the square of the curvature then we have approximately at $t = 0$:

$$\phi_0 = \frac{\kappa}{24\pi} \left( \frac{M(0)}{R} + \frac{3Nm}{2A} \right) \quad (2.4.15)$$

$$= \frac{\kappa c^2}{24\pi} \frac{M'}{R'} \quad (2.4.16)$$

where $M' = M(0) + Nm$, and $R'$ is (an average radius of curvature) defined by (2.4.15).

We have shown that forces between massive bodies are repulsive if the universe is conformally flat. Thus a scalar theory of gravitation only allows an expanding system of galaxies.

2.5 Tensor theory of gravitation.

In order that we may introduce attractive forces between particles we will have to assume that the universe is not conformally flat. This means that the tensor $b_{ik}$ as defined in (2.1.13) cannot be zero. Suppose we have a body at the origin destroying the conformal flatness of space time. Our metric in this general case will have the form

$$ds^2 = \phi^2 \gamma_{ik} dx^i dx^k \quad (2.5.1)$$

Neglecting the square of the curvature we have approximately

$$\phi \approx \phi_0 \quad (2.5.2)$$

where $\phi_0$ is given by (2.4.16).
The tensor density $\mathcal{M}^i_k(\phi)$ defined in (2.2.9) is of order $R^{-2}$ (from (2.2.12)), and is therefore negligible in our approximation. The field equations (2.2.8) now become

$$B^i_k - \frac{1}{2} \delta^i_k B - k\phi - \frac{1}{c^2} \oint_k \frac{\mathcal{J}^i}{c^2} = - \frac{24\pi R^i}{M^i} \oint_k \frac{\mathcal{J}^i}{c^2}$$  \hspace{1cm} (2.5.3)

It is interesting to note that this equation is similar to Einstein's original field equations. The static solution, having spherical symmetry, for the empty space surrounding a gravitating point particle of mass $m$ at the origin is

$$\Psi_{\mu\nu} = - \delta^\mu_\nu - \frac{am}{r} \left( \frac{1}{1 - am/r} \right) \frac{x^\mu x^\nu}{r^2}$$

$$\Psi_{44} = 1 - \frac{am}{r} \hspace{1cm} \Psi_{4\mu} = 0$$  \hspace{1cm} (2.5.4)

where Greek letters are restricted to the space coordinates 1, 2, 3, and $\alpha = 6R'/M'$.

So for $b^i_k$, we have

$$b_{\mu\nu} = - \frac{am}{r} \left( \frac{1}{1 - am/r} \right) \frac{x^\mu x^\nu}{r^2}$$

$$b_{44} = - \frac{am}{r} \hspace{1cm} b_{4\mu} = 0$$  \hspace{1cm} (2.5.5)

The Newtonian gravitational 'constant' is given by

$$G = \frac{1}{2} c^2 \alpha = 3c^2 R'/M'$$  \hspace{1cm} (2.5.6)
Hence $G$ is dependent on the cosmic distribution of matter in the universe, and it will in general be space and time dependent. We will discuss a theory of gravitation based on a variable gravitation 'constant' in section (2.7).

The author considered the case in which the deviation from conformal flatness is small. That is we neglect squares and products of the $b_{ij}$ defined in (2.1.13) and those of their derivatives. We neglect the square of the curvature so that (2.5.2) and (2.5.3) hold. We have

$$\gamma_{ik} = n_{ik} + b_{ik} \quad (2.5.7)$$

and to the correct order of approximation

$$B^i_j = \frac{1}{2} n_{kl} \frac{\partial^2 b^i_j}{\partial x^k \partial x^l} = -\frac{1}{2} \nabla^2 (n) b^i_j \quad (2.5.8)$$

with the auxiliary conditions

$$\frac{\partial}{\partial x^n} (b^n_i - \frac{1}{2} n_i^j b) = 0 \quad (2.5.9)$$

From (2.5.3) we get

$$B^i_j = -\kappa \phi_0^{-1} \left( \nabla^2 j^i_j - \frac{1}{2} \delta^i_j \right) \quad (2.5.10)$$

hence from (2.5.8) we obtain

$$\nabla^2 b^i_j = 2\kappa \phi_0^{-1} \left( \nabla^2 j^i_j - \frac{1}{2} \delta^i_j \right) \quad (2.5.11)$$
having the solution

\[ b^i_{j} = -\frac{\kappa \phi_0^{-1}}{2\pi} \int \left[ \frac{\gamma^i_{j} - \frac{1}{2} \delta^i_{j}}{r} \right] dV \]  

(2.5.12)

From the form of the tensor \( \gamma^i_{j} \), defined in appendix 6, we have for a static mass point \( m \) at the origin of coordinates

\[ J^i_4 = mc^2 \delta^i_4(r) \]  

(2.5.13)

the other components being zero.

Hence from (2.5.12) we obtain

\[ b_{ii} = -\frac{\kappa \phi_0^{-1} c^2}{4\pi} \frac{m}{r} = -\frac{am}{r} \quad (i = 1, \ldots 4) \]

(2.5.14)

\[ b_{ij} = 0 \quad i \neq j \]

This solution satisfies the Machian boundary condition

\[ \lim_{r \to \infty} b_{ik} = 0 \]

From (2.5.3) we find that in the approximation of neglecting the square of the curvature, the equations \( \gamma^k_{i;jk} = 0 \) will be satisfied (the covariant derivative being associated with the \( \gamma^k_{ij} \)). Hence the auxiliary conditions (2.5.9) will be satisfied since

\[ \frac{3}{8\pi} (b^n_i - \frac{1}{2} n^i n b) = \frac{\kappa \phi_0^{-1}}{2\pi} \int \left[ \frac{\partial \gamma^n_i / \partial x^n}{r} \right] dV \]

and \( \gamma^n_{i;jn} = \partial \gamma^n_i / \partial x^n \) to the correct order of approximation.
The metric of space-time takes the form

$$ds^2 = \phi_0^2 \left[ (1 - \frac{om}{c^2}) c^2 dt^2 - (1 + \frac{om}{c^2})(dx^2 + dy^2 + dz^2) \right] \quad (2.5.15)$$

The equation of motion of a test particle, using (2.2.18), is

$$\frac{d}{dt} \left( \phi_0 \frac{dx^i}{dt} \right) = -\phi_0 \{ i \} \frac{\partial}{\partial t} \frac{dx^j}{dt} \frac{dx^k}{dt} \quad (2.5.16)$$

where

$$\frac{dl^2}{dt} = \gamma_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt}$$

Hence \( \left( \frac{dl}{dt} \right)^2 = (1 - \frac{om}{c^2}) c^2 - (1 + \frac{om}{c^2}) \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \)

On neglecting squares of the spatial coordinate velocities, we have

$$\left( \frac{dl}{dt} \right)^2 = (1 - \frac{om}{c^2}) c^2 = \gamma_{ik} c^2$$

Now from (2.5.16) we obtain

$$\gamma_{pi} \frac{d}{dt} \left( \phi_0 \frac{dx^i}{dt} \right) = -\gamma_{pi} \phi_0 \{ i \} \frac{\partial}{\partial t} \frac{dx^j}{dt} \frac{dx^k}{dt}$$

or

$$\frac{d}{dt} \left( \phi_0 \gamma_{pi} \frac{dx^i}{dt} \right) = -\phi_0 \{ p, ik \} \frac{dx^j}{dt} \frac{dx^k}{dt} + \phi_0 \frac{\partial \gamma_{pi}}{\partial x^s} \frac{dx^s}{dt} \frac{dx^i}{dt}$$

where \( [ p, ik ] \) is the Christoffel symbol of the first kind constructed out of the \( \gamma_{ik} \). This last equation then becomes

$$\frac{d}{dt} \left( \phi_0 \gamma_{pi} \frac{dx^i}{dt} \right) = \frac{1}{2} \phi_j \frac{\partial \gamma_{ks}}{\partial x^p} \frac{dx^k}{dt} \frac{dx^s}{dt} \quad (2.5.17)$$

From (2.5.14) we have \( \gamma_{ik} = 0 \) for \( i \neq k \) and neglecting squares and products of the spatial coordinate velocities, (2.5.17)
becomes, for \( p = 1, 2, 3 \)

\[
\frac{d}{dt} \frac{d}{dt} \left( \phi_0 \gamma_{pp} \frac{dt}{dt} \frac{dx^p}{dt} \right) = \frac{1}{2} \phi_0 \frac{\partial^2 \gamma_{44}}{\partial x^p \partial x^p} \left( \frac{dt}{dt} \right)^2 c^2
\]

To the correct order of approximation this becomes

\[
\frac{d}{dt} \left( \phi_0 \gamma_{pp} \gamma_{44} \frac{\partial x^p}{\partial t} \right) = -\frac{1}{2} \phi_0 \alpha c^2 \frac{3}{3x^p} \left( \frac{\gamma}{r} \right)
\]

or

\[
\frac{d}{dt} \left[ \phi_0 (1 + \frac{3}{2} \frac{am}{r}) \gamma \right] = \frac{1}{2} \phi_0 \alpha c^2 \text{grad} \left( \frac{\gamma}{r} \right) \quad (2.5.18)
\]

in vector form.

The inertial mass is proportional to \( \phi_0 (1 + \frac{3}{2} \frac{am}{r}) \), and is therefore dependent on the distribution of matter in the universe. There is a correction term proportional to \( 1/r \) which denotes the inertial contribution of the mass destroying the conformally-flat space-time. The inertial mass of a particle will increase as it approaches the body at the origin. We further note that in an empty universe that is \( \phi_0 = 0 \), (2.5.18) will be satisfied when the velocity is an arbitrary function of time, which is not in agreement with Newton's first law of motion. Hence, in accordance with Mach's principle, a test particle will not possess inertia in an empty universe.

We note from (2.5.18) that the force acting on the test particle is attractive. Our equations (2.5.3) become linear when we consider small deviations away from conformal flatness, whereas the
original Einstein equations only become linear in the weak field approximation. Furthermore we note that \( \kappa \phi_0^{-1} \) plays the role of the gravitational constant instead of \( \kappa \) in the weak field case. From (2.5.6) we find \( GM'/R'c^2 \) is of the order unity in agreement with equation (1.1.16) of Sciama's theory.

2.6 Asymptotic behaviour of the Schwarzschild solution.

The usual static solution of Einstein's field equations

\[
\mathbf{R}_{ij} - \frac{1}{2} g_{ij} \mathbf{R} = - \kappa T_{ij}
\]

having spherical symmetry for a point mass \( m \) in empty space is

\[
ds^2 = \left(1 - \frac{\kappa m}{4\pi R} \right) c^2 dt^2 - \left(\delta_{\mu\nu} + \frac{\kappa m \chi_{\mu\nu}}{4\pi R^2} \right) dx^\mu dx^\nu \quad (2.6.1)
\]

We had obtained the solution for a point mass \( m \) superimposed on a conformally flat cosmological background as

\[
ds^2 = \phi^2_0 \gamma_{ik} dx^i dx^k \quad (2.6.2)
\]

where we neglected the square of the curvature.

Using (2.5.4) we can write this as

\[
ds^2 = \left(\frac{\kappa c^2}{24\pi} \frac{M'}{R'} \right)^2 \left\{ \left(1 - \frac{6R'}{M'} \frac{m}{R} \right) c^2 dt^2 + \left(\delta_{\mu\nu} + \frac{6R'}{M'} \frac{m \chi_{\mu\nu}}{R^2} \right) dx^\mu dx^\nu \right\} \quad (2.6.3)
\]
From (2.4.13) we see that $\phi_0 \to 0$ as $r \to \infty$, and hence the metric tensor $g_{ik} = \phi_0^2 \gamma_{ik}$ tends to zero as $r \to \infty$. Comparing this result with the Schwarzschild solution (2.6.1) which tends asymptotically to the metric of special relativity, we see that according to the Machian solution (2.6.3) a test particle will not possess inertial properties away from matter, whereas the Schwarzschild solution, being asymptotically flat, implied that a test particle would possess inertial properties at large distances from the point mass.

From (2.6.3) we have

$$\lim_{M' \to 0 \atop m \to 0} ds^2 = 0$$

This means that in the absence of matter there is no geometry. In other words there are no vacuum solutions of Einstein's equations. Since from (2.5.18) we found the inertial mass of a test particle was proportional to $M'/R'$, it follows that inertial mass would vanish in an empty universe.

Clearly the formulation of boundary conditions on the metric tensor, at spatial infinity, has enabled one to point out many aspects of Mach's principle within the framework of general relativity. Wheeler (1964) has also suggested that Mach's principle provides a boundary condition, selecting only physically admissible solutions of the field equations.

The solutions of the field equations provide three different types of geometry:
(i) spatially open
(ii) a geometry which is somewhere singular
(iii) spatially closed and free of singularity.

We would select a closed geometry as our boundary condition for physically admissible solutions, since Mach's ideas only correspond to a finite universe, bounded in space. For example in the case of negative spatial curvature, that is when $R$ as defined in (2.1.5) is imaginary, the integral (2.3.4) would diverge, and no relation of the form (2.3.6) would be obtained, furthermore the gravitational constant as in (2.5.6) would then be zero. An infinite universe would be possible only if the mean density of matter in space vanishes, and although this is possible, a finite mean density of matter in the universe is more probable. Hence a quasi-Euclidean infinite solution of the field equations, for example the Schwarzschild solution, will be rejected as a Machian solution because of our restrictive closed geometry. One can hardly attribute the cause of flat space at infinity to the central point mass, and hence inertial properties at infinity do not appear to be determined by any masses.

The Schwarzschild solution, being incompatible with Mach's principle because of its asymptotic flatness, can be made physically acceptable by considering the limit of a "Schwarzschild zone" in a lattice universe (Wheeler & Lindquist 1957).

The lattice universe consists of a number of mass concentrations so distributed in space and of approximately the same
magnitude such that the zone of influence of each mass can be reasonably approximated by a sphere. The number of mass centres and their distribution in space is so chosen as to curve the space into closure. When we have infinitely many infinitely small cells one goes over to the Friedmann universe, which is a closed dust-filled universe of uniform density and negligible pressure. The Schwarzschild metric holds inside each cell, and the equation of motion of the boundary between two zones is found to be identical with that of a freely falling particle. The interface between two zones moves outwards from the attracting masses on either side of it, and then falls back again. At the moment of maximum expansion the dynamics of the lattice universe agree approximately with the dynamics of the uniform Friedmann model. It is also true that in the limit as a typical cell size goes to infinity, the number of cells becomes infinite.

We now envisage the Schwarzschild geometry as the limit of the geometry of a closed lattice universe when the cell radius tends to infinity. The Schwarzschild geometry is a piece of a closed geometry, for which our Machian boundary condition is satisfied. However our reformulation of Einstein's equations in terms of the quantities $\gamma_{ik}$ and $\phi$ certainly rules out the asymptotic flatness of the Schwarzschild solution, since the metric becomes degenerate at infinity.
One further point concerns the origin of inertial forces. Gursey has shown that these forces may be interpreted as gravitational forces exerted by the cosmological background, in accordance with the interpretation of Sciama (1953) and Davidson (1957). However, Davidson's work was restricted to the weak field case, whereas Gursey's result is general in that one considers a conformal transformation, corresponding to an acceleration $a$, from an inertial frame into a non-inertial frame. For a particle at rest in the non-inertial frame one finds an inertial force proportional to $-a$ acting on the particle. This is explained by the fact that the distribution of cosmological matter relative to the non-inertial frame will not be uniform and isotropic, and this will cause gravitational effects which we call inertial forces.

2.7 Theory of gravitation based on a scalar field in a Riemannian Geometry.

This theory was specifically designed to incorporate Mach's principle into general relativity (Brans & Dicke 1961). Einstein's original field equations are modified by the introduction of a new scalar field $\phi$ which describes gravitational effects as well as the metric tensor $g_{ij}$. The theory is based on the assumption that the gravitational 'constant' varies with position, since by Mach's principle the gravitational 'constant' is dependent on the mass distribution in
the universe and any theory which assumes a varying universe must consider the possibility of a varying gravitational 'constant'.

The gravitational 'constant' $G$ is assumed to be a function of the scalar variable $\theta$. The new variable $\theta$ is introduced since the geometrical scalars which can be formed from the curvature tensor $R^m_{i j k}$ are all primarily determined by nearby matter, whereas we want $G$ to be determined mainly by distant matter.

The generalization of Einstein's theory is carried out by first considering the variational principle for general relativity

$$\delta \int \left[ R + \frac{16\pi G}{c^2} L \right] \sqrt{-g} \, d^4x = 0 \quad (2.7.1)$$

where $R$ is the scalar curvature and $L$ is the Lagrangian density of matter. One obtains Einstein's field equations by varying the components of the metric tensor and its first derivatives, and further, the geodesic equations of motion for a test particle are obtained by varying the matter variables. We generalize (2.7.1) by dividing through by $G$, and then by adding a Lagrangian density of the scalar field $\theta$ inside the bracket, and replacing $G^{-1}$ by $\theta$. This gives

$$\delta \int \left[ \theta R + \frac{16\pi}{c^2} L + w(\theta^{-1} \partial_i \theta \theta^i) \right] \sqrt{-g} \, d^4x = 0 \quad (2.7.2)$$

where $\theta^i = \partial \theta/\partial x^i$ and $w$ is a constant expressing the strength of the coupling of the scalar field $\theta$ to the metric tensor $g_{ij}$.

From Sciama's theory we obtained the relation

$$G^{-1} \sim M/Rc^2 \quad (2.7.3)$$
where $M$ denotes the finite mass of the visible universe, and $R$ is the effective radius of the visible universe. This suggests that since we have assumed $G^{-1}$ varies as $\theta$, we would expect a wave equation for $\theta$ with the scalar mass density as source giving a solution for $\theta$ in the form

$$\theta \sim \frac{M}{Rc^2} \quad (2.7.4)$$

Comparing the two variational principles (2.7.1), (2.7.2) we see that the term involving the Lagrangian density of matter is the same in both cases, implying that the equation of motion of a free particle in the new theory will be a geodesic in the four dimensional manifold as in general relativity. Further since $L$ is assumed to be a function of $g_{ij}$ only (not of $\theta$) we will still have the conservation equations

$$T^{jk}_{\ ;k} = 0 \text{ where } T^{jk} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{jk}} (\sqrt{-g} L)$$

By varying $\theta$ and $\theta^i$ in (2.7.2) we obtain

$$2\omega^{-1} \Box (g) \theta - \omega^{-2} \theta^i \theta_{,i} + R = 0 \quad (2.7.5)$$

and by varying the metric tensor and its first derivatives we obtain the field equations for this theory.
From (2.7.5), (2.7.6) we get the wave equation for $\theta$

$$\Box(g) \theta = \frac{8\pi}{(3 + 2w)c^2} T$$  (2.7.7)

with the contracted energy-momentum tensor as source.

Further analysis shows that

$$\theta(x_0) \sim M/Rc^2$$  (2.7.8)

which means that $\theta$, at the point $x_0$, is determined by the distribution of matter and energy in the universe, each element of mass contributing to a wavelet which propagates to the point $x_0$. This is just the interpretation of Mach's principle required.

As far as restriction on the coupling constant $w$ is concerned, the perihelion rotation of Mercury requires $w \geq 6$.

In this theory Einstein's equations were modified by the introduction of a scalar field which is part of the gravitational field, and the new field equations were found to be compatible with Mach's principle. However it has been shown (Dicke 1962) that a coordinate dependent transformation of units, allows the gravitational field to be solely characterised by the metric tensor, the scalar field $\theta$ appearing.
as a non-gravitational field. This means that the field equations in transformed units will be the same as Einstein's original field equations. Furthermore the gravitational constant is coordinate independent in this new formulation whereas the rest masses of all elementary particles vary with position, being functions of \( \theta \), although mass ratios remain constant. From (2.7.3) we see that if \( G \) is to be constant then \( M/R \) will have to remain fixed. We can explain this by pointing out that the masses of elementary particles adjust themselves through the scalar field \( \theta \) generated by all other matter in such a way that \( M/R \) remains fixed.

It is interesting to note that the variational principle (2.7.1) written in terms of quantities \( \phi \) and \( \gamma_{ik} \) of Gursey's theory, is identical to the variational principle (2.7.2) of Brans and Dicke's theory. From (2.2.4) we have

\[
R = \phi^{-2} B - 6\phi^{-3} \nabla (\gamma) \phi
\]  
(2.7.9)

and from (2.1.8) we get

\[
\sqrt{-g} = \phi^4
\]  
(2.7.10)

The variational principle (2.7.1) now becomes

\[
\delta \int \left[ \phi^{-2} B + \frac{16\pi G}{c^2} L - 6\phi^{-3} \nabla (\gamma) \phi \right] \phi^4 d^4x = 0
\]  
(2.7.11)

Now

\[
\phi \nabla (\gamma) \phi = \phi \frac{3}{2x^i} (\phi^i) = \frac{3}{2x^i} (\phi \dot{\phi}^i) - \phi \dot{\phi}^i
\]
The part of the integrand involving the term $\frac{\partial (\phi \phi^i)}{\partial x^i}$ can be transformed into a surface integral over a hypersurface surrounding the four-volume by Gauss' theorem. From (2.1.5) we see that $\phi(t^2)$ vanishes for large $r$, when $t$ is kept constant, and it therefore follows that $\phi$ will vanish at infinity in the inertial system. The surface integral will be zero, and if we further define a new Lagrangian density by

$$\mathcal{L} = \phi^3 L$$  \hspace{1cm} (2.7.12)

then (2.7.11) becomes

$$\delta \int \left[ \phi^2 B + \frac{16\pi G}{c^2} \phi \int + 6(\phi \phi^i) \right] d^4x = 0$$ \hspace{1cm} (2.7.13)

If in Brans and Dicke's modification of Einstein's variational principle one first divided by $G^2$ and then replaced $G^{-1}$ by $\theta$, a variational principle identical to (2.7.13) would have been obtained provided $\sqrt{-g} = 1$ and $w = 6$. 

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Chapter 3.

3.1 The steady-state theory.

We intend to give a physical interpretation of the potentials $\phi, A$ in general relativity as they were defined in section (1.3) (Davidson 1957). We shall discuss the Robertson-Walker metric in connection with the isotropic expanding or contracting types of cosmological models. We take the metric in the form

$$ds^2 = c^2 dt^2 - e^\beta(t) \frac{(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)}{(1 + kr^2/4)^2} (3.1.1)$$

where $k = 0, \pm 1$ corresponding to the 3-space $t = \text{constant}$ being Euclidean, spherical, or hyperbolic respectively, and where the fundamental particles have constant $r, \theta, \phi$.

With respect to the metric (3.1.1) the field equations

$$R_{ij} - \frac{1}{2} g_{ij} R = - \kappa T_{ij} (3.1.2)$$

give, for a material density $\rho(t)$ and an isotropic pressure $p(t)$:

$$k \rho = - k c^2 e^{-\beta(t)} - \ddot{\beta} - \frac{3}{4} \dot{\beta}^2$$

$$k c^2 p = 3 k c^2 e^{-\beta(t)} + \frac{3}{4} \ddot{\beta}^2 (3.1.3)$$

The cosmological mass density for isotropic cosmological models is given by

$$\sigma = \rho + 3p/c^2 (3.1.4)$$
Putting
\[
\kappa = \frac{8\pi G}{c^2}
\]
in (3.1.3) we get
\[
8\pi G\sigma = -3(\bar{\beta} + \frac{1}{2} \beta^2)
\]  (3.1.5)

For cosmoological models represented by the metric (3.1.1) we find the proper distance \( \ell \) to the particle \((r, \theta, \phi)\), measured in the simultaneity of the fundamental observers at cosmological time \( t \), from \( r = 0 \) is
\[
\ell = e^{\frac{1}{2}\beta(t)} \int_0^R \frac{dr}{(1 + kr^2/4)}
\]  (3.1.6)

Therefore the radial velocity of the particle is
\[
\dot{r} = \frac{1}{2} \beta \ell
\]  (3.1.7)

Now \(|\dot{\beta}| = c\) when \( \ell = 2c/|\dot{\beta}| \), and so the effective radius \( R \) of the model is
\[
R = 2c/|\dot{\beta}|
\]  (3.1.8)

For \( \dot{\beta} > 0 \) we have \( \dot{\beta} = \frac{2c}{R} \) and \( \ddot{\beta} = -\frac{2c\dot{R}}{R^2} \n\)

and substituting in (3.1.5) we get
From (3.1.7) we have $\xi = \pm \alpha \phi / R$ according as $\beta > 0$ or $\beta < 0$. In either case we obtain from (3.1.9) the Newtonian type equation

$$\ddot{\xi} = \frac{4 \pi}{3} \pi G \xi$$

(3.1.10)

With the gravitational force on unit mass being defined as the proper acceleration relative to the space origin, we see that (3.1.10) relates gravitational force and proper distance $\xi$ at cosmological time $t$. Comparing (3.1.10) with the ponderomotive equation, (1.3.3) we see that $A_z = 0$ and $\nabla \phi = \frac{4 \pi}{3} \pi G \xi$, for an observer whose radial space coordinate is the proper distance $\xi$, and whose time is the cosmological time $t$.

By analogy with Newtonian theory we define the gravitational "work" done by the field when a particle of unit mass is moved from proper distance $\xi$ to the horizon of the model as

$$\phi_\xi = -\frac{4 \pi}{3} \pi G \int_\xi^R \sigma \xi \, d\xi$$

(3.1.11)

hence

$$\phi_\xi = \int_\xi^R \xi \, d\xi = \left[ \frac{1}{2} \frac{c^2 \xi^2}{R^2} \right]_\xi^R$$

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\[ \phi_0 = \frac{1}{2} c^2 \] (3.1.13)

This gives a physical interpretation of the potential \( \phi_0 \) arising in (1.3.5), as the static potential, at the origin, of the whole visible universe.

We now consider the steady state metric proposed by Bondi and Gold (1948)

\[ ds^2 = c^2 dt^2 - e^{2ct/R} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \] (3.1.14)

By means of the transformation

\[ t = re^{ct/R}, \quad \tau = t - \frac{R}{2c} \log \left(1 - \frac{\xi^2}{R^2}\right) \]

the metric (3.1.14) is transformed into static coordinates

\[ ds^2 = c^2 (1 - \xi^2/R^2) \, dt^2 - \frac{d\xi^2}{1 - \xi^2/R^2} - \xi^2 d\theta^2 - \xi^2 \sin^2 \theta d\phi^2 \] (3.1.15)

We note that in these coordinates our reference frame is locally inertial, and from (1.3.2) we have

\[ \phi = \frac{1}{2} c^2 (1 - \xi^2/R^2) \]

in agreement with (3.1.12).

For the potential of unit mass, at distance \( \xi \) from the mass \( \delta V \) constantly in the volume \( dV \), we have the Newtonian type integral.

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\[ d\phi = -G \int_{l}^{R} \frac{g dV}{l^2} \, dl \]

\[ = -G \sigma V \left( \frac{1}{l} - \frac{1}{R} \right) \]

since \( \sigma \) and \( R \) are constant for the steady-state. Hence the gravitational potential at the origin arising from all the matter in the visible universe is

\[ \phi_0 = -G \sigma \int_{0}^{R} \left( \frac{1}{l} - \frac{1}{R} \right) 4\pi l^2 \, dl \]

\[ = -\frac{2\pi G \sigma R^2}{3} \]

For the steady-state (3.1.9) gives \( G\sigma R^2 = -3c^2/4\pi \) and so

\[ \phi_0 = \frac{1}{2} c^2 \]

This justifies our interpretation of \( \phi_0 \) in (1.3.5) as the gravitational potential of all matter in the universe apparent to an observer at the origin.

Since \( n_{44} = 2\phi_0 \), \( n_{11} = 2\phi_0/c^2 \), we make the tentative assumption that the Galilean values of \( g_{ij} \) are associated with world gravitation. In connection with inertial mass, we note that in the case of rectilinear motion described in section (1.3), the application of the pondermotive equation (1.3.4) shows that the inertial mass is apparently proportional to the Galilean \( g_{11} \), that is \( n_{11} \). This is
because a non-zero $A$ arose by a transformation of the $g_{ik}$ which involved multiplying the Galilean $g_{ij}$ in the original frame by $V$. Since we have made the tentative inference that the Galilean $g_{ij}$ are related to world gravitation it would follow that inertial mass is influenced by the whole universe in accordance with Mach's principle.

3.2 Mach's inertial frame as deduced from normal relativistic cosmology.

Mach's principle implies that the inertial frame of reference is one in which the distant astronomical objects are non-rotating. In other words inertial forces are observed only when the distant matter is rotating relative to the reference frame in question. In the derivation of the Robertson-Walker cosmological line element

$$ds^2 = dt^2 - R^2(t) \frac{(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)}{(1 + kr^2/4)^2} \quad (3.2.1)$$

we have assumed

(i) the world lines of matter form a normal congruence (that is hypersurface orthogonal) of time-like geodesics. This specifies a cosmic time, the proper time measured along these geodesics from the given hypersurface.
(ii) the hypersurfaces \( t = \text{constant} \) are isotropic, which implies that these 3-spaces are spaces of constant curvature.

A transformation \( r' = r/(1 + kr^2/4) \) then puts the line element (3.2.1) in the form (after dropping primes)

\[
ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{3.2.2}
\]

where \( k = 0, \pm 1, -1 \) for space of uniform zero, positive, and negative curvature respectively.

Now consider a transformation

\[
l = r R(t), \quad \tau = t + \frac{1}{2} \frac{\dot{R}}{R} \xi^2 \tag{3.2.3}
\]

Then correct to the order \( \xi^2 \), (3.2.2) becomes:

\[
ds^2 = (1 - \xi^2 \frac{\ddot{R}}{R}) \, dr^2 - \frac{d\xi^2}{(1 - \frac{\dot{R}^2}{R^2} - \xi^2 \frac{\dot{R}}{R^2})} - \xi^2 \, d\theta^2 - \xi^2 \sin^2 \theta d\phi^2 \tag{3.2.4}
\]

For small \( \xi \) we have approximately

\[
ds^2 = d\tau^2 - (d\xi^2 + \xi^2 d\theta^2 + \xi^2 \sin^2 \theta d\phi^2) \tag{3.2.5}
\]

which is the metric of special relativity. This means that the coordinates \( \xi, \theta, \phi, \tau \) represent a reference frame which is locally inertial. Furthermore the coordinates \( \theta, \phi \) to be associated with any particular galaxy are independent of \( \tau \), since \( \theta, \phi \) remain unchanged by
the transformation, and so the distant galaxies are non-rotating relative to this locally inertial frame.

This is in full agreement with the expectations of Mach's principle, but this result was obtained only because we introduced the two postulates of relativistic cosmology extraneous to general relativity. The transformed line element (3.2.4) is exact when \( R(t) = e^{Ht} \) and \( k = 0 \), in this case it will be the static form of the de Sitter line element.

Suppose we measure the speed of rotation of the earth by two independent methods, a dynamical determination with say a Foucault pendulum which measures the speed of rotation relative to a locally inertial frame, and an astronomical determination relative to the 'fixed' stars. It has been verified experimentally that these two methods give the same result, implying that the locally inertial frame is one in which the distant astronomical objects are non-rotating.

We have shown above that this coincidence follows from normal relativistic cosmology. One now asks whether general relativity can explain this coincidence, and we will see in the next section that the introduction of Hoyle's C-field into general relativity does improve the situation.

Further acceptable results in terms of Mach's principle, in the sense that the local inertial frame seems neatly tied to distant matter, have been put forward by Pachner (1963). Although
considered in the classical approximation, Pachner showed that, a reference frame at rest relative to the expanding system of galaxies, and whose origin coincides with the centre of gravity of a galaxy or galaxy cluster, is dynamically preferred for the non-existence of Centrifugal and Coriolis forces.

3.3 The creation field.

We consider the field equations of general relativity.

\[ R_{ij} - \frac{1}{2} g_{ij} R = - \kappa T_{ij} \]  (3.3.1)

Mach's principle asserts that the inertial field is determined by the distribution of matter in the universe. Keeping in mind the analysis of section (3.2), it would be interesting to know if general relativity would lead uniquely to the line element (3.2.1) given the distribution of matter in co-moving coordinates, viz

\[ T^{ij} = (\rho + p) \frac{dx^i}{ds} \frac{dx^j}{ds} - pg^{ij} \]

where

\[ \frac{dx^i}{ds} = \delta^i_4 \]  (3.3.2)

Unfortunately general relativity will not satisfy this requirement, the Godel solution is an exact solution of the field
equations with a matter distribution given by (3.3.2), which is fundamentally different from the line element (3.2.1) since it represents a rotation of distant matter relative to a locally inertial reference frame. (see section (4.1)).

It is clear that, in the framework of general relativity, we will have to define some boundary conditions on the metric tensor. This is the initial value, or Cauchy, problem for a system of equations of motion. Given a three-dimensional hypersurface imbedded in the space-time manifold, and knowing the dynamical situation and the metric tensor, including the quantities $\partial g_{ij}/\partial x^k$, $\partial^2 g_{ij}/\partial x^k \partial x^l$, on the initial hypersurface, one can calculate the dynamical situation and the metric tensor off the initial hypersurface. We could choose these initial conditions, consistently with (3.3.1), so that the line element turns out to be of the form (3.2.1).

One usually supposes that these initial conditions were imposed at the origin of the universe. However was it just by chance that these initial conditions leading to (3.2.1) were chosen, since it seems equally likely that the Godel solution should have turned up. Hence as far as general relativity is concerned, it appears as though it is just coincidence that the locally inertial reference frame turns out to be the one in which the distant astronomical objects are non-rotating.

In order to dispense with the need for initial conditions,
the C-field was introduced into general relativity (Hoyle 1948, 1949). By modifying the field equations with this creation field, the solution for the inertial field tends to the steady-state solution as $t \to \infty$.

The modified field equations are

$$R_{ij} - \frac{1}{2} g_{ij} R = -\kappa \left[ T_{ik} - f(C_i^i - \frac{1}{2} g_{ik} C^k_i) \right]$$  \hspace{1cm} (3.3.3)

where $C_i^i = \partial C / \partial x^i$. Of course this modification implies that the conservation law $T_{ik}^{ik} = 0$ must be dropped in order to allow for creation or destruction of matter. We also have

$$T_{ik}^{ik} = f \frac{C_i^i}{C_k^k}$$  \hspace{1cm} (3.3.4)

and

$$C_i^i = \frac{j_i^i}{f}$$  \hspace{1cm} (3.3.5)

where $j_i^i = \rho dx^i / ds$ the mass current, and $f$ is a coupling constant.

For the Robertson-Walker line element (3.2.2) we get

$$C_i^i = \dot{C} + 3 \frac{\dot{R}}{R} \dot{C}$$  \hspace{1cm} (3.3.6)

$$j_i^i = \dot{\rho} + 3 \frac{\dot{R}}{R} \rho$$  \hspace{1cm} (3.3.7)

$$T_{ik}^{ik} = (\dot{\rho} + 3 \frac{\dot{R}}{R} \rho) \delta_i^i$$  \hspace{1cm} (3.3.8)

where we have assumed that the C field is a function of $t$ only in the homogeneous isotropic case so that $C_i^i = (0, 0, 0, \dot{C})$. We also have...
\[ R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = -(2\ddot{R} + \dot{R}^2 + k) \delta^\mu_\nu \]

where \( \mu, \nu = 1, 2, 3 \)

and

\[ R^4_4 - \frac{1}{2} R = -\frac{3(R^2 + k)}{R^2} \]

The equations (3.3.3) with (3.3.9) give

\[ 3\left(\frac{\dot{R}^2 + k}{R^2}\right) = \kappa(\rho - \frac{1}{2} f \dot{C}^2) \]  

(3.3.10)

\[ \frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = \frac{1}{2} \kappa f \dot{C}^2 \]

Equations (3.3.4) and (3.3.5) give respectively

\[ \dot{C}(\ddot{C} + 3 \frac{\dot{R}}{R} \dot{C}) = \frac{1}{T} (\dot{\rho} + 3 \dot{\frac{R}{R}} \rho) \]  

(3.3.11)

\[ \ddot{C} + 3 \frac{\dot{R}}{R} \dot{C} = \frac{1}{T} (\dot{\rho} + 3 \dot{\frac{R}{R}} \rho) \]  

(3.3.12)

From (3.3.11), (3.3.12) we see that \( \ddot{C} = 1 \) if \( C_{i i} = 0 \), and so

\( C = t \) for all \( t \). Hence the hypersurfaces \( t = \) constant are just the surfaces \( C = \) constant. Now (3.3.12) leads to

\[ \rho = f(1 - \frac{A}{R^3}) \), \text{ A constant} \]  

(3.3.13)

and adding the two equations (3.3.10) we have when \( k = 0 \)

\[ \frac{2}{3R} \frac{d^2}{dt^2} (R^3) = \kappa \rho \]  

(3.3.14)

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combining this with (3.3.13) we obtain a solution of the type

\[ R^3 = A(1 + \cosh 3Ht) \]  

(3.3.15)

where \( H = \sqrt{\frac{f}{6}} \).

From (3.3.15) we see that as \( t \to \infty \), \( R \to e^{\frac{3H}{6}} \), and from (3.3.13) we have \( \rho \to f \).

Hence the steady state solution follows as an asymptotic case, that is creation and expansion are in exact balance.

At present it does not seem as though the introduction of the C-field has any advantages over general relativity. Given the distribution of matter on a particular C-surface we would still have to specify the metric tensor and its derivatives on this C-surface in order to obtain the line element (3.2.1) off the initial surface.

However it is possible that the initial choice of the metric tensor is irrelevant in such a way that the resulting line element will always tend to the steady state form

\[ ds^2 = dt^2 - e^{2Ht} \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] \]  

(3.3.16)

It has been shown (Hoyle & Narlikar 1963) that if the initial conditions on the metric tensor are taken as a perturbation of (3.3.16), that is

\[ g_{\mu\nu} = -(r_{\mu\nu} + h_{\mu\nu}) e^{2Ht} \]  

(3.3.17)

\[ g_{4\mu} = 0 \quad g_{44} = 1 + h_{44} \]
where we neglect squares and products of the $h_{ij}$ and those of its derivatives, then the perturbations gradually die away so that the line element (3.3.16) results. We have chosen the lines $x^\mu = \text{constant}$ to be orthogonal to the surface $C = t = \text{constant}$ so that the $g_{\mu\nu}$ terms do not appear. The solutions for the $h_{ij}$ are of the form

$$h_{\mu\nu} = a_{\mu\nu} + \beta_{\mu\nu} e^{-2Ht} + \gamma_{\mu\nu} e^{-3Ht} + \delta_{\mu\nu} e^{-5Ht}$$

$$h_{\mu\nu} = 0$$

(3.3.18)

where $a_{\mu\nu}$, $\beta_{\mu\nu}$, $\gamma_{\mu\nu}$, $\delta_{\mu\nu}$ are functions of $x^\mu$ only. As $t$ increases we have $h_{\mu\nu} \rightarrow a_{\mu\nu}$, and since the variation of the $a_{\mu\nu}$ over any given proper volume becomes smaller and smaller because of expansion, the $h_{\mu\nu}$ effectively tend to zero since the constant values of the $a_{\mu\nu}$ can be absorbed trivially into the $x^\mu$.

Since squares and higher powers of the $h_{ij}$ were neglected we cannot say, for arbitrarily chosen initial conditions, whether the line element will tend to the steady state form or not. However we can conclude for this restrictive case that the introduction of the C-field into general relativity allows for the continuous creation of matter in such a way that any initial anisotropy or inhomogeneity is 'smoothed' away over any specified proper volume.

The fact that the inertial frame is one in which the
distant galaxies are non-rotating, could be explained by this modified form of general relativity since the resulting line element may be independent of the initial conditions imposed at the origin of the universe.
Chapter 4.

4.1 Godel's universe.

It was mentioned that the Godel universe was a solution of the field equations fundamentally different from (3.2.1), since the matter in this universe is in a state of absolute rotation.

Consider the metric

$$ds^2 = dt^2 + 2e^{\alpha x^1} dt \, dx^2 - (dx^1)^2 + \frac{1}{2} e^{2\alpha x^1} (dx^2)^2 - (dx^3)^2 \quad (4.1.1)$$

We will show that this metric is an exact solution of Einstein's field equations compatible with an incoherent matter distribution (Godel 1949).

The non-zero Christoffel symbols are

$$\left\{ \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right\} = \frac{1}{2} e^{2\alpha x^1}, \left\{ \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \right\} = \frac{1}{2} e^{\alpha x^1}, \left\{ \begin{array}{c} 2 \\ 4 \end{array} \right\} = -\alpha e^{-\alpha x^1}$$

$$\left\{ \begin{array}{c} 4 \\ 1 \\ 2 \end{array} \right\} = \frac{1}{2} e^{\alpha x^1}, \left\{ \begin{array}{c} 4 \\ 1 \\ 4 \end{array} \right\} = \alpha$$

and for the non-zero components of the Ricci tensor

$$R_{22} = -\alpha^2 e^{2\alpha x^1},$$

$$R_{24} = R_{42} = -\alpha e^{\alpha x^1}$$

$$R_{44} = -\alpha^2.$$
We then obtain for the curvature invariant

\[ R = g^{ij} R_{ij} = -\alpha^2 \]

For an incoherent matter field in comoving coordinates we have the energy-stress-momentum tensor \( T_{ij} \) given by (3.3.2) and so

\[
T_{ij} = \begin{pmatrix}
  p & 0 & 0 & 0 \\
  0 & (\rho + \frac{1}{2} \rho) e^{2\alpha x^1} & 0 & \rho e^{\alpha x^1} \\
  0 & 0 & p & 0 \\
  0 & \rho e^{\alpha x^1} & 0 & \rho
\end{pmatrix}
\] (4.1.2)

Including the cosmological constant \( \Lambda \) in the field equations

\[
R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = -\kappa T_{ij} \] (4.1.3)

and neglecting the cosmic pressure \( \rho \) in (4.1.2) we obtain the relations

\[
\Lambda = -\frac{\alpha^2}{2} , \quad \alpha^2 = \rho \kappa \] (4.1.4)

If we assume the cosmological constant to be zero, then we cannot neglect the pressure \( \rho \), and so (4.1.2), (4.1.3) lead to

\[
\rho = p , \quad \kappa p = \frac{\alpha^2}{2} \] (4.1.5)

The important property of Godel's metric is that the world lines which characterize matter at rest in the co-moving system \((x^1, x^2, x^3, t)\) cannot be everywhere orthogonal to a one-parameter family of 3 dimensional hypersurfaces. Consider a family \( \mathcal{F} \) of
hypersurfaces which are parametrized by \( \lambda \) and which have equations of the form

\[ F(x^i) - \lambda = 0 \quad (4.1.6) \]

One specific normal vector to the member of \( \mathcal{F} \) which contains the world point \( x^i \) is \( \frac{\partial F}{\partial x^i} (x^i) \). Thus any arbitrary vector field \( u_i \) which is everywhere orthogonal to the members of \( \mathcal{F} \) may be written as

\[ u_i = \ell \frac{\partial F}{\partial x^i} \quad (4.1.7) \]

where \( \ell \) is an arbitrary scalar function.

Now consider the asymmetric tensor \( \omega_{ik} \) defined by

\[ \omega_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x^k} - \frac{\partial u_k}{\partial x^i} \right) \quad (4.1.8) \]

The components of this tensor will vanish if \((4.1.7)\) is satisfied. In other words if the world lines of matter are everywhere orthogonal to a family of hypersurfaces the tensor \( \omega_{ik} \) must vanish identically. For the Godel solution

\[ u^i = \frac{dx^i}{ds} = (0, 0, 0, 1) \]

and so

\[ u_1 = g_{1k} u^k = g_{11} = (0, e^{\alpha x^1}, 0, 1) \quad (4.1.9) \]

Now

\[ \frac{\partial u_i}{\partial x^k} = \begin{cases} \alpha e^{\alpha x^1} & i = 2, k = 1 \\ 0 & \text{otherwise} \end{cases} \]
hence the non-zero components of $w_{ik}$ are

$$w_{12} = -\frac{1}{2}ae^{x_1}, \quad w_{21} = -w_{12} \quad (4.1.10)$$

Thus the world lines of matter are not orthogonal to the hypersurfaces $t = \text{constant}$.

We see that for an incoherent matter distribution given by (3.3.2) we have two basically different solutions for the field equations, the Godel solution given by (4.1.1), and the Robertson-Walker metric given by (3.2.1). This means that given the distribution of matter, the field equations do not lead to a unique geometry as required by Mach's principle.

Defining the scalar angular velocity $w$ as

$$2w^2 = g^{ik}g^{ln}w_{ik}w_{ln} \quad (4.1.11)$$

we obtain from (4.1.10):

$$w = \frac{a}{\sqrt{2}}.$$  

From the definition of the three-vector $\mathbf{w}$:

$$\mathbf{w} = (w_{23}, w_{31}, w_{12})$$

we have

$$\mathbf{w} = (0, 0, -\frac{1}{2}e^{x_1}) \quad (4.1.12)$$

Hence the co-moving matter of the Godel universe possesses a constant intrinsic angular velocity $w = a/\sqrt{2}$ about the $x^3$ axis.

This rotation manifests itself in the motion of test bodies in the Godel universe. If we shoot a test particle from the origin along...
the $x^1$-axis in a nonrotating universe, for example a Friedmann universe, the particle will continue to move along the $x^1$-axis. However in the Godel universe the particle will spiral outward relative to the matter in the universe. If we have the initial conditions

$$x^4 = 0, \frac{dx^4}{ds} = 1, x^1 = 0, \frac{dx^1}{ds} = \beta$$

(4.1.13)

$$x^2 = 0, \frac{dx^2}{ds} = 0, x^3 = 0, \frac{dx^3}{ds} = 0$$

Then in a non-rotating universe we expect a solution of the form

$$x^i(s) = (\beta s, 0, 0, s)$$

However for the Godel universe we obtain, correct to first order in $\alpha$, the solution: $x^i(s) = (\beta s, \alpha s^2, 0, s - \alpha s^2)$, hence the particle deviates from the ray $x^2 = x^3 = 0$ and spirals outward.

Another property of Godel's metric is that there exist closed, everywhere time-like world lines, which means that it is possible for a person to travel into his own past. Further from (4.1.5) we see that if $A = 0$ then it is necessary that $\rho = p$, which we know is unreasonable from observation. On physical grounds the Godel universe could be rejected, however it is an exact solution of the field equations not satisfying Mach's principle, and so it seems as though general relativity will have to be supplemented with some global conditions if it is to incorporate Mach's principle.
Another spatially homogeneous 'rotating' solution of the field equations has been found, this solution differing from the Godel solution in that closed time-like world lines do not exist. (Ozsvath & Schucking 1962). For this solution the 3-space \( t = \text{constant} \) is closed.

4.2 An empty space solution.

We now consider a solution of Einstein's field equations for which the geometry of space is curved, but in which there is no matter (Ozsvath & Schucking 1962).

For the metric

\[
ds^2 = dt^2 + 2t^2(dx^3)^2 + (dx^1)^2 + 2dx^2dx^3 - 4tdx^1dx^3 \quad (4.2.1)
\]

the non-zero Christoffel symbols are

\[
\begin{pmatrix}
1 \\
3 \\
4
\end{pmatrix}
= -1 \quad \begin{pmatrix}
2 \\
1 \\
4
\end{pmatrix}
= -1
\]

\[
\begin{pmatrix}
4 \\
1 \\
3
\end{pmatrix}
= 1 \quad \begin{pmatrix}
4 \\
3 \\
3
\end{pmatrix}
= -2t
\]

and the non-zero components of the curvature tensor are

\[
R_{43\bar{3}4} = 1, \quad R_{31\bar{1}3} = -1.
\]

Every component of Ricci tensor is zero

\[
R_{ij} = 0 \quad \text{all } i, j \quad (4.2.2)
\]
From the field equations

\[ R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij} \]

we see that if (4.2.2) holds then \( T_{ij} = 0 \) all \( i, j \).

Hence we have an empty space solution of the field equations, for which the geometry is curved, since there exist non-zero components of the curvature tensor. The solution is free of singularities.

Clearly this solution is not compatible with Mach's principle since it implies that a test particle in an otherwise empty universe has inertial properties. If the cosmological constant is included in the field equations then (4.1.3) gives for \( R_{ij} = 0 \)

\[ -\kappa T_{ij} = \Lambda g_{ij} \quad (4.2.3) \]

so that all the components of the energy-stress-momentum tensor would not be zero.

Einstein first introduced the cosmological constant into general relativity in the hope that the field equations would have no solution for the \( g_{ij} \) when \( T_{ij} = 0 \). However the de Sitter solution

\[ ds^2 = (1 - r^2/R^2) dt^2 - \frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]

is a well-known vacuum solution of the field equations provided

\[ \frac{1}{R^2} = \frac{1}{3} \Lambda \]
So the introduction of the $A$ term turned out to be a failure in this respect. Whether the cosmological constant is included or not, the field equations still possess solutions in the absence of matter which is not in accordance with Mach's principle.

However Pachner (1963) has suggested, that provided we accept that the vanishing of the matter tensor does not signify the absolute absence of matter, the existence of centrifugal and Coriolis force fields appearing in the case when a single material body is rotating in an infinite absolutely empty space, is explained by the hypothesis that this empty space-time is to be considered as a Minkowski universe. The Minkowski universe is a world model with infinite total mass but vanishing mean mass density. Other exact solutions of the field equations, such as that of Ozsvath & Schucking discussed in this section, with vanishing matter tensor, which are free of singularities are then to be considered as "self-excited" states of the Minkowski universe.
Chapter 5.

5.1 Inertial mass in an expanding universe.

A consequence of Mach's principle was that inertial mass was not an intrinsic property of matter but was determined by the matter in the universe and its distribution. Therefore we would expect that in a homogeneous, isotropic, expanding universe the inertial mass of a particle would depend on time (Rosen 1965).

We consider a homogeneous closed model of the universe which is described by the line element

\[ ds^2 = dt^2 - \frac{R^2(t)}{(1 + r^2/4R_0^2)^2} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \]  

(5.1.1)

in co-moving coordinates.

In the neighbourhood of the space origin we neglect the curvature so that (5.1.1) can be written as

\[ ds^2 = dt^2 - R^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \]  

(5.1.2)

The geodesic equations of motion of a test particle are

\[ \frac{d}{ds} \left( R^2 \frac{dx^\alpha}{ds} \right) = 0 \quad \alpha = 1, 2, 3 \]  

(5.1.3)

\[ \frac{d^2 t}{ds^2} + R \frac{d}{ds} \left( \frac{dx}{ds} \right)^2 = 0 \]  

(5.1.4)
where \((\frac{dx}{ds})^2 = (\frac{dx^1}{ds})^2 + (\frac{dx^2}{ds})^2 + (\frac{dx^3}{ds})^2\)

\[= \left(\frac{dt}{ds}\right)^2 - 1]/R^2\]

So (5.1.4) can now be written in the form

\[R \frac{d^2 \tau}{ds^2} + \dot{R} \left(\frac{dt}{ds}\right)^2 = \ddot{R}\]

or

\[\frac{d}{ds} \left( R \frac{dt}{ds} \right) = \ddot{R} \quad (5.1.5)\]

Multiplying both sides of (5.1.5) by \(R \frac{dt}{ds}\) we obtain

\[\frac{d}{ds} \left( R \frac{dt}{ds} \right)^2 = \frac{d}{ds} (R^2) \quad (5.1.6)\]

and integrating:

\[R^2 \left(\frac{dt}{ds}\right)^2 = R^2 + A \quad (5.1.7)\]

where \(A\) is a constant.

Now according to (5.1.2), at a given instant of time an observer will measure a distance \(R x^a\) corresponding to a coordinate differential \(dx^a\). We now introduce 'observers' coordinates

\[\bar{x}^a = R x^a, \quad \bar{r} = R r \quad (5.1.8)\]

where \(r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2\).

In other words the proper distance to the particle
(x^1, x^2, x^3) measured in the simultaneity of fundamental observers at cosmological time t from r = 0 is Rr. We must remember that we are restricting ourselves to a neighbourhood of the space origin where space is approximately flat. If we take into account the curvature of space the corresponding proper distance would be

\[ R(t) \int_0^r \frac{dr}{1 + r^2/4R_0^2} \]

We further define the observer's 4-velocity

\[ \overline{u}^a = R \frac{dx^a}{ds}, \quad \overline{u}^t = \frac{dt}{ds} \tag{5.1.9} \]

which satisfy the relation

\[ \eta_{ik} \overline{u}^i \overline{u}^k = 1 \]

We define the 4-momentum P^k of a free particle

\[ P^k = m \overline{u}^k \tag{5.1.10} \]

where

\[ m = m_0 R(t) \tag{5.1.11} \]

m_0 being a constant.

From these definitions equation (5.1.3) gives

\[ \frac{d}{ds} (P^\alpha) = 0 \quad \alpha = 1, 2, 3 \tag{5.1.12} \]

This equation implies the conservation of momentum P^\alpha for a free particle. It is therefore natural to regard m as the inertial mass.
of the particle, and (5.1.11) shows that the inertial mass is time
dependent and actually increases in an expanding universe.

From (5.1.5) we have

$$\frac{d}{ds} (P^4) = \frac{cm}{ct}$$  \hspace{1cm} (5.1.13)

and (5.1.7) gives

$$P^4 = (m^2 + Am^2_0)^{\frac{1}{2}}$$  \hspace{1cm} (5.1.14)

Regarding $P^4$ as the energy of the particle we see
that energy is not conserved in a time dependent field.

Since in an expanding universe the matter distribution will change with time, Mach's principle will require the inertial mass to depend on time as we have shown in (5.1.11).

5.2 Inertial mass of a particle in the gravitational field
of a massive body superposed on the expanding universe.

The motion of a particle in the field of a massive body superposed on the expanding universe, is described by McVittie's metric. This metric is effectively the Schwarzschild solution for the field of a body of gravitational mass $M_0$ expressed in cosmical coordinates.

(McVittie 1933).
\[ ds^2 = \left[ \frac{1 - \mu(t)/2r_0 \cdot (1 + r^2/4R_0^2)^{1/2}}{1 + \mu(t)/2r \cdot (1 + r^2/4R_0^2)^{3/2}} \right] dt^2 + \left[ \frac{1 + \mu(t)/2r \cdot (1 + r^2/4R_0^2)^{3/2}}{1 + r^2/4R_0^2} \right]^4 R^2((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \]  

Neglecting the curvature of space locally we have

\[ ds^2 = \left( \frac{1 - \mu/2r}{1 + \mu/2r} \right)^2 dt^2 - R^2(1 + \mu/2r)^4((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \]  

(5.2.2)

where \( \frac{\mu}{\dot{R}} = -\frac{1}{2} \frac{\dot{R}}{r} \)

and \( M_0 = \mu(t) R(t) \).

At large distances away from the body (5.2.1) reduces to the Robertson-Walker form (5.1.1) and near the body (5.2.1) reduces to the isotropic Schwarzschild form if we transform to static coordinates.

Correct to first order in \( \mu/r \) and second order in the spatial coordinate velocities \( dx^a/dt \), the geodesic equations corresponding to the metric (5.2.2) are:

\[ \frac{d}{ds} \left( R^2 \frac{dx^a}{ds} \right) + \frac{\mu}{r^3} x^a (\frac{dt}{ds})^2 = 0 \]  

(5.2.4)

From the metric (5.2.2) we obtain, to the correct order of approximation
\[ \left( \frac{dc}{ds} \right)^2 = 1 + \frac{2m}{r} + \frac{r^2}{R} \left( \frac{dx}{dt} \right)^2 \]  

(5.2.5)

where

\[ \left( \frac{dx}{dt} \right)^2 = \delta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \]

and so (5.2.4) becomes

\[ \frac{d}{ds} (m_0 R^2 \frac{dx^\alpha}{ds}) = -\frac{m_0 M_0 x^\alpha}{R r^3} \]  

(5.2.6)

where \( m_0 \) is a constant.

Using observer's coordinates as defined in (5.1.8) and the observer's 4-velocity as in (5.1.9) we can rewrite (5.2.6) as

\[ \frac{d}{ds} (m_0 R^2 \frac{dx^\alpha}{ds}) = -\frac{m M_0 x^\alpha}{R r^3} \]  

(5.2.7)

where again

\[ m = m_0 R \]  

(5.2.8)

We see that both inertial mass and passive gravitational mass are given by (5.2.8), and so vary with time in the same way, in accordance with the equivalence principle. Hence inertial mass is dependent on the mass distribution in accordance with Mach's principle.

We note that one might have defined inertial mass differently. For example we might have defined the 4-momentum \( p^k \) by

\[ p^k = m_0 \frac{dx^k}{ds} \]

instead of (5.1.10), but this would not lead to conservation of momentum in a spatially homogeneous universe, which is not in accordance with the outlook of modern physics.
6.1 The field equations in integral form.

Four dimensional Green's functions play an essential role in the propagation of action not only in Minkowski space-time but also in curved space time. We propose to rewrite Einstein's equations in integral form by means of retarded bi-tensor Green's functions. (Lynden-Bell 1967). Since Mach's principle requires the total inertial field, the $g_{ik}$, to be determined by the distribution of matter in the universe, we will write the field equations in integral form in such a way, that the metric tensor is expressed in terms of the energy-stress-momentum tensor.

We would expect inertia at a point to be expressed as a sum over its sources. Since the sources are represented by the energy-stress-momentum tensor, we cannot sum this tensor at different points as the sum would not be a tensor anywhere. However we can get over this problem by the use of bi-tensors of parallel geodesic transport (Synge). Consider two points $P(x)$ and $Q(y)$, and the geodesic curve joining them. The vector $\lambda^a$ at $Q$ determines by parallel transport the vector $\lambda^i$ at $P$, and since the $\lambda^i$ are linear homogeneous functions of $\lambda^a$ we may write

$$\lambda^i = \bar{g}_{ia} \lambda^a \quad (6.1.1)$$
(We have let Latin indices $i$, $k \ldots$ refer to the point $x$, and Greek letters $\alpha$, $\beta \ldots$ refer to the point $y$).

The two-point tensor $\tilde{g}_{ia}(x,y)$ is called the parallel propagator, and is only dependent on the points $P$ and $Q$. In the limit as $Q$ tends to $P$ the parallel propagator becomes the ordinary metric tensor at the point $P$.

Hence in order to obtain the sum tensor at the point $x$, the tensors $T_{\alpha\beta}(y)$ are transported parallelly to $x$ along the geodesic joining $x$ and $y$ becoming tensors at $x$, and then the integral tensor at $x$ can be formulated.

However the field equations are non-linearly dependent on the sources $T_{ij}$, so we should not expect the inertia at a point to be just a sum over its sources. But the propagator of inertial influence is dependent on the geometry and so is implicitly dependent on the sources, since the geometry is curved by the matter in it. Hence the inertia integral, although explicitly a linear sum over the sources is implicitly non-linear, being dependent on the sources a second time via the geometry. These ideas are incorporated into Hoyle and Narlikar's theory of gravitation (section 6.3), by assuming that inertial effects add linearly when they are considered as propagated over the space-time geometry.

We first define the Einstein operator $E_{ik}^{mn}$ which acts on the symmetric potentials $\phi_{mn}$:
where
\[ A_{ik} \phi_{mn} = \frac{1}{2} g_{(i} g_{k)} q_{pq} \] \hspace{1cm} (6.1.3)

and
\[ E_{ik} = g_{(i} g_{k)} - \frac{1}{2} g_{ik} R_{ik} - \frac{1}{2} g_{ik} \phi_{mn} + \frac{1}{4} R_{ik} \phi_{mn} \] \hspace{1cm} (6.1.4)

Round brackets denote symmetrization in the sense
\[ Q(a\beta) = \frac{1}{2} (Q_{a\beta} + Q_{a\beta}) \]

Now
\[ E_{ik} (- g_{mn}) = R_{ik} - \frac{1}{2} g_{ik} R \] \hspace{1cm} (6.1.5)

therefore
\[ E_{ik} (- g_{mn}) = R_{ik} - \frac{1}{2} g_{ik} R = - \kappa T_{ik} \] \hspace{1cm} (6.1.6)

hence \(- g_{mn}\) is a solution of the differential equation
\[ E_{ik} \phi_{mn} = - \kappa T_{ik} \] \hspace{1cm} (6.1.7)

The retarded solution of (6.1.7) with \( T_{ik} \) regarded as the source of the potential \( \phi_{mn} \) is
\[ \phi_{mn}(x) = - \kappa \int G_{mn}^{a\beta}(x,y) T_{a\beta}(y) \sqrt{- g(y)} \, d^4 y \] \hspace{1cm} (6.1.8)

where \( G_{mn}^{a\beta}(x,y) \) is a two-point tensor Green's function, or propagator,
which satisfies

$$E_{ik}^{mn}(x) G_{mn}^{ab}(x,y) = \frac{\alpha}{g_i^a g_k^b} \delta^i(x-y)/\sqrt{g} \tag{6.1.9}$$

with the retarded condition that $G_{mn}^{ab}(x,y)$ is zero whenever the point $x$ lies outside the future light cone of the point $y$.

Now $g_{mn}$ is a solution of (6.1.7) and so we must have

$$g_{mn}(x) = \kappa \int G_{mn}^{ab}(x,y) T_{ab}(y)/\sqrt{g(y)} \, d^4y \tag{6.1.10}$$

the domain of integration is over all points belonging to the past of time-like or null geodesics through $x$.

This is just the interpretation of Mach's principle we require. Equation (6.1.10) shows how the metric tensor at a given point in space-time is caused by the pieces of matter in it. The Green's function effectively propagates the inertial influence of matter over the space-time geometry to the point $x$.

To determine which universes satisfy the Mach's principle (6.1.10), for example the Robertson-Walker type, one could write out the Einstein operator for this universe, solve (6.1.9) for the Green's function, and then see whether (6.1.10) is satisfied.

6.2 Machian boundary conditions.

We now formulate boundary conditions arising from (6.1.10) which enable us to test whether a particular cosmological model...
is compatible with the formulation of Mach's principle given in section (6.1). (Al'tshuler 1967).

Now from equation (6.1.9) we have

\[ \sum_{ik} A_{ik} g^a \Delta_{mn;pq} g^{ik} + B_{ik} g^a \Delta_{mn} g^{ik} = g^{ik} g_{ik} \delta^a \delta^b (x - y) \sqrt{-g} \]  

(6.2.1)

but since

\[ B_{ik} g^{ik} = -R_{mn} + \frac{1}{2} g_{mn} = \kappa T_{mn} \]

we have

\[ \int A_{ik} g^a \Delta^{mn;pq} g^{ik} \sqrt{-g(x)} \, d^4x + \int T_{mn} g^a \Delta^{mn} \sqrt{-g(x)} \, d^4x = \int g^a (y) \delta^4(x - y) \, d^4x \]

Using (6.1.10) we then obtain

\[ \int A_{ik} g^a \Delta_{mn;pq} g^{ik} \sqrt{-g(x)} \, d^4x = 0 \]  

(6.2.2)

and by (6.1.3) this reduces to:

\[ \int \left[ \frac{g_{pq}(x) G_{mn;p}^{a b}(x,y)}{\varepsilon} \right] \sqrt{-g(x)} \, d^4x = 0 \]  

(6.2.3)

Now the integral (6.2.3), which is an integral over a 4-volume can be transformed into an integral over a hypersurface by means of Gauss' theorem. We then have

\[ \int g_{pq}(x) G_{mn;p}^{a b}(x,y) \sqrt{-g(x)} \, ds_q = 0 \]  

(6.2.4)
If we choose the surface of integration to be the hypersurface \( t = \text{constant} \), then (6.2.4) becomes

\[
\int g^{\mu
\nu}(x) \mathcal{G}^{\alpha \beta}_{\mu \nu}(x,y) \sqrt{-g(x)} \, d^3x = 0 \quad (6.2.5)
\]

By pushing our space-like hypersurface \( x^4 = t = \text{constant} \) back in time we may regard (6.2.5) as a condition on the singularities at the start of the universe, or for non-singular universes as a condition on the infinite past.

Hence we have

\[
\lim \int g^{\mu
\nu}(x) \mathcal{G}^{\alpha \beta}_{\mu \nu}(x,y) \sqrt{-g(x)} \, d^3x = 0 \quad (6.2.6)
\]

for \( x^4 \to -\infty \) or at the initial time in a cosmological model.

Assuming that (6.1.10) is not satisfied we write

\[
\phi_{\mu \nu}(x) = \kappa \int \mathcal{G}^{\alpha \beta}_{\mu \nu}(x,y) T_{\alpha \beta}(y) \sqrt{-g(y)} \, d^4y \quad (6.2.7)
\]

and by (6.1.9) we therefore have

\[
\kappa_{\mu \nu \rho \sigma} \phi_{\rho \sigma} + B_{\mu \nu} \phi_{\mu \nu} = \kappa T_{\mu \nu} \quad (6.2.8)
\]

We note that \( \phi_{\mu \nu}(x) = \mathcal{G}^{\mu \nu}(x) \) is the retarded inhomogeneous solution of (6.2.8).

We consider the case of a gravitational field which is static. The tensor \( \phi_{\mu \nu} \) will not depend on time, and so all the derivatives with respect to time in (6.2.8) will drop out and the condition (6.2.3) may be written as an integral over the corresponding
3-volume, which, by Gauss' theorem, may be expressed as

$$\int g^{Dq}(x) \frac{\partial \Pi_{i,j}}{\partial x^p} \sqrt{-g(x)} \, df_{4q} = 0 \quad (6.2.9)$$

the integral is over the surface inclosing the 3-volume, and

$$D_{ik}^G(x,y)$$ is the Green's function of the system (6.2.8) in 3-dimensional space. Since the covariant derivative of the Green's function in (6.2.9) is with respect to the point $$x$$, we can further write (6.2.9) as

$$\int g^{Dq}(x) \frac{1}{3x^p} \left( D_{ik}^{G}(x,y) \right) \sqrt{-g(x)} \, df_{4q} = 0 \quad (6.2.10)$$

For the Schwarzschild solution we have matter in a finite region of space, the metric is spherically symmetric and asymptotically flat. In the asymptotically flat region (6.1.9) becomes the ordinary Laplace equation, and so the components of the Green's function are proportional to $$1/r_x$$ where $$r_x = x^1$$ is the radial coordinate of the point $$x$$. Taking the integral (6.2.10) over the surface of a sphere at "infinity", we then have, due to the spherically symmetric nature of the integrand

$$\lim_{r_x \to \infty} \left[ \frac{1}{3} \left( D_{ik}^{G}(x,y) \right) g^{ij}(x) \sqrt{-g(x)} \right] = 0 \quad (6.2.11)$$

In the asymptotically flat region the metric is approximately

$$ds^2 = dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (6.2.12)$$
We also have

\[ \frac{\partial}{\partial r} \left( \frac{\partial^{m} \Phi(x,y)}{m!} \right) \sim - \frac{A}{r^2} \]

where \( A \) is a constant, and so the left-hand side of (6.2.11) becomes

\[ \lim_{r \to \infty} A \]

which is in general non-zero, so that (6.2.11) is not satisfied.

Hence the Schwarzschild metric does not satisfy our Machian analysis. This is not surprising since in the asymptotically flat region we have no matter causing this 'flatness' as Mach's principle would require.

A similar analysis shows that the steady-state metric in static coordinates is compatible with the Machian condition (6.2.10), in accordance with the tentative analysis of section (3.1). Further it has been shown that the nature of the singularity of the Friedmann universe violates our Machian boundary condition.

6.3 Action-at-a-distance theory of gravitation.

In Newtonian gravitational theory one can write down the relation between particles and fields in two different ways. We can formulate Poisson's equation which relates the Laplacian of the gravitational potential to the density of matter in space, or one can
write down directly the gravitational potential as being proportional to the sum $\Sigma m/r$, where $m$ is the mass of the particle at distance $r$ from the point where the potential is being evaluated. We have Poisson's equation

$$\nabla^2 \phi = 4\pi G \rho$$

(6.3.1)

or

$$\phi = Gm/r$$

(6.3.2)

Now (6.3.1) is a second-order differential equation for the potential and to obtain the solution (6.3.2) from it, we require boundary conditions on the potential.

Considering the similar situation in general relativity, we have the field equations corresponding to the Poisson equation in Newtonian theory, but since we do not know what the universe is like far away, it is difficult to provide suitable boundary conditions on the metric tensor in order to obtain a formulation of the type (6.3.2) for the metric tensor.

Hoyle & Narlikar (1964) have presented a $\Sigma m/r$ formulation, or particle formulation, theory of gravitation. The interest in this action-at-a-distance formulation would no doubt be encouraged by the fact that Wheeler and Feynman found an action-at-a-distance formulation of electromagnetism after Maxwell formulated the field theory of electromagnetism. The problem of a gravitational theory, however, is much more difficult since it will be a non-linear theory, whereas
Electromagnetic theory is linear, and a Dirac formulation is much easier in the linear case.

The equations of Hoyle & Narlikar's theory are derived from the principle of direct interparticle action. The action function $J$ is written as

$$J = \sum \int G(A,B) \, da \, db \tag{6.3.3}$$

The sum is over pairs of particles labelled by $a, b \ldots$. $A$ is any point on the world line of particle $a$, and has coordinates $a^i$ ($i = 1, 2, 3, 4$). The proper time at $A$ measured along the world line is given by $\tau$, where $d\tau^2 = g_{ik} \, da^i \, da^k$.

$G(A,B)$ is a scalar Green's function denoting the inertial reaction between $A$ and $B$, and satisfies the generalized wave equation

$$g^{ik} \frac{\partial^2 G(X,Y)}{\partial X^i \partial Y^k} + \frac{1}{6} R(X) G(X,Y) = -\delta^4(X - Y) / \sqrt{-g} \tag{6.3.4}$$

where the indices, $i, k$ refer to the point $X$.

We now introduce the 'mass' functions. The mass at $X$ due to world line $a$ is defined by

$$m^{(a)}(X) = -\int G(X,A) \, da \tag{6.3.5}$$

and the mass of particle $a$ at $A$ is given by

$$m_a(A) = \sum_{b \neq a} m^{(b)}(A) \tag{6.3.6}$$

hence

$$J = -\frac{1}{2} \sum_a m_a \, da \tag{6.3.7}$$
The action given by (6.3.7) is formally the same as the usual inertial contribution to the action, hence $m_a$ plays the role of inertial mass. Equation (6.3.6) is in accordance with Mach's principle in that the inertial mass of a particle is dependent on all the other particles in the universe.

The equations of motion are obtained by variation of the world lines, while the field equations are obtained by varying the geometry. In the smooth fluid approximation the field equations go over to the equations of general relativity. (Hoyle & Narlikar 1964).
Conclusion.

One could possibly separate three lines of thought in regard to the relation of Mach's principle to general relativity:

(i) Mach's principle is of philosophical content rather than of a physical nature, and therefore is relatively unimportant because it is not possible to test its validity experimentally.

(ii) Mach's principle provides boundary conditions for the field equations of general relativity, but does not directly bear on the field equations of that theory.

(iii) One must have a theory which entirely incorporates Mach's principle in every detail.

The first view, probably Mach's own outlook due to his position in philosophy, does not seem plausible, since Mach's principle does have considerable physical significance. For example the anisotropic distribution of matter in our Galaxy, relative to the solar system, should lead to a small anisotropy of inertia. A variable gravitational 'constant' implies observable effects concerning the history and present state of the solar system. It is therefore clear that Mach's principle is capable of experimental verification and is not only of philosophical importance.

The author agrees with the second school of thought, concerned with boundary conditions. This means that general relativity is alright and does not need to be modified. After one embeds general relativity
into a model universe, corresponding to our own, then Mach's principle will emerge. In Gursey's reformulation of general relativity (chapter 2) we stated boundary conditions in a coordinate system in which the cosmological background is described by a conformally flat metric, and the modified field equations (2.2.8) enabled us to find solutions which were previously not physically acceptable. For instance the Schwarzschild solution now becomes 'degenerate' at infinity rather than flat. We also found that inertial mass is dependent on the cosmological structure and that the origin of local inertial forces is due to the gravitational interaction of distant matter. The scalar density $\Phi$ (defined in (2.1.8)), will change under a coordinate transformation, and hence will change when there is a redistribution of matter in the universe. The new equations, in terms of the tensor density $\gamma_{ik}$, become linear even for large $g_{ik}$ provided deviations from conformal flatness are small, whereas in Brans and Dicke's theory the D'Alembert operator (equation (2.7.7)) is associated with the metric tensor $g_{ik}$ and only becomes linear in the weak field case.

Certainly there is no theory, as yet, which can claim to satisfy the third viewpoint. General relativity is not acceptable, since in the absence of matter the metric tensor describes a flat space which therefore possesses inertial properties, and further the Godel solution is not acceptable to Mach's principle because matter, in this model, is in a state of absolute rotation. There is no reason why one should
Discriminate, in our universe, between Hoyle and Narlikar’s many-particle theory and general relativity, just because general relativity follows from their theory in the smooth fluid approximation.

Traces of Mach’s principle appear all through general relativity. Although restricted to the weak field approximation, Davidson’s work attributes the origin of inertial forces to an inductive effect of distant matter, as in Sciama’s vector theory of inertia. This provides a simple physical picture for the origin of the Newtonian fictitious force fields. Hoyle’s scalar creation field appears favourable to the relation between Mach’s principle and the steady-state theory, as does Davidson’s work (section 3.1) in the framework of general relativity.

Mach’s principle does not really explain why there is a fundamental distinction between uniform and accelerated motion. Inertial effects arise from interactions with distant matter, but we do not know why this interaction is velocity independent and further, why various states of uniform motion, relative to the distant matter, are indistinguishable while various states of accelerated motion, relative to them, are distinguishable. Newton’s ‘absolute’ space has therefore only been replaced by ‘fixed’ stars.

However, any theory requires more than a set of differential equations. For complete specification one requires initial-value conditions or boundary-value conditions. The initial condition imposed by the integral form of Einstein’s field equations (chapter 6) is too
restrictive, since it is violated by the nature of the singularity of the Friedmann universe. The boundary conditions imposed by Gursey's reformulation of general relativity, appear to exhibit Mach's principle in its strong form in a very satisfactory manner. Inertia must be closely linked with cosmology, and the problems of cosmology are difficult from both the philosophical and physical points of view, because our universe is so unique.
Appendix.

1. General solution of the field equations for the case of a weak non-stationary field.

We consider a weak gravitational field, produced by arbitrary bodies. We have

$$g_{ij} = h_{ij} + h_{ij} \quad (1.1)$$

where the $n_{ij}$ are the metric coefficients of special relativity, and where we neglect squares and products of the $h_{ij}$ and those of its derivatives. Let Latin letters denote space-time indices 1, 2, 3, 4, and Greek letters denote space coordinates 1, 2, 3.

Writing Einstein's field equations in the form

$$R_{ij} = -\kappa(T_{ij} - \frac{1}{2} g_{ij}T) \quad (1.2)$$

to the specified approximation we find

$$R_{ij} = \frac{1}{2} \kappa \frac{\delta^2 h_{ij}}{\delta x^k \delta x^l} \quad (1.3)$$

provided that

$$\frac{\partial}{\partial x^k} (h_{ij} \frac{\partial}{\partial x^j} - \frac{1}{2} \delta^2_{ij} h) = 0 \quad (1.4)$$

Combining (1.2) and (1.3) we have

$$h_{ij} = 2\kappa(T_{ij} - \frac{1}{2} g_{ij}T) \quad (1.5)$$
which has the solution

$$h_{ij} = -\frac{\kappa}{2\pi} \int \frac{[T_{ij} - \frac{1}{2} \eta_{ij} T]}{r} \, dV \quad (1.6)$$

The integrand is taken over a finite distribution of matter in the neighbourhood of the space origin. The conservation equations $T_{ij} = 0$ ensure that (1.4) is satisfied to the correct order of approximation.

Assuming the stress components of the energy momentum tensor to be small compared with the density and momentum components we have

$$T_{ij} - \frac{1}{2} \eta_{ij} T = \begin{pmatrix} \rho/2 & 0 & 0 & -\rho u^1 \\ 0 & \rho/2 & 0 & -\rho u^2 \\ 0 & 0 & \rho/2 & -\rho u^3 \\ -\rho u^1 & -\rho u^2 & -\rho u^3 & \rho/2 \end{pmatrix}$$

Then from (1.6) we get

$$h_{ii} = -\frac{\kappa}{4\pi} \int \frac{[\rho]}{r} \, dV$$

$$h_{4\alpha} = \frac{\kappa}{2\pi} \int \frac{[\rho u^\alpha]}{r} \, dV$$

$$h_{\alpha\beta} = 0 \quad \alpha \neq \beta$$

where $\rho$ is the mass density, $u^\alpha$ the space velocity of the element of mass in volume $dV$ at distance $r$ from the point where the $h_{ij}$ are evaluated, all quantities being measured by observers at rest in the reference frame.
2. **Rate of change of a vector moving relative to moving axes.**

Consider two frames of reference $s_1$ and $s_2$ the former fixed while the latter is moving relatively to it. We consider a rotation of $s_2$, with angular velocity $\omega$, about a point $O$ fixed in $s_1$, the point $O$ being common to both frames.

Let $\mathbf{r}$ be a variable vector, the position vector relative to $O$ of some moving point $P$. During time interval $\delta t$ the vector $\mathbf{r}$ changes from $\mathbf{OP}$ to $\mathbf{OB}$, then $\mathbf{PB}$ is the increment $\delta \mathbf{r}$ relative to $s_1$.

Also during this interval the point of the frame $s_2$ that was initially at $P$ has moved to $Q$ where $\mathbf{PQ} = (\omega \wedge \mathbf{R}) \delta t$, $\mathbf{R}$ being the position vector of the moving point relative to $s_2$. Hence $\mathbf{QR}$ is the increment $\delta \mathbf{R}$ of $\mathbf{R}$ relative to $s_2$.

Now, $\mathbf{PR} = \mathbf{PQ} + \mathbf{QR}$

$$\delta \mathbf{r} = (\omega \wedge \mathbf{R}) \delta t + \delta \mathbf{R}$$

for $\delta t \to 0$ we obtain

$$\mathbf{\dot{r}} = \omega \wedge \mathbf{R} + \mathbf{\ddot{r}}$$

If the origin of $s_2$ now has variable velocity $\mathbf{V}$ relative to $O$, then we have further

$$\mathbf{\ddot{r}} = \mathbf{V} + \mathbf{\dot{R}} + \omega \wedge \mathbf{R}$$

From this we obtain the relation between the accelerations as
\[ \vec{r} = \vec{v} + w \wedge \vec{v} + 2w \wedge \vec{a} + \dot{w} \wedge \vec{R} + w \wedge (\vec{w} \wedge \vec{R}) + \dot{\vec{R}} \]

3. **Robertson-Walker cosmological line element.**

The derivation of this line element is based on two geometrical assumptions. Firstly, on the large scale, space-time should be separable into space and a cosmic time orthogonal thereto in such a way that the line element could be written in the form

\[ ds^2 = dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3) \]

Secondly space-time should be spatially homogeneous and isotropic when looked at from a large scale point of view.

The space \( t = \text{constant} \) is

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \]

which must be of the form

\[ ds^2 = R^2(t) \, ds_0^2 / R^2(t_0) \]

where \( ds_0 \) is the metric of space at \( t = t_0 \). That is

\[ ds_0^2 = \epsilon_{\alpha\beta} dx^\alpha dx^\beta \]

where \( \epsilon_{\alpha\beta} \) are independent of \( t \) (due to homogeneity and isotropy).

The 3-space \( ds_0^2 \) must be a space of constant curvature, according to a well known theorem of differential geometry. Hence we have

105.
\[ ds^2 = dt^2 - \frac{R^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]}{(1 + \alpha x^2/4)^2} \]

where \( k = -1, 0, 1 \) corresponding to negative, zero, positive curvature of the 3-space \( t = \text{const}. \) The system of coordinates is co-moving since the fundamental particles have constant \( x^1, x^2, x^3. \)


If the fundamental tensors \( g_{ij} \) and \( \gamma_{ij} \) are conformally related by

\[ g_{ij} = e^{2\sigma} \gamma_{ij} \]

then we have

\[ R_{ij} = B_{ij} + 2\gamma_{ij} + \gamma_{ij}[\Delta_2 \sigma + 2\Delta_1 \sigma] \quad (4.1) \]

and

\[ R = e^{-2\sigma}[B + 6\Delta_2 \sigma + 6\Delta_1 \sigma] \quad (4.2) \]

where \( R_{ij} \) and \( B_{ij} \) are the Ricci tensors corresponding to \( g_{ij} \) and \( \gamma_{ij} \) respectively and

\[ \Delta_1 \sigma = \gamma^{ij}_{\sigma;ij}, \]

\[ \Delta_2 \sigma = \gamma^{ij}_{\sigma;ij} \quad (4.3) \]

\[ q_{ij} = \sigma_{;ij} - \sigma_{;i} \sigma_{;j} \]
From (2.1.12) we have $\sigma = \log \phi$, hence $\sigma_i = \phi^{-1} \frac{\partial \phi}{\partial x^i}$

Substituting the relations (4.3) in (4.1) and (4.2) we obtain

$$R_{ij} - \frac{1}{2} g_{ij} R = B_{ij} - \frac{1}{2} \gamma_{ij} B - 4\phi^{-2} \left[ \phi_i \phi_j - \frac{1}{4} \gamma_{ij} \phi P \right]$$

$$+ 2\phi^{-1} \left[ \phi_i \phi_j - \gamma_{ij} \square \phi \right]$$

where

$$\phi_i = \frac{\partial \phi}{\partial x^i} \quad \text{and} \quad \square \phi = \gamma^{ik} \phi_{ik}$$

We let $\Gamma^i_{jk}$ and $\left\{ \gamma^i_{jk} \right\}$ denote the Christoffel symbols constructed out of the $g_{ij}$ and $\gamma_{ij}$ respectively.

We have

$$\Gamma^i_{ij} = \left\{ \delta^i_{i} \right\} + \delta^i_{j} \sigma_{i;j} + \delta^i_{j} \sigma_{i;j} - g_{ij} g^{km} \sigma_{;m} \quad (4.4)$$

Defining

$$dl^2 = \gamma_{ik} dx^i dx^k$$

so that

$$ds = \phi dl$$

the geodesic equations in terms of the $g_{ij}$

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

now become
\[
\phi - \frac{1}{d} \left( \frac{-1}{\phi} \frac{dx}{dx} \right) + \left\{ \frac{i}{j} \right\} \phi - 2 \frac{dx}{dx} \frac{dx}{dx} + 2\phi - 3 \frac{dx}{dx} \frac{dx}{dx} \phi_k
\]

rearranging we get

\[
\frac{d}{dx} \left( \phi \frac{dx}{dx} \right) = ip \phi - \phi \left\{ \frac{i}{j} \right\} \frac{dx}{dx} \frac{dx}{dx}
\]

5. Potential of a uniform sphere.

We first consider the potential of a uniform circular disc of radius \(a\) at a point on its axis at distance \(x\) from the disc. This will be

\[
\phi_x = \rho \int_{r=0}^{a} \frac{2\pi r}{r^2 + x^2} \, dr
\]

where \(\rho\) is mass density.

Hence \(\phi_x = 2\pi \rho (\sqrt{a^2 + x^2} - x)\).

We now consider the potential of a uniform sphere of mass \(M\) and radius \(A\). We let \(\phi_r\) denote the potential at distance \(r\) from the centre. For \(r < A\) we have

\[
\frac{1}{2\pi \rho} \phi_r = \int_0^{A-r} \left[ \frac{A}{\sqrt{A^2 + 2\pi r - y^2}} - y \right] dy + \int_0^{A-r} \left[ \frac{A}{\sqrt{A^2 - 2\pi r - y^2}} - y \right] dy
\]
since
\[ M = 4\pi A^3 \rho /3 \]
we obtain
\[ \phi_r = M(3A^2 - r^2)/2A^3 \]

For \( r \geq A \) we get
\[ \frac{1}{2\pi} \phi_r = \left[ \sqrt{A^2 + 2ry - r^2} - y \right] dy \]
\[ \phi_r = M/r \]

6. **Energy momentum tensor density \( \mathbf{J}_k^i \).**

We justify the introduction of the tensor density \( \mathbf{J}_k^i \) defined in (2.2.7) by considering the energy momentum tensor for a system of mass points in flat space. In flat space we have
\[ t_k^i = \sum_i m_i c^2 \left\{ du_i \delta^4(x - z_i(u_i)) \frac{dz_i^j}{du_i} \frac{dz_i^m}{du_i} n_{mk} \right\} \quad (6.1) \]
where
\[ du^2 = n_{mn} dz^m dz^n \]
is the line element in flat space. (Thirring 1961).

Substituting \( dl \) and \( \gamma_{mk} \), as defined in (2.2.19), for \( du \) and \( n_{mk} \) respectively, in the expression (6.1) we get
\[ \mathbf{J}_k^i = \sum_i m_i c^2 \left\{ dl_i \delta^4(x - z_i(l_i)) \frac{dz_i^j}{dl_i} \frac{dz_i^m}{dl_i} \gamma_{mk} \right\} \quad (6.2) \]
which is a tensor density of weight 3/4, and since we have the asymptotic conditions $dl \to du$ and $\gamma_{mk} \to \eta_{mk}$, the expression (6.2) asymptotically goes over to the expression (6.1) for flat space.
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