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The Behaviour of Sea Ice in Ocean Waves

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A thesis submitted for the degree of Doctor of Philosophy at the University of Otago, Dunedin, New Zealand.

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Abstract

The entry of ocean waves from the open sea into pack ice is a feature of the marginal ice zone which has important consequences for navigation and the construction of offshore structures in ice-infested seas. In turn it is largely the action of waves which creates the marginal ice zone as it is the wave action which is responsible for the floe size distribution within the ice cover.

In this thesis a two-dimensional model for the behaviour of a single ice floe in ocean waves is developed using a Green's function formulation. This model allows us to calculate the reflection and transmission coefficients of a single floe. It predicts that there will be frequency-dependent critical floe lengths at which the reflection is zero, analogous to electromagnetic wave propagation through a homogeneous slab. It is also found that floe bending increases as a function of floe length until a critical length is reached, above which the strain is essentially constant. The model is successfully validated, at least for elastic sheets floating on water, by experiments performed on a polypropylene sheet. The single floe theory may also be synthesized approximately by an extension of the model developed by Fox and Squire [1990, 1991] for the interaction of waves with a semi-infinite sheet. This acts as an independent check on both theories.

The solution for a single floe may be extended to many floes as a full solution or as an approximate solution. It is shown that the approximate solution is sufficiently accurate in nearly all situations. This allows the development of a simple model for ocean wave propagation through a cover composed of many discrete floes. This model predicts that a field of pack ice will low pass filter incoming ocean waves. The model also predicts that there will be a narrowing of directional spectra with propagation through an ice cover.

Finally the model is extended so that the surge response, a frequently measured property of ice floes, may be predicted. The surge response agrees with that found by Rottier [1992] and is a strong function of floe length.

A different model for the motion of a single floe developed by Shen and Ackley [1991] is also investigated. This model is applicable to small ice floes and is related to Morrison's equation which is used extensively in problems of offshore structures. The Shen and Ackley model is shown to predict that in most physical cases all floes will tend to the same drift velocity which will be a function almost exclusively of wave amplitude.
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Chapter 1

Introduction

Where polar and temperate climate systems meet there is a transition region between the open water and the pack ice which is referred to as the marginal ice zone (MIZ). An MIZ is made up of discrete ice floes with dimensions varying from metres to hundreds of metres across, and a few tens of centimetres to metres in draft. Because an MIZ is a transition it is characterized physically by strong gradients in oceanic, meteorological, and sea ice properties. As an MIZ borders the ice-free ocean it is strongly affected by especially energetic open ocean processes. The ice edge undergoes considerable variation with the seasonal cycle. The yearly variation of the Antarctic ice cover, for example, is from 4 to $20 \times 10^6$ km$^2$.

The MIZ, especially in the Arctic, has been the subject of much recent investigation, including a sequence of Marginal Ice Zone Experiments (MIZEX) and the more recent Coordinated Eastern Arctic Experiment (CEAREX) (see Johannessen [1987] and Muench et al. [1991] and the various papers contained within these Journal of Geophysical Research special issues). There has been an increasing awareness of the importance of the MIZ to resource development, global climatology, and a wide variety of other problems.

Measurements made in the MIZ have shown that in general the mean size of floes increases with distance from the open water, while the mean spacing decreases [Wadhams, 1981]. The MIZ has also been observed to be composed of zones [Wadhams, 1981, 1986] within which the concentration and floe size are roughly uniform. A feature of the pack ice is the presence of open water in certain well-defined localities. These polynyas lie just beyond the edge of the fast ice and may persist for days or weeks [Wadhams, 1986].
1.1 Wave-Ice Interaction

Since the MIZ borders the open ocean, surface gravity ocean waves will penetrate the broken ice cover. The interaction between ocean waves and sea ice is a relevant issue to a number of fields of ice edge research; for example the break up of ice floes [Squire and Martin, 1980], the erosion of floe edges [McKenna and Crocker, 1990] and the generation of under water noise [Dyer, 1984]. Practical problems of navigation and the construction of offshore structures in ice-infested seas requires knowledge of the wave amplitudes within the ice cover. As well as ordinary ship or structure interactions with waves, the presence of discrete ice floes with their own momentum causes problems unique to the MIZ.

It is generally observed that the wave amplitudes at the ice edge are those of a typical open ocean and that there is exponential decay in amplitude with distance from the ice edge [Wadhams, 1986]. The decay rate also shows a decrease with wave period. This result is typical of most wave penetration problems since the lower period waves are more energetic and hence are more likely to be attenuated. Wave action has been observed [Squire and Martin, 1980] to break floes, so that it may be inferred that the wave spectrum will affect the floe size distribution. Correspondingly it is apparent that the presence of discrete floes will affect the wave spectrum as it propagates into the ice cover. Thus we are left with a problem in which the floe size distribution and the wave spectrum both influence each other. The full solution of such a problem is difficult and remains a major question in wave-ice research.

1.1.1 Theoretical Models for Waves Influencing Floes

The first theoretical attempt to model ice-coupled waves was by Greenhill [1887]. In that paper Greenhill modelled the ice as a thin elastic beam and solved the dispersion equation for water waves with an elastic covering. In this model, he assumed that the water flow was inviscid and that the waves were of small amplitude, so that the problem is completely linear. The assumption that the ice cover is a thin elastic beam or plate has been the standard model for linear water waves with an ice cover ever since. In a series of experiments using explosives Ewing and Crary [1934] and Ewing et al. [1934] confirmed the dispersion equation was correct. The general dispersion equation has found many applications and most recently was used by Squire et al. [1994] to explain the strain waves recorded in the Erebus glacier tongue in Antarctica.

The problem of the interaction of waves with an elastic sheet on water is not limited to ice. It has been posed, for example, in the context of breakwaters by Stoker [1957] and by Tayler [1986]. Both of these authors solved the problem in limiting cases, Stoker for the case of shallow water and Tayler for the case of very slightly elastic bodies. In both cases only two dimensional waves were considered. In fact, Stoker's assumption of shallow water reduces the dimension of the problem to
one. Wadhams [1986] found an approximate solution to the problem by matching potentials only at one depth, again reducing the dimension of the problem to one, and Squire [1982] used a finite element technique which produced a numerical solution. The first full solution was by Fox and Squire [1990, 1991] for the problem of wave interaction with a semi-infinite elastic ice sheet. Their solution involved matching potentials at all depths, and it is the most significant recent work in the area. The problem of a two dimensional finite elastic covering, that is a floe, has been solved approximately by an extension of the Fox and Squire theory [Meylan and Squire, 1994b] and by a full precise solution using a Green's function formulation [Meylan and Squire, 1993, 1994a]. In all of the above models it was assumed that the floe submergence was negligible compared to the wavelength so that any surge response was ignored.

If the floe length is assumed considerably smaller than the wavelength then the floe may be considered rigid and only the rigid body responses need be included. The simplest assumption is that the floe will behave as a particle in the fluid. This will certainly be the case for sufficiently small floses. If the floe length is still considered small then the Froude-Kriloff assumption [Froude, 1861; Kriloff, 1898], named after the 19th century naval architects, may be applied. This theory neglects any influence by the floating body on the wave motion. Thus one need only consider the linearized potential due to the wave in the absence of the body to calculate the body motion. Under this assumption the rigid body response of a floe may be calculated straightforwardly as was done by Rottier [1992].

Even under the assumption of pure linear waves the motion of an individual rigid floe with full interaction is difficult to solve. But the problem of a floating body has obvious applications outside the MIZ and the techniques used to solve the problem have been established. The problem is solved by use of a Green's function as described by John [1949, 1950] in two landmark papers. Although the techniques for solution are outlined, the computation is significantly complicated that full solution for the case of a rectangular floe has not appeared in the literature. The generalized techniques have also found much application in problems of offshore structures and are outlined in Sarpkaya and Isaacson [1981].

All of the above problems have been formulated as linear, and the time dependence has been removed by the consideration of only one frequency. Recently non-linear problems have been considered in the context of floating bodies and this has become a major area of research (see Ng and Isaacson [1993] and the other papers in International Offshore and Polar Engineers conference 1993 on higher order effects). Since the expansion approach has been so successful with waves (e.g., the Stokes wave), this has been the standard approach used. The only non-linear theory which includes the effects of flexure is due to Fox [1992], using an extension of the earlier Fox and Squire theory.

In problems of wave forcing where the characteristic body dimension is considerably less than
the wavelength the standard method of calculating the wave force is via Morrison's equation [Morrison et al., 1950; Sarpkaya and Isaacson, 1981]. In this formulation the wave is considered to be unaltered by the presence of the body and the force is assumed to be due to two components. The first force is due to the drag and is proportional to the square of the velocity. The second force is due to the inertia and is proportional to the acceleration of the fluid. Such a formulation has not been applied to the case of a floe but a similar equation has been developed by Rumer et al. [1979] in the context of ice transportation in the great lakes. These authors decomposed the wave force into two components, the drag force of Morrison's equation and a sliding force due to the body moving under gravity driven by the slope of the wave. This model for the wave force has the advantage that it is explicitly time-dependent and is not linear. Thus it explicitly predicts a long term drift velocity and it may be included in a more complicated forcing regime in which forces due to the collisions of floes are included. This is the reason that the Rumer model was modified and used by Shen and Ackley [1991] to predict the motion of an array of ice floes including a colliding force. This work was motivated by a desire to understand a process of sea ice formation in which the sea first freezes into pancake ice floes which are herded together by waves and freeze to form a composite sheet. This process was observed in the 1986 Weddell Sea Project. A further sensitivity study using this model was carried out by Frankeninstein and Shen [1993] to investigate the role of various coefficients in the equations in determining the collision frequency and the drift velocity.

1.1.2 Theoretical Models for the Influence of Many Floes on Waves

The approximate solution of Wadhams [1986] to the problem of a single floe was devised so that he could predict the propagation of waves through a sea of many floes. He supposed that the wave energy reflected and transmitted by each floe could be added, and introduced an arbitrary condition that each wave was only reflected twice so that the system was not conservative. This type of model has been employed by Meylan and Squire [1994c, 1994d] who used an extension of their full solution for a single floe to construct a model for the behaviour of a distribution of floes. They found that the presence of a discrete ice cover low pass filters incoming waves. Other authors have constructed models which attempt to approximate the ice cover of discrete floes as a whole. The pack ice is considered as a continuum, modelled as a thin elastic (or viscoelastic) plate onto which some hydrodynamic or alternative lossiness is imposed. This is the method used by Liu and Mollo-Christensen [1988] who considered that such mechanisms as anelastic floe collisions and wave breaking will cause damping of the transmitted wave. They introduce a phenomenological parameter, the eddy viscosity, to parametrize the wave attenuation in the viscous boundary layer. Squire [1993] used a similar formulation, which he applied specifically to the problem of wave propagation through zones of different ice properties. He considered these zones as defining regions
in which the refractive index for wave propagation is different, thereby leading to a corresponding change in the wave spectrum.

### 1.1.3 Experimental Work

There has been considerable experimental work on the relationship between waves and an individual floe. Squire and Martin [1980] measured the response and subsequent fracture of a single ice floe in the Bering Sea. Their strain measurements established that there is significant flexure of large floes and that the floe fracture which occurred was directly attributable to the strain in the floe caused by wave action. Floe acceleration has been measured by both Rottier [1992] and by McKenna and Crocker [1992] who were both investigating individual floe motions and collisions, and the role that waves played in them.

Measurements have also been made of wave propagation through pack ice and the subsequent change in the spectrum of the wave energy. Wadhams et al. [1988], for example, report results for a number of experiments carried out by the Scott Polar Research Institute between 1978 and 1983 in the MIZs of the Greenland and Bering Seas. They measured the wave energy at a series of locations within the ice cover and measured the floe size distribution by aerial photography. They found that in general the wave energy decayed exponentially and that the coefficient of decay decreased with increasing wave period. Similar results were found by Squire and Moore [1980] in the Bering Sea. Wadhams et al. [1986] reported measurements made during MIZEX-84 in the Greenland Sea. The main focus of this work was the measurement of directional spectra and the way in which these changed as the waves propagated through the ice cover. They found that the short period wind wave spectrum rapidly broadens with propagation through the ice and quickly becomes isotropic. Long period swell spectra initially narrow and then broaden to eventually also become isotropic. Measurements were also made which showed that there was significant reflection from the ice edge. These observations were explained by a scattering theory. Wadhams et al. [1988] compared their measurements to theoretical models and found some agreement, but making measurements to validate subtle differences in models of this kind is extremely logistically difficult.

### 1.2 This Thesis

In this thesis we will be concerned mainly with solving the problem of an elastic raft of finite length in waves. The model will be developed initially for infinite depth and then extended to finite depth. Although the method of solution of this problem is very different to that used by Fox and Squire, the problems are closely related, differing only in the fact that the ice sheet is semi-infinite for the Fox and Squire solution and finite for the problem we shall be considering.
Thus we shall find that the solutions may be related by a straightforward extension of the Fox and Squire theory. We shall then include the submergence of the raft and present an experimental verification of the model. The model will then be extended to the case of a distribution of floes and the effect on a wave spectrum will be established. We will also solve for the surge response of a finite floe. Finally we shall consider the Shen and Ackley [1991] model based on the formulation of Rumer et al. [1979] and investigate some of its mathematical consequences.
Chapter 2

A Single Flexible Ice Floe in Water of Infinite Depth

2.1 Introduction

Few experiments have been reported which provide sound data for the seakeeping motions and flexure of single ice floes under various wave conditions. Likewise no precise theoretical model is available, even for the simplest of geometries (e.g., two dimensional rectangular rafts); most current models derive from an approximation which is mathematically incomplete (see for example Wadhams [1986]). In this approximation the wave potentials in the open water can be matched to those beneath the ice floe at the surface only or at two discrete depths, as an infinite number of harmonic solutions are omitted from the analysis and these are required for a precise match at all depths. Similar inadequacies in theory were noted by Fox and Squire [1990, 1991] who solved the equivalent problem of wave propagation into a semi-infinite, continuous sheet of shore fast sea ice. We develop in this chapter a precise, though linearized, model for an ice floe of finite length moving under the action of ocean waves.

2.2 Formulation of the Problem

A solitary ice floe occupying the region $0 \leq x \leq L, z = 0$ (Figure 2.1) is modelled as an elastic raft floating on the equilibrium surface of the ocean which is assumed to be infinitely deep. The following is based on the results of Meylan and Squire [1993, 1994a]. We consider the two dimensional motion and flexure of the raft when it is acted upon by a train of small amplitude, surface gravity waves propagating from left to right in the direction of the positive $x$-axis. The raft is assumed
Figure 2.1: Schematic diagram showing geometry of problem and the coordinate system used.
to be in contact with the water at all points for all time, and thus displacement and pressure are continuous.

The linearized boundary value problem for the velocity potential $\Phi(x, z, t)$ of the water, assuming irrotational and inviscid flow, is:

\[
\begin{align*}
\nabla^2 \Phi &= 0, & 0 < z < \infty, \\
\frac{\partial \Phi}{\partial z} &= 0, & z \to \infty, \\
\frac{\partial \Phi}{\partial z} &= -\frac{\partial w}{\partial t}, & z = 0, \\
-\rho \left( g \frac{\partial \Phi}{\partial z} - \frac{\partial^2 \Phi}{\partial t^2} \right) &= -\frac{\partial p}{\partial t}, & z = 0, \\
\end{align*}
\]

(2.1)

where the equations at $z = 0$, appearing here and subsequently, derive from truncated expansions about the mean water level in the usual way [Stokes, 1847]. In equations (2.1) $w$ is the surface displacement of the water, $p$ is the pressure on the water surface, $\rho$ is the density of the water, and $g$ is the acceleration due to gravity. We also require appropriate conditions to be met as $x \to \pm \infty$ and at $t = 0$. Except over the raft ($0 < x < L$) we assume $p$ is constant. Over the raft we use the Bernoulli-Euler model for the deflection, neglecting gravity effects, as follows:

\[
\frac{\partial^2 w}{\partial t^2} + \mu^2 \frac{\partial^4 w}{\partial x^4} = \frac{p}{\rho' h}, \quad 0 < x < L, \quad 0 < x < \infty,
\]

(2.2)

where $\rho'$ is the density of the raft, $h$ is its thickness, and $\mu^2 = E h^2/12 \rho'(1 - \nu^2)$ where $E$ is the effective Young’s modulus and $\nu$ is Poisson’s ratio. Thus the boundary condition beneath the raft is:

\[
-\rho \left( g \frac{\partial \Phi}{\partial z} - \frac{\partial^2 \Phi}{\partial t^2} \right) = \rho' h \left( \frac{\partial^3 \Phi}{\partial t^2 \partial z} + \mu^2 \frac{\partial^5 \Phi}{\partial x^4 \partial z} \right), \quad z = 0, \quad 0 < x < L,
\]

(2.3)

whereas that in the open sea is:

\[
-\rho \left( g \frac{\partial \Phi}{\partial z} - \frac{\partial^2 \Phi}{\partial t^2} \right) = 0, \quad z = 0, \quad -\infty < x < 0, \quad L < x < \infty.
\]

(2.4)

Additional transition conditions, namely that the bending moment and shear must vanish at the ends of the raft, make the problem well-posed:

\[
\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0, \quad \text{at } x = 0, \quad \text{and } x = L.
\]

(2.5)

The system (2.1) together with boundary conditions (2.3), (2.4) and (2.5) is now non-dimensionalized using,

\[
\bar{t} = \frac{t}{L^2}, \quad \bar{\Phi} = \frac{\Phi}{L \sqrt{\rho L}}, \quad \bar{z} = \frac{z}{L}, \quad \bar{w} = \frac{w}{L^2}, \quad \bar{\Phi} = \frac{\Phi}{L^2 \sqrt{\rho L}},
\]

and defining,

\[
\gamma = \frac{\rho' h}{\rho L}, \quad \beta = \frac{\rho' \mu^2}{g \rho L^2}.
\]
Assuming the velocity potential $\Phi(x, z, t)$ is separable and that it is periodic in time with non-dimensional radian frequency $\sqrt{\alpha}$, we may write $\Phi(x, z, t) = \tilde{\phi}(x, z) e^{i\sqrt{\alpha}t}$ and hence the boundary value problem to be solved becomes:

\[
\begin{align*}
\nabla^2 \phi &= 0, & \infty < z < 0, \\
\frac{\partial \phi}{\partial z} &= 0, & z \to \infty, \\
\frac{\partial \phi}{\partial z} + \alpha \phi &= 0, & z = 0, \quad -\infty < x < 0, \quad 1 < x < \infty, \\
\frac{\partial^2 \phi}{\partial x^2} + \alpha \phi &= \alpha \frac{\partial \phi}{\partial z} - \beta \frac{\partial^2 \phi}{\partial x^2 \partial z}, & z = 0, \quad 0 < x < 1, \\
\frac{\partial^2 \phi}{\partial z^2} &= 0, & \text{at } x = 0, \text{ and } x = 1,
\end{align*}
\]

(2.6)

where the overbar has been omitted with the understanding that subsequently all variables have been non-dimensionalized.

Conditions at $x \to \pm\infty$ are now imposed. At negative infinity we suppose an input whose velocity potential $\phi$ is of unit amplitude. A proportion of the energy associated with this wave will be transmitted by the raft with the remaining energy being reflected. Thus we write the asymptotic potentials as follows:

\[
\lim_{x \to -\infty} \phi = Te^{-i\alpha x - \alpha z}, \quad \text{and} \quad \lim_{x \to -\infty} \phi = e^{-i\alpha x - \alpha z} + Re^{i\alpha x - \alpha z},
\]

(2.7)

where $R$ and $T$ are the reflection and transmission coefficients respectively.

### 2.2.1 An Integral Equation

The boundary condition beneath the raft for $z = 0$ in system (2.6) may be rewritten,

\[
\frac{d^4 \phi_z}{dx^4} - \zeta^4 \phi_z = -\frac{\alpha}{\beta} \phi, \quad z = 0, \quad 0 < x < 1,
\]

(2.8)

where it is understood that $\phi_z$ denotes $\partial \phi / \partial z |_{z=0}$, and $\zeta^4 = (\alpha \gamma - 1) / \beta$ which is negative for typical ice floe parameters. Similarly the conditions at the ends of the raft become,

\[
\frac{d^2 \phi_z}{dx^2} \bigg|_{x=0} = \frac{d^2 \phi_z}{dx^2} \bigg|_{x=1} = 0, \quad \text{and} \quad \frac{d^3 \phi_z}{dx^3} \bigg|_{x=0} = \frac{d^3 \phi_z}{dx^3} \bigg|_{x=1} = 0.
\]

(2.9)

We now construct a Green's function $g(\xi, x)$ to convert (2.8) and (2.9) into an integral equation; $g(\xi, x)$ must satisfy,

\[
\frac{d^4 g(\xi, x)}{d\xi^4} - \zeta^4 g(\xi, x) = \delta(\xi - x),
\]

(2.10)

together with the boundary conditions:

\[
g_{\xi}(0, x) = g_{\xi}(1, x) = g_{\xi\xi}(0, x) = g_{\xi\xi}(1, x) = 0.
\]

(2.11)
Equation (2.10) has general solution,
\[ \frac{\alpha}{\beta} g(x, \xi) = \begin{cases} A_1 e^{i\xi} + B_1 e^{-i\xi} + C_1 e^{i\xi} + D_1 e^{-i\xi}, & 0 < \xi < x < 1, \\ A_2 e^{i\xi} + B_2 e^{-i\xi} + C_2 e^{i\xi} + D_2 e^{-i\xi}, & 0 < x < \xi < 1. \end{cases} \tag{2.12} \]

On applying the end conditions (2.11) together with the usual matching and jump conditions across \( \xi = x \) we may write down a matrix equation for the unknown functions of \( x \): \( A_1, B_1, \ldots \) as follows:
\[
\begin{pmatrix}
-1 & -1 & 1 & 1 \\
-i & i & 1 & -1 \\
-e^{i\xi} & -e^{-i\xi} & e^i & e^{-i}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
-\frac{\alpha}{4\beta^2}
\end{pmatrix}
\begin{pmatrix}
e^{i\xi(1-x)} + ie^{-i\xi(1-x)} + e^\zeta(1-x) - e^{-\zeta(1-x)} \\
e^{i\xi(1-x)} + ie^{-i\xi(1-x)} + e^\zeta(1-x) + e^{-\zeta(1-x)}
\end{pmatrix}, \tag{2.13}
\]

together with equations linking \( A_1, B_1, \ldots \) and \( A_2, B_2, \ldots \)
\[
\begin{align*}
A_2 &= A_1 + \frac{i\alpha}{4\beta^2} e^{-i\zeta x}, \\
B_2 &= B_1 - \frac{i\alpha}{4\beta^2} e^{i\zeta x}, \\
C_2 &= C_1 + \frac{\alpha}{4\beta^2} e^{-\zeta x}, \\
D_2 &= D_1 - \frac{\alpha}{4\beta^2} e^{\zeta x}.
\end{align*} \tag{2.14}
\]

This system may be solved straightforwardly for the unknown coefficients.

To obtain \( \phi_z(x, 0) \) in terms of \( g(\xi, x) \) we integrate over the raft in the usual manner:
\[ \phi_z(x, 0) = -\frac{\alpha}{\beta} \int_0^1 g(\xi, x) \phi(\xi, 0) d\xi, \quad z = 0, \quad 0 < x < 1, \tag{2.15} \]

so that system (2.6), together with our asymptotic conditions (2.7), becomes:
\[
\begin{align*}
\nabla^2 \phi &= 0, \quad \infty < z < 0, \quad \frac{\partial \phi}{\partial z} = 0, \quad z \to \infty, \\
\frac{\partial \phi}{\partial z} + \alpha \phi &= 0, \quad z = 0, \quad -\infty < z < 0, \quad 1 < z < \infty, \\
\phi_z(x, 0) &= -\frac{\alpha}{\beta} \int_0^1 g(\xi, x) \phi(\xi, 0) d\xi, \quad z = 0, \quad 0 < x < 1,
\end{align*} \tag{2.16}
\]

\[ \lim_{x \to -\infty} \phi = T e^{-i\alpha x - \alpha z}, \quad \text{and} \quad \lim_{x \to -\infty} \phi = e^{-i\alpha x - \alpha z} + R e^{i\alpha x - \alpha z}. \]
2.3 Method of Solution

2.3.1 Green’s Function for the Half Space

A Green’s function $G(\xi, \eta; x, z)$ for the half space which satisfies the open water boundary conditions is now found, i.e., $G$ is the solution of the system,

\[
\begin{align*}
\nabla^2 G = \delta(\xi - x)\delta(\eta - z), & \quad -\infty < \xi, x < \infty, \quad 0 < \eta, z < \infty, \\
\frac{\partial G}{\partial \eta} + \alpha G = 0, & \quad \eta = 0, \quad -\infty < x < \infty, \\
\frac{\partial G}{\partial \eta} \rightarrow 0, & \quad \text{as } \eta \rightarrow \infty,
\end{align*}
\]

(2.17)

where $\nabla^2$ is now with respect to $\xi$ and $\eta$. To find $G(\xi, \eta; x, z)$ we take the Fourier transform with respect to $x-\xi$ so that equations (2.17) become,

\[
\begin{align*}
\frac{d^2 \hat{G}}{d\eta^2} - \omega^2 \hat{G} = \delta(\eta - z), & \quad -\infty < \xi, x < \infty, \quad 0 < \eta, z < \infty, \\
\frac{d \hat{G}}{d\eta} + \alpha \hat{G} = 0, & \quad \eta = 0, \quad -\infty < x < \infty, \\
\frac{d \hat{G}}{d\eta} \rightarrow 0, & \quad \text{as } \eta \rightarrow \infty,
\end{align*}
\]

(2.18)

where $\hat{G}$ is the Green’s function in Fourier transform space. Solving these equations we obtain the following,

\[
\hat{G}(\omega, z, \eta) = \frac{-1}{2|\omega|} \left( e^{-|\omega|(|\eta-z|)} + \left( \frac{|\omega| + \alpha}{|\omega| - \alpha} \right) e^{-|\omega|(|\eta+z|)} \right). 
\]

(2.19)

We know that the free space fundamental solution to the equation,

\[
\nabla^2 G = \delta(x - \xi)\delta(z - \eta), 
\]

(2.20)

is [Greenwood, 1971],

\[
G = \frac{1}{4\pi} \ln \left( (x - \xi)^2 + (z - \eta)^2 \right). 
\]

(2.21)

We also know from Appendix A that the Fourier transform of $\ln(x^2 - a^2)$ is,

\[
\int_{-\infty}^{\infty} \ln(x^2 - a^2) e^{i \omega x} dx = -\frac{2\pi e^{-|\omega|a}}{|\omega|}. 
\]

(2.22)

Therefore taking the inverse Fourier transform of equation (2.19) we obtain the following expression for the Green’s function,

\[
G(\xi, \eta; x, z) = \frac{1}{4\pi} \ln \left( (\xi - x)^2 + (\eta - z)^2 \right) - \frac{1}{4\pi} \ln \left( (\xi - x)^2 + (\eta + z)^2 \right) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\omega| - \alpha} e^{-|\omega|(|\eta+z|)} e^{i \omega (\xi - z)} d\omega. 
\]

(2.23)
This Green's function is identical to that derived by John [1950], a result we shall show formally for the more general case of a Green's function for water of finite depth. The Green's function must be calculated numerically and the techniques for this are outlined in Appendix E.

### 2.3.2 Solving for the Boundary Conditions

Green's theorem in the plane is now applied to the rectangle \( \Delta \) with sides \( \xi = -\xi_0, \xi = \xi_0, \eta = 0, \eta = \eta_0 \), where \( \xi_0 \) and \( \eta_0 \) are taken sufficiently large to include the point \((x, z)\) properly (i.e., \((x, z)\) may not be on the boundary, effectively requiring that \(z > 0\)). This gives us the following equation,

\[
\phi(x, z) = \int_{\Delta} \left( \frac{\partial G}{\partial n}(\xi, \eta; x, z)\phi(\xi, \eta) - \frac{\partial \phi}{\partial n}(\xi, \eta)G(\xi, \eta; x, z) \right) ds, \tag{2.24}
\]

where \( \partial / \partial n \) denotes differentiation with respect to the outward normal. We now consider the limit as \( \eta_0 \to \infty \) and \( \xi_0 \to \infty \) and integrate along each side separately, using the boundary conditions for \( G(\xi, \eta; x, z) \) and \( \phi(\xi, \eta) \) where appropriate. These boundary conditions give trivial solutions on \( \eta = 0 \) and \( \eta = \eta_0 \) but we must be more careful on \( \xi = -\xi_0 \) and \( \xi = \xi_0 \). From Appendix C we know that,

\[
\lim_{\xi \to \pm\infty} G = \pm e^{-a(\eta+z)} \sin(a(\xi - x)), \tag{2.25}
\]

\[
\lim_{\xi \to \pm\infty} G_\xi = \pm ae^{-a(\eta+z)} \cos(a(\xi - x)). \tag{2.26}
\]

We now consider the contribution due to the integral along \( \xi = \xi_0 \), taking the infinite limits and substituting the asymptotic potentials (equations (2.7)) to obtain,

\[
\lim_{\xi_0 \to \infty} \int_0^\infty (\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G)_{|\xi=\xi_0} d\eta = \int_0^\infty aT e^{-iax-2a\eta} d\eta = \frac{T}{2} e^{-iax-az}, \tag{2.27}
\]

and similarly,

\[
\lim_{\xi_0 \to -\infty} \int_0^\infty (\phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G)_{|\xi=\xi_0} d\eta = \frac{1}{2} e^{-iax-az} + \frac{R}{2} e^{iax-az}. \tag{2.28}
\]

Thus we have the following integral equation for the velocity potential,

\[
\phi(x, z) = \frac{R}{2} e^{iax-az} + \left(1 + \frac{T}{2}\right) e^{-iax-az} \tag{2.29}
\]

\[
+ \int_0^1 \left( \frac{\partial G}{\partial n}(\xi, 0; x, z)\phi(\xi, 0) - G(\xi, 0; x, z)\frac{\partial \phi}{\partial n}(\xi, 0) \right) d\xi.
\]

To construct an integral equation for the potential under the raft we want to set \(z = 0\). This cannot be done simply since we have assumed that \(z > 0\) in deriving equation (2.24). This was not strictly a requirement but the condition on the normal derivative of the Green's function at the water surface, \( G_n = \alpha G \), cannot hold in the case \(\eta = z = 0\) since \(G_n\) must also have a \(1/x\) type singularity. This condition arises in the derivation of the Green's function since we assume that the delta function \(\delta(z-\eta)\) is not at the water surface. This condition was noted by John
I. J. 1950 who only considered the case \( 0 < \eta < z < \infty \). We proceed as follows; first we make the substitution,

\[
\frac{\partial G}{\partial n}(\xi, 0; x, z) = \alpha G(\xi, 0; x, z),
\]

and consider the limit as \( z \) tends to zero. Since the singularities in the Green's function are logarithmic the integral is absolutely convergent and the limit may be taken inside the integral. This is not the case for the normal derivative of the Green's function since the singularity is of order \( 1/x \) and the integral of such a singularity exists only by the Cauchy principal value. We also substitute the integral equation (2.15) and obtain the following,

\[
\lim_{z \to 0} \frac{1}{z} \int_0^z \left( \frac{\partial G}{\partial n}(\xi, 0; x, z) \phi(\xi, 0) - G(\xi, 0; x, z) \frac{\partial \phi}{\partial n}(\xi, 0) \right) d\xi = \\
\int_0^1 \alpha G(\xi, 0; x, z) \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \varsigma) \phi(\varsigma, 0) d\varsigma \right) d\xi. \tag{2.30}
\]

Also the Green's function (2.23) at the surface is,

\[
G(\xi, 0; x, 0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\omega| - \alpha} e^{i\omega(\xi - x)} d\omega.
\]

This gives us the following integral equation for the potential at the surface of the water under the raft,

\[
\phi(x, 0) = R e^{i\alpha x} + \left( \frac{1 + T}{2} \right) e^{-i\alpha x} \\
+ \int_0^1 \alpha G(\xi, 0; x, 0) \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \varsigma) \phi(\varsigma, 0) d\varsigma \right) d\xi.
\tag{2.31}
\]

Before we can solve equation (2.31) we must substitute for \( R \) and \( T \). We do this using the asymptotic limits of the Green's function. From expression (2.7),

\[
\lim_{x \to \infty} \phi(x, 0) = T e^{-i\alpha x}, \quad \text{and} \quad \lim_{x \to -\infty} \phi(x, 0) = e^{-i\alpha x} + Re^{i\alpha x},
\]

and we know from Appendix C that,

\[
\lim_{x \to \pm \infty} G(\xi, 0; x, 0) = \frac{\pm i}{2} \left( e^{i\alpha(\xi - x)} - e^{-i\alpha(\xi - x)} \right).
\]

Therefore taking the limit \( x \to \infty \) of equation (2.31) gives us,

\[
T e^{-i\alpha x} = \frac{R}{2} e^{i\alpha x} + \left( \frac{1 + T}{2} \right) e^{-i\alpha x} \\
+ \frac{e^{-i\alpha x}}{2} \int_0^1 i\alpha e^{i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \varsigma) \phi(\varsigma, 0) d\varsigma \right) d\xi \\
- \frac{e^{i\alpha x}}{2} \int_0^1 i\alpha e^{-i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \varsigma) \phi(\varsigma, 0) d\varsigma \right) d\xi.
\tag{2.32}
\]

So that it follows that,

\[
R = i\alpha \frac{1}{\beta} \int_0^1 e^{-i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \varsigma) \phi(\varsigma, 0) d\varsigma \right) d\xi.
\tag{2.33}
\]
and,

\[ T = 1 + i\alpha \int_0^1 e^{i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \zeta)\phi(\zeta, 0) d\zeta \right) d\xi. \] \hspace{1cm} (2.34)

These expressions for the transmission and reflection coefficients may be derived by an alternative means without using the Green’s function and this is outlined in Appendix B.

Substituting these results into equation (2.31) we finally obtain the integral equation for \( \phi(x, 0) \),

\[ \phi(x, 0) = e^{i\alpha x} + \alpha \int_0^1 \left( G(\xi, 0; x, 0) + i \cos(\alpha(\xi - x)) \right) \times \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \zeta)\phi(\zeta, 0) d\zeta \right) d\xi. \] \hspace{1cm} (2.35)

The Green’s function has a singularity at \( x = \xi \) so that the kernel of the integral equation also has a singularity there.

Integral equation (2.35) is a linear Fredholm equation of the second kind. We solve the integral equation by the Nystrom or quadrature method using Simpson’s rule as our quadrature scheme. The singularity is of the order \( \ln(1/x) \) as would be expected for a Green’s function of Laplace’s equation, and thus it may be removed by a first order subtraction. Taylor [1986] has solved this equation in certain limiting cases and we may use his solution to check our derivation. This is outlined in Appendix D.
Chapter 3

Results for a Single Ice Floe in Infinitely Deep Water

In this section we present some relevant predictions of the solitary floe theory in infinitely deep water. Throughout, the following values are used for model parameters: effective Young's modulus for sea ice $E = 6 \text{ GPa}$, Poisson's ratio for sea ice $\nu = 0.3$, density of sea water $\rho = 1025.0 \text{ kg m}^{-3}$, and density of sea ice $\rho' = 922.5 \text{ kg m}^{-3}$.

3.1 Reflection and Transmission Coefficients

The reflection and transmission coefficients defined by expressions (2.7), (2.33) and (2.34) are related by the simple energy flux expression $|R|^2 + |T|^2 = 1$. Thus for waves of a particular wavelength it is straightforward to find $|T|$ given $|R|$ for a specific floe diameter. $|R|$ is plotted as a function of floe diameter for several ice thicknesses in Figure 3.1. Each curve has the same basic structure, comprising a series of convex-up segments which drop rapidly to zero at their ends. At these zero points, which we shall loosely call resonances, the wave tunes perfectly to the floe's length with the result that $|R| = 0$ and hence the transmission coefficient $|T| = 1$.

The fine structure of each curve is strongly dependent on thickness. Thicker ice leads to higher reflection coefficient maxima but also, since the wavelength within the ice increases with thickness, the number of zeros for a specific floe diameter varies inversely with floe thickness. The separation of the zeros tends to half the wavelength in the ice as the length of the floe increases, exhibiting characteristics analogous to electromagnetic wave propagation through a homogeneous slab. As we shall see later the resonances have especially important consequences to strain.
Figure 3.1: Reflection coefficient $|R|$ versus floe diameter at a wavelength of 100 m for $h = 1$ m (solid curve), 2 m (dashed curve), and 5 m (dotted curve). Perfect transmission (resonance) occurs when $|R| = 0$. 
3.2 Potential

Having solved the integral equation (2.35) we can then substitute the potential under the raft into equation (2.29) and calculate the potential throughout the space. Figure 3.2 shows the absolute value of potential plotted for a 100 m ice floe in waves of 100 m wavelength. The raft occupies the position \(0 < x < 100, z = 0\), and we have calculated the potential 100 m in front and behind the raft. The potential rapidly assumes the asymptotic limit away from the raft and reveals its most complicated behaviour under the raft. The potential decays exponentially with depth at all positions as expected.

3.3 Viscoelastic Ice Floes

Although we are mainly limiting our discussion to perfectly elastic ice, we have nowhere restricted our development of the equations to this constitutive behaviour and the inclusion of linear viscoelastic effects is straightforward by making the Young's modulus complex. The choice of exactly what values to give the Young's modulus is complicated since the exact mechanical properties of sea ice are strongly dependent on temperature, growth history, etc. We have chosen one early set of measurements made by Tabata [1955]. Tabata chose the standard model of sea ice as a 4-parameter fluid which gives us the following expression for the complex Young's modulus, \(\tilde{E}\), taking into account the fact that all variables are proportional to \(e^{i\sqrt{\alpha_1}}\),

\[
\tilde{E} = \left( \frac{1}{E_1} + \frac{1}{i\sqrt{\eta_1}} + \frac{1}{E_2 + i\sqrt{\eta_2}} \right)^{-1}.
\]

We use the following values for the various parameters: \(E_1 = 6 \times 10^8\) Pa, \(E_2 = 2.8 \times 10^9\) Pa, \(\eta_1 = 1.4 \times 10^{12}\) Pas and \(\eta_2 = 1.2 \times 10^{11}\) Pas, from Tabata. The reflection coefficient for a 200\times 1 m floe for waves of 4 to 10 s is then shown in Figure 3.3b. Also plotted is the amount of energy that the raft absorbs (Figure 3.3a), i.e., the difference between the energy input and the energy in the reflected and transmitted waves. It is clear from Figure 3.3 that the peaks in energy absorption are at precisely those frequencies where transmission is perfect. When the period becomes large the absorption tends to zero and we no longer see the peaks.

3.4 Heave and Roll

To compute the vertical (heave) and angular (roll) rigid body responses of the ice floe it is necessary to decouple these motions from the flexural response. To do this we may make the floe infinitely stiff, then no bending will occur and the floe will perform only rigid body motions. In the context of the present theory this may be done in two ways; either by considering the limit as the effective Young's modulus \(E \to \infty\), or from first principles.
Figure 3.2: The absolute value of potential in the domain $-100 < x < 200$, $0 < z < 50$ for a $100 \times 1$ m floe in $100$ m wavelength waves. The floe occupies the position, $z = 0$, $0 < x < 100$. 
Figure 3.3: a) Proportion of energy absorbed by the floe, and b) The magnitude of the reflection coefficient. Both are plotted against wave period for a 200 × 1 m ice floe with a complex Young’s modulus.
To consider the limit as $\zeta \to \infty$ we expand system (2.13) as a power series in $\zeta$ and substitute back into (2.12). Then, remembering that $\beta = (\alpha \gamma - 1)/\zeta^4$, we may take the limit as $\zeta \to 0$ in a second power series expansion to obtain,

$$-\frac{\alpha}{\beta} g(x, \xi) = \frac{\alpha}{\alpha \gamma - 1} \left( 4 - 6(x + \xi) + 12x \xi \right),$$  

(3.2)

and hence finally from (2.15),

$$\phi_x(x, 0) = \frac{\alpha}{\alpha \gamma - 1} \int_0^1 \left( 4 - 6(\xi + x) + 12\xi x \right) \phi(\xi, 0) \, d\xi.$$  

(3.3)

To derive this equation from first principles we use the equations derived by Stoker [1957], with some modification, assuming a priori that the raft is a rigid board. We allow two motions, the heave and roll (we shall use dimensional coordinates initially). The displacement is thus given by,

$$w(x, t) = (A + (x - L/2)\Theta) e^{i\sqrt{\alpha} t},$$  

(3.4)

where $A$ corresponds to heave and $\Theta$ to roll. This means that the outward normal derivative of the velocity potential is,

$$\phi_n = i\sqrt{\alpha} (A + (x - L/2)\Theta) e^{i\sqrt{\alpha} t}.$$  

(3.5)

We know that the pressure is given (from Bernoulli's equation) by,

$$i\sqrt{\alpha} P = -\rho g \phi_n + \alpha \rho \phi.$$  

(3.6)

We now must use the dynamical equations of motion of the board, which are Newton's 2nd law of linear motion,

$$\int_0^L P(\xi) \, d\xi = -\rho' h L \alpha \Lambda,$$  

(3.7)

and Newton's 2nd law of angular motion,

$$\int_0^L (\xi - L/2) P(\xi) \, d\xi = -\rho' h L^5 \alpha \Theta.$$  

(3.8)

Substituting the pressure equation (3.6) and equation (3.5) we obtain, ignoring the time dependent part,

$$\Lambda = \frac{i\sqrt{\alpha} \rho}{\rho' h L \alpha - L \rho g} \int_0^L \phi(\xi) \, d\xi,$$  

(3.9)

and,

$$\Theta = \frac{12 - i\sqrt{\alpha} \rho}{\rho' h L^5 \alpha - L^5 \rho g} \int_0^L (\xi - L/2) \phi(\xi) \, d\xi.$$  

(3.10)

These equations are now non-dimensionalized and combined to give,

$$\phi_n = \frac{\alpha}{1 - \alpha \gamma} \left( \int_0^1 \phi(\xi) \left( 4 - 6(x + \xi) + 12x \xi \right) \, d\xi \right),$$  

(3.11)

which agrees with (3.3).
Response amplitude operators (RAOs), which are the values of $A$ and $\Theta$ as a function of frequency assuming a unit wave input, are shown in Figures 3.4 and 3.5 for heave ($A$) and roll ($\Theta$) respectively, both calculated as above by considering the limit of infinite stiffness. All RAOs are shown for 1 m thickness of ice, for floes of 1 m, 10 m, 50 m, 100 m, and 200 m. (The 1 m ice cake curve is included to demonstrate rigid body resonance since a floating body which has a length/thickness aspect ratio of about unity will exhibit very high $Q$ [Hussey, 1983] resonance.)

The heave RAOs (Figure 3.4) show exactly what would be expected: a very high $Q$ peak occurring at just over 2 s for a 1 m x 1 m ice cake, with the curve then decreasing asymptotically to perfect response for long waves; a slight maximum for the 10 m x 1 m floe centred at about 7 s before the curve again decreases to its long wave asymptote; and monotonically increasing RAOs for all floes of greater aspect ratio, the curves again tending to perfect response when the wave period is long. These results suggest that care must be taken in the interpretation of acceleration data obtained from instruments deployed on a floe. The model agrees well with the predictions of Lee [1976] for the heave motion of a rigid floating raft. The roll RAOs (Figure 3.5), which have been multiplied by length so as to make them easier to interpret, all have the same skewed "bell-shaped" form with their respective maxima depending on the floe's aspect ratio. The 1 m x 1 m ice cake again undergoes a very sharply peaked response at a period of just over 2 s; the other curves peak at successively longer periods. For very long waves no roll is induced because the curvature of the sea surface is very small.

The infinite stiffness limit is now relaxed and we return to the complete theory with Young's modulus $E = 6$ GPa to discuss the strain induced in ice floes of various sizes.

### 3.5 Strain

A wave propagating beneath an ice floe will cause it to bend, giving rise to alternating extensional and compressional strains at the floe's upper surface. The magnitude of these strains will depend on the fraction of incident wave energy reflected by the floe, as this will determine the amplitude of the wave beneath the floe, and on the wavelength, since long waves will obviously lead to less curvature and hence to smaller strains. Surface strain may be found from the expression,

$$
\varepsilon(x, f) = \frac{h}{2} \frac{\partial^2 w}{\partial x^2} = -\frac{h}{2i\sqrt{\alpha}} \frac{\partial^3 \phi(x, 0)}{\partial x^2 \partial z},
$$

where $f$ is the (dimensional) linear frequency. Hence we may find the magnitude of the surface strain at all points in an ice floe of a particular length and thickness when it is subjected to waves of a particular period. By virtue of the edge conditions (2.5) the strain magnitude will vanish at $x = 0$ and $x = L$, i.e., at the two ends of the floe, but in between it will in general be nonzero. At one or more points within $0 < x < L$ the surface strain field will reach its maximum value.
Figure 3.4: The heave response (RAO) for ice floes of different length (marked on each curve) at a constant thickness of 1 m.
Figure 3.5: The roll response (RAO) multiplied by the length of the floe for ice floes of different length (marked on each curve) at a constant thickness of 1 m.
In Figures 3.6a-3.9a this maximum strain is plotted against wave period for 1m thick ice at various floe diameters (10m, 50m, 100m, and 200m respectively). The plots are accompanied (as Figures 3.6b-3.9b) by their associated reflection coefficients. Since the incident wave amplitude is normalized to 1 m, Figures 3.6a-3.9a represent the strain magnitude transfer functions for the specific ice floe being modelled.

As period increases the maximum strain curve for a $10 \times 1$ m floe (Figure 3.6a) decreases monotonically from its value at short periods. One zero (resonance) occurs in Figure 3.6b at just over 4 s but its effect on strain is not marked because reflection coefficients are quite small in any case. Overall, strain magnitudes are relatively low, suggesting that 10 m floes of 1m thickness do not bend very much as the wavelength beneath the ice is long relative to the length of the floe even for the shortest waves considered. The situation for a $50 \times 1$ m floe is similar although the strain magnitudes are a little larger. Again in Figure 3.7a we see a monotonic decrease in strain and again the zero in the reflection coefficient curve, in this case near 10s (Figure 3.7b), has little effect because it occurs when the coefficient is small.

As period increases the maximum strain curve of Figure 3.8a for a $100 \times 1$ m floe rises rapidly to a sharp peak, after which it decreases monotonically with further increase in period. The monotonic decrease occurs because long waves bend the ice less than short waves of the same height. To interpret the peak we must again consider the associated reflection coefficient of Figure 3.8b. Omitting for the moment any discussion of the curves fine structure, $|R|$ decreases as period increases. Thus we expect the curve of Figure 3.8a to have a maximum, since at short periods the curvature will be reduced by a large reflection coefficient whereas at long periods reflection will not be significant. The situation is, however, somewhat more complicated due to the zeros corresponding to perfect transmission which we encountered in Figure 3.1. Two of these resonant zeros appear in Figure 3.8b, and it is the first of these at about 6.5 s period which gives rise to the sharply-peaked transfer function of Figure 3.8a. The second zero, occurring at about 13.5 s, has negligible effect because transmission is close to perfect at these wavelengths in any case. Thus we see that the transfer function for strain in a $100 \times 1$ m floe is influenced markedly by the reflection/transmission characteristics of the ice floe, and that strain maxima are likely to be found in the vicinity of wave periods where $|T| = 1$.

Figure 3.9 shows the equivalent curves to Figures 3.6-3.8 for a $200 \times 1$ m ice floe. Here the resonance effect referred to above is even more pronounced, as a second zero occurs in $|R|$ before it has dropped to insignificant amplitude (Figure 3.9b). This zero causes the small hummock in the strain curve between 8 and 9 s.
Figure 3.6: a) The maximum surface strain magnitude experienced by the floe, and b) The magnitude of the reflection coefficient, both plotted against wave period for a $10 \times 1$ m ice floe.
Figure 3.7: Equivalent plots to Figure 3.6 for a $50 \times 1$ m ice floe.
Figure 3.8: Equivalent plots to Figure 3.6 for a 100 × 1 m ice floe.
Figure 3.9: Equivalent plots to Figure 3.6 for a 200 \times 1 \text{ m} ice floe.
3.6 Spectral Forcing

We now consider the effect of a sea composed of a spectrum of waves on a solitary ice floe. We assume the simplest possible open water wave spectra, namely the Pierson-Moskowitz spectrum, though of course any theoretical or measured spectrum may be substituted. The Pierson-Moskowitz spectrum has the form,

\[ E(f) = \alpha g^2 (2\pi)^{-4} f^{-5} \exp\left\{ -\frac{5}{4} \left( \frac{f_m}{f} \right)^4 \right\}, \] (3.13)

[Phillips, 1977] where \( \alpha \) is the Phillip’s constant and \( f_m \) is the peak frequency. The value of Phillips constant is found experimentally to be approximately \( \alpha = 1.2 \times 10^{-2} \). We assume that the peak frequency is 0.125 Hz, but any one dimensional spectrum could be used instead. The strain density spectrum at any point along the floe \( \mathcal{E}(x, f) \) is given by,

\[ \mathcal{E}(x, f) = |\varepsilon(x, f)|^2 E(f). \] (3.14)

Like \( \varepsilon(x, f) \), the quantity \( \mathcal{E}(x, f) \) will vanish at the ends of the floe because of (2.5) and will reach one or more local maxima between. It is the maxima that we are interested in as these will break the ice floe if the incoming seas are of sufficient amplitude. In Figure 3.10 we have plotted for a fixed thickness of 1 m the strain spectral density \( \mathcal{E}(f) = \mathcal{E}_{\text{max}}(x, f) \) for a set of floes of various length running from 10 m up to 120 m. Lengths from 40 m to 120 m are marked on the curves; smaller floes lie beneath the 40 m curve. As length is increased up to about 100 m the curves increase in height. The smallest floes and cakes do not bend very much, but as floe length is increased so the waves of significant energy in \( E(f) \) begin to affect the floe more and more. Maximum strain densities are reached when floes are about 102 m. Then the energy in the open sea is present at wavelengths which are most favourable to the bending of the floe. Resonances also occur at higher frequencies, appearing as fine structural detail in the 60 m, 90 m, 100 m and 110 m curves especially, and causing spectral broadening in the 70 m, 80 m and 120 m curves, but there \( E(f) \) is smaller to begin with. For floes longer than 102 m in length the peaks begin to decrease in height again slightly. But multiple-cycle tuning can of course occur so that equivalently large strain density maxima occur for larger floe diameters still. For a 1 m thick ice floe, once a length of 102 m has been exceeded strain spectral density peaks are always a significant fraction of those occurring in the 102 m curve with the peak never dropping below that for 80 m floes. The mean strains integrated over the bandwidth 0.04–0.4 Hz reveal a similar picture. As floe size is increased mean strain increases monotonically up to about \( 6.2 \times 10^{-4} \) which is generated in floes of about 100 m across. Beyond this mean strain does not change a great deal, its lowest subsequent value being about \( 5.9 \times 10^{-4} \). These results suggest a distinct upper bound to floe size: we would not expect to see a gradual change in floe diameter in the MIZ composed of ice of constant 1 m thickness; we would expect either no floes above about 80 m to exist or vast floes.
Figure 3.10: Strain spectral densities for a 1 m thick ice floe at different lengths due to Pierson-Moskowitz spectral forcing centred at 0.125 Hz (8 s). The numbers marked on each curve represent the floe length in metres.
Aerial photography collected over various MIZs does in fact suggest that the floe size distribution changes abruptly at particular penetrations into the ice cover, thereby forming zones within the pack ice as described in Squire and Moore [1980]. Within each zone floe diameter is bounded above because floes above the critical size cannot exist in the ambient wave field occurring at that time. Large areas of homogeneous floe size distribution are a common feature of the MIZ.
Chapter 4

A Single Flexible Ice Floe in Water of Arbitrary Depth

4.1 Introduction

In the previous chapters we have only considered water of infinite depth and it is natural to pursue the equivalent theory for water of any depth. Firstly it is wise to clarify precisely what infinite, shallow, and finite water depth actually means. Clearly no water has infinite depth so why make this approximation in the first place? In actual fact water may be considered to have infinite depth, to an accuracy of .37%, providing that the water depth is greater than half the wavelength [Kinsman, 1984]. It is apparent therefore that for the vast majority of the MIZ the infinite depth assumption is valid and it is questionable whether there is any value solving at finite depth. Apart from the possible application of this theory to other floating objects and intellectual curiosity, the finite depth theory will prove vital in the analysis of the experimental work reported in Chapter 5.

At the other extreme, the water may be considered shallow, to 5% accuracy, when the wavelength is greater than 20 times the water depth (again from Kinsman [1984]). Thus this approximation, covered in any case by the finite water depth theory, is highly restrictive and inappropriate for most applications. The case of an elastic raft in shallow water was solved by Stoker [1957].

The derivation of the boundary value problem is identical to that for infinitely deep water except that the condition of no flow across the bottom boundary is applied at the water depth $H$, 

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rather than at infinity. This gives us the following system of equations,
\[ \begin{align*}
\nabla^2 \phi &= 0, \quad \infty < z < 0, \\
\frac{\partial \phi}{\partial z} &= 0, \quad z = H, \\
\frac{\partial \phi}{\partial z} + \alpha \phi &= 0, \quad z = 0, \quad -\infty < x < 0, \quad 1 < x < \infty, \\
\phi_x(x, 0) &= -\frac{\alpha}{\beta} \int_0^1 g(\xi, x) \phi(\xi, 0) d\xi, \quad z = 0, \quad 0 < x < 1, \\
\lim_{z \to \infty} \phi &= T \frac{\cosh \omega_0 (z - H)}{\cosh \omega_0 H} e^{-i \omega_0 x}, \\
\lim_{z \to -\infty} \phi &= \frac{\cosh \omega_0 (z - H)}{\cosh \omega_0 H} e^{i \omega_0 x} + R \frac{\cosh \omega_0 (z - H)}{\cosh \omega_0 H} e^{i \omega_0 z},
\end{align*} \]

where \( \omega_0 \) is the positive root of the equation,
\[ \alpha = \omega \tanh(\omega H). \]

To proceed with the solution of this boundary value problem we must first derive the Green's function for water of finite depth.

### 4.2 The Green's Function for Water of Finite Depth

We seek a function \( G \) which satisfies the following conditions:
\[ \begin{align*}
\nabla^2 G &= \delta(\xi - x) \delta(\eta - z), \quad -\infty < \xi, x < \infty, \quad 0 < \eta, z < H, \\
\frac{\partial G}{\partial \eta} + \alpha G &= 0, \quad \eta = 0, \quad -\infty < x < \infty, \\
\frac{\partial G}{\partial \eta} &= 0, \quad \eta = H.
\end{align*} \]

As before we will take the Fourier Transform with respect to \( \xi - x \) and derive the following conditions,
\[ \begin{align*}
\frac{d^2 \hat{G}}{d\eta^2} - \omega^2 \hat{G} &= \delta(\eta - z), \quad -\infty < \xi, x < \infty, \quad 0 < \eta, z < \infty, \\
\frac{d\hat{G}}{d\eta} + \alpha \hat{G} &= 0, \quad \eta = 0, \quad -\infty < x < \infty, \\
\frac{d\hat{G}}{d\eta} &= 0, \quad \eta = H,
\end{align*} \]

where \( \hat{G} \) is the function in Fourier transform space. Solving these equations we obtain the result that,
\[ \hat{G} = \frac{1}{\gamma} \left[ \cosh(\omega(\eta + z - H)) + \cosh(\omega(\eta - z - H)) \right] \]
\[ + \frac{\alpha}{\omega \gamma} \left[ \sinh(\omega(|\eta - z| - H)) - \sinh(\omega(\eta + z - H)) \right], \]
where $\gamma$ is,

$$\gamma = 2\alpha \cosh(\omega H) - 2\omega \sinh(\omega H).$$

Although the Green's function for finite depth is more complicated than for infinite depth it does have the advantage that it is analytic. $\hat{G}$ has two first order singularities on the real axis at $\omega = \pm \omega_0$. The residue from the singularities will be,

$$\text{Res}(\pm \omega_0) = \pm \frac{2\pi i}{\gamma'(\omega_0)} \left[ -\cosh(\omega_0(\eta + z - H)) \right]$$

$$= \pm \frac{\alpha 2\pi i}{\omega_0 \gamma'(\omega_0)} \left[ \sinh(\omega_0(\eta - z) - H)) - \sinh(\omega_0(\eta + z - H)) \right].$$

John [1950] derived this Green's function (equation (4.4)) and he expressed it in a number of forms, the one which most closely resembles our form is as follows;

$$G(x, z; \eta, \xi) = -\frac{1}{\pi} \int_0^\infty p(\omega) \cos \omega(\xi - x) d\omega, \quad 0 < \eta < z < H,$$

where,

$$p(\omega) = \cosh(\omega(\eta + H)) \frac{\omega \cosh \omega z + \alpha \sinh \omega z}{\omega \sinh \omega z - \alpha \cosh \omega z}.$$

It is obvious that John has expressed his Green's function as a cosine transformation. Since from equation (4.4) $\hat{G}$ is even, we may express the inverse Fourier transform as a cosine transformation and, imposing John's conditions that $0 < \eta < z < H$, we may write,

$$G(x, z; \xi, \eta) = \frac{1}{\pi} \int_0^\infty \hat{G} \cos \omega(\xi - x) d\omega,$$

where,

$$\hat{G} = \frac{1}{\gamma} \left[ \cosh(\omega(\eta + z - H)) + \cosh(\omega(z - \eta - H)) \right]$$

$$+ \frac{\alpha}{\omega \gamma} \left[ \sinh(\omega(z - \eta - H)) - \sinh(\omega(\eta + z - H)) \right].$$

This expression may be rearranged by use of the hyperbolic identities to give,

$$-\frac{p(\omega)}{\omega} = \hat{G},$$

so that the two Green's functions (equations (4.8) and (4.7)) are proved identical.

### 4.3 The Boundary Conditions

Although we could guess the boundary conditions from a combination of common sense and knowledge of the infinite water depth case, we will derive them as follows. We consider a region $\Delta$ with sides $\eta = 0$, $\eta = H$, $\xi = -\xi_0$, $\xi = \xi_0$ where we choose $\xi_0$ large enough to include $(x, z)$ and consider the limit as $\xi_0 \to \infty$. We know that the integral across the sea floor will vanish, i.e.,

$$\lim_{\xi_0 \to \infty} \int_{-\xi}^{\xi} (\phi G_n - \phi_n G) |_{\eta = H} d\eta = 0.$$
We also know that,
\[
\lim_{\xi_0 \to -\infty} \int_{-\infty}^{\infty} G e^{i\omega (\xi_0 - z)} d\omega = \frac{\text{Res}}{2},
\]
(4.10)
where the residues are those defined in equation (4.6). We define,
\[
q(\eta, z) = [\cosh(\omega_0(\eta + z - H)) + \cosh(\omega_0(|\eta - z| - H))] + \frac{\alpha}{\omega_0} [\sinh(\omega_0(|\eta - z| - H)) - \sinh(\omega_0(\eta + z - H))],
\]
(4.11)
and obtain the results that,
\[
\lim_{\xi_0 \to -\infty} G(\xi_0, \eta; x, z) = -q(\eta, z) \sin(\omega_0(\xi_0 - x)),
\]
(4.12)
and,
\[
\lim_{\xi_0 \to -\infty} G_n(\xi_0, \eta; x, z) = -\omega_0 q(\eta, z) \cos(\omega_0(\xi_0 - x)).
\]
(4.13)
We know that,
\[
\lim_{\xi_0 \to -\infty} \phi = T \cosh \omega_0 (\eta - H) \frac{\cosh \omega_0 H}{\cosh \omega_0 H} e^{-i\omega_0 \xi_0},
\]
(4.14)
and,
\[
\lim_{\xi_0 \to -\infty} \phi_n = -i\omega_0 T \cosh \omega_0 (\eta - H) \frac{\cosh \omega_0 H}{\cosh \omega_0 H} e^{-i\omega_0 \xi_0}.
\]
(4.15)
Therefore,
\[
\lim_{\xi_0 \to -\infty} \int_0^H (\phi G_n - \phi_n G) d\eta = \frac{-\omega_0 T e^{-i\omega_0 x}}{\gamma'/\omega_0} \int_0^H \eta q(\eta, x) \cosh \omega_0 (\eta - H) d\eta.
\]
(4.16)
Now we substitute \(\alpha = \omega_0 \tanh \omega_0 H\) and integrate to obtain,
\[
\int_0^H q(\eta, z) \cosh \omega_0 (\eta - H) d\eta = \frac{\cosh \omega_0 (z - H)}{\omega_0 \cosh \omega_0 H} \left( H \omega_0 + \sinh \omega_0 H \cosh \omega_0 H \right),
\]
(4.17)
and we know that,
\[
\gamma'/\omega_0 \cosh \omega_0 H = -2\omega_0 H - 2 \sinh \omega_0 H \cosh \omega_0 H,
\]
(4.18)
so that it follows that,
\[
\lim_{\xi_0 \to -\infty} \int_0^H (\phi G_n - \phi_n G) d\eta = \frac{T \cosh \omega_0 (z - H)}{2 \cosh \omega_0 H} e^{-i\omega_0 x}.
\]
(4.19)
Likewise,
\[
\lim_{\xi_0 \to -\infty} \int_0^H (\phi G_n - \phi_n G) d\eta = \frac{1}{2} \cosh \omega_0 (z - H) e^{-i\omega_0 x} + \frac{R \cosh \omega_0 (z - H)}{2 \cosh \omega_0 H} e^{i\omega_0 x}.
\]
(4.20)
4.4 The Integral Equation

Taking the limit \( x \to 0 \) we obtain the following integral equation for the potential under the raft,

\[
\phi(x, 0) = \frac{R \cosh \omega_0 (x - H)}{2 \cosh \omega_0 H} e^{i\omega_0 x} + \frac{1 + T \cosh \omega_0 (x - H)}{2 \cosh \omega_0 H} e^{-i\omega_0 x} + \int_0^1 G(x, 0; \xi, 0) \left( \alpha \phi(\xi, 0) - \phi_n(\xi, 0) \right) d\xi. \tag{4.21}
\]

As before we now substitute for \( R \) and \( T \) by considering the limit as \( x \to \pm \infty \) of the Green's function. We know that,

\[
\lim_{x \to \pm \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega (z-\xi)}}{\alpha - \omega \tanh \omega H} d\omega = \frac{i}{2} \left( -e^{i\omega_0 (z-\xi)} + e^{-i\omega_0 (z-\xi)} \right). \tag{4.22}
\]

Therefore the expressions for \( R \) and \( T \) for finite depth are,

\[
R = \frac{i\alpha/2}{\tanh kH + kH \text{sech}^2 kH} \int_0^1 e^{-i\omega k} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \xi') \phi(\xi', 0) d\xi' \right) d\xi, \tag{4.23}
\]

\[
T = 1 + \frac{i\alpha/2}{\tanh kH + kH \text{sech}^2 kH} \int_0^1 e^{i\omega k} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \xi') \phi(\xi', 0) d\xi' \right) d\xi,
\]

giving finally an integral equation for \( \phi(x, 0) \), where we have substituted for \( \phi_n \),

\[
\phi(x, 0) = e^{i\alpha x} + \alpha \int_0^1 \left( G(\xi, 0; x, 0) + \frac{i \cos (k(x - \xi))}{\tanh kH + kH \text{sech}^2 kH} \right) \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \xi') \phi(\xi', 0) d\xi' \right) d\xi. \tag{4.24}
\]

This integral equation is solved in the same way as that for infinite depth. The methods used to calculate the Green's function are outlined in Appendix F.

The results for finite depth do not differ markedly from those for infinite depth. In Figure 4.1 the effect on \(|R|\) of a change in water depth is plotted for an ice thickness of 1 m. Depth alters \(|R|\) in two ways: first, and most significantly, shallower water causes the envelope of \(|R|\) to decrease; and second, because of its effect on the wavelength beneath the ice, a decrease in depth leads to a slight change in the separation of the resonant points where \(|R|\) drops to zero.
Figure 4.1: Reflection coefficient against floe length for a wavelength of 100 m and an ice thickness of 1 m at the water depths marked on curves.
Chapter 5

An Experiment to Verify the Theory of Wave Interaction with an Elastic Sheet

5.1 Introduction

An experiment was carried out in the wave flume in the Department of Civil and Environmental Engineering at Clarkson University, Potsdam, USA, to validate the model developed in the earlier chapters to describe the interaction of waves with a floating elastic body. Specifically the spatial pressure field surrounding a floating elastic body was monitored as the body was acted on by a train of surface gravity waves. Since the theory as posed applies to a generic floating body whose elastic behaviour is governed by the Bernoulli-Euler equations, the experiments sought only to verify theory. The data collected do not and cannot in any way prove that the Bernoulli-Euler equations apply to floating sea ice.

5.2 Experiment

The wave flume was six feet (1.83 m) wide and approximately sixty feet long (20 m) with a paddle-type wave-maker at one end. The geometry of the tank reduces the problem to two dimensions because the wave-maker generates waves uniform in one spatial direction, and the structure of the waves is maintained by the geometry of the tank. Thus the experiment is a direct comparison of the two dimensional theory.

The elastic material chosen for the experiment was a polypropylene sheet six by four feet (1.22 m
Two sheets were used, of thickness one eighth of an inch (3.175 mm) and one quarter of an inch (6.35 mm). The Young's modulus of the polypropylene was measured experimentally by its deflection under a torque and was found to be $E = 1.8 \pm 0.2 \times 10^9$ Pa. The density of polypropylene is 900 kg m$^{-3}$.

The elastic sheet was held in a fixed position horizontally so that it could not undergo any long term drift but was free to move vertically. Waves propagated towards the elastic sheet from the wave paddle and some energy was transmitted and some was reflected. Within a few wave periods an equilibrium was established between the waves generated by the paddle and those reflected and transmitted. Therefore there will exist a standing wave type pattern in front of the raft where there is a spatial beating of wave amplitude. The transmitted energy was absorbed by a beach at the far end of the tank so that there was minimal reflection of wave energy except by the raft.

The wave paddle was capable of producing waves of minimum period $T = 0.8$ s, but the wave amplitude was very dependent on water depth so that for low period waves the water depth in the tank had to be reduced. Pressure on the floor of the tank was measured using a sensitive pressure probe. The measurements were made at about forty positions in front of, under, and behind the polypropylene raft. In general the measurements were made every 0.1 m, from 2 m in front of the raft to 0.5 m behind, although the measurements were closer together when the periods were lower. The measurements were of one hundred seconds duration each and the sample rate was 100 Hz. These data were subsequently analysed to extract amplitudes. This analysis was done using the Welch method of power spectral estimation running under Matlab. This method uses the Fast Fourier Transform (FFT) algorithm to calculate spectral densities.

Experiments were performed on both the polypropylene sheets at the following periods and water depths: at a water depth of 0.25 m and periods of 2 s, 1.8 s, 1.6 s, 1.4 s, 1.2 s and 1.0 s; at a water depth of 0.2 m and periods of 1.0 s, 0.9 s and 0.8 s. The wavelengths therefore varied between 1 m and 3 m. It is obvious that we are in the finite depth regime and it is of course a requirement that this be the case for the experiment to work otherwise the variation of pressure on the floor of the tank will become negligible.

5.3 Theory

As just mentioned the experimental measurements are in the range covered by the finite depth case. The theory developed so far has allowed us to calculate the velocity potential $\phi$ directly under the raft at the water surface. Firstly we must show how the pressure is related to the velocity potential. Following Kinsman [1984] we know that in our linearized case the variation of the hydrostatic pressure is zero and therefore the variation in the pressure is due entirely to the
potential term in Bernoulli's equation. That is,
\[ P = \rho \phi_t = \rho \sqrt{\alpha \phi}. \] (5.1)

Thus we require the potential on the floor of the tank. From our Green's function formulation we know that the potential on the floor of the tank, i.e., at \( z = H \) is,
\[
\phi(x, H) = \frac{Re^{i\omega_0 x}}{2 \cosh \omega_0 H} + \frac{(1 + T)e^{-i\omega_0 x}}{2 \cosh \omega_0 H} + \int_0^1 G(x, \xi, H, 0) (\phi(\xi, 0) - \phi_n(\xi, 0)) d\xi.
\] (5.2)

In the above expression the reflection and transmission coefficients are known, as is the potential at the surface and its normal. Thus the potential on the floor of the tank can be calculated straightforwardly provided we know the Green's function. We know that,
\[
G(x, 0, H, 0) = \frac{1}{\pi} \int_0^\infty \frac{\cos \omega x}{\cosh \omega H (\alpha - \omega \tanh \omega H)} d\omega.
\] (5.3)

So the Green's function may be found easily by the techniques outlined in Appendix F.

The measured pressure amplitude is proportional to the absolute value of the complex potential predicted by theory and the only scaling that is required is with respect to the incident wave amplitude. Thus the experimental amplitudes are multiplied by an appropriate constant to match most closely with the theoretical amplitudes.

### 5.4 Comparison of Experiment and Theory

The agreement between experiment and theory is good for the case of the thicker raft at low periods. Figure 5.1 shows the experimental and theoretical amplitudes for a water depth of 200 mm and a wavelength of 1.07 m for the 6.37 mm raft. The agreement between the two curves is excellent except on the beach side of the raft (position greater than 1.2 m). The divergence of the curves on the beach side of the raft is a feature in many of the curves and is possibly due to reflection at the beach or some mechanism of energy absorption, both of which will be discussed later. Also shown in Figure 5.1 is the theoretical curve for the case of an infinitely stiff raft. This shows that the full elastic theory is needed to explain the experimental results as the correlation is much poorer for the infinitely stiff raft. Figures 5.2 to 5.5 show more experimental and theoretical results for a range of wavelengths, all for the 6.37 mm raft. Again the comparison is good, especially when the many sources of possible error are taken into account. Note also that there was a general worsening of results as the wavelength became larger.

The experiment had a number of sources of experimental error. Under long wavelength regimes, especially with the thinner raft, the reflection was so small as to be beyond the resolution of the
Figure 5.1: Normalized amplitude against position relative to the front of the raft for a water depth of 200 mm and a wavelength of 1.07 m, for the 6.35 mm raft. The raft is located between 0 and 1.2 m and the waves are propagating from the left. The curves are, theoretical (solid), experimental (dashed) and theoretical for an infinitely stiff raft (dotted).
Figure 5.2: Normalized amplitude against position relative to the front of the 6.35 mm raft for a water depth of 200 mm and a wavelength of 0.89 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
Figure 5.3: Normalized amplitude against position relative to the front of the 6.35 mm raft for a water depth of 200 mm and a wavelength of 1.18 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
Figure 5.4: Normalized amplitude against position relative to the front of the 6.35 mm raft for a water depth of 248 mm and a wavelength of 1.35 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
Figure 5.5: Normalized amplitude against position relative to the front of the 6.35 mm raft for a water depth of 248 mm and a wavelength of 1.66 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
pressure sensor. The other problem was that the beach was not absorbing all the energy and was reflecting a proportion. This was especially true at long wavelengths. Thus measurements made with wavelengths greater than 2 m were dominated by the reflection from the beach rather than from the raft. This is shown in Figure 5.6 which clearly shows a spatial standing wave pattern unexplained by the theory. Note also that the variation in the theoretical curve is quite small. Clearly it is difficult to construct a perfect beach for waves with a wavelength of 3 m in water of depth 0.25 m. It is relevant that the beach had been set up for water depths of around 0.5 m. It is also an unfortunate fact that even a small reflection of energy leads to a large effect in this experimental set up as the following example will explain:

Consider a beach which reflects 1% of the incoming energy absorbing the rest. The reflected wave will therefore have an amplitude of 10% of the incoming wave since energy is proportional to amplitude squared. The wave amplitude will beat spatially with an amplitude of ±10%, that is 20%. Therefore we have gone from an energy reflection of only 1% (which seems perfectly believable) to a 20% beating in amplitude. In fact in Figure 5.6 the beating is 0.80 ± 0.15, which implies the beach is reflecting about 3.5% of the incoming energy.

Figure 5.7 shows the experimental and theoretical results for the 3.175 mm raft in 200 mm of water with a wavelength of 0.88 m. The experiment was less successful for the thinner raft at low periods, but the reasons for this are not simple. The most likely reasons are that the amplitudes at these periods were too great for the thin raft, and that there was significant interference with the waves breaking over its upper surface. This wave breaking was certainly an observed phenomenon, although the extent to which it interfered with the experiment is unclear. Figure 5.7 does not show any strong evidence of a spatial standing wave pattern of any kind. Viscoelastic effects seem unlikely to provide an explanation since the relaxation time scale of polypropylene is in the order of hours.
Figure 5.6: Normalized amplitude against position relative to the front of the 6.35 mm raft for a water depth of 248 mm and a wavelength of 2.41 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
Figure 5.7: Normalized amplitude against position relative to the front of the 3.175 mm raft for a water depth of 200 mm and a wavelength of 0.88 m. The curves are, theoretical (solid) and experimental (dashed). The raft was located between 0 and 1.2 m and the waves propagated from the left.
Chapter 6

Comparison with the
Semi-Infinite Theory of Fox and Squire

6.1 Introduction

In this chapter we show that the model which we have developed for the reflection and transmission
of waves from a floe of finite length is related to the theory developed by Fox and Squire [1990, 1991]. The following account is based on work reported by Meylan and Squire [1994b]. We begin
with a brief review of the Fox and Squire theory.

6.2 The Semi-Infinite Model

The problem that Fox and Squire solved was the problem of the reflection and transmission of
waves from a semi-infinite ice sheet. They used exactly the same model for the ice sheet as we
have used and the two problems are very similarly posed. Perhaps the most important difference is
that, because of the method of solution used, they considered only finite depth. This of course does
not exclude the infinite depth case since one need only set the depth suitably large. Essentially
the problem they solved was the following (non-dimensionalizing as in section 2.2),

\[
\begin{align*}
\nabla^2 \phi &= 0, \quad -\infty < x < \infty, \quad 0 < z < H, \\
\frac{\partial \phi}{\partial z} &= 0, \quad -\infty < x < \infty, \quad z = H, \\
\frac{\partial \phi}{\partial z} + \alpha \phi &= 0, \quad -\infty < x < 0, \quad z = 0, \\
\frac{\partial^2 \phi}{\partial z^2} + \alpha \frac{\partial \phi}{\partial z} &= \alpha \gamma \frac{\partial \phi}{\partial z} - \beta \frac{\partial^2 \phi}{\partial x^2 \partial z}, \quad 0 < x < \infty, \quad z = 0, \\
\frac{\partial^3 \phi}{\partial z^3} &= \frac{\partial^4 \phi}{\partial x^4 \partial z} = 0, \quad x = 0^+, \quad z = 0,
\end{align*}
\] (6.1)

together with conditions as \( z \to \pm \infty \). These conditions are that there is an input wave of unit amplitude, and a reflected wave as \( x \to -\infty \) and a transmitted wave as \( x \to \infty \). Comparison of this system of equations and the system of equations (2.6) shows just how similar the two problems are.

The method of solution used by Fox and Squire involved the expansion of the solutions on either side of the ice-water interface into modes and their subsequent matching by variational techniques. The most important thing to realise in what follows is that their matching involved the use of so called evanescent modes. These modes correspond to solutions of the dispersion equation with imaginary roots which decay with horizontal distance from the ice edge. Although they carry no energy they are essential to get a potential match at all depths.

6.3 The Finite-Floe Model

Our starting point for the finite-floe model is the reflection/transmission coefficient pair, \( R \) and \( T \), as derived for the semi-infinite ice sheet above. These coefficients respectively denote the wave amplitude of the ocean wave reflected back from the ice edge and the amplitude of the ice-coupled wave proceeding into the ice, for an open water wave of unit amplitude incident on a semi-infinite ice sheet. We now consider the reverse problem, namely, finding the reflection and transmission coefficient pair for a wave impinging on the ice edge from within the ice sheet.

6.3.1 Reversed Time

We apply time reversal to the semi-infinite ice sheet problem; this is directly analogous to the Stokes time reversal that is used in optics. This must be applied with care, as our new wave functions will be proportional to \( e^{-i\omega t} \) and not \( e^{i\omega t} \). Since it is only the real part of the wave function that has physical meaning, we may represent a wave in one of two ways: either as \( \Re\{A e^{i(\omega t + kx)}\} \) or as \( \Re\{A^* e^{-i(\omega t + kx)}\} \), where the symbol \( \Re \) indicates the real part and the * denotes the complex conjugate. Previously we have assumed proportionality to \( e^{i\omega t} \), so it is therefore simplest to take complex conjugates. We now apply the water-to-ice reflection/transmission results to the "reflected
wave" which has magnitude $\mathcal{R}^*$. This wave will give rise to a transmitted component $T\mathcal{R}^*$ and a reflected component $|\mathcal{R}|^2$. Hence for consistency the "transmitted wave" must lead to a reflected component $-T\mathcal{R}^*$ and a transmitted component $1 - |\mathcal{R}|^2$. Thus the reflection and transmission coefficients for the transition from ice to water in terms of those from water to ice are,

$$-\frac{\mathcal{R}^*T}{T^*}, \quad \text{and} \quad \frac{1 - |\mathcal{R}|^2}{T^*},$$

respectively.

### 6.3.2 Determination of $R$ and $T$

We now consider an ice floe of finite length $l$, as shown in Figure 6.1. Implicit in the following discussion is that the floe is of sufficient length that the evanescent and damped modes generated at an ice edge exist only locally to that edge, and that by the time the far edge is reached these components have decayed to a negligible fraction of the principal transmitted wave. In practice, this means that the following analysis will be accurate when floes are longer than the wavelength in the ice, irrespective of the water depth. Because of the length of the floe there is a phase change between its ends given by $\kappa l$, where $\kappa$ is the wave number under the ice and must satisfy the following dispersion relation (in non-dimensionalized form),

$$\alpha = \frac{\beta \kappa^5 + \kappa}{\kappa \gamma + \coth \kappa H}.$$  

We may write down the following simultaneous equations for the amplitude coefficients,

$$R = \mathcal{R} + \frac{1 - |\mathcal{R}|^2}{T^*} b,$$

$$a = T - \frac{\mathcal{R}^*T}{T^*} b,$$

$$b = -\frac{\mathcal{R}^*T}{T^*} ae^{-2i\kappa l},$$

$$T = \frac{1 - |\mathcal{R}|^2}{T^*} ae^{i(k\kappa - \kappa l)}.$$

These equations have solution,

$$R = \mathcal{R} - \frac{\mathcal{R}^*T^2(1 - |\mathcal{R}|^2)}{T^*e^{2i\kappa l} - T^2\mathcal{R}^*},$$

and,

$$T = \frac{TT^*(1 - |R|^2)}{T^*e^{2i\kappa l} - T^2\mathcal{R}^*},$$

which satisfy the energy balance equation, $|R|^2 + |T|^2 = 1$, as required.

### 6.3.3 Resonance

If we now substitute,

$$\psi = \kappa l + \text{Arg}(\mathcal{R}) - \text{Arg}(T/T^*),$$


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Figure 6.1: Schematic diagram showing geometry of problem and the coordinate system used to solve the semi-infinite problem. For the finite-floe solution the region is effectively divided into three domains, $-\infty < x < 0, 0 < z < H$, $0 \leq z \leq l, 0 < z < H$, and $l < z < \infty, 0 < z < H$. 
into equation (6.6), we obtain the following expression for $R$ in terms of $\mathcal{R}$,

$$\frac{R}{\mathcal{R}} = \frac{e^{2i\psi} - 1}{e^{2i\psi} - |\mathcal{R}|^2}. \quad (6.8)$$

The right-hand side of equation (6.8) is a biharmonic transformation which maps the unit circle $e^{2i\psi}$ to a circle with centre at $1/(1 + |\mathcal{R}|^2)$ and radius $1/(1 + |\mathcal{R}|^2)$. It is clear from equation (6.8) that the finite-floe reflection coefficient $R$ vanishes when $\psi = n\pi$, where $n$ is an integer, and that for a fixed floe thickness, $R$ will be a periodic function of floe length. Thus a discrete spectrum of frequencies yields no reflected wave. The zeros in $R$ correspond to perfect transmission; a situation which is directly analogous to the transmission of light through a finite homogeneous slab. Like the optics case, perfect transmission occurs at half-wavelength intervals.

The maximum value of the finite-floe reflection coefficient $R$ is $2R/(1 + |\mathcal{R}|^2)$, which occurs when $\psi = (2n - 1)\pi/2$.

### 6.3.4 Comparison with the Finite Floe Theory

We now compare the theory developed so far with the full finite floe theory of Chapter 2. Figures 6.2, 6.3, and 6.4 make the comparison for a 1 m, 2 m, and 5 m ice floe respectively; incident wavelength is 100 m, and the water is assumed to be infinitely deep. In the 1 m case the two solutions merge at about 80 m, i.e., floes of length greater than 80 m will be closely modelled by the semi-infinite theory using equations (6.5) and (6.6). The approximation becomes worse as floe thickness is increased: for 2 m of ice the fit is excellent for lengths beyond 130 m; for 5 m, floes must be longer than 225 m to achieve an excellent fit. This is to be expected, as the wavelength in ice increases with ice thickness. For very large floe lengths there is some suggestion that the curves diverge slightly, particularly in Figures 6.2 and 6.3. However, this is actually an indication that the finite floe theory is becoming imprecise, rather than imprecision in the value of $|\mathcal{R}|$ calculated by equations (6.5); the semi-infinite theory naturally works best for very long ice floes. The imprecision in the finite floe model is due to the discretization employed to solve the theory’s final integral equation, which requires more points as the floe is made longer.

The comparison made in Figures 6.2, 6.3, and 6.4 is very stringent since, although only $|\mathcal{R}|$ is plotted, the phase of the two models must be identical for the curves to overlay so precisely.
Figure 6.2: The absolute value of the reflection coefficient $|R|$ as a function of floe length for an incident open water wavelength of 100 m. Floe thickness is 1 m. The solid curve represents $|R|$ as computed by the semi-infinite method, while the dashed curve is found using the finite floe method. The wavelength beneath the ice is about 127 m. The curves coalesce when the floe length is greater than 100 m and this corresponds to the point at which the first evanescent mode in the ice has decayed to become negligible. The slight difference at large floe length is due to numerical error.
Figure 6.3: Equivalent curves to Figure 6.2 for 2 m ice thickness. The wavelength beneath the ice is about 170 m. The curves coalesce for floe lengths greater than 140 m and this corresponds to the point at which the first evanescent mode in the ice has decayed to become negligible.
Figure 6.4: Equivalent curves to Figure 6.2 for 5 m ice thickness. The wavelength beneath the ice is about 271 m. The curves coalesce for floe lengths greater than 230 m and this corresponds to the point at which the first evanescent mode in the ice has decayed to become negligible.
Chapter 7

Wave Interaction with more than one Floe.

7.1 Introduction

The theory for the interaction of ocean waves with one solitary floe derived in earlier chapters can be extended to include two or more floes. This may be done precisely, taking into account interactions between one or more floes, or as an approximation based on the single floe theory developed so far applied serially.

7.1.1 Complete Theory

The method of solution for one floe may be generalized to two or more floes. Here we only generalize to two floes; the n-floe case follows straightforwardly. Consider two adjacent floes, scaling so that the first is between $0 < x < 1$ and the second lies between $d < x < d + l$ (clearly we must have $1 < d$). The integral equation for the two floes will then be,

$$
\phi(x,0) = \frac{R}{2} e^{i\alpha x} + \frac{1 + T}{2} e^{-i\alpha x} + \int_0^1 G(\xi,0;x,0)(\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi
$$

$$
+ \int_d^{d+l} G(\xi,0;x,0)(\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi.
$$

(7.1)

As before we substitute for $R$ and $T$ to obtain,

$$
R = i\alpha \left( \int_0^1 e^{-i\alpha \xi} (\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi + \int_d^{d+l} e^{-i\alpha \xi} (\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi \right),
$$

(7.2)

and,

$$
T = 1 + i\alpha \left( \int_0^1 e^{i\alpha \xi} (\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi + \int_d^{d+l} e^{i\alpha \xi} (\alpha \phi(\xi,0) - \phi_n(\xi,0))d\xi \right),
$$

(7.3)
so that the integral equation becomes,

$$
\phi(x,0) = e^{-i\alpha x} + \alpha \int_0^1 \left( G(\xi,0;x,0) + i \cos(\alpha(\xi - x)) \right) \\
\times \left( \phi(\xi,0) - \frac{1}{\beta} \int_0^1 g(\xi,\zeta)\phi(\zeta,0) d\zeta \right) d\xi \\
+ \alpha \int_d^{d+l} \left( G(\xi,0;x,0) + i \cos(\alpha(\xi - x)) \right) \\
\times \left( \phi(\xi,0) - \frac{1}{\beta} \int_d^{d+l} g(\xi,\zeta)\phi(\zeta,0) d\zeta \right) d\xi.
$$

(7.4)

This is again a linear Fredholm equation of the second kind for the potential,

$$
\phi(x,0), \quad 0 < x < 1, \quad \text{or} \quad d < x < d + l,
$$

and solution is straightforward using a Simpson’s rule, provided that we construct our integration rule in two parts, i.e., from $0$ to $1$ and from $d$ to $d + l$.

The solution at finite depth is similar, with each $i \cos \alpha(\xi - x)$ term divided by a factor $(\tanh kH + kH \text{sech}^2 kH)$.

### 7.2 Coherent Addition of Amplitude

We can construct a simple approximate theory for the reflection and transmission from two floes by treating them as though they have no interaction. The results presented here are based on those of Meylan and Squire [1994c 1994d]. We begin by considering a single floe subjected to waves of amplitude $t_{j-1}$ from the left and of amplitude $r_j$ from the right, which gives rise to waves of amplitude $r_{j-1}$ on the left and $t_j$ on the right. The floe is considered to have reflection and transmission coefficients $R_j$ and $T_j$ respectively. The front of the raft is assumed to be at $x = d_j$ and the far end at $x = d_j + l_j$. This is shown in Figure 7.1. We then derive the following system of equations,

$$
\begin{align*}
    r_{j-1} &= R_j e^{-2i\alpha d_j} t_{j-1} + T_j r_j, \\
    t_j &= T_j t_{j-1} + R_j e^{2i\alpha(d_j+l_j)} r_j,
\end{align*}
$$

(7.5)

where $\alpha$ is the non-dimensionalized wavenumber as before. We write these equations as the following matrix equation,

$$
\begin{pmatrix}
    T_j & 0 \\
    -R_j e^{-2i\alpha d_j} & 1
\end{pmatrix}
\begin{pmatrix}
    t_{j-1} \\
    r_{j-1}
\end{pmatrix} =
\begin{pmatrix}
    1 & -R_j e^{2i\alpha(d_j+l_j)} \\
    0 & T_j
\end{pmatrix}
\begin{pmatrix}
    t_j \\
    r_j
\end{pmatrix},
$$

(7.6)

or,

$$
\begin{pmatrix}
    t_{j-1} \\
    r_{j-1}
\end{pmatrix} =
\frac{1}{T_j}
\begin{pmatrix}
    1 & -R_j e^{2i\alpha(d_j+l_j)} \\
    R_j e^{-2i\alpha d_j} & T_j^2 - R_j^2 e^{2i\alpha l_j}
\end{pmatrix}
\begin{pmatrix}
    t_j \\
    r_j
\end{pmatrix}.
$$

(7.7)
Figure 7.1: Schematic diagram showing a single floe subject to waves from all directions.
It is clear how this result can be generalized to \( n \) floes. We simply consider the matrix relationship for the various waves between the floes and impose the conditions that \( t_0 = 1 \) and \( r_n = 0 \). For the case of two floes we may write down the following expressions for the reflection and transmission coefficients, \( R \) and \( T \), for the two floes combined,

\[
\begin{align*}
R &= R_1 + \frac{R_2 T_2 e^{-2\imath \alpha d}}{1 - e^{2\imath \alpha (1-d) R_1 R_2}}, \\
T &= \frac{T_1 T_2}{1 - e^{2\imath \alpha (1-d) R_1 R_2}}.
\end{align*}
\] (7.8)

### 7.3 Some Results

To illustrate an application of the complete two-floe theory the magnitude of the reflection coefficient, \(|R|\), is plotted against floe separation for a pair of 100 m floes, a 100 m and a 50 m floe, and a 100 m and a 200 m floe in Figure 7.2. The wave period chosen was 10 s so the wavelength is 156 m. In the case of two floes of the same length zeros occur in \(|R|\) corresponding to perfect transmission, rather like those for the solitary floe. When two floes of different length are modelled no zeros occur, but \(|R|\) oscillates between two (positive) values.

The anomaly between the complete two-floe theory (equation (7.4)) and the approximate non-interactive theory (equations (7.8)) is shown in Figure 7.3. The ordinate axis represents the absolute value of the difference in the reflection coefficients as computed by the two theories for two identical 100 m ice floes, and the wavelength is again 156 m. It is clear that the non-interactive theory is amply precise except when the floes are very close together. Even at 10 m separation the anomaly in \(|R|\) is only about \(1.8 \times 10^{-3}\). While different floe geometries will lead to different anomalies, the difference in the predictions of the two theories is always small.

### 7.4 The Incoherent Addition of Energy

We now consider the case where the energy adds incoherently. That is, instead of adding the wave amplitudes, we will add the wave energies. This will eliminate the variation in reflection due to the separation of the floes and is the approach used by Wadhams [1986]. We allow for some dissipation of energy by reflecting only a proportion \( \nu \) of the energy that would be reflected in a perfectly conservative system. Thus the equations are (where the coefficients now refer to the energies not amplitudes),

\[
\begin{align*}
r_{j-1} &= \nu R_j t_{j-1} + T_j r_j, \\
t_j &= T_j t_{j-1} - \nu R_j r_j,
\end{align*}
\] (7.9) (7.10)

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Figure 7.2: Reflection coefficient $|R|$ against floe separation for a pair of identical 100 m ice floes (solid curve), a pair of floes of length 50 m and 100 m (dashed curve), and a pair of length 200 m and 100 m (dotted curve). Ice thickness is 1 m and the wavelength is 156 m. The coefficient is calculated by the full interaction theory.
Figure 7.3: Difference between the complete two-floe theory and the non-interactive theory. The floes were both 1 m × 100 m. The difference between the two models is small even when the floes are practically touching. The decay in the anomaly plot corresponds to the decay in the first evanescent mode in the water.
so that,

$$
\begin{pmatrix}
 t_j
 \\
 r_j
\end{pmatrix}
 =
 \frac{1}{T_j}
 \begin{pmatrix}
 1
 & -\nu R_j
 \\
 \nu R_j & T_j^2 - \nu R_j^2
\end{pmatrix}
 \begin{pmatrix}
 t_{j-1}
 \\
 r_{j-1}
\end{pmatrix}.
$$

(7.11)

In fact we will deal only with the two cases $\nu = 1$ and $\nu = 0$, that is perfect reflection and no reflection. In what follows we set $t_0 = 1$, $r_n = 0$ and solve for the transmitted energy after the waves have passed through $n$ floes, $t_n$, and the energy reflected from the ice edge, $r_0$.

### 7.5 Results for a Distribution of Floe Sizes

Practical problems of navigation and the construction of offshore structures in ice infested seas require knowledge of the wave spectrum inside the ice cover. This will allow the prediction of ice floe induced forces as well as wave forces. For this reason the extension of the theory developed so far for a single floe to the case of many floes is important. We consider a distribution of floes and calculate the wave propagation as a function of frequency and ice cover. The treatment will at first be theoretical and then we shall consider more physically realistic problems.

#### 7.5.1 Theoretical Distribution

We consider a distribution of floe lengths given by the Rayleigh distribution which has probability function,

$$
f(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}, \ x \geq 0,
$$

[Hoffman and Karst, 1975] and mean $\mu = \alpha \sqrt{\pi/2}$. The density function is plotted in Figure 7.4 for an average floe size of 20 m, 50 m and 100 m. The distribution rises to a maximum and then tapers off. The Rayleigh distribution was chosen because it has the same basic shape as the floe size distributions measured by Wadhams et al. [1983] and as we shall see, fits with our measured data. In practice this distribution did not give significantly different results for the reflection and transmission coefficients than those obtained from a uniform distribution with the same mean.

#### 7.5.2 Incoherence and Coherence

Our first consideration is exactly which of the scattering theories would be most appropriate to use to calculate the effect of many floes. In Figure 7.5 we have plotted the transmitted energy, $t_n$, as a function of frequency for the same random Rayleigh distribution of 100 floes of average length 100 m calculated by three different techniques. The dashed line corresponds to coherent addition of amplitude, the solid line to incoherent addition of energy with full reflection and the chained line to incoherent addition of energy allowing no reflection. For the case of coherent addition we assumed that the floe separation was distributed randomly between 0 and 10 m. All the curves
Figure 7.4: The Rayleigh distribution of floe sizes with average sizes of 20 m, 50 m and 100 m as shown.
The transmitted energy, $t_{nt}$, after the waves have passed through an ice cover of 100 floes with Rayleigh distributed average length 100 m and thickness 1 m, as a function of frequency calculated using three techniques. The solid line is found by assuming the incoherent addition of energy with full reflection, the dashed line by considering the coherent addition of amplitude, and the chained line by considering the incoherent addition of energy allowing no reflection. The same Rayleigh distribution of 100 floes was used in all cases.
have the behaviour expected, with perfect transmission at low frequencies, no transmission at high frequencies, and a transition region between. The incoherent transmission functions are both smooth curves, with the case of no reflection always laying beneath that of perfect reflection. The coherent curve shows a chaotic transition region which varies markedly from one distribution to the other. This is due to the fact that in the coherent model there are many critical frequencies at which the transmitted energy, $t_n$, is unity. These critical frequencies lie across the transition band and because the resolution at which we can calculate $t_n$ is not sufficient we see only a series of apparently random spikes. This situation is made clearer in Figure 7.6 where we have reduced the number of floes to 4. In this case although the spikes do not go to unity due to the insufficient resolution their structure is clear. As the number of floes increases the spikes become closer together and more numerous. The existence of these critical frequencies appears quite unphysical and justifies the use of the incoherent model in most published work [Wadhams, 1986] and in what follows.

### 7.5.3 MIZ Filtering

In Figure 7.7 we assume an average floe length of 100 m, i.e. the floe lengths were chosen randomly from a Rayleigh distribution with mean 100 m. We then plot the transmitted energy for 20, 50, 200, and 400 floes. For each value we have plotted 10 curves to give some measure of statistical variation. As expected this variation decreases markedly as $n$ increases. The basic low pass filter structure of the transmitted energy is the same for all curves, falling from 1 to 0 in the frequency range plotted. As the number of floes is increased so the transition band occurs at a lower frequency and with greater sharpness.

Figure 7.8 presents similar results except that the number of floes is fixed at 100 and the average floe size is varied. It is apparent that there is a marked change in $t_n$ as the floe size is increased until the floe size rises above 100 m. The transition band of the filter becomes sharper with increasing floe size and $t_n$ is reduced generally. Above 100 m the average floe size does not strongly affect the transmitted energy, since the 100 m and 400 m curves are virtually indistinguishable. Note that 100 floes of average size 100 m will occupy considerably less space than 100 floes of average size 400 m.

In Figure 7.9 we have considered the effect of floe thickness on $t_n$. We have assumed a Rayleigh distribution of 100 floes of mean length 100 m and have chosen floes of thickness 0.5 m, 1 m, 2 m and 5 m. It is clear that the floe thickness has a marked effect on the transmitted energy, with the centre frequency of the transition band decreasing with increasing floe thickness.

Figure 7.10 shows the the transmitted energy 5 km into the ice cover as a function of average floe length for floes of 1 m thickness. Waves of frequencies 0.08 Hz, 0.1 Hz, 0.12 Hz and 0.14 Hz
Figure 7.6: The transmitted energy, $t_n$, after the waves have passed through an ice cover of 4 floes with Rayleigh distributed average length 100 m and thickness 1 m, as a function of frequency calculated using three techniques. The solid line is found by assuming the incoherent addition of energy with full reflection, the dashed line by considering the coherent addition of amplitude, and the chained line by considering the incoherent addition of energy allowing no reflection. The same Rayleigh distribution of 4 floes was used in all cases. The hump at 0.15 Hz in the coherent curves is a coincidence due to the particular floe geometry.
Figure 7.7: The transmitted energy, $t_n$, as a function of frequency calculated 10 times with the number of floes shown. The floes were assumed to have thickness 1 m and their lengths were determined randomly from a Rayleigh distribution with 100 m mean. The incoherent theory was used assuming perfect reflection.
Figure 7.8: The transmitted energy, $t_n$, as a function of frequency calculated 10 times with the average floe length shown. There were assumed to be 100 floes of 1 m thickness whose lengths were determined randomly from a Rayleigh distribution with mean given by the values attached to the curves. The incoherent theory was used assuming no dissipation.
Figure 7.9: The transmitted energy, $t_n$, as a function of frequency for floes of thickness shown on the curves. There were assumed to be 100 floes whose lengths were determined randomly from a Rayleigh distribution with a mean floe length of 100 m. The incoherent theory was used assuming no dissipation.
Figure 7.10: The transmitted energy, $t_n$, as a function of average floe length, for waves of frequency shown on the curves. The distance into the ice cover was fixed as 5 km, and the appropriate number of floes of 1 m thickness were chosen from a Rayleigh distribution. The calculations were performed 100 times and averaged to give some smoothing. Incoherent theory was used assuming no dissipation.
were considered. Since these frequencies lie in the transition band for floes of this thickness there
was a marked variation with frequency. As the floe size is increased from zero, $t_n$ falls because
the larger floes have a lower transmitted energy. But once the floe length exceeds a critical value
$T_n$ rises again because the total number of floes within the 5 km stretch of the ice cover decreases
significantly. The results for very small floes are complicated by the small peak in the individual
floe reflection and transmission coefficients visible in Figure 3.1. This is essentially due to the fact
that for small floes the Rayleigh distribution is not sufficiently wide to smooth out the sharp peaks
in the individual floe reflection and transmission coefficients, again visible in Figure 3.1.

7.5.4 Spectral Evolution

We now apply these results to some spectra using the Pierson-Moskowitz spectrum outlined in
section 3.6. We assume that the peak frequency is at 0.1 Hz, i.e., a peak period of 10 s. Figure
7.11 shows the effect on a spectrum of a Rayleigh distribution of floes of average length 100 m
and thickness 1 m. The number of floes was 10, 50 and 100, and the results were calculated using
incoherent addition of energy with no dissipation. In each case the spectrum was reduced, the
reduction being negligible at low frequencies and almost 100% at high frequencies.

7.5.5 Directional Changes

We now consider the effect that an MIZ may have on the directional characteristics of the wave
spectrum. We assume a sea whose spectrum is isotropic in direction, i.e., in the open ocean the
waves are travelling in all directions equally. It is then assumed that the wave amplitude within
the ice cover is due to the transmitted energy, $t_n$, calculated for waves which have travelled the
appropriate distance. Waves whose direction of propagation is far from the normal to the ice edge
will have to travel further and hence will have a lower amplitude. Thus the directional nature of
the sea will evolve with penetration into the ice cover. This is shown in Figure 7.12 where we have
considered the effect on an isotropic spectrum propagating through a Rayleigh distributed MIZ
with mean length 100 m and floes of thickness 1 m. It is clear that waves whose angle of propagation
is large compared to the normal are attenuated greatly. There is also an increasing dissipation
with distance into the ice cover. Figure 7.13 is a similar plot except that the distance into the
ice cover is fixed at 5 km and frequencies of 0.08 Hz, 0.1 Hz, 0.12 Hz and 0.14 Hz are considered.
Again since these frequencies straddle the transition band the variation in frequency has a marked
effect, with the transmitted energy showing a marked decrease with increasing frequency.

7.5.6 Data from a Photograph

We consider a distribution of floe sizes taken from an aerial photograph, shown in Figure 7.14. A
Figure 7.11: A Pierson-Moskowitz spectrum with peak frequency 0.1 Hz (solid line) together with the attenuated spectrum after passage through 10 floes (dashed line), 50 floes (dotted line) and 100 floes (chained line). The floe sizes were determined by a Rayleigh distribution with average length 100 m and thickness 1 m.
Figure 7.12: The transmitted energy, $t_n$, as a function of angle from the normal, for the distances into the ice cover shown on the curves. The floes were Rayleigh distributed with an average length 100 m, and thickness 1 m. The frequency was 0.1 Hz and the calculations were performed 100 times and averaged to give some smoothing. The incoherent theory was used assuming no dissipation.
Figure 7.13: The transmitted energy, \( t_n \), as a function of angle from the normal, for the frequencies shown on the curves. The fies were Rayleigh distributed with an average length 100 m, and thickness 1 m. The distance into the ice cover was 5 km and the calculations were performed 100 times and averaged to give some smoothing. The incoherent theory was used assuming no dissipation.
Figure 7.14: An aerial photograph of ice floes from which data concerning the distribution of floe size was obtained.
number of lines were placed across the photograph and the floe size information was measured. The histogram of the data obtained is shown in Figure 7.15 for the measurements made from the photograph. The average floe size is approximately 25 m. The floe size distribution decays exponentially with increasing floe size and it also appears to show a decrease as the floe size goes to zero. This is in basic agreement with the Rayleigh distribution used previously. In Figure 7.16 we have constructed a distribution of floes by sampling randomly from our measured distribution, applying it to the Pierson-Moskowitz spectrum used previously. We have considered a distribution of 10, 50 and 100 floes and as before used the conservative incoherent theory. In this case, since the average floe size is less, there is not the same shift in the peak frequency but there is marked dissipation of high frequency energy and there is an increasing dissipation with a larger number of floes.

7.6 Incoherent Addition with \( n \) Identical Floes

Having calculated the effect of 200 floes of random length we see that there is little variation from one sample to the next. Thus we may consider 400 floes as being composed of two groups of 200 floes and apply our results to these two groups, treating them as defining a single unit. It would also be possible to introduce a damping mechanism at this stage. We will consider in this section the problem of \( n \) identical floes, or what is more physically real, \( n \) identical groups of floes. We shall assume that full reflection occurs, and that the floes or groups of floes have reflection and transmission coefficients \( R \) and \( T \) respectively (again referring to energy). Therefore equation (7.11) becomes,

\[
\begin{pmatrix} t_{j-1} \\ r_{j-1} \end{pmatrix} = \frac{1}{T} \begin{pmatrix} 1 & -R \\ R & T^2 - R^2 \end{pmatrix} \begin{pmatrix} t_j \\ r_j \end{pmatrix}.
\]

We define,

\[
M = \frac{1}{T} \begin{pmatrix} 1 & -1 + T \\ 1 - T & 2T - 1 \end{pmatrix},
\]

or using a similarity transformation,

\[
M = \begin{pmatrix} 1 & 0 \\ 1 & T/(T - 1) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & T/(T - 1) \end{pmatrix}^{-1}.
\]

Therefore if we consider \( n \) identical groups of floes, we obtain the following system:

\[
\begin{pmatrix} 1 \\ r_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & T/(T - 1) \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & T/(T - 1) \end{pmatrix}^{-1} \begin{pmatrix} t \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 - n + n/T & n - n/T \\ -n + n/T & 1 + n - n/T \end{pmatrix} \begin{pmatrix} t_n \\ 0 \end{pmatrix},
\]

which gives,

\[
t_n = \frac{1}{1 - n + n/T},
\]
Figure 7.15: The histogram of the floe sizes as measured from the aerial photograph.
Figure 7.16: A Pierson-Moskowitz spectrum with peak frequency 0.1 Hz (solid line) together with the attenuated spectrum after passage through 10 floes (dashed line), 50 floes (dotted line) and 100 floes (chained line). The floe sizes were determined by random sampling from the data from an aerial photograph.
and,

$$r_0 = \frac{-n + n/T}{1 - n + n/T}. \quad (7.18)$$

Hence when $n$ is large,

$$t_n \approx \frac{1}{n} \left( \frac{T}{1 - T} \right). \quad (7.19)$$

This suggests that the presence of floes, even with small reflections and no dissipative mechanism, will effectively block the propagation of waves into the ice cover provided there are enough of them.
Chapter 8

Submergence of the Raft

8.1 Introduction

In all of the previous work we have assumed that the floe's submergence in the water is negligible compared to the wavelength, so that the boundary condition under the raft may be assumed to be at the surface of the water. This assumption was made by both Stoker [1957] and Tayler [1986]. It is clear that this assumption will not be valid in all situations and it is of interest to calculate the motion of the floe allowing for non-zero submergence. It will also be possible with this formulation to calculate the surge motion of the floe which is a frequently measured property (e.g. McKenna and Crocker [1992]). As before we use the linear theory and assume that the first order motions may be taken on the boundary defined by the zero order motions (i.e., the submerged position).

We assume that the wave amplitude and body motions are sufficiently small that products of these terms are negligible so that the problem reduces to that of integrating the pressure over the floe in its rest position [Newman, 1980]. We then integrate around the boundary of the body and the water surfaces as before. The surface around which we integrate, $\Gamma$, is shown in Figure 8.1, where $h'$ is the draft. The conditions at $x = \pm \infty$ will be the same as before, so that we may derive the following equation from Green's theorem, provided $x \notin \Gamma$ but is within the region of integration (i.e., the water):

$$\phi(x, z) = \frac{R}{2} e^{iax - az} + \frac{1 + T}{2} e^{-iax - az} + \int_{\Gamma} (\phi G_n - \phi_n G) ds, \quad x \notin \Gamma. \quad (8.1)$$

When $x \in \Gamma$ the $\delta$ function is on the border of the domain of integration and we must introduce a factor of a half so that equation (8.1) becomes,

$$\frac{1}{2} \phi(x, z) = \frac{R}{2} e^{iax - az} + \frac{1 + T}{2} e^{-iax - az} + \int_{\Gamma} (\phi G_n - \phi_n G) ds \quad x \in \Gamma. \quad (8.2)$$
Figure 8.1: Schematic diagram showing the lines of integration, $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma$.

\[ \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \]
We now consider suitable boundary conditions for the floe. Since we envisage this as an extension of our previous work we impose the same boundary condition as before under the floe. We also begin initially by imposing the condition of zero surge motion on the raft which implies that across the boundaries \( \Gamma_1 \) and \( \Gamma_3 \) we have \( \phi_n = 0 \).

### 8.2 Substituting for \( R \) and \( T \)

As before we use the infinite limit of the Green’s function and our knowledge about the boundary conditions at \( \pm \infty \) to express the reflection and transmission coefficients in terms of the integration around the surface \( \Gamma \). We now consider that \( \Gamma \) is composed of three lines \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \), as shown in Figure 8.1. Then,

\[
R = \int_{\Gamma_1} \phi \alpha e^{-\alpha \eta} \, ds + i \int_{\Gamma_2} (\alpha \phi - \phi_n) e^{-\alpha h'} e^{-i \alpha \xi} \, ds - \int_{\Gamma_3} \phi \alpha e^{-\alpha \eta} e^{-i \alpha} \, ds, \tag{8.3}
\]

and,

\[
T = 1 - \int_{\Gamma_1} \phi \alpha e^{-\alpha \eta} \, ds + i \int_{\Gamma_2} (\alpha \phi - \phi_n) e^{-\alpha h'} e^{i \alpha \xi} \, ds + \int_{\Gamma_3} \phi \alpha e^{-\alpha \eta} e^{i \alpha} \, ds. \tag{8.4}
\]

If we consider \( x \in \Gamma \) then from equation (8.2) we derive the following system of Fredholm integral equations,

\[
\frac{1}{2} \phi(x, z) = e^{-i \alpha x - \alpha z} + \int_{\eta=0}^{h'} \phi(0, \eta) \left( i \alpha e^{-\alpha (\eta + z)} \sin \alpha x + G_n(x, z; 0, \eta) \right) \, d\eta \\
+ \int_{\xi=0}^{1} \phi(\xi, h') \left( i \alpha e^{-\alpha (\xi + h')} \cos \alpha (x - \xi) + G_n(x, z; \xi, h') \right) \, d\xi \\
- \int_{\xi=0}^{1} \phi_n(\xi, h') \left( i e^{-\alpha (\xi + h')} \cos \alpha (x - \xi) + G(x, z; \xi, h') \right) \, d\xi \\
+ \int_{\eta=0}^{h'} \phi(1, \eta) \left( i \alpha e^{-\alpha (\eta + z)} \sin \alpha (1 - x) + G_n(x, z; 1, \eta) \right) \, d\eta, \tag{8.5}
\]

and,

\[
\begin{align*}
\phi_n(x) &= 0, \quad x \in \Gamma_1, \text{ and } x \in \Gamma_2, \\
\phi_n(x) &= \frac{\alpha}{\beta} \int_0^1 g(\xi, x) \phi(\xi, h') \, d\xi, \quad x \in \Gamma_2. \tag{8.6}
\end{align*}
\]

The calculation of the Green’s function and its normal derivative follows from Appendix E and the solution of the integral equations is straightforward.

### 8.3 Surge

The restriction of the motion to zero surge is an obvious approximation which we would like to remove. We will consider only surge motion and ignore any roll-type effects on the end of the raft.
Thus the boundary conditions on \( \Gamma_1 \) and \( \Gamma_3 \) become,

\[
\phi_n = A, \quad x \in \Gamma_1, \quad \text{and} \quad \phi_n = -A, \quad x \in \Gamma_3,
\]

(8.7)

where \( A \) is a constant to be determined from the equations of motion. We shall ignore effects from \( \Gamma_2 \) on the surge motion following Newman [1980] and consider Bernoulli’s equation integrated over the vertical surfaces, \( \Gamma_1 \) and \( \Gamma_3 \). We know that the pressure is given by,

\[
p(\xi, \eta) = -i \omega \rho \phi(\xi, \eta) + pg\eta.
\]

(8.8)

The hydrostatic pressure will be the same at either end of the raft due to our assumptions that we integrate the pressure field over the boundary of the zero order motions to calculate the force. We know that,

\[
a = i \omega \phi_n, \quad \text{on} \ \Gamma_1,
\]

(8.9)

where \( a \) is the horizontal acceleration. Therefore Newton’s second law gives,

\[
-i \omega \rho \int_0^{h'} (\phi(0, \eta) - \phi(1, \eta)) \, d\eta = i \omega h \rho' LA.
\]

(8.10)

Non-dimensionalizing gives,

\[
\phi_n(x) = A = -\frac{1}{\gamma} \int_0^{h'} (\phi(0, \eta) - \phi(1, \eta)) \, d\eta, \quad x \in \Gamma_1.
\]

(8.11)

Again we substitute for the reflection and transmission coefficients and derive the following system of Fredholm integral equations for the potential,

\[
\frac{1}{2} \phi(x, z) = e^{-i \alpha x - \alpha z} + \int_{\eta=0}^{h'} \phi(0, \eta) \left( ie^{-\alpha (\eta + z)} \sin \alpha x + G_n(x, z; 0, \eta) \right) \, d\eta
\]

- \( \int_{\eta=0}^{h'} \phi_n(0, \eta) \left( ie^{-\alpha (\eta + z)} \cos \alpha x + G(x, z; 0, \eta) \right) \, d\eta
\]

+ \( \int_{\xi=0}^1 \phi_n(\xi, h) \left( ie^{-\alpha (\eta + z)} \cos \alpha (x - \xi) + G_n(x, z; \xi, h') \right) \, d\xi
\]

- \( \int_{\xi=0}^1 \phi_n(\xi, h) \left( ie^{-\alpha (\eta + z)} \cos \alpha (x - \xi) + G(x, z; \xi, h') \right) \, d\xi
\]

+ \( \int_{\eta=0}^{h'} \phi(1, \eta) \left( ie^{-\alpha (\eta + z)} \sin \alpha (1 - x) + G_n(x, z; 1, \eta) \right) \, d\eta
\]

- \( \int_{\eta=0}^{h'} \phi_n(0, \eta) \left( ie^{-\alpha (\eta + z)} \cos \alpha (1 - x) + G(x, z; 1, \eta) \right) \, d\eta,
\]

(8.12)

and,

\[
\phi_n(x) = -\frac{1}{\gamma} \int_0^{h'} (\phi(0, \eta) - \phi(1, \eta)) \, d\eta, \quad x \in \Gamma_1,
\]

\[
\phi_n(x) = \frac{\alpha}{\beta} \int_0^1 g(\xi, x) \phi(\xi, h') \, d\xi, \quad x \in \Gamma_2,
\]

\[
\phi_n(x) = \frac{1}{\gamma} \int_0^{h'} (\phi(0, \eta) - \phi(1, \eta)) \, d\eta, \quad x \in \Gamma_2.
\]

(8.13)
8.4 The Froude-Kriloff Limit

In naval architecture a common assumption is the so called Froude-Kriloff approximation [Froude, 1861; Kriloff, 1898] which assumes that the wave potential is unaltered by the presence of the obstacle. This effectively assumes that the size of the obstacle is considerably smaller than the wavelength, and that the driving force is due to the potential. Therefore the pressure due to the wave, assuming without loss of generality that it is of unit amplitude and ignoring the hydrostatic term, is (non-dimensionalized),

\[
P = -p_0 e^{-iax - az},
\]

where \(a\) is the wavenumber as before. If we solve for the surge response of the floe using the Froude-Kriloff assumption and assume that the surge force is due entirely to the integration of the potential across the ends of the floe, then we find the surge response \(S\) as follows:

\[
S = \frac{1 - e^{iaL}}{aL} \frac{1 - e^{-ah'}}{ah'}.
\]  

This is identical to the surge response found by Rottier [1992] and confirms that the approximation used to derive equation (8.13) will give answers that are right at least to first order.

8.5 Results

8.5.1 Surge

Figure 8.2 shows the surge response for floes of 1 m thickness and 10 m, 20 m, 50 m and 100 m length respectively for waves of period 2 to 20 s. It is clear in the long wavelength limit that the surge response is 1. That is, the floe responds as a particle in the wave. It is also apparent that there are zeros in the surge response at certain critical frequencies which are a function of the length of the floe. As the period decreases there is a decrease in surge response and the surge response is lower for longer floes.

Figure 8.3 compares the surge response predicted by the full theory to that predicted by the Froude-Kriloff assumption of Rottier [1992] for 10 by 1 m and 100 by 1 m floes. In the long period limit the two curves coalesce, though there is still a slight difference even at periods up to 20 s due to diffraction. The comparison is close and even at low periods the full theory surge response seems to follow the envelope defined by the Froude-Kriloff theory. This is not surprising as the Froude-Kriloff theory is a good estimator of the order of magnitude of the wave forces.

8.5.2 Reflection

Figure 8.4 shows the reflection coefficient for a 10 by 1 m floe for periods 2 to 20 s. The solid curve
Figure 8.2: The absolute value of surge response, $|S|$, versus period for a 100 by 1 m floe (solid), a 50 by 1 m floe (dashed), a 20 by 1 m floe (dotted) and a 10 by 1 m floe (chained).
Figure 8.3: The absolute value of surge response, $|S|$, versus period for a 10 by 1 m floe and a 100 by 1 m floe calculated by the full theory (solid) and the Froude-Kriloff assumption (dashed).
Figure 8.4: The absolute value of reflection coefficient, $|R|$, versus period for a 10 by 1 m floe calculated by allowing surge (solid), assuming zero surge (dashed), and assuming there is no submergence (chained).
was calculated allowing the floe to surge, the dashed line by setting the surge response to zero, and the chained line with the assumption that the floe lies on the surface. It is apparent that in the limit of long wavelengths the reflection coefficients for zero surge and the zero submergence are the same but that the reflection for surge motion is rather less. Remember that at these frequencies the wavelength is considerably greater than the floe length and there is a lowering of reflection when the floe is allowed to surge with the waves. At lower periods the curves behave similarly but the curve for no surge does not exhibit the first zero of the other curves. Figures 8.5 and 8.6 show equivalent plots for floes of wavelength 50 m and 100 m respectively. Again the zero surge and zero submergence curves are the same in the high period limit but the surge curve is less. The curves also follow similar patterns at low periods. The zero surge curve does not show the highest period zero that the other curves do. It is also apparent in these figures, especially in Figure 8.6 that at low periods the zero surge and surge curves coalesce. This is due to the fact that when the floe length is considerably larger than the wavelength the surge response becomes small. It is also clear that the reflection coefficient at low periods is greater for cases where the raft submergence has been included, than for zero submergence. This is an expected result since when the depth of submergence becomes comparable to the wavelength greater reflection will result.

Figure 8.7 shows the reflection coefficient for floes of thickness 5 m varying the length from 1 m to 500 m for waves of wavelength 100 m. The solid curve is the reflection for a submerged floe allowed to surge, the dashed curve is for a submerged floe with zero surge, and the chained curve is calculated with the assumption of zero submergence. It is clear that for long wavelengths the zero and non-zero surge curves come together, although there is still some residual difference. It is also clear that the first zero in reflection is missing for the case of zero submergence. When the floe length is large the reflection coefficients are greater when submergence is allowed for than when zero submergence is assumed.

It is possible from these results to calculate a correction to the reflection coefficient calculated by Fox and Squire [1990], [1991] allowing for the submergence of their infinite ice sheet.
Figure 8.5: The absolute value of reflection coefficient, $|\mathcal{R}|$, versus period for a 50 by 1 m floe calculated by allowing surge (solid), assuming zero surge (dashed), and assuming there is no submergence (chained).
Figure 8.6: The absolute value of reflection coefficient, |R|, versus period for a 100 by 1 m floe calculated by allowing surge (solid), assuming zero surge (dashed), and assuming there is no submergence (chained).
Figure 8.7: The absolute value of reflection coefficient, $|R|$, versus floe length for a wavelength of 100 m calculated by allowing surge (solid), assuming zero surge (dashed), and assuming there is no submergence (chained).
Chapter 9

The Rumer et. al. Model

9.1 Introduction

If the assumption is made that the floe is small, and therefore the wave is not affected by its presence there is some question as to what equation should be used to govern the floe's motion. The Froude-Kriloff approach used by Rottier [1992] is certainly valid in many regimes, but it is linear so will not show any long term drift and it is not time-dependent. The slope-sliding model of Rumer et al. [1979] has been used recently by Shen and Ackley [1991] and by Frankensteiun and Shen [1993] to model the collisions of ice floes. This model may have a similar region of validity to Morrison's equation [Morrison et al., 1950] to which it is somewhat analogous. Experimental work is also being planned which may determine the validity of the approximation. The mathematical structure of the model will form the subject of this chapter.

9.2 Transformation into an Autonomous System

Shen and Ackley [1991] developed a model for the motion of a single floe based on the slope sliding model of Rumer et al. [1979]. In this model the force on the floe due to wave action is considered as being due to two components. The first component of the force is directly analogous to the drag force of Morrison's equation and is proportional to the square of the velocity. The second component of the force is due to the body moving under gravity driven by the slope of the wave. This model is explicitly time-dependent and is not linear. It allows the time dependent motion of a floe in waves to be calculated. The equation of motion for the Shen and Ackley model is as follows,

\[
(1 + C_m) \frac{dV}{dt} = \frac{gA \cos(kx - \omega t)}{1 + (kA \cos(kx - \omega t))^2}
\]

\[
+ \frac{\rho C_w}{\rho h} \omega A \sin(kx - \omega t) - V(\omega A \sin(kx - \omega t) - V),
\]

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\[ \frac{dx}{dt} = V, \]

where \( V \) is the floe velocity, \( C_m \) is the coefficient of added mass, \( h \) is the floe thickness, \( A \) is the wave amplitude, and \( C_w \) is the drag coefficient. We non-dimensionalize as follows,

\[ \tilde{t} = t \sqrt{\frac{g}{A}}, \quad \tilde{x} = \frac{x}{A}, \quad \text{(9.2)} \]

so that the equation (9.1) becomes,

\[ (1 + C_m) \frac{d\tilde{V}}{dt} = -\frac{\tilde{k} \cos(\tilde{k}\tilde{x} - \omega \tilde{t})}{1 + (kA \cos(\tilde{k}\tilde{x} - \tilde{\theta})^2)} + \rho C_w A \frac{\omega \sin(\tilde{k}\tilde{x} - \tilde{\theta})}{\rho' h} - \tilde{V} \left( \tilde{\omega} \sin(\tilde{k}\tilde{x} - \tilde{\theta}) - \tilde{V} \right) - V, \quad \text{(9.3)} \]

Dropping the bar and defining,

\[ \tau = \frac{1}{1 + C_m}, \quad \sigma = \frac{\rho C_w A}{\rho' h}, \quad \text{and} \quad \theta = kx - \omega t, \quad \text{(9.4)} \]

we transform equation (9.3) into the following 2-dimensional autonomous system,

\[ \frac{dV}{dt} = P(V, \theta) = -\tau \frac{\omega^2 \cos \theta}{1 + \omega^2 \cos^2 \theta} + \tau \sigma \omega \sin \theta - V \left( \omega \sin \theta - V \right), \]

\[ \frac{d\theta}{dt} = Q(V, \theta) = \omega^2 V - \omega. \quad \text{(9.5)} \]

Equation (9.5) is periodic in \( \theta \), with period \( 2\pi \). It will thus be of value to consider this system as defined on a cylindrical manifold \( \mathbb{R} \times S^1 \). In fact it is on this manifold that the equation (9.5) will exhibit its long term behaviour most clearly.

### 9.3 Deductions from the Governing Equations

#### 9.3.1 Bendixson's Theorem and Criterion

The first fact we can deduce about the system follows from the fact that when \( |V| \) is large equation (9.5) implies that,

\[ \frac{dV}{dt} \approx -\sigma |V| V. \]

Therefore solutions with high \( V \) will move down the cylinder (decreasing the velocity) and solutions with low \( V \) will move up the cylinder (increasing the velocity). Thus solutions are trapped in a bounded region of the cylinder. This essentially says nothing more than that floes with large positive or negative velocities will be slowed (by the drag force of the water). The Poincare-Bendixson theorem [Perko, 1991] applied to a cylinder then implies that the solution must either tend towards an equilibrium point or to a limit cycle. Importantly a new kind of limit cycle can
exist on a cylinder, a limit cycle which encircles the cylinder itself. We shall refer to such a cycle as an orbiting limit cycle. Although powerful the Poincare-Bendixson theorem does not guarantee that all solutions must tend towards the same limit cycle. Despite the presence of the absolute value in the expression for $P$, $P$ does have continuous first partial derivatives. We use Bendixson's criterion \cite{Perko, 1991} which is based on Green's theorem to show that there can exist at most one limit cycle and that this limit cycle is orbiting. We know that,

$$\frac{\partial P}{\partial V} + \frac{\partial Q}{\partial \theta} = -2\pi \alpha |\omega \sin(\theta) - V|,$$

(9.6)

which is always less than or equal to zero and is only zero on a line, not in any region. It therefore follows from Bendixson's criterion (again as applied to a cylinder) that an ordinary limit cycle cannot exist and that only one orbiting limit cycle can exist. Thus if we can show that the system has no equilibrium points then we can say that all solutions must tend to the same orbiting limit cycle.

### 9.3.2 Equilibrium Points

We now turn to a consideration of the equilibrium points. These require $P = Q = 0$. $Q = 0$ implies that,

$$V = \frac{1}{\omega} = \text{wave velocity}.$$  

(9.7)

This means that if the floe motion tends to an equilibrium point then the floe must be travelling at the phase speed of the wave. The requirement that $P = 0$ is that the force on the floe is zero, so that for an equilibrium point we must have a stationary phase where the force is zero. It is obvious that for small waves the drag force for a floe travelling at the wave speed will be far greater than the sliding force, and thus no equilibrium points can exist. Formally we say that if the non-dimensionalized angular wave velocity is small then it will follow that,

$$P(1/\omega, \theta) \approx -\sigma \frac{1}{\omega^2},$$

(9.8)

which is always less than zero. Therefore in the case of small waves all floe motions must tend to the same orbiting limit cycle. In fact it is only for the case of extremely steep waves and low drag coefficients that there will exist any fixed points. We derive the following sufficient condition for no fixed points to exist,

$$\frac{\omega^2}{1 + \omega^4} < \sigma (\omega - 1/\omega)^2.$$  

(9.9)

(While it is possible for $1/\omega < 1$ this will not occur for physically realistic parameters.)

We now consider the case when there are equilibrium points. We know that,

$$P(1/\omega, \theta) < 0, \quad \text{if} \quad 0 < \theta < \pi/2, \quad \text{or} \quad 3\pi/2 < \theta < 2\pi.$$  

(9.10)
From a graphical analysis there can exist at most two equilibrium points and, ignoring the case
where there is only one equilibrium point (this will only occur at certain exact values), we can say
the following about the equilibrium points (which we shall label \( \theta_1 \) and \( \theta_2 \)),

\[
\frac{\pi}{2} < \theta_1 < \theta_2 < \frac{3\pi}{2}.
\]  

(9.11)

Therefore it must follow from equation (9.10) that,

\[
P_\theta(1/\omega, \theta_1) > 0 \text{ and } P_\theta(1/\omega, \theta_2) < 0.
\]  

(9.12)

Thus it follows from the equilibrium point classification theorems [Perko, 1991] that the first
equilibrium point \( \theta_1 \) is a saddle point and that the second equilibrium point \( \theta_2 \) is either an attracting
node or a spiral. Thus we have a saddle node bifurcation.

Since these equilibrium points correspond to the points on the wave where the force is zero
if you are travelling at the same speed as the wave, it can be easily seen why such points will
come in pairs, only one of which will be stable. This stable equilibrium point corresponds to a floe
"surfing" with the wave. It is also readily seen that such surfing will only occur for extreme wave
steepness.

The behaviour of the separatrices or unstable manifolds of the saddle point will dictate the
behaviour of the entire system since the separatrices divide the phase space into regions where the
long term behaviour of the solutions are identical. Any further conclusions will have to follow from
numerical solutions.

9.4 An Approximate Solution

We now construct an approximate solution to equations (9.5) assuming small \( \omega \). We assume that
the velocity \( V \) is of order \( \omega \) (the approximate solution will have this property), and make the
following approximation,

\[
\frac{d\theta}{dt} = \omega V \approx -\omega,
\]

so that,

\[
\frac{dV}{dt} = -\omega \frac{dV}{d\theta}.
\]

We assume further that,

\[
\frac{\omega^2 \cos \theta}{1 + (\omega^2 \cos \theta)^2} \approx \omega^2 \cos \theta.
\]  

(9.13)

The final assumption is that the added mass term may be neglected, so that \( \tau = 1 \). Unfortunately
this approximation does not necessarily become more valid as we make \( \omega \) smaller, although it is
probable. These assumptions give us the following approximate solution,

\[
V = \omega \sin \theta,
\]  

(9.14)
or putting this equation in dimensional form and eliminating $\theta$,

$$V = \omega A \sin(kx - \omega t).$$

(9.15)

Unfortunately this equation may be solved only by numerical methods. This velocity is of course nothing more than the linearized surface particle velocity, thus confirming the model's validity. Therefore in the limit of large wavelengths the floe will move as a particle in the fluid, a commonly used approximation. It is interesting to note that while equation (9.15) is linear, its integration will lead to a drift velocity; in fact any forcing function which is a function of phase will lead in general to a drift motion.

### 9.5 Numerical Solution

The floe motion is a function of three input parameters, $\omega$, $\sigma$ and $\tau$. Of these only the first two are significant since the variation in $\tau$, which is proportional to the added mass, is at most about 30% and $\tau$ does not affect the equilibrium points or general structure of the solution. We are then faced with the problem of determining realistic values for $\sigma$ and $\omega$. Since realistic values for the wavelength and amplitude are known it is simpler for $\omega$. Unfortunately choosing the physical magnitude of $\sigma$ is not straightforward since $\sigma$ is proportional to $C_w$, which has not been determined accurately. We follow Shen and Ackley (1991) and assume that $C_w$ is of the order $0.1 \text{ ms}^{-1}$. In all work that follows the autonomous system (equations 9.5) is solved by the Runga-Kutta 4-5 method.

#### 9.5.1 Phase Plots

Figure 9.1 shows the phase plot for $\omega=0.5$, $\sigma = 0.15$ and $\tau = 0.89$, together with the orbiting limit cycle. In this case there are no equilibrium points and all solutions must tend towards the orbiting limit cycle. This picture is a projection of a cylinder and thus there is a mapping from the right edge to the left edge of the picture. The orbiting limit cycle can be seen to have the basic form of the approximate solution, $V = w \sin \theta$. Figure 9.2 shows the phase plot for $\omega=0.75$, $\sigma = 0.15$ and $\tau=0.89$, together with the orbiting limit cycle (solid curve) and separatrices (dashed curves). The separatrices divide the cylinder into two regions, which we have labelled 1 and 2. In region 2 all solutions tend to the orbiting limit cycle, and in region 1 all solutions tend to the attracting node which can be readily seen at about (3.5,1.33) and is marked with a circle. Likewise the saddle point is marked at (1.5,1.33). The situation where all the solutions are attracted to the stable equilibrium point does exist but as we shall see, not for physically realistic parameters.
Figure 9.1: The phase plot for $\omega=0.5$, $\sigma=0.15$ and $\tau=0.89$. The solid curve is the orbiting limit cycle that all solutions will tend to.
Figure 9.2: The phase plot for $\omega=0.75$, $\sigma=0.15$ and $\tau=0.89$. The solid curve is the orbiting limit cycle, and the dashed curves are the separatrices which divide the cylinder into two regions. The equilibrium points are marked with circles. In region 2 all solutions tend to the orbiting limit cycle and in region 1 all solutions tend to the attracting node (the circle at $(1.33, 3.5)$).
9.5.2 Bifurcation Diagram

In Figure 9.3 we have plotted the bifurcation diagram for the system taking as fixed $\tau=0.89$. Note that we have considered extreme values of both $\sigma$ and $\omega$ to show certain effects. The bifurcation diagram shows three regions. In region 1 all solutions tend to an orbiting limit cycle. In region 2 there exists two solutions, both an attracting node and an orbiting limit cycle. In region 3 all solutions tend to an attracting node, which can be seen to occur only for physically unrealistic parameters as stated earlier, i.e. if $\omega = 2$ then the ratio of wave amplitude to wavelength, $A/\lambda$, is equal to $2/\pi$.

9.5.3 Drift Velocity

In the following work we present dimensional solutions varying the wave amplitude and $C_w$, and taking as fixed the wavelength, $\lambda = 10$, $\tau = 0.87$ and the floe thickness $D = 0.1$. We shall allow the wave amplitude maximum to be 1, and assume $C_w \approx 0.03$. These values are chosen because they correspond to those used by Shen and Ackley [1991]. Once the floe behaviour has reached the steady solution it will undergo long term drift. The rate of this drift is of interest since it is of both physical significance and is an easily measured quantity. In Figure 9.4 we have plotted the drift velocity as a function of wave amplitude for three values of $C_w$, 0.005, 0.03, and 0.1. The drift velocity rises with increasing amplitude. The drift velocity decreases with increasing $C_w$, but this only occurs for large values of wave amplitude, $A$. Also plotted is the approximate solution (equation (9.14)), which is not a function of $C_w$. It can be seen that the approximate solution does show the same general behaviour as the exact solutions but it differs even at small $A$ because of the assumption of zero added mass ($\tau = 1$) in the approximate solution. Shown in Figure 9.5 is the plot of drift velocity as a function of $C_w$ for three values of wave amplitude, $A = 0.2$, 0.4 and 0.6. It is clear again from these plots that the drift velocity is not a strong function of $C_w$ and depends mainly on the wave amplitude, $A$. It is also clear that increasing $C_w$ lowers the drift velocity.
Figure 9.3: The bifurcation diagram for the system taking as fixed $\tau = 0.89$. The $\sigma$-$\omega$ plane is divided into three regions. In region 1 all solutions tend to the orbiting limit cycle. In region 2 there exists two solutions, both an attracting node and an orbiting limit cycle. In region 3 all solutions tend to an attracting node. The upper values of $\omega$ in this plot are not physically realistic.
Figure 9.4: The drift velocity plotted against wave amplitude for the three values of $C_w$ shown and the approximate solution.
Figure 9.5: The drift velocity plotted against the drag coefficient $C_w$ for the three values of wave amplitude $A$ shown.
Chapter 10

Summary and Conclusions

We began this thesis by considering a single ice floe modelled as a two-dimensional, viscoelastic raft floating in water. The raft was then allowed to move under the action of a train of ocean waves. This problem has existed in the literature for some time (Stoker [1957] and Tayler [1986]) but to date has only been solved in an approximate way. By means of a Green's function formulation we were able to transform the problem into an integral equation and then solve it for infinite depth. The solution allowed us to predict the strain and reflection from an arbitrary floe in waves.

The first result established concerned infinitely stiff floes where the roll and heave response were calculated as a function of frequency. We also established the existence of critical frequencies at which there are peaks in the heave and roll response functions which are dependent on floe length. We then considered rafts with typical sea ice properties (i.e., capable of flexure) and established the existence of critical wavelengths at which the reflection coefficient is zero and where there is complete transmission. It was also established that the zeros in reflection correspond to maxima in the floe strain. Next we considered the effect of a spectral forcing on the maximum strain which was found to increase with floe length until a critical floe length was reached above which the maximum strain was constant. Therefore, if the wave spectrum has sufficient amplitude to break floes then it will do so for all floes of a critical size or greater. This is an observed phenomenon.

The initial theory was developed only for water of infinite depth, an approximation which is certainly valid for most of the MIZ. The theory was then extended to water of arbitrary depth and it was found that the reflection coefficient decreased with decreasing water depth.

A similar theory to the finite floe model developed so far had been conceived by Fox and Squire [1990, 1991] and a relationship between the two theories was reported. It was found that for floes of sufficient length it was possible to calculate the reflection and transmission coefficients for a floe of finite length from the Fox and Squire theory for semi-infinite ice sheets. In fact, for floes of long length compared to the wavelength it is preferable to use the theory of Fox and Squire. Since the
solution techniques for these two problems were different this acted as an independent check on the theory developed so far.

As a further check an experiment was carried out in the wave flume at Clarkson University to evaluate the theory. The experiment was performed using a polypropylene sheet, thereby testing the theory's application to an elastic sheet on water but not its applicability to sea ice. As with most experimental work the results were imperfect and there is probably justification for further experimental work, but some of the results did show some good correlation with theory. Certainly many of the observed measurements were explained by the theory.

The original approximate solution of the finite floe problem made by Wadhams [1986] was motivated by a desire to understand the effect of a number of floes on a wave spectrum. We began by extending our complete solution to the finite floe problem to the complete solution for two finite floes. This solution was shown to differ only slightly from an approximate solution which neglected any interaction between the two floes. A further approximate solution was developed for the problem of many floes, assuming that there is incoherent addition of energy. This is the standard model for multiple floe scattering problems. With this model random distributions of floes, distributed in length according to the Rayleigh distribution, were considered. It was found that with sufficient floes an MIZ will low pass filter incoming waves. The cut off frequency decreased with an increase in the number of floes, and the transition band became sharper. Further investigations of the effect of distributions of floes on wave spectra were considered, where the floe size distribution was determined by measurements from an aerial photograph.

In all of the considerations so far and in the previous theoretical work, the assumption was made that floe submergence was negligible compared to wavelength. This assumption was no doubt true in many cases, but it does not allow surge response to be calculated and it is valuable to study the range of validity of the assumption. For this reason the theory was extended to consider a floe with non-zero submergence. The surge response was found and shown to agree with the approximate solution of Rottier [1992]. It was found to be a strong function of floe length and critical ratios of floe length to wavelength where the surge vanished were found.

Finally, an associated model for the motion of floes in waves which has been developed by Shen and Ackley [1991] and further extended by Frankenestein and Shen [1993] was considered. This model is applicable to small floes and was originally developed for the problem of pancake ice freeze up in Antarctica. The model was shown to predict that all floes will tend to the same drift velocity in the same waves. The existence of a stationary solution in which the floes are carried forward with the wave was shown, provided that the wave was sufficiently steep. This solution coexisted with an orbiting limit cycle solution. The drift velocity as a function of wave amplitude and drag coefficient was calculated, and it is hoped that experiments will be performed to compare with this model.
The principal aim of this thesis was to model the propagation of waves through the MIZ and to understand the way in which ice floes respond to wave action. This thesis presented the first full solution to the linearized problem of a viscoelastic beam on water. From this solution a more accurate understanding of the problems of wave propagation in ice infested seas and of the wave induced strain in ice floes is possible. This in turns leads us to a better understanding of the general problem of wave ice interaction in the MIZ.
Appendix A

The Fourier Transform of

\[ \ln(\xi^2 + a^2) \]

We wish to evaluate the Fourier transform of, \[ \ln(\xi^2 + a^2). \]

To accomplish this we first differentiate the Fourier transform with respect to \( a \) to obtain,

\[ \frac{d}{da} \int_{-\infty}^{\infty} \ln(\xi^2 + a^2)e^{i\omega\xi}d\omega = 2a \int_{-\infty}^{\infty} \frac{1}{\xi^2 + a^2}e^{i\omega\xi}d\omega. \]  
(A.1)

We know [Erdelyi et.al., 1954] that,

\[ \int_{-\infty}^{\infty} \frac{1}{\xi^2 + a^2}e^{i\omega\xi}d\omega = \frac{\pi e^{-a|\omega|}}{\omega}, \quad \Re(a) > 0, \]  
(A.2)

(where \( \Re \) denotes the real part) so that,

\[ \frac{d}{da} \int_{-\infty}^{\infty} \ln(\xi^2 + a^2)e^{i\omega\xi}d\omega = 2\pi e^{-a|\omega|}, \quad \Re(a) > 0. \]  
(A.3)

Now we integrate with respect to \( a \) to obtain,

\[ \int_{-\infty}^{\infty} \ln(\xi^2 + a^2)e^{i\omega\xi}d\omega = -\frac{2\pi e^{-a|\omega|}}{|\omega|} + c(\omega), \]  
(A.4)

where \( c(\omega) \) is the constant of integration. We also know [Lighthill, 1958] that,

\[ \int_{-\infty}^{\infty} \ln |\xi|e^{i\omega\xi}d\omega = \frac{\pi}{|\omega|}, \]  
(A.5)

and it follows that,

\[ \int_{-\infty}^{\infty} \ln(\xi^2)e^{i\omega\xi}d\omega = 2 \int_{-\infty}^{\infty} \ln |\xi|e^{i\omega\xi}d\omega \]  
(A.6)

\[ = -\frac{2\pi}{|\omega|}. \]
Therefore \( c(\omega) = 0 \) and we obtain the result that,

\[
\int_{-\infty}^{\infty} \ln(\xi^2 + a^2)e^{i\omega \xi} d\omega = \frac{-2\pi e^{-|\omega|}}{|\omega|}, \quad \Re(a) > 0.
\]  \hspace{1cm} (A.7)
Appendix B

Alternative Derivation of $T$.

It is possible to calculate the transmission and reflection coefficients without explicitly deriving the Green's function. We shall derive the transmission coefficient, and the reflection coefficient can be derived similarly. We use Green's theorem and consider the same infinite half space $\Delta$ as used in Chapter 2. Let $\phi$ be the velocity potential which satisfies the boundary value problem outlined in section 2.2 and define,

$$\phi' = e^{i\alpha\xi - \alpha\eta}. \quad (B.1)$$

Then from Green's theorem,

$$\int_{\Delta} (\phi'_{\eta} - \phi_{\eta}\phi') ds = 0, \quad (B.2)$$

where $\Delta$ is a rectangle with sides $\eta = 0, \eta = \eta_0, \xi = \xi_0$ and $\xi = -\xi_0$. We now consider the limit as $\eta_0 \to \infty$ and $\xi_0 \to \infty$, and consider the four sides of $\Delta$ separately. At large $\xi$, i.e., as $\xi \to \infty$, the wave field is due to the transmitted wave alone, i.e.,

$$\lim_{\xi \to \infty} \phi = T e^{-i\alpha\xi - \alpha\eta}, \quad (B.3)$$

and we know that on the contour $\xi = \xi_0$,

$$\phi'_{\eta} = \frac{\partial \phi'}{\partial \xi} = i\alpha \phi', \quad \text{and} \quad \lim_{\xi \to \infty} \phi_n = -i\alpha \phi, \quad (B.4)$$

which gives us,

$$\int_{0}^{\infty} \lim_{\xi \to -\infty} (\phi'_{\eta} - \phi_{\eta}\phi') d\eta = \int_{0}^{\infty} \lim_{\xi \to -\infty} T e^{-i\alpha\xi - \alpha\eta} (\phi'_{\eta} + i\alpha \phi) d\eta = 2i\alpha \int_{0}^{\infty} T e^{-2\alpha\eta} d\eta = iT. \quad (B.5)$$

At large negative $\xi$, i.e., as $\xi \to -\infty$, on the other hand the wave field is composed of a unit wave input and a reflected wave, i.e.,

$$\lim_{\xi \to -\infty} \phi = e^{-i\alpha\xi - \alpha\eta} + Re^{i\alpha\xi - \alpha\eta}, \quad (B.6)$$
and we know that on the surface $\xi = -\xi_0$,

$$\phi_n' = -\frac{\partial \phi}{\partial \xi} = -i\alpha \phi', \quad \text{and} \quad \lim_{\xi \to -\infty} \phi_n = i\alpha e^{-i\alpha \xi - i\alpha \eta} - i\alpha Re^{i\alpha \xi - i\alpha \eta}. \quad (B.7)$$

This gives,

$$\int_0^\infty \lim_{\xi \to -\infty} (\phi \phi_n' - \phi' \phi_n) \, d\eta = \int_0^\infty \lim_{\xi \to -\infty} \left[ e^{-i\alpha \xi - i\alpha \eta} (\phi_n' - i\alpha \phi') + Re^{i\alpha \xi - i\alpha \eta} (\phi_n' + i\alpha \phi') \right] \, d\eta$$

$$= -2i\alpha \int_0^\infty e^{-2\alpha \eta} \, d\eta = -i. \quad (B.8)$$

The boundary conditions also tell us that the integral along $\eta_0$ must vanish as $\eta \to \infty$. From the boundary conditions at $\eta = 0$ we obtain the result that,

$$\int_{-\infty}^{\infty} (\phi \phi_n' - \phi' \phi_n) |_{\eta = 0} \, d\xi = \int_0^1 \phi' (\alpha \phi - \phi_n) |_{\eta = 0} \, d\xi. \quad (B.9)$$

Substituting the integral equation relationship between $\phi$ and $\phi_n$ derived in section 2.2, following the notation used there, we obtain,

$$\int_0^1 \phi' (\alpha \phi - \phi_n) |_{\eta = 0} \, d\xi = i\alpha \int_0^1 e^{i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \zeta) \phi(\zeta, 0) \, d\zeta \right) \, d\xi, \quad (B.10)$$

which gives from equation B.2 the result:

$$T = 1 + i\alpha \int_0^1 e^{i\alpha \xi} \left( \phi(\xi, 0) - \frac{1}{\beta} \int_0^1 g(\xi, \zeta) \phi(\zeta, 0) \, d\zeta \right) \, d\xi. \quad (B.11)$$

This expression is identical to that derived in subsection 2.3.2.
Appendix C

The Asymptotic Limits of the Green’s Functions.

We wish to evaluate expressions of the form,

\[ \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{|\omega| - \alpha} e^{i \omega \xi} d\omega. \]  
\hspace{1cm} (C.1)

First we consider,

\[ \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{1}{x} e^{i \xi} dx \quad (\alpha > 0). \]  
\hspace{1cm} (C.2)

By the Riemann-Lebesgue lemma [Lighthill, 1958] we know that for integrable functions \( f \):

\[ \lim_{\xi \to \pm \infty} \int_{a}^{b} f(x) e^{\pm i \xi} dx = 0. \]  
\hspace{1cm} (C.3)

Thus in the limit as \( \xi \to \pm \infty \) we may disregard any of the region of integration where \( f(x) \) is integrable.

We will evaluate the integral (equation C.2) by calculating its Cauchy principal value,

\[ \int_{-\infty}^{\infty} \frac{1}{x} e^{i \xi} dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left( \frac{1}{x} e^{i \xi} - \frac{1}{x} e^{-i \xi} \right) dx \]
\[ = \int_{0}^{\infty} \frac{2i}{x} \sin(x \xi) dx \]
\[ = 2i \int_{0}^{\infty} \frac{\sin x}{x} dx. \]  
\hspace{1cm} (C.4)

Hence,

\[ \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{1}{x} e^{i \xi} dx = \lim_{\xi \to \pm \infty} 2i \int_{0}^{\infty} \frac{\sin x}{x} dx \]
\[ = 2i \int_{0}^{\infty} \frac{\sin x}{x} dx = i \pi. \]  
\hspace{1cm} (C.5)
Now consider an $f(x)$ which is integrable, and also has an integrable derivative:

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(x)}{x} e^{i \alpha} dx = \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(x) - f(0)}{x} e^{i \alpha} dx + \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(0)}{x} e^{i \alpha} dx. \tag{C.6}
\]

The first integral on the R.H.S. vanishes since,

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x} = f'(0), \tag{C.7}
\]

and $f'(x)$ is integrable hence,

\[
\frac{f(x) - f(0)}{x},
\]

is integrable throughout the domain $(-a, a)$. Thus the integral vanishes by the Riemann-Lebesgue lemma. Hence,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(x)}{x} e^{i \alpha} dx = i \pi f(0). \tag{C.8}
\]

We also require,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{1}{x} e^{i \alpha} dx = 2i \int_{0}^{\infty} \frac{\sin x}{x} dx \quad (a > 0)
\]

\[
= 2i \int_{0}^{\infty} \sin x \frac{dx}{x} = -i \pi. \tag{C.9}
\]

Thus,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(x)}{x} e^{i \alpha} dx = -i \pi f(0). \tag{C.10}
\]

We now apply these results to equation (C.1). Consider the following,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega - \alpha} e^{i \omega} d\omega = \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega - \alpha} e^{i \omega \xi} d\omega + \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega - \alpha} e^{i \omega} d\omega
\]

\[
= \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega + \alpha)}{\omega} e^{(\omega + \alpha) \xi} d\omega - \lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega - \alpha)}{\omega} e^{(\omega - \alpha) \xi} d\omega
\]

\[
= \lim_{\xi \to \pm \infty} e^{i \alpha \xi} \int_{-\infty}^{\infty} \frac{f(\omega + \alpha)}{\omega} e^{i \omega \xi} d\omega - \lim_{\xi \to \pm \infty} e^{i \alpha \xi} \int_{-\infty}^{\infty} \frac{f(\omega - \alpha)}{\omega} e^{i \omega \xi} d\omega, \tag{C.11}
\]

where we have disregarded the rest of the integral by the Riemann-Lebesgue lemma. Applying the above results we obtain,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega - \alpha} e^{i \omega} d\omega = i \pi \left( f(\alpha) e^{i \alpha} - f(-\alpha) e^{-i \alpha} \right). \tag{C.12}
\]

Similarly the negative limit is,

\[
\lim_{\xi \to \pm \infty} \int_{-\infty}^{\infty} \frac{f(\omega)}{|\omega| - \alpha} e^{i \omega} d\omega = -i \pi \left( f(\alpha) e^{i \alpha} - f(-\alpha) e^{-i \alpha} \right). \tag{C.13}
\]

We now apply these results to our Green's function,

\[
\lim_{\xi \to \pm \infty} G(x, z; \xi, \eta) = \lim_{\xi \to \pm \infty} -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\omega| - \alpha} e^{-|\omega|(|\eta - z|) e^{i \omega (\xi - z)} d\omega
\]

\[
= \pm e^{-\alpha (\eta + z)} \sinh \alpha (\xi - z), \tag{C.14}
\]

and likewise,

\[
\lim_{\xi \to \pm \infty} G_\xi = \pm e^{-\alpha (\eta + z)} \cos \alpha (\xi - z). \tag{C.15}
\]
Appendix D

Tayler’s Solution

Tayler [1986] solved the problem in the following case. He assumed that $\gamma$ and $|\beta|$ are small so that the potential function was basically unchanged by the raft. He therefore approximated the potential function by the unit Stokes wave, i.e.,

$$\phi \approx e^{-i\alpha z - \alpha z}. \quad \text{(D.1)}$$

Applying Tayler’s simplification to the boundary condition over the raft we obtain,

$$\gamma \frac{\partial \phi}{\partial x} - \beta \frac{\partial^2 \phi}{\partial x^2 \partial z} \bigg|_{z=0, \ 0 < x < 1} = (\gamma \alpha^2 - \beta \alpha^3) e^{-i\alpha x} = Ae^{-i\alpha x}, \quad \text{(D.2)}$$

where $A = -\gamma \alpha^2 + \beta \alpha^3$. Therefore the potential function satisfies the boundary value problem,

$$\begin{cases}
\nabla^2 \phi = 0, & \infty < z < 0, \\
\frac{\partial \phi}{\partial z} = 0, & z \to \infty,
\end{cases} \quad \text{(D.3)}$$

$$\begin{align*}
\frac{\partial \phi}{\partial z} + \alpha \phi &= 0, & z &= 0, & -\infty < x < 0, & 1 < x < \infty, \\
\frac{\partial \phi}{\partial z} + \alpha \phi &= Ae^{-i\alpha z}, & z &= 0, & 0 < x < 1,
\end{align*}$$

together with the asymptotic conditions as $x \to \pm \infty$ of a unit wave input and a reflected, and transmitted wave respectively, i.e.,

$$\lim_{x \to \infty} \phi = Te^{-i\alpha x - \alpha z} \text{ and } \lim_{x \to -\infty} \phi = e^{-i\alpha x - \alpha z} + Re^{i\alpha x - \alpha z}. \quad \text{(D.4)}$$

Solution of these equations for the reflection and transmission coefficients is a trivial consequence of equations (2.33) and (2.34) and gives us

$$T = 1 + iA, \quad \text{and} \quad R = \frac{A}{2\alpha} \left( 1 - e^{-2i\alpha} \right), \quad \text{(D.5)}$$

in agreement with the results of Tayler.
Appendix E

Numerical Calculation of the Infinite Depth Green’s Function

To calculate the Green's function (equation (2.23)) we require a method of numerically calculating the following,

$$\int_{-\infty}^{\infty} \frac{1}{|\omega| - \alpha} e^{-|\omega|(\eta + x)} e^{i\omega(\xi - z)} d\omega.$$  \hspace{1cm} (E.1)

The first step is to note that the the integrand is even in $\omega$ and to write the equation in the following form,

$$\int_{-\infty}^{\infty} \frac{1}{|\omega| - \alpha} e^{-|\omega|(\eta + x)} e^{i\omega(\xi - z)} d\omega = \Re \left[ 2 \int_{0}^{\infty} \frac{e^{-\omega(\eta + x)} e^{i\omega(\xi - z)}}{\omega - \alpha} d\omega \right],$$  \hspace{1cm} (E.2)

where $\Re$ denotes the real part. However the evaluation of this expression is still not straightforward as there is a singularity in the integrand. We therefore assume that $x > \xi$ and move the line of integration from 0 to $\infty$ to 0 to $-\infty$, making the substitution $\theta = i(\xi - x) - \eta - z$. Therefore we have,

$$\int_{0}^{\infty} \frac{e^{\theta \omega}}{\omega - \alpha} d\omega = \int_{0}^{\infty} \frac{e^{\theta \omega}}{\omega - \alpha} d\omega - i\pi e^{\alpha \theta},$$  \hspace{1cm} (E.3)

where we have allowed for half the residue since the integral on the left hand side is to be understood as the Cauchy principal value. Now,

$$\int_{0}^{-\infty} e^{\theta \omega} \frac{d\omega}{\omega - \alpha} = \int_{0}^{-\theta \infty} \frac{e^{t \alpha \theta}}{t + i\alpha \theta} dt = \int_{0}^{\infty} \frac{e^{t \alpha \theta}}{t + i\alpha \theta} dt,$$  \hspace{1cm} (E.4)

(the last step being by another change in the line of integration). Here it is to be understood that the notation $-\theta \infty$ means the line integral is the complex plane is to $\infty$ in the direction $-\theta$. We now introduce the following notation,

$$f(x) = \int_{0}^{\infty} \frac{\sin \omega}{\omega + x} d\omega \quad \Re(x) > 0,$$
and

\[ g(x) = \int_0^\infty \frac{\cos \omega}{\omega + x} \, d\omega \quad \Re(x) > 0. \]

These functions are listed in *Abramowitz and Stegun* [1965] as the auxiliary functions to the Sine and Cosine integrals. We calculate their value by an asymptotic expansion for large \( x \), by integration for medium sized \( x \), and by their representation as cosine and sine integrals for small \( x \). The three solutions are plotted in Figure E.1, the solution by numerical integration is the solid line and is in fact valid for large and medium sized \( x \). The dashed line is a solution by an asymptotic expansion and is valid for large \( x \). The chained curve is a solution by a series expansion of the Sine and Cosine integrals valid for small \( x \). Therefore we have,

\[
\int_0^\infty \frac{e^{i\omega}}{\omega - \alpha} \, d\omega = g(i\alpha \theta) + if(i\alpha \theta) - i\pi e^{\alpha \theta}, \quad \Re(\theta) < 0, \text{ and } \Im(\theta) < 0, \tag{E.5}
\]

where \( \Im \) denotes the imaginary part (When \( \Im(\theta) > 0 \) we simply consider the complex conjugate).
Figure E.1: Solution of the function $f(x)$ by three methods. The solid curve is a solution by numerical integration, the dashed curve is found from an asymptotic expansion and the chained curve is a solution obtained by a series expansion of the Sine and Cosine integrals.
Appendix F

Numerical Calculation of the Finite Depth Green’s Function

For our numerical work we need to calculate the Green’s function when \(z = \eta = 0\). Here \(\tilde{G}\) is,

\[
\tilde{G} = \frac{1}{\alpha - \omega \tanh \omega H}.
\]  

(F.1)

Since this function is even in \(\omega\) our problem is to calculate,

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\cos \omega x}{\alpha - \omega \tanh \omega H} d\omega.
\]  

(F.2)

We write this integral as (ignoring the \(1/\pi\)),

\[
\Re \left[ \int_{0}^{\infty} \frac{e^{i\omega x}}{\alpha - \omega \tanh \omega H} d\omega \right].
\]  

(F.3)

We now use complex calculus to transform this equation to one that does not have the singularity and which will converge more rapidly (i.e., decay exponentially and not oscillate). Note that the singularities are on the real and imaginary axes. We therefore use,

\[
\int_{0}^{\infty} \frac{e^{i\omega x}}{\alpha - \omega \tanh \omega H} d\omega = \frac{-i\pi e^{i\omega_0 x}}{\tanh \omega_0 H + \omega_0 H \text{sech}^2 \omega_0 H} + \int_{0}^{(1-i)\infty} \frac{ie^{-\omega x}}{\alpha + \omega \tan \omega H} d\omega.
\]  

(F.4)

We will calculate this integral by integration for medium sized \(x\), a series expansion for small \(x\) and by an asymptotic expansion for large \(x\), as before.

F.0.4 Numerical Integration

We rewrite the integral as,

\[
\int_{0}^{(1-i)\infty} \frac{ie^{-\omega x}}{\alpha + \omega \tan \omega H} d\omega = \int_{0}^{\infty} \frac{(1 + i)e^{(-1+i)\omega}}{\alpha x + (1 - i)\omega \tan((1 - i)\omega H/x)} d\omega.
\]  

(F.5)

The above expression may be evaluated by a simple numerical integration procedure.
F.0.5 Asymptotic Expansion

We know that for small $\omega$,

$$\frac{1}{\alpha + \omega \tan \omega H} \sim \sum_{j=0}^{n} a_j \omega^j \text{ as } \omega \to 0,$$  \hspace{1cm} (F.6)

where $a_j$ are the coefficients of the series expansion. Therefore it follows from Watson's lemma [Hinch, 1991] that for large values of $x$,

$$\int_{0}^{(1-i)\infty} \frac{e^{-\omega x}}{\alpha + \omega \tan \omega H} d\omega \sim \sum_{j=0}^{n} \frac{a_j}{x^{j+1}}.$$  \hspace{1cm} (F.7)

For our case the asymptotic expansion will be pure imaginary and thus will be zero.

F.0.6 Series Solution

When $x$ is small we break the integral up as follows,

$$\int_{0}^{(1-i)\infty} \frac{e^{-\omega x}}{\alpha + \omega \tan \omega H} d\omega = \int_{0}^{(1-i)\infty} \frac{e^{-\omega x/H}}{H\alpha + \omega \tan \omega} d\omega.$$  \hspace{1cm} (F.8)

Now we assume that for some $M$, if $\omega > (1 - i)M$ then $\tan \omega = -i$ so that,

$$\int_{0}^{(1-i)\infty} \frac{e^{-\omega x/H}}{H\alpha + \omega \tan \omega} d\omega = \int_{0}^{(1-i)M} \frac{e^{-\omega x/H}}{H\alpha + \omega \tan \omega} d\omega + \int_{(1-i)M}^{(1-i)\infty} \frac{e^{-\omega x/H}}{H\alpha - i\omega} d\omega.$$  \hspace{1cm} (F.9)

We calculate the first integral numerically and the second one we convert to,

$$\int_{(1-i)M}^{(1-i)\infty} \frac{e^{-\omega x/H}}{H\alpha - i\omega} d\omega = \int_{0}^{(1-i)\infty} \frac{e^{-(1-i)Mx/H} e^{-\omega x/H}}{H\alpha - (1 + i)M - i\omega} d\omega,$$  \hspace{1cm} (F.10)

and,

$$\int_{0}^{(1-i)\infty} \frac{e^{-\omega x/H}}{H\alpha - (1 + i)M - i\omega} d\omega = \int_{0}^{\infty} \frac{ie^{\omega x/H}}{\omega - H\alpha + (1 + i)M} d\omega.$$  \hspace{1cm} (F.11)

The above expression can be found using the numerical method developed for the infinite depth Green's function as outlined in Appendix E.
References


