Equilibrium Asset Prices and Variance Risk Premia

by

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Abstract

This thesis studies equilibrium asset prices and variance risk premia (VRP) with three classes of models: consumption-based (Chapter 2), production-based (Chapter 3) and demand-based (Chapter 4) asset pricing models.

In Chapter 2, we provide a complete solution to the problem of equilibrium asset pricing in a pure exchange economy with two types of heterogeneous investors having higher/lower risk aversion. Using a perturbation method, we obtain analytical approximate formulas for the optimal consumption-sharing rule, which is numerically justified to be accurate for a large risk aversion and heterogeneity. We present analytical formulas for the equilibrium pricing function, Sharpe ratio, risk-free rate, stock price and optimal trading strategies. We then analyse the properties of the equilibrium and derive some testable hypotheses, which enhance our understanding on the economics of financial markets.

In Chapter 3, we provide a production-based equilibrium model with a recursive-preferences investor, which successfully explains the equity premium puzzle with very low risk aversion, and theoretically generates the negative sign of the diffusive volatility risk premium. The empirical results show that all models can perfectly explain the equity premium puzzle, and that the stochastic volatility with contemporaneous jumps (SVCJ) model and the stochastic volatility with jumps in volatility (SVJV) model built on our cost-free production economy can well capture both the large equity and variance risk premiums only if the annualized equity premium is at or larger than 11% (e.g., the periods, 1990–1999 and 2010–2016).

Chapter 4 is the first to provide a demand-based equilibrium model of volatility trading with three kinds of traders – dealers, asset managers and leveraged funds – which complements Eraker and Wu’s (2017) consumption-based equilibrium model. Our theoretical results are consistent with existing empirical observations, and two
endogenous cases reach the same conclusion. Our novel model links together risk
aversion, market price of the volatility risk, VRP, VIX futures price and return and
futures trading activities. This allows us to test empirically the impact of the three
traders’ net positions on the VRP and the VIX futures return.
First and foremost, I would like to thank my supervisor, Professor Jin Zhang, for his methodical inspiration, insightful guidance and continuous encouragement. His methodical inspiration gave me the key to open the door of financial research. His insightful guidance let me efficiently become a good PhD student in finance, and his continuous encouragement has made me enjoy the journey of becoming a scholar. Over a period of three years, we have produced two published papers and nine working papers together. His carefully designed training on both research and teaching has given me the confidence to be a good teacher and scholar in the future.

Second, I would like to thank all my colleagues at the University of Otago, including Professor Timothy Falcon Crack, Associate Professor Ivan Diaz-Rainey, Dr Xing Han, Professor David Lont, Dr Zheyao Pan, Associate Professor I M Premachandra, Dr Helen Roberts and Dr Eric Tan; our group members, Fang Zhen, Nhu Nguyen, Sebastian Gehricke and Tian Yue; and our proof editor, Marianne Lown.

Third, of course, I acknowledge the generous support from the University of Otago Doctoral Scholarship, which has been essential for me to finish the PhD program smoothly.

My sincere and indebted thanks goes to my parents, who have provided endless support to me. Last but not least, I would like to express my great gratitude to my wife, Ms. Jiexiang Huang, for her longlasting support and constant love. I may have doubted my research topics at times, but I have never doubted the unwavering love from my wife, which was undeniably the bedrock upon which the past five years of my life have been built.
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Chapter 1

Introduction

During the period between September 2011 and June 2014, I was studying in a Master of Management, Operations Research and Management program, specializing in Financial Mathematics and Financial Engineering at Southwestern University of Finance and Economics (SWUFE) in P. R. China. At the end of the first semester in 2012, I made up my mind to pursue a career of being a scholar, as I thought it would be an enjoyable life. With this aspiration, I worked very hard in doing research. After reading a large number of papers, I published 10 papers in good SCI-indexed journals, including *Applied Mathematics and Computation*, *Journal of Computational and Applied Mathematics*, and *Communications in Mathematical Sciences*. In April 2013, I first learned the name of Professor Jin Zhang, when I was reading his much-cited paper: Zhang, Zhao, and Chang (2012). In order to deeply understand his paper, I asked him some research questions via emails. After several rounds of email communications, I expressed my interest in being a PhD student under his supervision. Finally, with an academic career in my mind and his warm encouragement, I decided to apply to enter the PhD program at the University of Otago (UO). I was awarded the UO Doctoral Scholarship in August 2014, and then I accepted and started my PhD program in November 2014.

After I enrolled in UO, I started to read and search PhD topics. Equilibrium
asset pricing is an active research topic in continuous-time finance. The topic is difficult due to its high demand for advanced mathematics. Finance scholars have been working very hard in explaining the large equity premium puzzle and negative variance risk premium by developing equilibrium models in different kinds of economy. After a few rounds of discussion with my supervisor, and judging from my previous experience of research in the area of applied mathematics, we determined the topic of my PhD thesis: *Equilibrium Asset Prices and Variance Risk Premia*. Along the process of doing this topic, I have studied three different classes of asset pricing models (i.e., consumption-based, production-based and demand-based models).

### 1.1 Background

Asset pricing models have been developed, starting with the capital asset pricing models (CAPM) of Sharpe (1964); Lintner (1965), and Mossin (1966). The most popular are the consumption-based asset pricing models, which assume that investors attempt to smooth consumption, given their information on the future distribution of asset returns and then use marginal rates of substitution (i.e., stochastic discount factor) to determine the prices of assets. The classical equilibrium models study a representative (single-agent) economy with the same aggregate consumption series as the heterogeneous-agent economy and the same asset price functions (e.g., Lucas (1978); Mehra and Prescott (1985); Bansal and Yaron (2004)). Wang (1996) studies a simple pure exchange economy with two classes of investors who have time-additive, state-independent, constant relative risk aversion (CRRA) preferences with risk-aversion coefficients and uses it to discuss the term structure of interest rates. Later on, several papers are along this extension (e.g., Bhamra and Uppal (2014); Chabakauri (2015)). In Chapter 2, by using a perturbation method, we provide the complete solution with very good accuracy to the equilibrium in a pure exchange
Chapter 1. Introduction

3 economy with two heterogeneous investors.

The consumption-based asset pricing models are built in a pure exchange or endowment economy, in which the consumption processes are exogenously given. An important application of these models is to explain the equity premium puzzle (see, e.g., Mehra and Prescott (1985); Rietz (1988); Bansal and Yaron (2004)). Recently, several papers further uses it to explain the VRP (see, e.g., Bollerslev, Tauchen, and Zhou (2009); Drechsler and Yaron (2011); Drechsler (2013); Jin (2015)). In production-based asset pricing models, the consumption processes are endogenously solved. Following the same application of the consumption-based asset pricing models, these production-based models are also used to explain the equity premium puzzle, (e.g., Constantinides (1990); Cochrane (1991); Zhang et al. (2012); Jahan-Parvar and Liu (2014); Hirshleifer, Li, and Yu (2015)). However, there is little literature studying the VRP in a production economy. Chapter 3 fills this gap.

There are few papers studying the demand-based equilibrium model. For example, Garleanu, Pedersen, and Poteshman (2009) propose a demand-based equilibrium model (which is a two-trader model) for option pricing. Chapter 4 develops their two-trader model into a three-trader model, in which there are three kinds of traders (i.e., dealers, asset managers and leveraged funds) and uses it to analyse how the traders’ demand influences the volatility market.

1.2 Structure of this PhD thesis

Chapter 2-Chapter 4 in this thesis comprise three independent but related papers. Table 1.1 summarizes the details of the three chapters and the contribution made by the candidate.
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<td>Undertook formula derivation, model interpretation and writing. Coauthor provided guidance on literature and support on theoretical techniques, writing skills, submitting and revising the paper.</td>
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<td>Presented at 2016 Auckland Finance Meeting, and submitted for publication</td>
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<td>4</td>
<td>A Demand-Based Equilibrium Model of Volatility Trading</td>
<td>Xinfeng Ruan and Jin E. Zhang</td>
<td>Undertook model construction, solution derivation, empirical analyses and writing. Coauthor provided support on literature, submitting and revising the paper.</td>
<td>Accepted for presentation at 2017 Auckland Finance Meeting</td>
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I previously focused on the Cox, Ingersoll, and Ross (1985a) production economy. However, the main stream finance literature focuses more on an exchange economy, i.e., consumption-based asset pricing. After a comprehensive study of the literature on consumption-based models, my supervisor and I decided to focus our attention on MIT Professor Jiang Wang’s (1996) problem, i.e., modelling equilibrium asset pricing in a pure exchange economy with two types of heterogeneous investors having higher/lower risk aversion. Using a perturbation method initially developed in engineering science, we obtained an analytical approximate formula for the consumption-sharing rule between two types of investors. Compared with Wang’s (1996) closed-form solution for two specific risk-aversion coefficients and Bhamra and Uppal’s (2014) analytical solution in a series form, our approximate formula gives accurate numerical values of consumption for a wide range of model parameters. Because of the complexity of their consumption rules, Wang (1996) only studies bond price model, and Bhamra and Uppal (2014) only handles a stock with a single dividend. Our solution is more intuitive and easier to use. With this advantage, we are able to solve the general equilibrium for the stock model completely. In August 2015, I finished a paper jointly with my supervisor, Asset Pricing in a Pure Exchange Economy with Heterogeneous Investors, and submitted it to the Auckland Finance Meeting for presentation. The paper was accepted (75 out of 200 submissions were accepted) and presented in December 2015. Currently, the paper is under review by a good journal for publication. This is the first essay of my PhD thesis.

After finishing a paper on the consumption-based asset pricing model, I continued my research journey on studying a production-based asset pricing model and it’s application on VRP. The recent papers on this topic published in top journals, including Bollerslev et al. (2009); Drechsler (2013); Jin (2015) are mainly consumption-based equilibrium models with long-run risks and stochastic volatility. I also studied
the recent development on production-based Q-theory models, e.g., Columbia Professor Neng Wang’s papers, which feature more on illiquid capital and have a less clean result in explaining either equity or VRP. Judging from the status quo, after a discussion with my supervisor, we decided to extend our own cost-free production-based model (Zhang et al. (2012)), from constant relative risk aversion to Epstein and Zin (1991) recursive preference, and from constant volatility to stochastic volatility with jumps in both stock and volatility that was popular in finance literature, see e.g., Broadie et al. (2007). With some effort, we obtained closed-form solutions for both instantaneous equity risk premium and VRP. After estimating model parameters in physical measure using Markov Chain Monte-Carlo (MCMC) simulation, we were able to examine the impact of risk-aversion and elasticity of intertemporal substitution on the VRP. In August 2016, I finished a paper jointly with my supervisor, Equilibrium Equity and Variance Risk Premiums in a Cost-Free Production Economy, and submitted it to the Auckland Finance Meeting for presentation. The paper was accepted (85 out of 200 submissions were accepted) and presented in December 2016. Currently, the paper is under review by a top journal for publication. This is the second essay of my PhD thesis.

In August 2016, I was invited to review a paper on VIX futures markets. During the process of reviewing the paper, I took note of the data of the three main kinds of traders, dealers (market makers), asset managers (hedgers) and leveraged funds (speculators), published by U.S. Commodity Futures Trading Commission (CFTC) in the Commitments of Traders reports, which are available from the CFTC. I got excited about the trading mechanism between the traders, and wanted to develop a demand-based equilibrium model using a setup similar to Kyle (1985) model. I therefore obtained closed-form solutions for three different equilibrium settings. Three demand-based equilibrium models, respectively, link the market price of volatility risk with the traders’ positions, which allows us to empirically test the impact of
the three traders’ net positions on the VRP and the VIX futures return. In March 2017, I finished a paper jointly with my supervisor, *A Demand-Based Equilibrium Model of Volatility Trading*. We revised the paper after collecting feedback from seminar presentations at a few international universities, and then submitted it to the 2017 Auckland Finance Meeting for presentation. The paper now has been accepted by the meeting. This is the third essay of my PhD thesis.

Three working papers discussed above can be summarized as follows.


[3] Ruan, Xinfeng, and Jin E. Zhang, 2016, A Demand-Based Equilibrium Model of Volatility Trading, Accepted for presentation at 2017 Auckland Finance Meeting. [Chapter 4]

1.3 Contributions of this PhD thesis

Compared with the existing closed-form solutions of the equilibrium model with two agents, in Chapter 2, we give an explicit solution for all variables, including the stock price and the trading strategies and derive an aggregate risk aversion formula that reveals the aggregate mechanism of two investors. The simple solution in the chapter has more explanatory power than the complicated and long closed-form solutions which lose the original economic intuition.
Chapter 1. Introduction

There are at least two contributions in Chapter 3. First, we develop a cost-free production-based equilibrium model as the first production-based equilibrium model to explain the equity premium puzzle and the large negative VRP. Second, we provide guidance on the sign of the diffusive volatility risk premium (DVRP) based on the SVCJ model.

In Chapter 4, the first contribution is that this is the first chapter to provide a demand-based equilibrium model of volatility trading with three kinds of traders (i.e., dealers, asset managers and leveraged funds) that fully supports the existing empirical results. Due to our novel model, the second contribution we make is that this chapter is the first to test the impact of the three main traders’ net positions on the VRP and the VIX futures return.
Chapter 2

Asset Pricing in a Pure Exchange Economy with Heterogeneous Investors

This chapter is joint work with Jin E. Zhang. Its earlier version was presented at 2015 Auckland Finance Meeting, 17-19 December 2015, AUT, Auckland, New Zealand; and 2016 New Zealand Finance Colloquium, 11-12 February 2016, University of Otago, Queenstown, New Zealand.

2.1 Introduction

This chapter provides an intuitive solution to the problem of asset pricing in a pure exchange economy with two types of heterogeneous investors. Unlike the classical equilibrium model with one representative investor (e.g., Lucas (1978)), the two agents share risks by exchanging their goods in the economy. Hence, there exist optimal trading strategies and equilibrium asset prices. This equilibrium model has many economic implications, such as modelling a competitive market in microeconomics, the evaluation of economic policy in macroeconomics and asset pricing in
finance. Our neat analytical expressions intuitively provide comparative statics for the effects of heterogeneity in risk-aversion and heterogeneity in initial wealth.

There are already several papers available that provide a closed-form solution. Wang (1996) provides a solution for specific sets of risk-aversion coefficients (i.e., 0.5 and 1), with the stock price given as the solution to an ordinary differential equation. Longstaff and Wang (2012) extend Wang’s (1996) model into a more general setting of risk-aversion coefficients (i.e., $\gamma$ and $2\gamma$) and solve the solution in terms of hyper-geometric functions. Cvitanić, Jouini, Malamud, and Napp (2012) discuss the bounds of the equilibrium in a pure exchange economy in which agents differ with respect to both beliefs and their preference parameters. Bhamra and Uppal (2014) provide a closed-form solution for the economy which is similar to Cvitanić et al.’s (2012). Their solution is based on infinite Taylor expansions and yields solutions based on hyper-geometric functions. Moreover, Chabakauri (2013, 2015) provide solutions for economies with heterogeneity in risk-aversion and portfolio constraints which are typically based on solutions to hypergeometric functions.¹

Compared with the above existing closed-form solutions, we do make several contributions. Firstly, compared with Wang (1996), we give an explicit solution for the stock price and explicitly solve the trading strategies for heterogeneous investors. In addition, we generalize the range of risk aversion in Wang (1996); Longstaff and Wang (2012). Secondly, we give a complete solution of the equilibrium rather than the limit bounds in Cvitanić et al. (2012). Thirdly, in Bhamra and Uppal (2014), as the limitation of their complicated solution, they only solve the price of dividend strip. In the chapter, we give a very intuitive solution for the price of the stock yielding multiple or continuous dividends. Fourthly, in contrast to Chabakauri

Chapter 2. Asset Pricing in a Pure Exchange Economy with Heterogeneous Investors

Our crystal clear aggregate risk aversion formula reveals the aggregate mechanism of two investors. The simple solution in the chapter has more explanatory power than the complicated and long closed-form solutions which lose the original economic intuition. Finally, besides discussing the effect of the risk-aversion heterogeneity, this chapter also discusses the effect of the size of investors, that reveals how the heterogeneity and the heterogeneous investors’ proportion influence the equilibrium together.

Using a perturbation method,\(^2\) we provide the complete solution with very good accuracy to the optimal consumption-sharing rule, pricing function, Sharpe ratio, risk-free rate, stock price (in a long-lived version) and optimal trading strategies in a pure exchange economy with two heterogeneous investors. All perturbation expansions for the equilibrium are simple and tractable. The effects of risk aversion heterogeneity and the size of investors on the equilibrium are clearly explained by using our approximate solutions.

The remainder of this chapter is organized as follows. Section 2.2 presents our model of an exchange economy with heterogeneous agents and the definition of the equilibrium. Section 2.3 presents our solution to the equilibrium and analyses its properties. Section 2.4 concludes. Appendix 2.5.1 presents the explicit solution for our maximization problem using Bhamra and Uppal’s (2014) method. Appendix 2.5.2 collects all proofs and Appendix 2.5.3 provides an extension of the results.

2.2 Model setup and market equilibrium

2.2.1 Model setup

We consider a pure exchange economy with an infinite horizon. There are two assets in this economy: one risk-free asset (a bond) in zero net supply with the risk-free

\(^2\)The same method is used by Kogan and Uppal (2001) to analyse a heterogenous-agent economy in the presence of portfolio constraints.
interest rate $r_t$ determined in the equilibrium, and one risky asset (a stock) $S_t$ in net supply of one unit. In addition, the risky asset produces a consumption good paid as a dividend $D_t$ at time $t \in [0, \infty)$. We assume the flow of the dividend $D_t$ follows a geometric Brownian motion,

$$dD_t = D_t (\mu dt + \sigma D dB_t),$$  \hspace{1cm} (2.1)$$

where $B_t$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The initial value $D_0 > 0$, the drift $\mu \geq 0$ and the volatility $\sigma_D > 0$ are constant.

There are two types of heterogeneous investors, $H$ and $L$. They have constant relative risk aversion (CRRA) preferences with different risk aversions $\gamma_i$ ($i \in \{H, L\}$),

$$u_i(c_{i,t}) = e^{-\delta t} \frac{c_{i,t}^{1-\gamma_i}}{1-\gamma_i},$$  \hspace{1cm} (2.2)$$

where $\delta > 0$ is the time discount parameter and is the same across the two types of heterogeneous investors, and $c_{i,t}$ is the consumption of the type $i$ investors. In addition, we assume risk aversion $\gamma_H > \gamma_L > 0$. Following Gârleanu and Pedersen (2011) and Rytchkov (2014), one implication of our model is that type $H$ agents can be regarded as retail investors who are averse to risk and of type $L$ agents as institutional investors who are risk-seeking. Based on the following analysis, the heterogeneity in investor preferences plays a crucial role in the economy.

In this market, heterogeneous investors continuously trade in the two assets. Given initial wealth $W_{i,0} > 0$, the type $i$ investors choose their consumption-trading strategies $\{c_{i,t}, \phi_{i,t}, \psi_{i,t}\}$ in consumption, stock and bond, respectively. The wealth process of type $i$ agents, $W_{i,t}$, at time $t$ is subject to,

$$dW_{i,t} = \psi_{i,t} r_t dt + \phi_{i,t} (D_t dt + dS_t) - c_{i,t} dt,$$  \hspace{1cm} (2.3)$$
with initial wealth \( W_{H,0} = (1 - \beta)S_0 \) and \( W_{L,0} = \beta S_0 \), where \( \beta \in (0, 1) \). The parameter \( \beta \) represents the (initial) fraction or size of type \( L \) investors in the economy.

Type \( i \) investors maximize their expected objective function by choosing consumption-trading strategies \( \{c_i, \phi_i, \psi_i\} \),

\[
\max_{c_i,\phi_i,\psi_i} E_t \left[ \int_t^{\infty} e^{-\delta u} \frac{c_1^{1-\gamma_i}}{1-\gamma_i} du \right] \quad \text{s.t.} \quad \text{Equation (2.3)}. \quad (2.4)
\]

As \( c_{H,t} \) and \( c_{L,t} \) are aggregate equilibrium consumption streams of the two types of investors, the sum of them is less than the total consumption good (paid as a dividend) \( D_t \). In other words, \( c_{H,t} + c_{L,t} \leq D_t \) for \( t \in [0, \infty) \).

### 2.2.2 Market equilibrium

We follow the definition of a market equilibrium in Wang (1996) and present it as follows:

**Definition 2.2.1 (market equilibrium (Wang (1996)))** Equilibrium in our economy is defined in a standard way: equilibrium consumption-trading strategies \( \{c_i, \phi_i, \psi_i\} \) and the pair of asset prices \( \{S_t, r_t\} \) are such that type \( i \) investors maximize their expected objective function in (2.4), and markets clear, that is,

\[
\phi_{H,t} + \phi_{L,t} = 1, \quad (2.5)
\]

\[
\psi_{H,t} + \psi_{L,t} = 0. \quad (2.6)
\]

In order to solve this equilibrium, following Wang (1996), we first solve the Pareto-optimal allocations and then, given Pareto-optimal allocations, an Arrow-Debreu equilibrium can be derived that supports the allocations.

When both types of investors have positive initial wealth \( W_{i,0} > 0 \), a consumption pair \( \{c_H, c_L\} \) is Pareto optimal if and only if there is a constant \( \lambda \in (0, 1) \) such
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that \( \{c_H, c_L\} \) solves the welfare problem, \(^3\)

\[
\max_{c_H, c_L} E_t \left[ \int_t^\infty e^{-\delta u} \left( \lambda \frac{c_H^{1-\gamma_H}}{1-\gamma_H} + (1-\lambda) \frac{c_L^{1-\gamma_L}}{1-\gamma_L} \right) du \right], \quad \text{s.t. } c_{H,t} + c_{L,t} \leq D_t. \quad (2.7)
\]

Here, the parameter \( \lambda \) is the weight of type \( H \) investors in the welfare function to be maximized. Wang (1996) states that maximizing the above expected intertemporal welfare function is equivalent to maximizing the welfare function period by period and state by state. Thus, the above maximization problem can be changed to

\[
\max_{c_{H,t} + c_{L,t} \leq D_t} e^{-\delta t} \left[ \frac{1}{\lambda} \frac{c_{H,t}^{1-\gamma_H}}{1-\gamma_H} + (1-\lambda) \frac{c_{L,t}^{1-\gamma_L}}{1-\gamma_L} \right]. \quad (2.8)
\]

Denoting \( b = \frac{1-\lambda}{\lambda} \in (0, +\infty) \), which is the ratio of the weight of type \( L \) investors in the welfare function to the weight of type \( H \) investors in the welfare function, the maximization problem (2.8) takes the following form: \(^4\)

\[
\max_{c_{H,t} + c_{L,t} \leq D_t} e^{-\delta t} \left[ \frac{1}{\lambda} \frac{c_{H,t}^{1-\gamma_H}}{1-\gamma_H} + b \frac{c_{L,t}^{1-\gamma_L}}{1-\gamma_L} \right]. \quad (2.9)
\]

Following Wang (1996), we define a representative investor who has the same utility function as in the maximization problem (2.9), that is,

\[
u_r(c_{H,t}, c_{L,t}) = e^{-\delta t} \left[ \frac{1}{\lambda} \frac{c_{H,t}^{1-\gamma_H}}{1-\gamma_H} + b \frac{c_{L,t}^{1-\gamma_L}}{1-\gamma_L} \right]. \quad (2.10)
\]

In order to define an Arrow-Debreu equilibrium, we assume that, given \( b \in (0, +\infty) \) and the corresponding Pareto-optimal allocations \( \{\hat{c}_H, \hat{c}_L\} \) for the maximization problem (2.9), the marginal utility of the representative investor can be

---

\(^3\)Unlike the incomplete information model (Basak (2005)) in which \( \lambda \) is stochastic as investors face the different state price densities, in our complete information model, \( \lambda \) should be constant (e.g., Wang (1996); Bhamra and Uppal (2009)).

\(^4\)A solution to (2.9), based upon the method of Bhamra and Uppal (2014), is provided in Appendix 2.5.1.
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\[ M_t = \frac{\partial u_r(\hat{c}_{H,t}, \hat{c}_{L,t})}{\partial D_t} = e^{-\delta t} \left( \hat{c}_{H,t} \frac{\partial \hat{c}_{H,t}}{\partial D_t} + b \hat{c}_{L,t} \frac{\partial \hat{c}_{L,t}}{\partial D_t} \right). \] (2.11)

In addition, the stochastic process of the marginal utility can be written as

\[ \frac{dM_t}{M_t} = -r_t dt - \theta_t dB_t, \] (2.12)

where \( \theta_t \) is the Sharpe ratio.

**Definition 2.2.2 (Arrow-Debreu equilibrium (Wang (1996)))** Given \( b \in (0, +\infty) \) and the corresponding Pareto-optimal allocations \( \{\hat{c}_H, \hat{c}_L\} \) for the maximization problem (2.9), there exists an Arrow-Debreu equilibrium that leads to the same allocation, with the pricing function given by \( p_t = M_t/M_0, t \in [0, \infty) \). Moreover, in this Arrow-Debreu equilibrium, there exists a dynamic implementation in which prices of traded securities are given by

\[ S_t = E_t \left[ \int_t^\infty \left( \frac{M_s}{M_t} \right) D_s ds \right], \quad r_t = -\frac{1}{dt} E_t \left[ \frac{dM_t}{M_t} \right]. \] (2.13)

Investors optimally choose the consumption plan \( \{\hat{c}_H, \hat{c}_L\} \), financed respectively by budget-feasible trading strategies and the securities market clears.

### 2.3 Main results

#### 2.3.1 Optimal consumptions

In terms of the maximization problem (2.9) with heterogeneous risk aversion, its first-order condition becomes \( \hat{c}^{\gamma_H}_{H,t} = b (D_t - c_{H,t})^{-\gamma_L} \). Although Bhamra and Uppal (2014) have employed Lagrange’s theorem to solve the type \( H \) investors’ consumption \( c_{H,t} \), their explicit expression is too complicated because it is written in terms of
infinite Taylor expansions (see Appendix 2.5.1). Our target is to analyse the equilibrium asset prices and trading strategies in a pure exchange economy with two heterogeneous investors. Based on this, we use perturbation methods which rely on there being a dimensionless parameter in the problem that is relatively small: $1 \gg \varepsilon > 0$. We assume that $\gamma_H = \gamma(1 + \varepsilon), \gamma_L = \gamma(1 - \varepsilon)$. Here $\varepsilon$ is the risk-averse heterogeneity and $\gamma$ is the mean risk aversion in the market. Furthermore, we assume $\gamma$ and $\delta$ satisfy $\delta - \mu(1 - \gamma_i) + \frac{1}{2}\sigma^2 D \gamma(1 - \gamma) > 0$ to guarantee $S_t > 0$ for $t \in [0, T]$. In addition, $\delta - \mu(1 - \gamma_i) + \frac{1}{2}\sigma^2 D \gamma(1 - \gamma_i) > 0$ for $i \in \{H, L\}$ as both the cases where the type $H$ or $L$ investors accumulate all the wealth must be considered. Then the maximization problem (2.9) becomes:

$$\max_{c_{H,t} + c_{L,t} \leq D_t} e^{-\delta t} \left[ \frac{c_{H,t}^{1-\gamma(1+\varepsilon)}}{1 - \gamma(1 + \varepsilon)} + b \frac{c_{L,t}^{1-\gamma(1-\varepsilon)}}{1 - \gamma(1 - \varepsilon)} \right],$$

(2.14)

The first-order condition derives

$$D_t - c_{H,t} = b^{1/\gamma} ((D_t - c_{H,t})c_{H,t})^\varepsilon c_{H,t}. $$

(2.15)

By using perturbation methods, the solutions of Equation (2.15) are given by the following proposition.

**Proposition 2.3.1 (Pareto-optimal consumption allocations)** Given $b \in (0, +\infty)$, the corresponding Pareto-optimal consumption allocations $\{\tilde{c}_H, \tilde{c}_L\}$ for the maximization
problem (2.14) are given by

\[ \hat{c}_{H,t} = \frac{1}{a + 1} D_t - \varepsilon \frac{ah_t}{(a + 1)^2} D_t - \varepsilon^2 \frac{a(1 - a)(h_t^2 + 2h_t)}{2(a + 1)^3} D_t + O(\varepsilon^3). \] (2.16)

and

\[ \hat{c}_{L,t} = \frac{a}{a + 1} D_t + \varepsilon \frac{ah_t}{(a + 1)^2} D_t + \varepsilon^2 \frac{a(1 - a)(h_t^2 + 2h_t)}{2(a + 1)^3} D_t + O(\varepsilon^3). \] (2.17)

where

\[ h_t = \ln \left( \frac{a}{(a + 1)^2} \right) + 2 \ln D_t, \quad a = b^{1/\gamma}. \]

Pareto-optimal consumption allocations in Proposition 2.3.1 are in second-order. If we remove \( \varepsilon^2 \) terms, they will become first-order perturbation solutions. In order to verify the accuracy of our solutions in Proposition 2.3.1, we compare them with Wang’s (1996) exact solutions with particular parameters. Following Wang’s (1996) parameters, we set \( \gamma = 3/4, \varepsilon = 1/3 \) (i.e., \( \gamma_H = 1, \gamma_L = 1/2 \)). Moreover, we take \( b = 1.7 \) as \( \beta = 67\% \). Without loss of generality, we only compare the consumption of type H investors, which is given in Figure 2.1.

---

8Following Wang (1996), we focus on the consumption rate rather than the consumption over dividend (income) ratio. Based on the solutions in equations (16)-(17), it does not make any difference using either one.

9According to Aghion, Van Reenen, and Zingales (2013), the proportion of institutional investors in the U.S. public equity market in 2010 is 67%.

10The optimal consumption for investor A in Wang (1996) is

\[ \hat{c}_{H,t} = \frac{1}{2b^2} (\sqrt{1 + 4b^2 D_t} - 1). \]

and Bhamra and Uppal’s (2014) solution is given in Appendix 2.5.1.
Figure 2.1: The consumption of type \( H \) investors based on Wang’s (1996) setup.

This figure shows the value of type \( H \) investors’ consumption by using first-order and second-order approximations in our chapter and using Wang’s (1996) and Bhamra and Uppal’s (2014) solutions. Dividend is in amount. The model parameters are as follows: \( \gamma = 3/4, \epsilon = 1/3 \) and \( b = 1.7 \).

Figure 2.1 shows the second-order perturbation solution is better than the first-order one, but the first-order perturbation solution is good enough to approximate
Wang’s (1996) exact solution against small dividend value (less than 2.5) in Panel A. The second-order perturbation solution works so well even for large dividend value (around 40) in Panel B, Figure 2.1. Based on that, we can claim that our second-order perturbation solution is a very good approximation for Wang’s (1996) problem.

In order to examine whether our perturbation solutions can handle the large risk-averse heterogeneity cases, we keep $\varepsilon = 1/3$ and then $\gamma_H$ and $\gamma_L$ are scaled by $\gamma$. Here $\varepsilon = 1/3$ indicates that $\gamma_H$ is the double of $\gamma_L$. For example, $\gamma_H = 20$ and $\gamma_L = 10$. Figure 2.3 gives the numerical results. From Panel A-D, Figure 2.3, even for large dividend value (around 40), the second-order perturbation solution gets a very accurate approximation for the explicit solution in Bhamra and Uppal (2014) and the accuracy is most likely not changed by $\gamma$. Thus, we can know that our perturbation solutions (e.g., the second-order solution) can well handle the large risk-averse heterogeneity cases.
This figure shows the value of type $H$ investors' consumption by using first-order and second-order approximations in our chapter and using Bhamra and Uppal's (2014) solutions. The model parameters are as follows: $\varepsilon = 1/3$, $\beta = 0.67$. We set $b$ as $\left(\frac{\beta}{1-\beta}\right)^{\gamma}$ where $\gamma = \frac{\gamma_H + \gamma_L}{2}$.

In addition, we want to investigate how large risk-averse heterogeneity our perturbation solutions can address. As the accuracy is most likely not changed by $\gamma$, without a loss of generality, we set $\gamma = 10$ and get Figure 2.5 which plots the perturbation solutions against risk-averse heterogeneity $\varepsilon$. In a small dividend value, based on Figure 2.5, both the first-order and second-order perturbation solutions can well fit the Bhamra and Uppal’s (2014) explicit solution. For both large risk-averse heterogeneity settings $\varepsilon = 0.2$ (i.e., $\gamma_H = 12$ and $\gamma_L = 8$) and $\varepsilon = 0.4$ (i.e., $\gamma_H = 14$ and $\gamma_L = 6$) in Figure 2.5, the perturbation solutions give very good approximations for $D_t < 2$. 

Figure 2.3: The consumption of type $H$ investors with different $\gamma$. 

Panel A: $\gamma_H = 1, \gamma_L = 0.5$

Panel B: $\gamma_H = 5, \gamma_L = 2.5$

Panel C: $\gamma_H = 10, \gamma_L = 5$

Panel D: $\gamma_H = 20, \gamma_L = 10$
This figure shows the value of type $H$ investors’ consumption by using first-order and second-order approximations in our chapter and using Bhamra and Uppal’s (2014) solutions. The model parameters are as follows: $\gamma = 10$ and $\beta = 0.67$. We set $b$ as $\left(\frac{\beta}{1-\beta}\right)^{\gamma}$.

Based on the above comparison, we conclude that our perturbation solutions can accurately approximate Wang’s (1996) and Bhamra and Uppal’s (2014) explicit solutions. In addition, our perturbation solutions can handle not only large but also
extremely large risk-averse heterogeneity cases. However, compared to closed-form solution in Wang (1996); Chabakauri (2013); Bhamra and Uppal (2014); Chabakauri (2015), in the next section, we will see that our perturbation solutions generate very intuitive solutions for other variables in the equilibrium.

2.3.2 Equilibrium

By using the Pareto-optimal consumption allocations, we derive an Arrow-Debreu equilibrium as defined in Definition 2. In this Arrow-Debreu equilibrium, equilibrium asset prices, the pricing function, the Sharpe ratio and optimal trading strategies can be analytically solved. Because the expressions in the second-order are very long, in order to reveal economic intuition, we have decided to present our perturbation solutions in the first-order in the main text of the chapter and leave the second-order solutions in Appendix 2.5.3.

Proposition 2.3.2 (Equilibrium) Given an Arrow-Debreu equilibrium as defined in Definition 2.2.2, there exists a market equilibrium in which the solutions for the equilibrium prices of traded securities are given by

\[ r = \delta + \mu \gamma - \frac{1}{2} \sigma_D^2 \gamma (\gamma + 1) + \varepsilon \gamma \left( \frac{1}{2} \sigma_D^2 (2 \gamma + 1) - \mu \right) \frac{a - 1}{a + 1} + O(\varepsilon^2), \tag{2.18} \]

\[ S_t = \left[ \frac{1}{\xi} + \varepsilon \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{\gamma (a - 1)}{a + 1} \frac{1}{\xi^2} \right] D_t + O(\varepsilon^2), \tag{2.19} \]

and the price-dividend ratio is

\[ V_t = \frac{S_t}{D_t} = \frac{1}{\xi} + \varepsilon \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{\gamma (a - 1)}{a + 1} \frac{1}{\xi^2} + O(\varepsilon^2), \]

where

\[ \xi = \delta - \mu (1 - \gamma) + \frac{1}{2} \sigma_D^2 (1 - \gamma); \]

The only difference between the first-order and second-order perturbation solutions is that the risk-free rate and the price-dividend ratio are no longer constant. More details see Appendix 2.5.3.
the marginal utility of the representative investor can be approximated by

$$M_t = e^{-\delta t} \left( \frac{1}{a+1} \right)^{-\gamma} D_t^{-\gamma} (1 + \varepsilon g_t) + O(\varepsilon^2), \quad (2.20)$$

where

$$g_t = \frac{\gamma}{a+1} \left( (a-1) \ln \frac{D_t}{a+1} + a \ln a \right),$$

and the pricing function $p_t = M_t/M_0$; the Sharpe ratio is

$$\theta_t = \left( 1 - \varepsilon \frac{a-1}{a+1} \right) \gamma \sigma_D + O(\varepsilon^2); \quad (2.21)$$

the solutions for heterogeneous investors’ optimal consumption strategies are

$$\hat{c}_{H,t} = \frac{1}{a+1} D_t - \varepsilon a D_t + O(\varepsilon^2), \quad (2.22)$$

$$\hat{c}_{L,t} = \frac{a}{a+1} D_t + \varepsilon a D_t + O(\varepsilon^2), \quad (2.23)$$

which are financed, respectively, by the following trading strategies:\textsuperscript{12}

$$\hat{\phi}_{H,t} = \frac{1}{a+1} - \varepsilon a \left( h_t + 2 + 2 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{\xi} \right) + O(\varepsilon^2), \quad (2.24)$$

$$\hat{\phi}_{L,t} = 1 - \hat{\phi}_{H,t}, \quad (2.25)$$

$$\hat{\psi}_{H,t} = 2\varepsilon \frac{a}{(a+1)^2} D_t + O(\varepsilon^2), \quad \hat{\psi}_{L,t} = -\hat{\psi}_{H,t}; \quad (2.26)$$

\textsuperscript{12}Corresponding to Wang (1996), the trading strategies are expressed by the unique state variable $D_t$ rather than $W_{i,t}$ or $S_t$. However, our solution are more explicit than Wang (1996) as $S_t$ is solved in (2.19).
investors’ consumption shares are

\[ s_H = \frac{\hat{c}_{H,t}}{D_t} = \frac{1}{a+1} - \varepsilon \frac{ah_t}{(a+1)^2} + \mathcal{O}(\varepsilon^2), \quad s_L = \frac{\hat{c}_{L,t}}{D_t} = \frac{a}{a+1} + \varepsilon \frac{ah_t}{(a+1)^2} + \mathcal{O}(\varepsilon^2); \tag{2.27} \]

the weight ratio \( b \) is restricted by\(^ {13} \)

\[ \beta = \frac{a}{a+1} + \varepsilon \frac{a}{(a+1)^2} \left( h_0 + 2 + 2 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{\xi} \right) + \mathcal{O}(\varepsilon^2), \tag{2.28} \]

where

\[ h_t = \ln \left( \frac{a}{(a+1)^2} \right) + 2 \ln D_t, \quad a = b^{1/\gamma}. \]

**Remark 2.3.1** Although we keep all perturbation solutions in first-order, our perturbation solutions lead to a complete solution to the equilibrium. Firstly, Wang (1996) obtains exact solutions for Pareto-optimal consumption allocations; however, the equilibrium stock price is given as a single integration following an ordinary differential equation, and it is not convenient to use Wang’s solution in discussing the market equilibrium. In addition, substituting Bhamra and Uppal’s (2014) solution for consumption strategies in Appendix 2.5.1 into the marginal utility of the representative investor defined by (2.11), we find the marginal utility is a power function in terms of infinite Taylor expansions, which is too difficult to solve for the price of the long-lived stock. In this chapter, using a perturbation method, we solve the two-agent equilibrium model completely and provide the first complete solution to the equilibrium, including the optimal consumption rule, pricing function, Sharpe ratio, risk-free interest rate, stock price and optimal trading strategies. Finally, this chapter combines two factors the size of investors and heterogeneity to study their influence on the equilibrium.

\(^ {13} \)Given \( \beta, \gamma \) and \( \varepsilon \), we can solve \( a \) from a quadratic equation in (2.28) so that \( b \) can be determined by \( \beta, \gamma \) and \( \varepsilon \).
Remark 2.3.2 In Proposition 2.3.2, the risk-free interest rate is constant.\textsuperscript{14} In the absence of risk-averse heterogeneity \( \varepsilon \), the risk-free interest rate \( r = \delta + \mu \gamma - \frac{1}{2} \sigma_D^2 \gamma (\gamma + 1) \), which corresponds to the benchmark case \( \varepsilon = 0 \). If \( \mu - \frac{1}{2} \sigma_D^2 (2 \gamma + 1) \cdot \frac{a + 1}{a + 1} > 0 \), \( r \) decreases with the risk-averse heterogeneity in the economy, and vice versa. One situation is that fixing \( \mu - \frac{1}{2} \sigma_D^2 (2 \gamma + 1) > 0 \) and setting \( b > 1 \) (equivalently, \( a > 1 \)), \( r \) decreases with \( \varepsilon \). In other words, in a financial market with a large presence of type \( L \) investors, higher risk-averse heterogeneity reduces the risk-free interest rate. If \( \mu - \frac{1}{2} \sigma_D^2 (2 \gamma + 1) > 0 \) is fixed, then \( r \) decreases with the weight ratio \( b \) in the economy.

Remark 2.3.3 The price-dividend ratio (or the stock price) is time-varying, and its volatility is constant and equals \( \sigma_D \), from Equation (2.19).\textsuperscript{15} Given \( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma > 0 \), if the weight ratio \( b > 1 \) (which mean type \( L \) investors relatively dominate the financial market), the price-dividend ratio increases with the risk-averse heterogeneity \( \varepsilon \), and vice versa. In addition, the weight ratio \( b \) always pushes up the price-dividend ratio because the size of type \( L \) investors determines the weight ratio which can be regarded as a measure of the size, and the price-dividend ratio is a good measure for the P/E ratio. Thus, we can understand it as the larger size of institutions (or firms) deriving higher P/E ratio.

Remark 2.3.4 The Sharpe ratio is constant in Equation (2.21). In the homogeneous case, the Sharpe ratio \( \theta = \gamma \sigma_D \), which corresponds to the benchmark case \( \varepsilon = 0 \). If \( b > 1 \) (equivalently, \( a > 1 \)), the Sharpe ratio decreases with the risk-averse heterogeneity, and vice versa. Furthermore, the Sharpe ratio decreases with the weight ratio \( b \) in the economy.

Remark 2.3.5 Given \( h_t + 2 + 2 ( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma ) \cdot \frac{1}{2} > 0 \), the stock investment of type \( L \) investors, \( \hat{\phi}_{L,t} \), goes up with an increase both the risk-averse heterogeneity and the

\textsuperscript{14} The risk-free interest rate in the second-order approximation is stochastic. More details see Appendix 2.5.3.

\textsuperscript{15} The volatility in the second-order approximation is not constant. More details see Appendix 2.5.3.
weight ratio $b$ in the economy. The risk-free asset investment of type $L$ investors is always negative for any $\varepsilon > 0$. This means that type $L$ investors are always levered. Moreover, higher risk-averse heterogeneity brings higher leverage (borrowing) for type $L$ investors from Equation (2.26).

**Remark 2.3.6** Investors’ optimal consumption shares are stochastic and are with respect to $h_t$. If $h_t < 0$ ($b$ is extremely large), type $L$ investors’ consumption share $s_L$ decreases with the risk-averse heterogeneity, and vice versa. Obviously, from Equation (2.27), type $L$ investors’ consumption share increases with the weight ratio.

**Remark 2.3.7** The weight ratio $b$ is restricted by Equation (2.28), which reveals the relationship among $\beta$, $\varepsilon$ and $b$. If $b = 1, \varepsilon > 0$, then $\beta > 0.5$. Alternatively, if $\beta = 0.5, \varepsilon > 0$, then $b > 1$. Given $\beta \in (0, 1)$, $b$ decreases with the risk-averse heterogeneity $\varepsilon$. Obviously, $b$ increases with $\beta$. All solutions in Proposition 2.3.2 are related to the weight ratio $b$, which is dominated by the size of type $L$ investors $\beta$. This is consistent with the empirical evidence that financial institutions (as type $L$ investors) matter for asset pricing (Allen (2001) and Blume and Keim (2012)).

### 2.3.3 Aggregate risk aversion.

As there exist two types of investors, how they aggregate their risk aversion is an interesting question. Based on the definition of the representative investor, her/his relative-risk aversion also can be well defined as,

$$
\gamma_A = \frac{-D_t \frac{\partial^2 u_r(c_{H,t},c_{L,t})}{\partial D_t^2}}{\frac{\partial u_r(c_{H,t},c_{L,t})}{\partial D_t}} = \frac{-D_t \frac{\partial M_t}{\partial D_t}}{M_t}. 
$$

(2.29)

Plugging (2.20) into (2.29), we get the aggregate risk aversion of the market,

$$
\gamma_A = \gamma - \frac{\gamma(a-1)\varepsilon}{a+1} + O(\varepsilon^2).
$$

(2.30)
Corollary 2.3.1 An asymptotic expression for the aggregate risk aversion is \(^{1617}\)

\[
\gamma_A = \gamma - \frac{\gamma(a - 1)}{a + 1} \varepsilon + O(\varepsilon^2) \\
= (1 - \beta)\gamma_H + \beta\gamma_L + O(\varepsilon^2).
\] (2.31)

Remark 2.3.8 Equation (2.31) provides an exact aggregate rule for \(\gamma_A\) and reveals an economic intuition that the weighted average risk aversion using initial wealth ratio as the weight. This aggregating mechanism is true for the second-order solution (see Appendix 2.5.3). In addition, if \(\beta \to 0\), then \(b \to 0\) and \(\gamma_A \to \gamma_H\); if \(\beta \to 1\), then \(b \to \infty\) and \(\gamma_A \to \gamma_L\).

Remark 2.3.9 In (2.31), the aggregate risk aversion \(\gamma_A\) decreases with \(b\). This shows that more risk-seeking type \(L\) investors in the market lead to lower aggregate risk aversion. Furthermore, when \(b > 1\) (equivalently, \(a > 1\)), \(\gamma_A\) decreases with the risk-averse heterogeneity \(\varepsilon\), and vice versa. In other words, the higher risk-averse heterogeneity changes the type \(L\) investors have lower risk aversion and the type \(H\) investors have a higher one. Due to there being more type \(L\) investors in the economy, the aggregate risk aversion \(\gamma_A\) decreases.

The formula of the aggregate risk aversion in Equation (2.31) can intuitively explain the effects of the size of type \(L\) investors and risk-averse heterogeneity. In the homogeneous (aggregate) risk-aversion in Lucas (1978), a higher risk aversion leads to a higher equity risk premium, a higher Sharpe ratio and a lower stock price. Equation

\(^{16}\)The second equality is proved by using the restricted equation of the weight ratio \(b\) in (2.28).

\(^{17}\)By using Equation (2.21) and the fact that the volatility of the stock is \(\sigma_D\), the equity premium in our model is \(\left(1 - \frac{\varepsilon}{a + 1}\right)\gamma\sigma_D^2 + O(\varepsilon^2) = \gamma_A\sigma_D^2 + O(\varepsilon^2)\) which is equivalent to the results solved in the classical equilibrium model with one representative investor (see Mehra and Prescott (1985)). It shows that our model can not fully explain the equity premium puzzle. However, the main target of the chapter is to explore how the size of type \(L\) investors and risk-averse heterogeneity influence the prices of the assets and investments.
(2.31) indicates a lower \( b \) or a lower (higher) \( \varepsilon \) with \( b > (\leq) 1 \) leads to a higher risk aversion, a higher Sharpe ratio and a lower stock price. It intuitively explains the effects of the size of type \( L \) investors and risk-averse heterogeneity in Figure 2.7 and Figure 2.11, respectively. Crucially, whether the price goes up or down depends on which type of the investors dominate.

### 2.3.4 Properties of the equilibrium

In this subsection, we analyse the equilibrium effects of the size of type \( L \) investors \( \beta \) and the risk-averse heterogeneity \( \varepsilon \) on asset prices and optimal consumption strategies to understand the economics in financial markets. In terms of the parameters in this chapter, we set \( D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \gamma = 3/4, \delta = 0.02, t = 20, D_t = 1.43 \).\(^{18}\)

#### Effects of the size of type \( L \) investors \( \beta \).

In the benchmark homogeneous case (i.e., \( \varepsilon = 0 \)), the risk-free interest rate and the Sharpe ratio are constant and not related to the size of type \( L \) investors. This means that if type \( H \) and \( L \) investors are homogeneous, the risk-free interest rate and the Sharpe ratio are only influenced by their common time discount parameter, common risk aversion and the parameters of the dividend process. Moreover, the price-dividend ratio in the benchmark case is constant. The weight ratio \( b = \left( \frac{\beta}{1 - \gamma} \right)^{\gamma} \) is a monotonically increasing function with respect to \( \beta \in (0, 1) \). The results are presented in Figure 2.7 with a dashed line.

\(^{18}\)Here we set \( E[D_t] = D_0 e^{\mu t} = 1.43 \) as the value of \( D_t \).
Panel A plots the ratio of the weight of type L investors in the welfare function over the weight of type H investors in the welfare function $b$ against the size of type L investors $\beta$. Panel B plots the equilibrium Sharpe ratio $\theta$ against the faction of type L investors in the economy. Panels C and D plot the equilibrium asset prices: risk-free interest rate $r$ and the price-dividend ratio $V$, respectively. The solid (blue) line corresponds to the equilibrium with heterogeneous investors; the dashed (red) line corresponds to an equilibrium in the benchmark homogeneous economy. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \varepsilon = 1/3, \gamma = 3/4, \delta = 0.02, t = 20$.

In an economy with two heterogeneous investors, Figure 2.7, Panel A shows that the larger size of type L investors (e.g., $\beta = 0.8$) makes them have higher weight in the welfare function than the smaller size of type L investors (e.g., $\beta = 0.2$). Figure 2.7, Panels B, C and D illustrate that when the size of type L investors $\beta$ increases, the equilibrium Sharpe ratio $\theta$ and risk-free interest rate $r$ decrease, but the price-dividend ratio $V$ goes up or the return of stock goes down. Due to the volatility of the stock being a constant, $\sigma_D$, the Sharpe ratio is a good proxy for the equity
premium (or the excess return). As we know that the return of the stock equals the equity premium plus the risk-free interest rate, Figure 2.7, Panels B, C and D tell us that the decrease of the stock return is caused by the decrease in both the equity premium and the risk-free interest rate. Actually, the prices of the assets are determined by buyers. We name this effect as “buyer pressure”. On the one hand, with the larger size of type $L$ investors, the type $L$ investors demand more stocks. However, since the risky stock is in a fixed supply, it pushes the stock price $V$ up and meanwhile the return of the stock goes down. On the other hand, the type $H$ investors want to buy more bonds in order to protect themselves so that the price of the bond goes up. That makes the risk-free interest rate $r$ decrease. This “buyer pressure” provides a buying power to determine the prices of both risk-free and risk assets.

In the homogeneous case, the optimal consumption shares of type $L$ investors and holdings of the stock are the same and equal to $\beta$. In addition, holdings of the risk-free asset are kept at zero (the dashed lines in Figure 2.9). However, when heterogeneity exists, the relationship between holdings of the stock and the size of the institution $\beta$ is nonlinear. The largest difference is that the plot of type $L$ investors shorting the risk-free asset (borrowing money) is bell-shaped (Figure 2.9, Panel F). With an increase in the size of type $L$ investors, the type $L$ investors are able to borrow more money to invest in the risky asset. At a certain point, as the type $L$ investors become larger, the size of the type $H$ investors shrinks, and therefore the borrowing of type $L$ investors is reduced in the economy. This is a very important illustration of how leverage in the economy depends on the size of the type $L$ investors. Actually, the plot of type $L$ investors longing the additional risk asset is bell-shaped as well (Figure 2.9, Panel D). The changes of the stock investment $\phi_L$ times the stock price should equal the total borrowed money, which has the same shape as Figure 2.9, Panel E.
Panels A, C and E, respectively, plot the consumption share, holdings of the risk-free asset and holdings of the risky asset of type $H$ investors. Panels B, D and F, respectively, plot that of type $L$ investors. The solid (blue) line corresponds to the equilibrium with heterogeneous investors; the dashed (red) line corresponds to an equilibrium in the benchmark homogeneous economy. The plots are typical. The models parameters are as follows:
\[ D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \varepsilon = 1/3, \gamma = 3/4, \delta = 0.02, t = 20, D_t = 1.43. \]

In addition, Figure 2.9, Panel B shows that with the small size of type $L$ investors, type $L$ investors prefer to consume more because they can easily borrow enough money from the type $H$ investors to buy the stock. At the same time, because their borrowing of type $L$ investors is reduced in the economy, type $L$ investors prefer to consume less to buy the stock.

**Testable hypotheses on the effects of the size of type $L$ investors:** In an Arrow-Debreu equilibrium with heterogeneous type $L$ investors,
(i) the weight ratio \( b \) (which is the ratio of the weight of type \( L \) investors over the weight of type \( H \) investors in the welfare function) increases with the size of type \( L \) investors, \( \beta \), in the economy;

(ii) the Sharpe ratio \( \theta \) and the risk-free interest rate \( r \) decrease with the size of type \( L \) investors;

(iii) the price-dividend ratio \( V \) increases with the size of type \( L \) investors \( \beta \);

(iv) the consumption share of type \( L \) investors \( s_L \) increases with \( \beta \);

(v) the stock investment of type \( L \) investors \( \hat{\phi}_L \) increases with the size of type \( L \) investors; and

(vi) for \( \beta \in (0, 1) \), the type \( L \) investor is always levered, \( \psi_L < 0 \).

Effects of risk-averse heterogeneity \( \varepsilon \).

In Figure 2.11, the weight ratio \( b \) falls with increasing type \( L \) investors’ risk aversion (Figure 2.11, Panel A), but there is no effect of the risk-averse heterogeneity on the weight ratio \( b \) when the size of the type \( L \) investors is zero (\( \beta = 0 \)) or one (\( \beta = 1 \)). The effects on the Sharpe ratio, the risk-free interest rate and the price-dividend ratio depend on the size of type \( L \) investors. In particular, compared with the price-dividend ratio, the Sharpe ratio and the risk-free interest rate have a negative response to the risk-averse heterogeneity for the large size of type \( L \) investors (e.g., \( \beta = 0.75 \)). Those effects are the same as the effects of the size of type \( L \) investors \( \beta \). When the risk-averse heterogeneity \( \varepsilon \) increases, the type \( L \) investors have low risk aversion, while the type \( H \) investors have more. That encourages the type \( L \) investors to buy the more risky asset and the type \( H \) investors to buy more risk-free asset, so that the prices of risky and risk-free assets increase. In other words, the
“buyer pressure” leads to the return of the stock price and the risk-free interest rate goes down.

![Figure 2.11: Effects of risk-averse heterogeneity $\varepsilon$.](image)

Panel A plots the ratio of the weight of type $L$ investors in the welfare function over the weight of type $H$ investors in the welfare function $b$ against the size of type $L$ investors $\beta$. Panel B plots the equilibrium Sharpe ratio $\theta$ against the faction of type $L$ investors in the economy. Panels C and D plot the equilibrium asset prices: risk-free interest rate $r$ and the price-dividend ratio $V$, respectively. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \gamma = 3/4, \delta = 0.02, t = 20$.

In Figure 2.13, Panel D, the risky asset investment of the type $L$ investors goes slightly up with the risk-averse heterogeneity. If we changes the stock investment $\phi_L$ times the stock price, its shape should be same to Figure 2.13, Panel E. Figure 2.13, Panel F reveals that except for the homogeneous case ($\beta = 0, 1$), higher risk-averse heterogeneity forces the type $L$ investors to borrow more money (i.e., higher leverage). Finally, in Figure 2.13, Panel B, relatively more risk-seeking type
L investors have less consumption to save more money to buy the risky asset in the presence of type L investors.

Panels A, C and E, respectively, plot the consumption share, holdings of the risk-free asset and holdings of the risky asset of type H investors. Panels B, D and F, respectively, plot that of type L investors. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \gamma = 3/4, \delta = 0.02, t = 20, D_t = 1.43$.

Testable hypotheses on effects of the risk-averse heterogeneity $\varepsilon$:

(i) the Sharpe ratio $\theta$ and the risk-free interest rate $r$ decrease for type L investors with a large size (e.g., $\beta = 0.75$) but increase for those with a small size (e.g., $\beta = 0.25$) with the increase of the risk-averse heterogeneity $\varepsilon$;

(ii) the price-dividend ratio $V$ increases for type L investors with a large size but
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decreases for those with a small size with the increase of the risk-averse heterogeneity $\varepsilon$;

(iii) the weight ratio $b$ and type $L$ investors’ consumption share $s_L$ slightly go down, while the stock investment of the type $L$ investors $\hat{\phi}_L$ slightly goes up with the increase of the risk-averse heterogeneity $\varepsilon$;

(iv) for $\beta \in (0, 1)$, the type $L$ investors are always levered, and the leverage rises with the $\varepsilon$ in the economy.

Based on the above analysis, our perturbation solutions for the equilibrium model in a pure exchange economy can perfectly explain the effects of the size of type $L$ investors and the risk-averse heterogeneity on the Sharpe ratio, the risk-free interest rate, the stock price and optimal trading strategies for institutional and type $H$ investors. To summarize, following the standard framework in Wang (1996), this chapter reveals the mechanism of how the size of type $L$ investors (the initial wealth) and risk-averse heterogeneity affect the weight ratio and the equilibrium.

2.4 Conclusion

In this chapter, we study a pure exchange economy in which there are two types of investors, with lower and higher risk aversion, each with CRRA utility. This chapter presents theoretical analysis and a complete solution to the equilibrium problem in an exchange economy with these two heterogeneous investors.

This chapter completely solves for the equilibrium in this economy and to identify the optimal consumption-sharing rule, pricing function, Sharpe ratio, risk-free rate, stock price and optimal trading strategies for each type of investor. Our solutions are written as functions of the size of type $L$ investors in a perturbation form, with risk-aversion heterogeneity as a small parameter. Numerical experiments show
that our solution is accurate. It is very convenient to use our solutions, as they are in a closed-form.

Taking advantage of the tractability of our solutions, we analyse the effects of the size of type L investors and the risk-aversion heterogeneity on the equilibrium. We fully discuss the effect of the size of type L investors and the effect of the risk-aversion heterogeneity. These effects help us to better understand the economics of financial markets.

2.5 Appendix

2.5.1 The solution for the maximization problem (2.9) using Bhamra and Uppal’s (2014) method

Here, we use Bhamra and Uppal’s (2014) method to solve the maximization problem (2.9). The first-order condition for the maximization problem (2.9) is

\[ c^{-\gamma_H} = bc^{-\gamma_L} \]

By denoting \( s_{H,t} := \frac{c_{H,t}}{D_t} \) and \( s_{L,t} := \frac{c_{L,t}}{D_t} \), then,

\[ D_t^{-\gamma_H} s_{H,t}^{-\gamma_H} = bD_t^{-\gamma_L} s_{L,t}^{-\gamma_L}. \]  \hspace{1cm} (2.32)

Equation (2.32) can be rewritten as

\[ s_{L,t} = A_t s_{H,t}^{-\eta}. \]  \hspace{1cm} (2.33)

where \( A_t = \left( \frac{bD_t^{-\gamma_L}}{D_t^{-\gamma_H}} \right)^{\frac{1}{\gamma_L}} = b^{\frac{\gamma_H}{\gamma_L}} D_t^{\frac{2H-\gamma_L}{\gamma_L}}, \eta = \frac{\gamma_H}{\gamma_L} \) and \( s_{H,t} + s_{L,t} = 1 \). In our chapter, we assume \( \gamma_L < \gamma_H \).
The solutions for Equation (2.33) are

\[
S_{L,t} = \begin{cases} 
1 + \sum_{n=1}^{\infty} \left( \frac{-A^{-1} \eta}{\eta} \right)^n \left( \frac{n}{n-1} \right), & A_t > \overline{R}, \\
-\sum_{n=1}^{\infty} \left( \frac{-A_t}{\eta} \right)^n \left( \frac{n\eta}{n-1} \right), & A_t < \overline{R}, 
\end{cases}
\]

and

\[
S_{H,t} = \begin{cases} 
-\sum_{n=1}^{\infty} \left( \frac{-A^{-1} \eta}{\eta} \right)^n \left( \frac{n}{n-1} \right), & A_t > \overline{R}, \\
1 + \sum_{n=1}^{\infty} \left( \frac{-A_t}{\eta} \right)^n \left( \frac{n\eta}{n-1} \right), & A_t < \overline{R}, 
\end{cases}
\]

where \( \overline{R} = \frac{(n-1)^{n-1}}{\eta^n} \); for \( z \in \mathbb{C} \) and \( k \in \mathbb{N} \), \( \left( z \right) = \prod_{j=1}^{k} \frac{z-k+j}{j} \) is the generalized binomial coefficient and for \( z, k \in \mathbb{R}^+ \), \( \left( z \right) = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\Gamma(k+1)} \), the Gamma function \( \Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \).

Finally, we have the binomial series expressions for the maximization problem (2.9),

\[
\hat{c}_{H,t} = \begin{cases} 
D_t \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{n+1} \left( b^{-1} D_t \gamma_{L} - \gamma_{H} \right)^{n} \left( \frac{n \gamma_{L}}{\gamma_{H}} \right), & D_t > b^{1} \gamma_{L} - \gamma_{H}, \\
D_t - D_t \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{n+1} \left( b D_t^{\gamma_{H} - \gamma_{L}} \right)^{n} \left( \frac{n \gamma_{L}}{\gamma_{H}} \right), & D_t < b^{1} \gamma_{L} - \gamma_{H},
\end{cases}
\]
and

\[
\hat{c}_{L, t} = \begin{cases} 
D_t - D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{b} D_t^{-\gamma_H} \left( \frac{n \gamma_L}{\gamma_H} \right) \left( \frac{n}{n - 1} \right) , & D_t > \frac{1}{\gamma_L - \gamma_H} \frac{\gamma_H - \gamma_L}{\gamma_L} \left( \frac{\gamma_H}{\gamma_L} \right)^{\gamma_H - \gamma_L}, \\
D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( b D_t^{\gamma_H} - \gamma_L \right) \left( \frac{n \gamma_H}{\gamma_L} \right) \left( \frac{n}{n - 1} \right) , & D_t < \frac{1}{\gamma_L - \gamma_H} \frac{\gamma_H - \gamma_L}{\gamma_L} \left( \frac{\gamma_H}{\gamma_L} \right)^{\gamma_H - \gamma_L}. 
\end{cases}
\]

### 2.5.2 Proofs for Propositions

**Proof of Proposition 2.3.1:**

According to perturbation methods (e.g., Nayfeh (2008)), we denote \( f(D_t; \varepsilon) := \hat{c}_{H, t} \).

For a small \( \varepsilon \), \( f(D_t; \varepsilon) \) can be approximated in second-order by

\[
f(D_t; \varepsilon) = f_{0,t} + f_{1,t} \varepsilon + f_{2,t} \varepsilon^2 + O(\varepsilon^3). \tag{2.34}
\]

Substituting Equation (2.34) into (2.15), we get

\[
\frac{1}{b^{1/\gamma}} \left( D_t - f_{0,t} - f_{1,t} \varepsilon - f_{2,t} \varepsilon^2 \right) \\
= f_{0,t} + (f_{0,t} \ln [(D_t - f_{0,t}) f_{0,t}]) + f_{1,t} \varepsilon \\
+ \left[ \frac{1}{2} f_{0,t} \ln^2 [(D_t - f_{0,t}) f_{0,t}] + \frac{-f_{1,t} f_{0,t} + (D_t - f_{0,t}) f_{1,t}}{D_t - f_{0,t}} + f_{1,t} \ln [(D_t - f_{0,t}) f_{0,t}] + f_{2,t} \right] \varepsilon^2 \\
+ O(\varepsilon^3).
\]
We can collect powers of

\[ O(\varepsilon^0) : \quad D_t - f_{0,t} = b^{1/\gamma} f_{0,t}, \]

\[ O(\varepsilon^1) : \quad -f_{1,t} = b^{1/\gamma} (f_{0,t} \ln [(D_t - f_{0,t}) f_{0,t}] + f_{1,t}), \]

\[ O(\varepsilon^2) : \quad -f_{2,t} = b^{1/\gamma} \left[ \frac{1}{2} f_{0,t} \ln^2 [(D_t - f_{0,t}) f_{0,t}] \right. \]

\[ + \left. \frac{-f_{1,t} f_{0,t} + (D_t - f_{0,t}) f_{1,t}}{D_t - f_{0,t}} + f_{1,t} \ln [(D_t - f_{0,t}) f_{0,t}] + f_{2,t} \right]. \]

Now we solve at each order,

\[ f_{0,t} = \frac{1}{a+1} D_t, \quad f_{1,t} = -\frac{a h_t f_{0,t}}{a+1} = -\frac{a h_t}{(a+1)^2} D_t, \]

and

\[ f_{2,t} = -\frac{a(1-a)(h_t^2 + 2h_t)}{2(a+1)^3} D_t, \]

where \( h_t = \ln [(D_t - f_{0,t}) f_{0,t}] = \ln \left( \frac{a}{(a+1)^2} \right) + 2 \ln D_t, a = b^{1/\gamma}. \)
Proof of Proposition 2.3.2:

Following the definition of the marginal utility of the representative investor in (2.11), the marginal utility in first-order approximation is

\[ M_t = e^{-\delta t} \hat{c}_{H,t} e^{-\gamma (1+\varepsilon)} \frac{\partial \hat{c}_{H,t}}{\partial D_t} + e^{-\delta t} b(\hat{c}_{L,t}) e^{-\gamma (1-\varepsilon)} \frac{\partial \hat{c}_{L,t}}{\partial D_t} \]

\[ = e^{-\delta t} \left[ \left( 1 - \varepsilon \frac{1}{a+1} h_t \right) \frac{1}{a+1} D_t \right]^{-\gamma - \gamma \varepsilon} \left( 1 - \varepsilon \frac{1}{a+1} (h_t + 2) \right) \frac{1}{a+1} \]

\[ + e^{-\delta t} b \left[ \left( 1 + \varepsilon \frac{1}{a+1} h_t \right) \frac{a}{a+1} D_t \right]^{-\gamma + \gamma \varepsilon} \left( 1 + \varepsilon \frac{1}{a+1} (h_t + 2) \right) \frac{a}{a+1} \]

\[ + O(\varepsilon^2) \]

\[ = e^{-\delta t} \left( \frac{1}{a+1} \right)^{-\gamma} D_t^{-\gamma} \left[ 1 + \varepsilon \frac{\gamma}{a+1} \left( (a-1) \ln \frac{D_t}{a+1} + a \ln a \right) \right] \]

\[ + O(\varepsilon^2). \]

Thus, the marginal utility of the representative investor can be approximated in first-order by

\[ M_t = e^{-\delta t} \left( \frac{1}{a+1} \right)^{-\gamma} D_t^{-\gamma} (1 + \varepsilon g_t) + O(\varepsilon^2). \]

where \( g_t = g_1 \ln D_t + g_0, g_1 = \frac{\gamma(a-1)}{a+1} \) and \( g_0 = \frac{\gamma}{a+1} (a \ln a - (a - 1) \ln(a + 1)) \). Then, the stochastic differential equation process of the marginal utility is

\[ \frac{dM_t}{M_t} = - \left[ \delta + \mu \gamma - \frac{1}{2} \sigma_D^2 \gamma (\gamma + 1) + \varepsilon g_1 \left( \frac{1}{2} \sigma_D^2 (2\gamma + 1) - \mu \right) \right] dt \]

\[ - (\gamma - \varepsilon g_1) \sigma_D dB_t, \]

and the first-order perturbation solution for the Sharpe ratio is

\[ \theta_t = (\gamma - \varepsilon g_1) \sigma_D + O(\varepsilon^2). \]
Since \( D_s = D_t e^{(\mu - \frac{1}{2} \sigma_D^2) (s-t) + \sigma_D (B_s - B_t)} \), then \( \ln D_s = \ln D_t + (\mu - \frac{1}{2} \sigma_D^2) (s-t) + \sigma_D (B_s - B_t) \). In addition, the moment generating function of \( B_s - B_t \) is \( F(\omega) := E_t[e^{\omega (B_s - B_t)}] = e^{\frac{1}{2} (s-t) \omega^2} \) for \( \omega \in \mathbb{R} \) and \( \frac{\partial F(\omega)}{\partial \omega} = e^{\frac{1}{2} (s-t) \omega^2} (s-t) \omega \). By using those, the stock price is

\[
S_t = E_t \left[ \int_t^\infty \left( \frac{M_s}{M_t} \right) D_s ds \right] = D_t \left[ \int_t^\infty e^{-\delta(s-t)} D_s^{1-\gamma} (1 + \varepsilon g_s) ds \right] + \mathcal{O}(\varepsilon^2)
\]

\[
= D_t \left[ \int_t^\infty e^{-\delta(s-t)} + (\mu - \frac{1}{2} \sigma_D^2)(s-t) + \sigma_D (1-\gamma)(B_s - B_t) \right] \left( 1 + \varepsilon g_t + \varepsilon g_1 (\mu - \frac{1}{2} \sigma_D^2)(s-t) \right) ds + \mathcal{O}(\varepsilon^2)
\]

\[
= D_t \left[ \int_t^\infty e^{-\xi(s-t)} \left( 1 + \varepsilon g_t + \varepsilon g_1 (\mu - \frac{1}{2} \sigma_D^2)(s-t) \right) ds + \mathcal{O}(\varepsilon^2) \right]
\]

\[
= D_t \int_t^\infty e^{-\xi(s-t)} ds + \varepsilon D_t g_1 (\mu - \frac{1}{2} \sigma_D^2) \int_t^\infty e^{-\xi(s-t)} (s-t) ds + \mathcal{O}(\varepsilon^2)
\]

\[
= \frac{1}{\xi} + \varepsilon g_1 \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1-\gamma) \right) \frac{1}{\xi^2} D_t + \mathcal{O}(\varepsilon^2).
\]
Similarly, the wealth of type $H$ investors is

$$W_{H,t} = E_t \left[ \int_t^\infty \left( \frac{M_s}{M_t} \right) \hat{c}_{H,s} ds \right]$$

$$= \frac{1}{a + 1} D_t^{-\gamma} (1 + \varepsilon g_t) E_t \left[ \int_t^\infty e^{-\delta(s-t)} D_s^{1-\gamma} (1 + \varepsilon g_s) \left( 1 - \varepsilon \frac{a}{a + 1} h_s \right) ds \right] + O(\varepsilon^2)$$

$$= \frac{D_t}{a + 1} \int_t^\infty E_t \left[ e^{-\delta(s-t)} + \left( 1 + \varepsilon g_t + \varepsilon g_1 (\mu - \frac{1}{2} \sigma_D^2) (s-t) + \varepsilon g_1 \sigma_D (B_s - B_t) \right) \left( 1 - \varepsilon \frac{a}{a + 1} h_t - 2\varepsilon \frac{a}{a + 1} (\mu - \frac{1}{2} \sigma_D^2) (s-t) \right) \right] ds + O(\varepsilon^2)$$

$$= \frac{D_t}{a + 1} \int_t^\infty E_t \left[ e^{-\delta(s-t)} + \left( 1 + \varepsilon g_1 (\mu - \frac{1}{2} \sigma_D^2) (s-t) - \varepsilon \frac{a}{a + 1} h_t - 2\varepsilon \frac{a}{a + 1} (\mu - \frac{1}{2} \sigma_D^2) (s-t) \right) \right] ds + O(\varepsilon^2)$$

$$= \frac{D_t}{a + 1} \int_t^\infty e^{-\xi(s-t)} \left( 1 - \varepsilon \frac{a}{a + 1} h_t + \varepsilon \left( g_1 - 2\frac{a}{a + 1} \right) (\mu - \frac{1}{2} \sigma_D^2) (s-t) \right) ds$$

$$+ \frac{D_t}{a + 1} \varepsilon \left( g_1 - 2\frac{a}{a + 1} \right) \sigma_D \int_t^\infty e^{-\xi(s-t)} (s-t) ds + O(\varepsilon^2)$$

$$= \frac{D_t}{a + 1} \left( 1 - \varepsilon \frac{a}{a + 1} h_t \right) \int_t^\infty e^{-\xi(s-t)} ds$$

$$+ \varepsilon (\mu - \frac{1}{2} \sigma_D^2) D_t \left( g_1 - 2\frac{a}{a + 1} \right) \int_t^\infty e^{-\xi(s-t)} (s-t) ds$$

$$+ \frac{D_t}{a + 1} \varepsilon \left( g_1 - 2\frac{a}{a + 1} \right) \sigma_D \int_t^\infty e^{-\xi(s-t)} \sigma_D (1 - \gamma) (s-t) ds + O(\varepsilon^2)$$

$$= \frac{D_t}{a + 1} \left( 1 - \varepsilon \frac{a}{a + 1} h_t \right) + \varepsilon \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{D_t}{a + 1} \gamma (a - 1) - 2a \frac{1}{\xi^2} + O(\varepsilon^2).$$
Then, the optimal investments in stock and bond are

\[
\hat{\phi}_{H,t} = \frac{dW_{H,t}}{dS_t} = \frac{dW_{H,t}}{dD_t} = \frac{dS_t}{dD_t}
\]

\[
= \frac{1}{a+1} \left( 1 - \varepsilon \frac{a}{a+1}(h_t + 2) \right)^{\frac{1}{\xi}} + \varepsilon \left( \mu + \sigma_D^2 \left( \frac{1}{2} - \gamma \right) \right) \frac{\gamma(a-1)}{(a+1)^2} + O(\varepsilon^2)
\]

\[
= \frac{1}{a+1} - \varepsilon \left[ \frac{a}{(a+1)^2} (h_t + 2) + \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{2a}{(a+1)^2} \right] \frac{1}{\xi} + O(\varepsilon^2),
\]

and

\[
\hat{\psi}_{H,t} = W_{H,t} - \hat{\phi}_{H,t} S_t = 2\varepsilon \frac{a}{(a+1)^2} \frac{1}{\xi} D_t + O(\varepsilon^2).
\]

Therefore, the portfolios of type \(L\) investors are

\[
\hat{\phi}_{L,t} = 1 - \hat{\phi}_{H,t}, \quad \hat{\psi}_{L,t} = -\hat{\psi}_{H,t}.
\]

Using initial wealth of type \(H\) investors, \(W_{A,0} = (1 - \beta)S_0\), the weight ratio \(b\) is restricted by

\[
1 - \beta = \hat{\phi}_{A,0} = \frac{1}{a+1} - \varepsilon \left[ \frac{a}{(a+1)^2} (h_0 + 2) + \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{2a}{(a+1)^2} \right] \frac{1}{\xi},
\]

where \(h_0 = \ln \left( \frac{a}{(a+1)^2} \right) + 2 \ln D_0\).

2.5.3 An extension to the second-order solutions

In this appendix, we shall present the second-order solutions to study the effects of the cash flow news \(D_t\) on the equilibrium. In addition, in this case, the volatility of the risk-free interest rate is no longer zero and the volatility of the stock market
return is no longer same to the dividend’s volatility, so that we will investigate how
the size of type L investors and the risk-averse heterogeneity affect the volatility of
assets. Taking on the same steps of the proof of the first-order solutions, we give the
solution for the equilibrium in the second-order in the following proposition.

**Proposition 2.5.1 (Equilibrium in second-order)** 19 When the economy is in equilib-
rium, the equilibrium risk-free interest rate is

\[
 r_t = \delta + \mu \gamma - \frac{1}{2} \sigma_D^2 \gamma (\gamma + 1) + \varepsilon g_1 \left( \frac{1}{2} \sigma_D^2 (2 \gamma + 1) - \mu \right) \\
+ 2 \varepsilon^2 \left( \frac{4 \gamma a}{(a + 1)^2} \ln D_t + k_1 - g_0 g_1 \right) \left( \frac{1}{2} \sigma_D^2 (2 \gamma + 1) - \mu \right) - k_2 \sigma_D^2 \right) + O(\varepsilon^3). 
\](2.35)

where

\[
g_1 = \frac{\gamma (a - 1)}{a + 1}, \quad g_0 = \frac{\gamma}{a + 1} (a \ln a - (a - 1) \ln(a + 1)), \\
k_2 = \frac{\gamma}{2(a + 1)^2} [\gamma (a - 1)^2 + 4a], \\
k_1 = \frac{\gamma}{2(a + 1)^2} [2 \gamma a (a - 1) \ln(a) - 2 \gamma (a - 1)^2 \ln(a + 1) + 4a \ln(a) \\
- 8a \ln(a + 1) + 8a],
\]

and its volatility is 20

\[
\sigma_r := \frac{2 \varepsilon^2 \gamma (\gamma (a - 1)^2 + 8a) \left( \frac{1}{2} \sigma_D^2 (2 \gamma + 1) - \mu \right)}{(a + 1)^2} \sigma_D + O(\varepsilon^3).
\]

19The proof is available upon request.

20Similarly to Bhamra and Uppal (2009), the non-zero volatility of the risk-free interest rate or the
stochastic risk-free interest rate is due to the more volatile consumptions in (2.16)-(2.17) in the second-
order. In addition, it leads to that the volatility of the stock price is no longer $\sigma_D$. 

In addition, the Sharpe ratio is

$$\theta_t = \left[ \gamma - \left( \varepsilon g_1 + 2\varepsilon^2 \left( \frac{4\gamma a}{(a+1)^2} \ln D_t + k_1 - g_0 g_1 \right) \right) \right] \sigma_D + O(\varepsilon^3). \quad (2.36)$$

The equilibrium price of the stock is given by

$$S_t = \frac{1}{\xi} D_t + \varepsilon g_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{\xi^2} D_t$$

$$+ \varepsilon^2 \left[ k_2 \left( 2 \ln D_t \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) + \sigma_D^2 \right) + k_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) 
- g_t g_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{\xi^2} D_t 
+ \varepsilon^2 k_2 \left( \mu - \frac{1}{2} \sigma_D^2 \right) \left( \mu - \frac{1}{2} \sigma_D^2 + 2 \sigma_D^2 (1 - \gamma) \right) + \sigma_D^2 (1 - \gamma)^2 \right] \frac{2}{\xi^3} D_t + O(\varepsilon^3),$$

and the volatility of stock market return \((dS_t/S_t)\) is

$$\sigma_S = \sigma_D + 4\varepsilon^2 \gamma \left( \gamma (a-1)^2 + 8a \right) \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{(a+1)^2} \sigma_D + O(\varepsilon^3),$$

and the equity premium is

$$E_t \left[ \frac{dS_t + D_t dt}{S_t} \right] - r_t = \theta_t \sigma_S$$

$$= \gamma \sigma_D^2 \left( 1 - \frac{a - 1}{a + 1} \right) 
- 2\varepsilon^2 \gamma \sigma_D^2 \left[ \left( \frac{4\gamma a}{(a+1)^2} \ln D_t + k_1 - g_0 g_1 \right) - \frac{2\gamma (\gamma (a-1)^2 + 8a) \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right)}{(a+1)^2} \right].$$
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The price-dividend ratio is

\[ V_t = \frac{S_t}{D_t} = \frac{1}{\xi} + \varepsilon g_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \frac{1}{\xi^2} \]

\[ + \varepsilon^2 \left[ k_2 \left( 2 \ln D_t \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) + \sigma_D^2 \right) + k_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \right. \]

\[ - g_1 \left( \mu + \frac{1}{2} \sigma_D^2 - \sigma_D^2 \gamma \right) \left] \frac{1}{\xi^2} \right. \]

\[ + \varepsilon^2 k_2 \left( \mu - \frac{1}{2} \sigma_D^2 \right) \left( \mu - \frac{1}{2} \sigma_D^2 + 2 \sigma_D^2 (1 - \gamma) \right) + \sigma_D^2 (1 - \gamma)^2 \right) \frac{2}{\xi^3} + \mathcal{O}(\varepsilon^3). \]

The trading strategies for type H investors is

\[ \hat{\phi}_{H,t} = \frac{1}{a + 1} - \frac{a}{(a + 1)^2} \left[ \varepsilon (h_t + 2) + 2 \varepsilon \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{1}{\xi} \right. \]

\[ + \varepsilon^2 g_1 (\mu - \frac{1}{2} \sigma_D^2) (h_t + 2) \frac{1}{\xi} + 2 \varepsilon^2 g_1 (\mu - \frac{1}{2} \sigma_D^2) \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{2}{\xi^2} \]

\[ + \varepsilon^2 g_1 \sigma_D^2 (1 - \gamma) (h_t + 2) + 2 \varepsilon^2 g_1 \sigma_D^2 (1 - \gamma) (\mu - \frac{1}{2} \sigma_D^2) \frac{1}{\xi} \]

\[ + 2 \varepsilon^2 g_1 \sigma_D^2 \frac{1}{\xi} + 2 \varepsilon^2 g_1 \sigma_D^2 (1 - \gamma)^2 \frac{2}{\xi^2} + \varepsilon^2 \left( 1 - a \right) (h_t^2 + 6 h_t + 4) \frac{1}{(a + 1)^2} \]

\[ + 8 \varepsilon^2 \left( 1 - a \right) (\ln D_t + 1) (\mu - \frac{1}{2} \sigma_D^2) \frac{1}{\xi} + 4 \varepsilon^2 \left( 1 - a \right) (\mu - \frac{1}{2} \sigma_D^2) \sigma_D^2 \frac{2}{\xi^2} \]

\[ + 8 \varepsilon^2 \left( 1 - a \right) (\ln D_t + 1) \sigma_D^2 (1 - \gamma) + 8 \varepsilon^2 \left( 1 - a \right) (\mu - \frac{1}{2} \sigma_D^2) \sigma_D^2 (1 - \gamma) \frac{1}{\xi} \]

\[ + 4 \varepsilon^2 \left( 1 - a \right) \sigma_D^2 \frac{1}{\xi} + 4 \varepsilon^2 \left( 1 - a \right) \sigma_D^2 (1 - \gamma)^2 \frac{2}{\xi^2} \]

\[ + 4 \varepsilon^2 \left( 1 - a \right) \frac{1}{(a + 1)^2} (\ln \left( \frac{a}{(a + 1)^2} \right) + 1) \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{1}{\xi} \]

\[ - \varepsilon^2 g_1 (h_t + 2) \frac{1}{\xi} + 2 \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{1}{\xi^2} \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \]

\[ + \mathcal{O}(\varepsilon^3), \]
and type $H$ investors’ wealth process is given by\footnote{Similar to Wang (1996), the wealth process can be expressed by the stock price and the divided. As there is only one state variable $D_t$, the wealth process $W_{H,t}$ is one of our solutions.}

\[
W_{H,t} = \frac{S_t}{a + 1} - \frac{aD_t}{(a + 1)^2} \left[ \epsilon h_t \frac{1}{\xi^2} + 2\epsilon \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{1}{\xi^2} \right. \\
+ \left. 2\epsilon^2 g_1 (\mu - \frac{1}{2} \sigma_D^2) h_t \frac{1}{\xi^2} + 2\epsilon^2 g_1 (\mu - \frac{1}{2} \sigma_D^2) \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{2}{\xi^3} \right] \\
+ \left. 2\epsilon^2 g_1 \sigma_D^2 (1 - \gamma) h_t \frac{1}{\xi^2} + 2\epsilon^2 g_1 \sigma_D^2 (1 - \gamma) (\mu - \frac{1}{2} \sigma_D^2) \frac{1}{\xi^2} \right] \\
+ \left. 8\epsilon^2 \frac{(1 - a)}{(a + 1)^2} \ln D_t (\mu - \frac{1}{2} \sigma_D^2) \frac{1}{\xi^2} + 4\epsilon^2 \left( \frac{1 - a}{(a + 1)^2} (\mu - \frac{1}{2} \sigma_D^2) \right)^2 \frac{2}{\xi^3} \right] \\
+ \left. 8\epsilon^2 \frac{(1 - a)}{(a + 1)^2} \ln D_t \sigma_D^2 (1 - \gamma) \frac{1}{\xi} + 8\epsilon^2 \frac{(1 - a)}{(a + 1)^2} (\mu - \frac{1}{2} \sigma_D^2) \sigma_D^2 (1 - \gamma) \frac{1}{\xi} \right] \\
+ \left. 4\epsilon^2 \frac{(1 - a)}{(a + 1)^2} \sigma_D^2 \frac{1}{\xi^2} + 4\epsilon^2 \frac{(1 - a)}{(a + 1)^2} \sigma_D^2 (1 - \gamma) \frac{2}{\xi^3} \right] \\
+ \left. 4\epsilon^2 \frac{(1 - a)}{(a + 1)^2} (\ln \frac{a}{(a + 1)^2} + 1) \left( \mu - \frac{1}{2} \sigma_D^2 + \sigma_D^2 (1 - \gamma) \right) \frac{1}{\xi^2} \right] + O(\epsilon^3). \]

Therefore, the bond investment of type $H$ investors and the portfolio of type $L$ investors are

\[
\hat{\psi}_{H,t} = W_{H,t} - \hat{\phi}_{H,t} S_t, \quad \hat{\phi}_{L,t} = 1 - \hat{\phi}_{H,t}, \quad \hat{\psi}_{L,t} = -\hat{\psi}_{H,t},
\]

and the weight ratio $b = a^\gamma$ is restricted by

\[
1 - \beta = \hat{\phi}_{H,0}. \quad (2.37)
\]
Finally, the aggregate risk aversion is:

\[
\gamma_A = \gamma - \frac{\gamma(a - 1)}{a + 1} \varepsilon - \frac{2 \gamma a (2 + h_t)}{(a + 1)^2} \varepsilon^2 + \mathcal{O}(\varepsilon^3)
\]

\[
=(1 - \beta)\gamma_H + \beta\gamma_L + \mathcal{O}(\varepsilon^3).
\]  

(2.38)

In Proposition 2.5.1, the risk-free interest rate is stochastic. If \( \mu - \frac{1}{2} \sigma_D^2 (2 \gamma + 1) > 0 \) is fixed, then \( r_t \) decreases with cash flow news \( D_t \) in the economy (e.g., see Figure 2.15, Panel A), and vice versa. In addition, the Sharpe ratio and the excess market return (the equity premium) are always decreasing with cash flow news \( D_t \) (e.g., see Figure 2.15, Panel B and C), while the price-dividend ratio \( S_t / D_t \) is always increasing with cash flow news \( D_t \) (e.g., see Figure 2.15, Panel D). Actually, the volatility of the stock is constant so that the excess return behaves analogously to the Sharpe ratio. Because the type \( L \) investors are overweighted in the risky stock relative to the retail investors, good cash flow news always produces a wealth transfer from the type \( H \) investors to the type \( L \) investors. So, the the Sharpe ratio is therefore decreasing in \( D_t \). In terms of price-dividend ratio, intuitively, if a good cash flow news is coming, the stock price will always increase.

\[^{22}\text{By using the restricted equation of the weight ratio } b \text{ in (2.37), we can get the second equality in (2.38). As } \gamma_H = \gamma(1 + \varepsilon) \text{ and } \gamma_H - \gamma(1 - \varepsilon), \text{ only constant and the first order terms of } \hat{\phi}_{H,0} \text{ contribute to the finally results.}\]
Panel A, B, C and D respectively plot the risk-free interest rate, Sharpe ratio, the excess return of the stock and the price-dividend ratio against the cash flow news $D_t$. The plots are typical. The models parameters are as follows: $\mu = 0.018, \sigma_D = 0.036, b = 1.7, \gamma = 3/4, \delta = 0.02$.

In addition, Figure 2.17 Panel B illustrates the response of the type $L$ investors’ equilibrium stock investment to cash flow news. In equilibrium, both types of investors have positive holdings of the risky stock (see Figure 2.9 and 2.13) so that good cash flow news derives each investor to buy more risky stocks. As we set $b = 1.7$ (or $\beta = 0.67$), the type $L$ investors dominates the market. Thus, as positive cash flow news arrives ($D_t$ increases), the type $L$ investors buy more risky asset from the type $H$ investors.

When an negative shock occurs (moving from the middle to the left hand side in Figure 2.17, Panel D), type $L$ investors will sell their risky asset and buy more bonds to reduce their lost, and consequently become less important, so that type
$L$ investors may increase their risk aversion. Further, because type $L$ investors are overweighted in the risky stock, the risk takers is hurt relatively more when an negative shock occurs.

For a good economy, type $L$ investors earn more. With good new coming, the price of the stock increases (moving from the middle to the right hand side in Figure 2.17, Panel D). As type $L$ investors own more risky stock, they buy more risky stocks so that they become wealthier and wealthier (Figure 2.17, Panel F). In contrast, retails investors loss more and more. Finally, they have to borrow money from type $L$ investors (Figure 2.17, Panel C).\footnote{Even though we set $\beta < 0.5$ (e.g., $\beta = 0.2$ or $b = 0.35$) which means the initial size type $L$ investors is quite small, the evolutions of Figure 2.17 are not changed. It tells us that with the economy developing better and better, the type $L$ investors become wealthier and wealthier. This is consistent to the fact that the size of type $L$ investors jumps sharply from 7-8% in 1950 to 67% in 2010. Our model does explain the phenomenon that type $L$ investors as risk takers become larger and larger in the financial market.}
Testable hypotheses on the effects of cash flow news: In an Arrow-Debreu equilibrium with heterogeneous type L investors,

(i) the Sharpe ratio $\theta$, the risk-free interest rate $r$ and the stock excess return decrease with the cash flow news;

(ii) the price-dividend ratio $V$ increases with the cash flow news $D_t$;

(iii) the stock investment and the wealth of type $L$ investors $\hat{\phi}_L$ increase with the cash flow news.
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In benchmark case, the volatilities of the risk-free interest rate and stock market return are not related to the size of type $L$ investors, while, in heterogeneous case, the plots of the asset volatility are bell-shaped (see Figure 2.19). Based on the above discussion of Figure 2.9, the bell-shaped diagram is because of effect of leverage on bond and stock markets. The wealth transfers between type $L$ investors and retails investors increase the fluctuation of assets. Because the type $H$ investors (who have higher risk aversion) are more sensitive to shocks, the effect of the two agents on bond and stock market volatility is not symmetric.

Figure 2.19: Effects of the size of type $L$ investors $\beta$.

Panels A and B, respectively, plot the relative volatilities of the risk-free interest rate and the risky asset against the fraction of type $L$ investors in the economy. The plots are typical. The models parameters are as follows: $D_0 = 1$, $\mu = 0.018$, $\sigma_D = 0.036$, $\gamma = 3/4$, $\delta = 0.02$.

Figure 2.21 reveals that with increasing of risk-averse heterogeneity, the volatility of the risk-free and risky assets goes up against different size of type $L$ investors.
It shows that risk-averse heterogeneity is one of sources of risk. This response is corresponding to Bhamra and Uppal (2009, 2014).

![Figure 2.21: Effects of risk-averse heterogeneity $\varepsilon$.](image)

Panels A and B, respectively, plot the relative volatilities of the risk-free interest rate and the risky asset against the risk-averse heterogeneity. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.018, \sigma_D = 0.036, \gamma = 3/4, \delta = 0.02$.

**Testable hypotheses on the volatility of assets:** In an Arrow-Debreu equilibrium with heterogeneous type $L$ investors,

(i) the volatility of the risk-free interest rate and the volatility of the stock are higher in the presence of type $L$ investors;

(ii) the volatility of the risk-free interest rate and the volatility of the stock increase with the increase of the risk-averse heterogeneity $\varepsilon$. 


By virtue of the second-order solutions, we further understand the underlying economic mechanism on how the cash flow news $D_t$ influences the equilibrium, e.g., the risk-free interest rate, price-dividend ratio, wealth process of investors and investors’ investments in both bond and stock markets. Furthermore, the stochastic risk-free interest rate and the stochastic stock produce their volatilities affected by both the size of type $L$ investors and risk-averse heterogeneity.
Chapter 3

Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

This chapter is joint work with Jin E. Zhang. Its earlier version was presented at 2016 Auckland Finance Meeting, 16-18 December 2016, AUT, Auckland, New Zealand.

3.1 Introduction

In this chapter, we construct an equilibrium model in a cost-free production economy with a representative investor who has recursive preferences. To simplify, we assume the process of the stock price follows a stochastic volatility with contemporaneous jumps (SVCJ) model. After solving the equilibrium, we conclude the SVCJ model and its degenerated models built on our cost-free production economy can perfectly capture the high equity risk premium (ERP). In addition, for some extremely large ERP periods in which the annualized ERP is at or larger than 11% (e.g., 1990–1999 and 2010–2016), the SVCJ model and the stochastic volatility with

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1The SVCJ is one of the most widely-used affine jump-diffusion (AJD) models (see Duffie, Pan, and Singleton (2000)).
jumps in volatility (SVJV) model built on our cost-free production economy can successfully explain the large ERP and the large negative VRP. To our knowledge, our model is the first production-based equilibrium model to explain the equity premium puzzle and the large negative VRP.

An explanation of the high negative VRP, defined as the realized variance (RV) minus the implied variance (IV), is a very important topic. Practically, the VRP is the practitioners’ cost to get protection against high realized variance via buying variance swaps. How much they need to pay, of course, is a very crucial issue for practitioners’ care. Empirically, Carr and Wu (2009) use the difference between the RV and this synthetic variance swap rate to quantify the VRP and find that there exists a large and negative mean of the VRP on five stock indexes and 35 individual stocks by using a large options data set. Todorov (2010); Bollerslev and Todorov (2011); Bollerslev, Todorov, and Xu (2015) use the rare events to account for the large average VRP. Recently, González-Urteaga and Rubio (2016) discuss and test the volatility risk premium at the individual and portfolio level. Barras and Malkhozov (2016) formally compare the market VRP inferred from equity and option markets and find that the average difference between the two VRPs is essentially zero. Aït-Sahalia, Karaman, and Mancini (2015); Li and Zinna (2017) examine the term structures of the VRP by using variance swap rates data. Dew-Becker, Giglio, Le, and Rodriguez (2017) use novel data on a wide range of variance swaps with maturities between one month and 10 years in the period 1996–2014 to analyse the pricing of variance swaps, and they find that news about future volatility is unpriced, while exposure to realized variance is strongly priced. These results present a challenge to all

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2The large negative VRP means the large average negative VRP. Our definition follows Carr and Wu (2009). Some papers propose the positive VRP as they define the VRP as the implied variance minus the realized variance.

3Mixon and Onur (2015) report that the gross vega notional outstanding for variance swaps, in 2014, is over USD 2 billion, with USD 1.5 billion in S&P 500 products. The volatility market has become particularly popular over last decade.

4The difference between the variance risk premium and the volatility risk premium is whether we take a square root of the variance.
existing asset pricing models, such as the intertemporal capital asset pricing model (ICAPM) and recent models with Epstein–Zin preferences and long-run risks. Theoretically, Bollerslev et al. (2009); Drechsler and Yaron (2011); Bollerslev, Sizova, and Tauchen (2012); Drechsler (2013); Jin (2015) adopt the long-run risks model (first proposed by Bansal and Yaron (2004)) to successfully explain the large average VRP. In addition, Buraschi, Trojani, and Vedolin (2014) use a two-tree Lucas (1978) economy with two heterogeneous investors to explain well the volatility risk premium. All previous models in the literature are built on a consumption economy. In this chapter, we construct a simple production-based equilibrium model and successfully explain the large ERP and VRP with a much lower level of relative risk aversion (RRA) of 1.008, when the annualized ERP is at or larger than 11% (e.g., the periods, 1990–1999 and 2010–2016). However, the existing literature, Drechsler and Yaron (2011) choose a value of 9.5 to explain the VRP and Drechsler (2013) sets it as 5.

The production-based equilibrium model adopted in this chapter is developed from the neoclassical growth model in Constantinides (1990), which is first studied by Cox et al. (1985a) and followed by Bates (1991, 1996); Vasicek (2005); Zhang et al. (2012); Fu and Yang (2012); Ruan, Zhu, Huang, and Zhang (2016). The main difference between our production-based equilibrium model and the classic consumption-based equilibrium model (e.g., Lucas (1978); Mehra and Prescott (1985); Bansal and Yaron (2004); Drechsler and Yaron (2011); Wachter (2013)) is that we construct our model starting from the index level (e.g., S&P 500), while the consumption-based model is based on the fundamentals (i.e., consumption and dividends). This difference does affect a lot. For example, in terms of the equity premium puzzle, based on a consumption-based equilibrium model, Mehra and Prescott (1985) find that the equity premium (6.18%) is the product of the coefficient of the RRA and the variance

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5The cost-free production economy is taken from Constantinides (1990). However, Constantinides (1990) only solves an investment problem instead of an equilibrium problem. Thus, we adopt the equilibrium conditions in Cox et al. (1985a) and others to extend the investment model in Constantinides (1990) into a cost-free production-based equilibrium model.
of the growth rate of consumption (3.57\%\textsuperscript{2}), which leads to the very large coefficient of RRA, 48.7, based on a sample of the U.S. economy from 1889 to 1978. On the other hand, with the same sample, the production-based model (see the second term in Equation (3.1) in Zhang et al. (2012)) implies a very small coefficient of the RRA, only 2.2. This is because the ERP in Zhang et al. (2012) is the product of the coefficient of the RRA and the variance of the real return on the S&P 500 (16.54\%\textsuperscript{2}). As the production-based equilibrium model works so well in explaining in the equity premium puzzle, in this chapter we use it to explain the VRP.

One reason we call our model the cost-free production-based equilibrium model is that it can be regarded as a special case of the AK production model, which is developed from the neoclassical investment model (see Hayashi (1982)) and recently well extended by Bolton, Chen, and Wang (2011); DeMarzo, Fishman, He, and Wang (2012); Wang, Wang, and Yang (2012); Bolton, Chen, and Wang (2013); Pindyck and Wang (2013) and others, in which there is no cost of installing capital (i.e., new investments) and the firm’s productivity (i.e., “A”) and the Tobin’ q equals one, so that the value of the capital stock (i.e., stock price) is the same as the capital stock (i.e., “K”). For example, if we assume the cost of installing capital is zero in Pindyck and Wang (2013), we will get the one-valued Tobin’ q and solve that the stock price is the capital stock. In order to emphasize the importance of our model, we name this specialized model the cost-free production-based equilibrium model.

Should the diffusive volatility risk premium (DVRP), which is defined as the mean-reverting speed of the volatility in the risk-neutral measure minus that in the physical probability measure,\(^6\) be theoretically positive or negative? As a detailed overview provided in Broadie et al. (2007), conflicting estimates of the DVRP have existed for a long time in the literature. For example, using the simple stochastic

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\(^6\)The similar concept is used by Heston (1993); Bates (2000); Pan (2002); Eraker (2004); Neumann, Prokopczuk, and Wese Simen (2016).
volatility model (SV), Jones (2003) estimate a positive DVRP, while Pan (2002); Eaker et al. (2003) observe a negative DVRP. Thus, regarding the sign of the DVRP, Broadie et al. (2007) argue that there is no theory providing guidance. Actually, there are a few studies on that. Cox, Ingersoll, and Ross (1985b); Bates (1991, 1996, 2000) show that the DVRP is negative as the volatility is negatively correlated to the S&P 500 index. Recently, Eraker and Wu (2017) have proposed a consumption-based equilibrium model and document that the DVRP is negative for any positive risk aversion coefficients. In order to qualify Broadie et al.’s (2007) argument, we use an equilibrium model to provide guidance on the sign of the DVRP based on the SVCJ model. Consistent with Cox et al. (1985b); Bates (1991, 1996, 2000); Eraker and Wu (2017), our production-based model documents again that the sign of the DVRP should be negative.

Finally, our model constructed in this chapter involves the recursive preferences, which are well studied by Weil (1989); Epstein and Zin (1989, 1991). Later on, Duffie and Epstein (1992b,a) develop it into the continuous-time version. Now the recursive preferences are popularly used in asset pricing models (e.g., Bansal and Yaron (2004); Benzoni, Collin-Dufresne, and Goldstein (2011); Wachter (2013)). The main advantage of the recursive preferences is separating the RRA and the elasticity of intertemporal substitution (EIS). In spite of increasing its complexity, we provide analytical expressions for all solutions, which are linked to the impact of the RRA and the EIS. Similar to the DVRP, conflicting estimates of the EIS exist in the literature. For example, Bansal and Yaron (2004) estimate EIS of 1.5 and Bansal, Gallant, and Tauchen (2007) estimate the EIS of 2, while Hall (1988); Epstein and Zin (1991) and others estimate that EIS is below 1. In this chapter, we provide an alternative parameter setting, $0 < EIS < 1$ and $1 < RRA < 2$, to explain the ERP and VRP based on our cost-free production-based equilibrium model. As we mentioned before, Dew-Becker et al. (2017) raise the challenge that Epstein–Zin preferences for
early resolution of uncertainty (e.g., \( RRA > 1/EIS \)) can not explain their empirical observations, i.e., news about future volatility is unpriced. In our chapter, we get a calibration with Epstein–Zin preferences for late resolution of uncertainty (e.g., \( RRA < 1/EIS \)).

This may explain the empirical observations in Dew-Becker et al. (2017).

This chapter makes at least two contributions: (i) We develop a cost-free production-based equilibrium model as the first production-based equilibrium model to explain the equity premium puzzle and the large negative VRP. (ii) We provide guidance on the sign of the DVRP based on the SVCJ model.

The remainder of our chapter is organized as follows. Section 3.2 presents the definition of the ERP and VRP. Section 3.3 presents and solves the equilibrium model. Section 3.4 provides the empirical analysis. Section 3.5 concludes. Appendix 3.6.1 collects all proofs. Appendix 3.6.2 gives a comparison of estimates and Appendix 3.6.3 presents an example.

### 3.2 Equity and Variance Risk Premiums

#### 3.2.1 Equity and Variance Risk Premiums

According to Bollerslev et al. (2009); Drechsler and Yaron (2011); Bollerslev et al. (2012); Drechsler (2013); Jin (2015), we purposely rely on the readily available squared VIX index as our measure for the risk-neutral expected variance. In addition, following Buraschi et al. (2014); González-Urteaga and Rubio (2016), we use daily returns of the S&P 500 to calculate the realized variance over 21-day post windows.

---

7 In this case, the marginal utility gain is very important with regard to risk aversion; the investor wishes to wait in order to get a potential increase in future utility. Then, he or she prefers the late resolution of uncertainty. Similarly, we can understand that the investor is not very risk-averse so that he or she does not want to know the realization of the randomness in the future. Overall, the late resolution of uncertainty indicates the investor is quite risk-taking. See Kreps and Porteus (1978); Eeckhoudt, Gollier, Treich, et al. (2005).
daily ERP is simply defined as the daily log returns of the S&P 500 minus the three-month Treasury bill adjusted by the U.S. inflation rate. All data are obtained from Bloomberg.  

Definition 3.2.1 (Equity and Variance Risk Premium) We define the (annualized) ERP as

$$ ERP_t = \left( R_t - \frac{r_t}{252} \right) \times 252, $$  

(3.1)

where $R_t = (\ln S_t - \ln S_{t-1}) \times 100$ is the daily percentage returns of the S&P 500 and $r_t$ is the real risk-free rate which is calculated from the daily three-month Treasury bill rate adjusted by the U.S. inflation rate. In addition, we define the one-month VRP as

$$ VRP_t = RV_t - VIX_t^2, $$  

(3.2)

where the one-month realized variance $RV_t$ is calculated by the annualized variance of the daily percentage returns of the S&P 500 over 21-day post windows at day $t$ and $VIX_t^2$ is the daily squared VIX index at day $t$ divided by 12.
Chapter 3. Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

Table 3.1: Summary statistics.

The data are from 02 January 1990 to 30 September 2016. We compute the daily equity risk premium with $R_t - r_t/252$ where $R_t$ is the daily percentage returns of the S&P 500 and $r_t$ is the three-month Treasury bill rate adjusted by the inflation rate. The one-month $VRP_t = RV_t - VIX_t^2$, where the one-month realized variance $RV_t$ calculated by the annualized variance of the daily percentage returns of the S&P 500 over 21-day post windows at day $t$ and $VIX_t^2$ is the daily squared VIX index at time $t$ divided by 12. All mean, standard deviation, minimum and maximum variables are reported in units of 1/12 of the annual value, except that $r_t$ and the mean of ERP is given in annualized terms, and $R$ and the standard deviation, minimum and maximum of ERP are reported in daily.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$R$</td>
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<td>ERP</td>
<td>RV</td>
<td>$VIX^2$</td>
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<tr>
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<tr>
<td>Std</td>
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<tr>
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<tr>
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<td>-0.4369</td>
<td>3.6861</td>
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Table 3.1 shows that the mean of the annualized ERP is 6.4299 and the mean of the one-month VRP in the sample period is $-10.7884$ per month, which is significantly negative, over the period from 02 January 1990 to 30 September 2016. The mean of the one-month realized variance is 26.7926 and the mean of the one-month $VIX^2$ is 37.5811. We provide the summary statistics for the same sample in Drechsler (2013) and find that the variables generated based on our definitions are close to Drechsler (2013). The means of $RV$, $VIX^2$ and $VRP$ in Table 3.1 are 28.8674, 40.1087 and $-11.2413$ and in Drechsler (2013) are 28.82, 39.32 and $-10.55$. Comparing different sample periods, the average ERP is more volatile than the average VRP. For example, the means of the VRP in the periods, 1990–2009 and 2010–2016 are very close, around $-10$, while, the mean of the ERP in the 2010–2016 period is more than double the mean in the 1990–2009 period.
The data are from 02 January 1990 to 30 September 2016. We define the one-month variance risk premium as $VRP_t = RV_t - VIX_t^2$, where the one-month realized variance $RV_t$ is calculated by the annualized variance of the daily percentage returns of the S&P 500 over 21-day post windows at day $t$ and $VIX_t^2$ is the daily squared VIX index at time $t$ divided by 12.

Some days have a positive $VRP$ (e.g., maximum is 470.7357), which indicates financial disasters. For example, in Figure 3.1, there are 41 days from August to November in 2008 whose $VRP$ is larger than 100. In addition, the maximum exists on 25 September 2008. More crashes (e.g., the 1997 Asian financial crisis, 1998 Russian financial crisis, stock market downturn of 2002, August 2011 stock market fall and 2015–16 Chinese stock market crash) can be found in the positive-$VRP$ periods. This suggests that the positive VRP can be an indicator of financial crashes.\footnote{For example, Chen, Shu, and Zhang (2016) use the sentiment factor to explain why the variance risk premium is positive in a financial crisis period.}
3.2.2 Model-implied Equity and Variance Risk Premiums

The affine jump-diffusion models are well studied by Duffie et al. (2000). In this chapter, we adopt the following SVCJ model (e.g., Eraker et al. (2003); Eraker (2004); Broadie et al. (2007)) to describe the joint dynamics of the stock price.\(^\text{12}\) More general models are discussed in Duffie et al. (2000).

Under the physical probability measure \(\mathbb{P}\), at time \(t\), the stock price \(S_t\) follows,

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{V_t} dB_{S,t} + (e^x - 1)dN_t - \lambda m dt, \\
\frac{dV_t}{V_t} &= \kappa (\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t} + y dN_t,
\end{align*}
\]

where \(B_{S,t}\) and \(B_{V,t}\) are a pair of correlated Brownian motions with correlation coefficient \(\rho\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\); \(N_t\) is a Poisson process with the intensity \(\lambda\); the jump size of the volatility has an exponential distribution \(y \sim \exp(1/\mu_V)\) with mean \(\mu_V\) and the jump size of the stock price is \(x \sim N(\mu_S, \sigma_S^2)\); the growth rate \(\mu_t = r + \phi_t\) where the risk-free rate \(r\) is constant and \(ERP_t = 100 \times \phi_t = 100 \times (\eta_S V_t + (m \lambda - m Q \lambda Q))\) is the (instantaneous) equity risk premium (ERP) contributed by the diffusive risk premium \(\eta_S V_t\) and the price jump risk premium \((m \lambda - m Q \lambda Q)\) scaled 100 (see Broadie et al. (2007));\(^{13}\) \(r\) is risk-free rate and \(\kappa, \theta, \sigma_V\) are constant and \(m = e^{\mu_S + \frac{1}{2} \sigma_S^2} - 1.\)\(^{15}\) The SVCJ model can be degenerated into two other

\(^{12}\)The SVCJ is the most popular AJD model used for option and other derivatives pricing (e.g., Eraker et al. (2003); Eraker (2004); Broadie et al. (2007); Zhu and Lian (2012); Zheng and Kwok (2014); Neumann et al. (2016) and others). They empirically document that the SVCJ works quite well to fit the S&P 500 index.

\(^{13}\)In our model, we assume the exogenous \(\mu_t\) have a special form \(\mu_t = r + \phi_t\), i.e., the sum of a constant exogenous risk-free rate \(r\) and the endogenous equity premium \(\phi_t\) which can be solved in the equilibrium.

\(^{14}\)The constant \(\eta_S\) can be estimated; for example, see Neumann et al. (2016).

\(^{15}\)The independence of jump sizes, \(x\) and \(y\), is consistent with the results of previous studies. For example, Eraker et al. (2003); Eraker (2004) report statistically insignificant correlations between the two jump sizes. In addition, this correlation primarily affects the conditional skewness of returns, \(\mu_V\) and the correlation between the two jump sizes play a very similar role. Broadie et al. (2007) show that it is difficult to estimate this parameter precisely. Following Broadie et al. (2007), we assume the two jump sizes are independent.
model specifications often employed in the literature, namely, the stochastic volatility model with jumps in prices (SVJP) \((\mu_V = 0)\) and the simple stochastic volatility model (SV) \((\mu_V = \mu_S = \sigma_S = \lambda = 0)\).\(^{16}\)

We specify the transition between the risk-neutral measure \(\mathbb{Q}\) and the physical probability measure \(\mathbb{P}\) by using the similar transformations to those applied in Eraker (2004); Broadie et al. (2007).\(^{17}\) Then, the price process in a risk-neutral probability \(\mathbb{Q}\) becomes

\[
\begin{align*}
\frac{dS_t}{S_t} &= r dt + \sqrt{V_t} dB_{S,t}^Q + (e^{x} - 1) dN_t - m^Q \lambda^Q dt, \\
dV_t &= (\kappa \theta - \kappa V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t}^Q + y dN_t,
\end{align*}
\]

(3.4)

where \(B_{S,t}^Q\) and \(B_{V,t}^Q\) are a pair of correlated Brownian motions with correlation coefficient \(\rho\) on a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\); Poisson process \(N_t\) in the \(\mathbb{Q}\) measure has the intensity of \(\lambda^Q\), and jumps sizes \(x \sim N(\mu_S^Q, \sigma_S^2)\) and \(y \sim \exp(1/\mu_V^Q)\), and \(m^Q = e^{\mu_S^Q + \frac{1}{2} \sigma_S^2} - 1\).\(^{18}\) In addition, \(\rho, \kappa \theta, \sigma_V, \sigma_S\) are the same across both measures (the detailed transformations are shown in Section 3.3). Hence, the variance risk premium can be defined as follows.

**Definition 3.2.2 (Model-implied Equity and Variance Risk Premiums)** *Having estimated the \(\mathbb{P}\) and \(\mathbb{Q}\) parameters, we can define the (instantaneous) ERP as*\(^{19}\)

\[
\frac{ERP_t}{100} = \frac{1}{dt} E^\mathbb{P} \left[ \frac{dS_t}{S_t} \right] - \frac{1}{dt} E^\mathbb{Q} \left[ \frac{dS_t}{S_t} \right] = \mu - r = \eta_S V_t + \left( m \lambda - m^Q \lambda^Q \right),
\]

(3.5)

\(^{16}\)Our SVJP model is same to SVJ model in Eraker et al. (2003); Broadie et al. (2007); Yun (2011); Neumann et al. (2016).

\(^{17}\)In Broadie et al. (2007), authors assume \(\lambda = \lambda^Q\). We set \(\lambda \neq \lambda^Q\) which corresponds to the equilibrium models in Bates (2000); Liu, Pan, and Wang (2005); Zhang et al. (2012); Ruan et al. (2016). In Section 3.3, we will confirm our setting.

\(^{18}\)Following Broadie et al. (2007), a correlation between jumps in prices and volatility would be difficult to identify under \(\mathbb{Q}\) because \(\mu_V^Q\) plays the same role in the conditional distribution of returns. In order to clearly define the volatility jump risk premium, it is necessary to assume that the correlation between jumps should be zero.

\(^{19}\)Based on Definition 3.2.2, for daily frequency data, the ERP is instantaneous. Thus, Equation (3.5) can model the daily frequency ERP.
where $\eta$ is a constant and can be determined by the equilibrium, and define the VRP as, based on SVCJ model,

$$
\frac{VRP_t}{100} = \frac{1}{\tau} E^P \left[ \int_t^{t+\tau} (d \ln S_t)^2 \right] - \frac{1}{\tau} E^Q \left[ \int_t^{t+\tau} (d \ln S_t)^2 \right] 
= \frac{1}{\tau} E^P \left[ \int_t^{t+\tau} V_s ds \right] - \frac{1}{\tau} E^Q \left[ \int_t^{t+\tau} V_s ds \right] + \lambda E^P \left[ \sigma^2 S - \lambda \right] - \lambda \sigma^2 S + \mu^2 S,
$$

(3.6)

where

$$A = \frac{1 - e^{-\kappa \tau}}{\kappa \tau}, \quad B = \left( \theta + \frac{\lambda \mu V}{\kappa} \right) (1 - A) + \lambda (\sigma^2 S + \mu^2 S),$$

(3.7)

and

$$A^Q = \frac{1 - e^{-\kappa^Q \tau}}{\kappa^Q \tau}, \quad B^Q = \left( \frac{\kappa \theta}{\kappa^Q} + \frac{\lambda^Q \mu^Q}{\kappa^Q} \right) (1 - A^Q) + \lambda^Q (\sigma^2 S + \mu^2 S).$$

(3.8)

As we are interested in the one-month (21 business days) horizon variance risk premium, we set $\tau = 21$ days.

Based on the above definition, the model-implied equity risk premium can be decomposed into two components: the ERP contributed by the diffusion in the price ($ERP^{PD}$) and the ERP contributed by the price jump ($ERP^{PJ}$),

$$ERP_t = ERP^{PD}_t + ERP^{PJ}_t,$$

(3.9)

where

$$\frac{ERP^{PD}_t}{100} = \eta_S V_t, \quad \frac{ERP^{PJ}_t}{100} = m\lambda - m^Q \lambda^Q.$$

(3.10)

---

20In the SVCJ model, according to Duan and Yeh (2010), $VIX^2 = \frac{1}{\tau} E^Q \left[ \int_t^{t+\tau} V_s ds \right] + 2\lambda E^Q \left[ e^x - x - 1 \right]$, while, $IV_t = \frac{1}{\tau} E^Q \left[ \int_t^{t+\tau} V_s ds \right] + \lambda E^Q \left[ x^2 \right]$. As $e^x = 1 + x + \frac{1}{2} x^2 + O(x^3)$, we have $VIX^2 = IV_t + O(x^3)$. So, in both theoretically and empirically, we find that the difference between $IV_t$ and $VIX^2$ is very small and that then $VIX^2$ can be regarded as a good proxy of the $IV_t$. 
Similarly, the model-implied variance risk premium can be decomposed into three components: the VRP contributed by the diffusion in volatility \((VRP_{VD})\), the VRP mainly contributed by the volatility jump \((VRP_{VJ})\) and the VRP contributed by the price jump \((VRP_{PJ})\).

\[
VRP_t = VRP_{VD}^t + VRP_{VJ}^t + VRP_{PJ}^t,
\]

where

\[
\frac{VRP_{VD}^t}{100^2} = \left( A - A^Q \right) \cdot V_t + \theta (1 - A) - \left( \frac{\kappa}{\kappa^Q} \right) (1 - A^Q),
\]

\[
\frac{VRP_{VJ}^t}{100^2} = \frac{\lambda}{\kappa} \mu_v (1 - A) - \frac{\lambda^Q}{\kappa^Q} \mu^Q_v (1 - A^Q),
\]

\[
\frac{VRP_{PJ}^t}{100^2} = \lambda \left( \sigma^2_S + \mu^2_S \right) - \lambda^Q \left( \sigma^2_S^Q + \mu^2_S^Q \right).
\]

From the expression (3.12), the \(VRP_{VD}\) is mainly from the contribution of the DVRP (i.e., \(\kappa^Q - \kappa\)). The jump intensity risk premium, \(\lambda - \lambda^Q\), influences both \(VRP_{VJ}\) and \(VRP_{PJ}\) in (3.13)-(3.14). In Broadie et al. (2007), authors conclude that the DVRP is insignificant in all SV, SVJP and SVCJ models because of the flat implied volatility term structure. In addition, they argue that even for the more efficient estimation procedure, they still can not confront the fact that the term structure is flat, which still implies that the DVRP is insignificant. In Section 3.3, we give an economic explanation that the DVRP is large and negative in reality, which suggests that researchers need to choose more efficient data (e.g., Zhu and Lian (2012)), in order to get a significant and negative DVRP.
3.3 A Cost-free Production-based Equilibrium Model

3.3.1 Model Setup

Our cost-free production economy follows Constantinides (1990). There exists only one production good, which is also consumption, and two production technologies. There is no cost for investments, so that we can call the two production technologies cost-free technologies and the economy the cost-free production economy. The risky technology has a stochastic return over the period $[t, t + dt]$,

$$\mu_t dt + \sqrt{V_t} dB_{S,t} + (e^x - 1) dN_t - \lambda m dt,$$

where

$$dV_t = \kappa(\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t} + y dN_t.$$

The riskless technology has constant returns $r dt$.

We suppose that there is a representative investor whose portfolio is $(u_t, 1 - u_t)$, which represents the fraction of wealth invested in the risky and riskless technology, respectively. The consumption rate of the investor is $c_t$. Then, the investor’s capital process $W_t$ with the initial capital $W_0 > 0$ satisfies a stochastic differential equation, as follows:

$$\begin{aligned}
\frac{dW_t}{W_t} &= \left[ r + (\phi_t - \lambda m) u_t - \sigma_{W_t} \right] dt + u_t \sqrt{V_t} dB_{S,t} + u_t (e^x - 1) dN_t, \\
dV_t &= \kappa(\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t} + y dN_t.
\end{aligned}$$

(3.15)

Similar assumptions for the representative-consumer production economy can be found in Cox et al. (1985a); Bates (1991, 1996); Vasicek (2005); Zhang et al. (2012); Fu and Yang (2012); Ruan et al. (2016).
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In addition, the representative consumer has recursive preferences (see Duffie and Epstein (1992b,a)) given by

\[ J_t = E_t \left[ \int_t^\infty f(c_s, J_s)ds \right], \quad (3.16) \]

with

\[ f(c, J) = \beta(1 - \gamma)J \left[ \frac{e^{1 - \psi^{-1}}}{((1 - \gamma)J)^\omega} - 1 \right] = \frac{\beta}{1 - \psi^{-1}} \frac{e^{1 - \psi^{-1}} - ((1 - \gamma)J)^\omega}{((1 - \gamma)J)^\omega - 1}, \quad (3.17) \]

where the subjective time discount factor is denoted by \( \beta \); \( \psi \) is the EIS and \( \gamma \) is the coefficient of RRA. In addition, \( \omega = (1 - \psi^{-1})/(1 - \gamma) \). Recursive utility allows us to separate the effects of RRA and EIS. The special case of power utility results by setting the EIS equal to the inverse of the RRA coefficient. In this case, only innovations to consumption are priced. In the general case \( \gamma \neq 1/\psi \), state variables carry a risk premium, too. We assume throughout that \( \beta > 0 \) and \( \gamma > 0 \). Most of the discussion focuses on the case \( \gamma > 1 \).

By choosing the investment \( u_t \) in the stock and the consumption rate \( c_t \), the representative investor maximizes his/her expected objective function (3.16). Based on Cox et al. (1985a); Bates (1991, 1996); Vasicek (2005); Zhang et al. (2012); Fu and Yang (2012); Ruan et al. (2016), we define the market equilibrium in a production economy as follows.

**Definition 3.3.1 (Market equilibrium)** Equilibrium in our cost-free production economy is defined in a standard way: equilibrium consumption-portfolio pairs \((u_t, c_t)\) are such that the representative investor maximizes his/her expected objective function in (3.16), and markets are clear, \( u_t = 1 \).
3.3.2 Solutions

After solving the equilibrium in Definition 3.3.1, we get the following proposition:

**Proposition 3.3.1** The value function is

\[ J(W, V) = e^{aV + bW^{1 - \gamma}} \frac{1}{1 - \gamma}, \]  

(3.18)

and the optimal consumption rate is

\[ c_t^* = \left( e^{-\psi(aV_t + b)\beta} \right) W_t, \]  

(3.19)

where constants \( a \) and \( b \) satisfy,\(^{22}\)

\[
\begin{align*}
0 &= -\frac{\beta(1 - \gamma)}{1 - \gamma} + \left( \frac{1}{1 - \gamma} - 1 \right) (1 - \gamma) (1 + \psi aV) e^{-\psi(a\gamma + b)\beta} \\
&\quad + r (1 - \gamma) + \kappa \theta a + \lambda E \left[ e^{a(y + (1 - \gamma)x)} - 1 \right] - (1 - \gamma) \lambda E \left[ e^{a(y - \gamma x)} (e^x - 1) \right], \\
0 &= -ae^{-\psi(aV_t + b)\beta} + \gamma (1 - \gamma) \frac{1}{2} - \kappa a + \frac{1}{2} \sigma^2 a^2,
\end{align*}
\]  

(3.20)

with the effective long-term mean of the variance \( \overline{V} = \frac{\kappa \theta + \lambda \mu}{\kappa} \).

The equity risk premium is solved as

\[
\frac{ERP_t}{100} = (\gamma - \sigma_V \rho a) V_t + \lambda E \left[ (1 - e^{-\gamma x + ay})(e^x - 1) \right] = (\gamma - \sigma_V \rho a) V_t + \lambda m - \lambda Q m Q. 
\]  

(3.21)

The state-price density is given by

\[
\frac{d\pi_t}{\pi_t} = -r dt - \gamma \sqrt{V_t} dB_{St} + a \sigma_V \sqrt{V_t} dB_{Vt} + (e^{-\gamma x + ay} - 1) dN_t - \lambda E \left( e^{-\gamma x + ay} - 1 \right) dt.
\]  

(3.22)

\(^{22}\)Note here we choose \( 0 < a < 1/\mu_V \) and \( \gamma > 1 \). Obviously, \( a = 0 \) when \( \gamma = 1 \).
The explicit transition between the risk-neutral measure $\mathbb{Q}$ and the physical probability measure $\mathbb{P}$ is given by

$$
\kappa^\mathbb{Q} = \kappa + (\rho \gamma - a \sigma_V) \sigma_V, \quad \mu^\mathbb{Q}_S = \mu_S - \gamma \sigma^2_S, \quad \mu^\mathbb{Q}_V = \frac{1}{1 - a \mu_V} \mu_V, \quad \lambda^\mathbb{Q} = \frac{\lambda e^{\frac{1}{2} \sigma^2_S \gamma^2 - \mu_S \gamma}}{1 - a \mu_V}.
$$

(3.23)

Following Constantinides (1990), if a firm has capital $K_t$ at time $t$ and it can be freely invested in two production technologies, we assume the firm invests $\delta_1 K_t$ capital in the risky technology and $(1 - \delta_1) K_t$ in the riskless one, where $\delta_1$ is constant. In addition, the firm is financed with the equity (stock) $S_t$ and risk-free bond $M_t$, where $dM_t/M_t = rdt$. We assume the leverage $\delta_2 = S_t/(S_t + M_t)$, which is a constant. Then, we have equality between the investment in two production technologies and the value of the firm.

$$
\begin{align*}
\left\{
\begin{array}{l}
dS_t + M_trdt = \delta_1 K_t (\mu_t dt + \sqrt{V_t} dB_{S,t} + (e^x - 1) dN_t - \lambda mdt) + (1 - \delta_1) K_t rdt, \\
dV_t = \kappa (\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t} + ydN_t,
\end{array}
\right.
\end{align*}
$$

(3.24)

which can be rewritten as

$$
\begin{align*}
\left\{
\begin{array}{l}
\frac{dS_t}{S_t} = \frac{\delta_1}{\delta_2} \left( \mu_t - r dt + \sqrt{V_t} dB_{S,t} + (e^x - 1) dN_t \right) - \lambda mdt + rdt, \\
\frac{dV_t}{V_t} = \kappa (\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t} + ydN_t,
\end{array}
\right.
\end{align*}
$$

(3.25)

In order to be consistent with our stock price process in Equation (3.3), following Constantinides (1990), we set $\frac{\delta_1}{\delta_2} = 1$. Thus, the process of the stock price (with dividends included) in (3.25) is the same as (3.3).

**Remark 3.3.1** Our production-based model can be regarded as an extension of the model in Cox et al. (1985a); Bates (1991, 1996, 2000); Vasicek (2005, 2013); Zhang et al. (2012); Fu and Yang (2012); Ruan et al. (2016), which only consider that the representative consumer has a constant relative risk aversion (CRRA) utility function. The
production-based equilibrium model studied in previous literature is widely used in derivative pricing, especially in option pricing. It is reasonable to employ the production-based model to explain the VRP because the VIX index can be regarded as the variance swap rates.

**Remark 3.3.2** In Broadie et al. (2007), the authors suggest that \( \frac{ERP_{100}}{100} = \eta_S V_t + (m\lambda - m^Q\lambda^Q) \) which is contributed by the diffusive risk premium \( \eta_S V_t \) and the price jump risk premium \( (m\lambda - m^Q\lambda^Q) \). The solution of the ERP in (3.21) supports their assumption. In addition, \( \eta_S \) can be solved as \( \gamma - \sigma_V \rho a \), which is constant, and \( \lambda m - \lambda^Q m^Q = \lambda E[(1 - e^{-\gamma x + ay})(e^x - 1)] \), which can be verified by using the transition in (3.23). If the volatility is a constant, \( \sigma \), the ERP will become \( \frac{ERP_{100}}{100} = \gamma \sigma^2 + \lambda E[(1 - e^{-\gamma x})(e^x - 1)] \), which has been studied in Zhang et al. (2012).

**Remark 3.3.3** In the equilibrium model, we find a stochastic density factor (i.e., state-price density) in (3.22), which provides a transition for parameters between the risk-neutral measure \( Q \) and the physical probability measure \( P \). The stochastic density factor in (3.22) captures all risks, which are two Brownian motion risks \( (B_{S,t} \text{ and } B_{V,t}) \) and one jump risk \( (N_t) \). In the transition (3.23), it shows that the jump intensity in two measures is not equal, i.e., \( \lambda \neq \lambda^Q \), which is consistent with Bates (1991, 1996, 2000); Liu et al. (2005); Zhang et al. (2012); Ruan et al. (2016). This suggests that we have to estimate \( \lambda^Q \) as an independent risk-neutral parameter. In addition, as \( \mu_S < 0, \gamma > 1 \) and \( 0 < a < 1/\mu_V \), we have \( \lambda < \lambda^Q \). In other words, the jump intensity should be larger in the risk-neutral probability measure than in the physical probability measure.

**Remark 3.3.4** The transition in (3.23) documents that \( \kappa^Q - \kappa = (\rho \gamma - a\sigma_V) \sigma_V < 0 \) for \( \rho < 0, \gamma > 1, 0 < a < 1/\mu_V \) and \( \sigma_V > 0 \). In other words, the equilibrium model can generate the negative DVRP. It is not surprising. Actually, in the same production-based framework, Cox et al. (1985b); Bates (1991, 1996, 2000) argue that
DVRP is negative because of the negative correlation between the volatility and the S&P 500 index. Recently, Eraker and Wu (2017) have proposed a consumption-based equilibrium model and get the similar result that \( \kappa^Q - \kappa < 0 \) for any \( \gamma > 0 \). The DVRP is also negative for the case \( \rho = 0 \) (if we assume \( \gamma > 1 \)). Its negative sign can thus not be attributed to the sign of \( \rho \) and thus to the “stock risk part” of variance risk.

**Remark 3.3.5** For any \( a > 0 \) and \( \gamma > 0 \), we have \( \mu^Q_S < \mu_S < 0 \) and \( \mu^Q_V > \mu_V > 0 \). This corresponds to the empirical results, e.g., Eraker (2004); Broadie et al. (2007); Neumann et al. (2016). It means that the jumps in the price and the volatility are larger in the risk-neutral probability measure than in the physical probability measure.

**Remark 3.3.6** In a special case with \( \gamma = 1 \), we can solve \( a = 0 \) and

\[
\kappa^Q = \kappa + \rho \sigma_V, \quad \mu^Q_S = \mu_S - \sigma_S^2, \quad \mu^Q_V = \mu_V, \quad \lambda^Q = \lambda e^{2\sigma^2 - \mu_S}.
\]

The risk-neutral parameters are not affected by the EIS.

### 3.3.3 Equilibrium Model-implied ERP and VRP

Plugging the explicit transition between the risk-neutral measure \( Q \) and the physical probability measure \( P \) given in (3.23) into Definition 3.2.2 gives us the model-implied VRP based on our cost-free production-based equilibrium model.

**Definition 3.3.2 (Equilibrium model-implied Equity and Variance Risk Premiums)**

Having estimated the \( P \) parameters and the recursive preferences parameters of the representative investor, based on SVCJ model, we define the equilibrium model-implied ERP as

\[
ERP_t = \left[ (\gamma - \sigma_V \rho a) V_t + \lambda E \left( (1 - e^{-\gamma x + ay})(e^x - 1) \right) \right] \times 100,
\]

(3.27)
and the equilibrium model-implied VRP is defined as

\[ VRP_t = (A_\Delta \cdot V_t + B_\Delta) \times 100^2, \]  

(3.28)

where

\[ A_\Delta = \frac{1 - e^{-\kappa \tau}}{\kappa \tau} - \frac{1 - e^{-(\kappa + (\rho \gamma - a \sigma_V) \sigma_V) \tau}}{(\kappa + (\rho \gamma - a \sigma_V) \sigma_V) \tau}, \]

and

\[ B_\Delta = \left( \theta + \frac{\lambda \mu_V}{\kappa} \right) \left( 1 - \frac{1 - e^{-\kappa \tau}}{\kappa \tau} \right) + \lambda \left( \sigma_S^2 + \mu_S^2 \right) \]

\[ - \frac{1}{\kappa + (\rho \gamma - a \sigma_V) \sigma_V} \left( \kappa \theta - \frac{\lambda \mu_V \sigma_S^2 \gamma^2 - \mu_S \gamma}{(1 - a \mu_V)^2} \right) \left[ 1 - \frac{1 - e^{-(\kappa + (\rho \gamma - a \sigma_V) \sigma_V) \tau}}{(\kappa + (\rho \gamma - a \sigma_V) \sigma_V) \tau} \right] \]

\[ + \frac{\lambda e^{\frac{1}{2} \sigma_S^2 \gamma^2 - \mu_S \gamma}}{1 - a \mu_V} \left( \sigma_S^2 + (\mu_S - \gamma \sigma_S^2)^2 \right). \]

Equation (3.28) shows that VRP is a function of the physical measure parameters \( \kappa, \theta, \sigma_V, \rho, \lambda, \mu_S, \sigma_S, \mu_V \) and the preferences parameters \( \beta, \gamma \) and \( \psi \). In addition, the average ERP, the average VRP, the risk-free rate \( r \), all physical parameters \( \kappa, \theta, \sigma_V, \rho, \lambda, \mu_S, \sigma_S, \mu_V \), and the preferences parameter \( \beta \), we have an equation system with respect to \( \gamma, \psi \) and \( a \),

\[
\begin{align*}
E[ERP_t] &= \left[ (\gamma - \sigma_V \rho a) \, EV + \lambda E \left[ \left( 1 - e^{-\gamma x + ay} \right) (e^x - 1) \right] \right] \times 100, \\
E[VRP_t] &= (A_\Delta \cdot EV + B_\Delta) \times 100^2, \\
0 &= -\frac{\beta(1-\gamma)}{1-\psi} + \left[ \frac{1}{1-\psi} - 1 \right] (1 - \gamma) + aV \left[ \frac{\gamma(1-\gamma)}{2a} - \kappa + \frac{1}{2} \sigma_V^2 a \right] \\
&+ r \left( 1 - \gamma \right) + \kappa \theta a + \lambda E \left[ e^{ay + (1-\gamma)x} - 1 \right] - (1 - \gamma) \lambda E \left[ e^{ay - \gamma x} (e^x - 1) \right],
\end{align*}
\]

(3.29)

where \( EV = E[V_t] \) is the average variance which is given as well.

There is a hot debate on whether the value of the EIS should be larger or lower

\[ ^{23} \text{The parameter } a \text{ is determined by } \kappa, \theta, \sigma_V, \rho, \lambda, \mu_S, \sigma_S, \mu_V, \beta, \gamma \text{ and } \psi \text{ in Equation (3.20). Based on Definition 3.3.2, we can further analyse the impacts of these parameters on the ERP and VRP.} \]
than 1. Hall (1988); Epstein and Zin (1991) and others estimate that EIS is below 1, while in order to fit the consumption data, Bansal and Yaron (2004); Drechsler and Yaron (2011); Drechsler (2013); Jin (2015) and others, who use a long-run risks model, have to choose an EIS which is larger than 1. This is a critical issue. We provide an alternative setting of EIS and RRA, in order to explain the higher ERP and VRP based on our production-based equilibrium model.

We set the SVCJ model parameters as $\kappa = 6.522$, $\theta = 0.01361$, $\sigma_V = 0.2016$, $\rho = -0.48$, $\lambda = 1.512$, $\mu_S = -0.0263$, $\sigma_S = 0.0289$, $\mu_V = 0.0373$, $r = 1.4821\%$, $EV = 0.0222$, $\beta = -\ln(0.999)$ and the targeting $ERP = 12.3619$ and $VRP = -14.5477$. As $E[ERP] > (\gamma - \sigma_V \rho a) EV \times 100 > \gamma EV \times 100$ and $EV = 2.22\%$, then based on our parameter setting, we have to choose $\gamma < 12.3619/2.22 = 5.57$. Here we consider the case $\gamma = 1.1$. Then, we adjust the EIS to fit the target ERP and VRP in the mean level. The results are shown in Figure 3.2.

$^{24}$All parameters are annualized estimates in Table 3.9. The mean of instantaneous volatility $EV = E[V_t]$ is based on 10,000 simulations, with each statistic calculated using a sample size equal to its VRP data counterpart. The mean of the VRP is from 1990 to 1999 which is the overlapping period between Eraker et al. (2003) and Broadie et al. (2007) after VIX launched. The risk-free rate $r = 1.8421$ is the average three-month Treasury bill adjusted by the U.S. inflation rate over the same period, 1990 to 1999. For summary statistics, see Table 3.1.

$^{25}$Actually, given all parameters, we are able to explicitly solve $a, \gamma$ and $\psi$ from the equation system (3.29). However, with restrictions $\gamma > 1, \psi > 0$ and $0 < a < 1/\mu_V$, the equation system (3.29) has no solution. We have to find some approximate solutions.
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Figure 3.2: Equilibrium ERP and VRP against EIS.

We set the model parameters as $\kappa = 6.522, \theta = 0.01361, \sigma_V = 0.2016, \rho = -0.48, \lambda = 1.512, \mu_S = -0.0263, \sigma_S = 0.0289, \mu_V = 0.0373, r = 1.4821\%, EV = 0.0222, \beta = -\ln(0.999)$ and the targeting $ERP = 12.3619$ and $VRP = -14.5477$.

Figure 3.2 shows that the parameter combination of $1 < \gamma < 2$ and $0 < EIS < 1$. 
(i.e., $\gamma = 1.1$ and $\psi$ is around 0.6) is able to explain the ERP and VRP based on our production-based model. This parameter choice corresponds to Epstein and Zin (1991). In contrast to long-run risks models in Bansal and Yaron (2004); Drechsler and Yaron (2011); Drechsler (2013); Jin (2015), our cost-free production economy provides an alternative way to explain the equity premium puzzle and high negative VRP.

In addition, if we set $\psi = 1/\gamma = 1/1.1 \approx 0.9$, then the model generates $E[ERP] = 26.8854$ and $E[VRP] = -65.6358$, both of which are too large. This is the reason why we have to use the Epstein-Zin recursive preferences. Using a small RRA, e.g., $\gamma = 1.1$, we adjust the EIS to fit the average ERP and VRP.

### 3.3.4 VRP Return Predictability

We calculate the conditional equity premium as

$$
E_t \left[ \frac{ERP_{t+1}}{100} \right] = (\gamma - \sigma V \rho a) E_t[V_{t+1}] + \lambda E \left[ (1 - e^{-\gamma x + ay})(e^x - 1) \right] \\
= (\gamma - \sigma V \rho a) \left[ e^{-\kappa} V_t + \left( \theta + \frac{\lambda \mu V}{\kappa} \right) (1 - e^{-\kappa}) \right] + \lambda E \left[ (1 - e^{-\gamma x + ay})(e^x - 1) \right] \\
= (\gamma - \sigma V \rho a) e^{-\kappa} V_t + (\gamma - \sigma V \rho a) \left( \theta + \frac{\lambda \mu V}{\kappa} \right) (1 - e^{-\kappa}) \\
+ \lambda E \left[ (1 - e^{-\gamma x + ay})(e^x - 1) \right].
$$

(3.30)

Equation (3.28) implies that $V_t = \frac{V_{ERP}}{100^2} - \frac{B_\Delta}{A_\Delta}$. Combining it with Equation (3.30), we can get the predictive regression for the equity premium (excess market return),

$$
\frac{ERP_{t+1}}{100} = \beta_{pred} \frac{VRP_t}{100^2} + \alpha_{pred} + \epsilon_{t+1}.
$$

(3.31)

---

26In this setting, we get $\kappa^Q = 5.7906$, $\lambda^Q = 3.9031$, $\mu^Q_S = -0.02722$ and $\mu^Q_V = 0.09348$. 

where

\[ \beta_{\text{pred}} = \frac{\gamma - \sigma_V \rho a}{A_\Delta} e^{-\kappa}, \]  

(3.32)

and

\[ \alpha_{\text{pred}} = (\gamma - \sigma_V \rho a) e^{-\kappa} \frac{-B_\Delta}{A_\Delta} + (\gamma - \sigma_V \rho a) \left( \theta + \frac{\lambda \mu_V}{\kappa} \right) (1 - e^{-\kappa}) + \lambda E \left[ (1 - e^{-\gamma x + ay})(e^x - 1) \right]. \]  

(3.33)

As \((\rho_\gamma - a \sigma_V) \sigma_V < 0\), which leads to \(A_\Delta < 0\), we have \(\beta_{\text{pred}} < 0\). Therefore, the predictive coefficient is negative (with respect to the \(\frac{\text{VIX}}{100}\) in Equation (3.6)). It corresponds to empirical results in Bollerslev et al. (2009); Drechsler and Yaron (2011); Jin (2015) and others. We report our empirical results from the daily predictive regressions of monthly equity premiums on variance risk premiums in Table 3.2. We find \(\beta_{\text{pred}}\) is significantly negative with very large t-statistics (\(\beta_{\text{pred}} = -13.52\) with \(-16.32\)). The predictive power from the volatility \(V_t\) determines the expected excess return in Equation (3.30) and the VRP in Equation (3.28). Thus, through the information of the volatility, the variance risk premium is able to predict the excess market return.

**Table 3.2: Return predictability.**

The data are from 02 January 1990 to 30 September 2016. This table reports empirical results from the monthly predictive regressions of the equity premiums (S&P 500 log returns minus the risk-free rate) on variance risk premiums (i.e., regression (3.31)). We define the monthly ERP as \((R_t - r_t/12) \times 12\) where \(R_t\) is the monthly percentage returns of the S&P 500 and \(r_t\) is the three-month Treasury bill rate. At the beginning of each month, the variance risk premium is \(\text{VRP}_t = \text{RV}_t - \text{VIX}_t^2\) where \(\text{RV}_t\) calculated by the daily percentage returns of the S&P 500 over 21-day post windows at day \(t\); \(\text{VIX}_t^2\) is the daily squared VIX index divided by 12 as one-month horizon at time \(t\). T-statistics are adjusted based on Newey-West (1987) standard errors. ***, ** and * represents statistical significance at the 1, 5 and 10% level.

<table>
<thead>
<tr>
<th>(\beta_{\text{pred}})</th>
<th>t-statistic</th>
<th>adj. (R^2) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.90***</td>
<td>-3.39</td>
<td>3.06</td>
</tr>
</tbody>
</table>
3.4 Empirical Analysis

Even though we have shown that the SVCJ model can perfectly explain the ERP and VRP based on the parameters in Broadie et al. (2007) (see Appendix 3.6.3), in this section, we will use longer data to robustly examine whether the SVCJ model built in a cost-free production economy can explain the large ERP and VRP.

3.4.1 Estimates of Physical Measure Parameters

The first step is using S&P 500 returns data to estimate the physical measure parameters by using a Markov chain Monte Carlo (MCMC) sampler. Eraker et al. (2003); Eraker (2004); Amengual (2009); Zhu and Lian (2012); Kaeck and Alexander (2012) show that (i) MCMC yields very accurate estimates for jump-diffusion models; (ii) MCMC provides estimates of the latent volatility; (iii) MCMC outperforms the generalized method of moments (GMM), the quantile maximum likelihood estimation (QMLE) and the efficient method of moments (EMM); (iv) MCMC can utilize priors to disentangle jumps from diffusions in an intuitive manner. Hence, we use the MCMC approach to estimate our affine jump-diffusion models. We present a time-discretization of Equation (3.3) by using the discrete scheme,

\[
\begin{align*}
R_t &= \alpha + \sqrt{V_t} \epsilon_t^S + J_t^S q_t, \\
V_t &= V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma^V \sqrt{V_{t-1}} \epsilon_t^V + J_t^V q_t,
\end{align*}
\]

(3.34)

where \( \alpha = r - \frac{1}{2} V_t + \eta S V_t - \lambda m; \epsilon_t^S \) and \( \epsilon_t^V \) are samples from two dependent standard normal distributions with correlation \( \rho \); \( R_t = (\ln S_t - \ln S_{t-1}) \times 100, y \sim \exp(1/\mu_V), \) \( x \sim N(\mu_S, \sigma_S^2) \) and \( q_t \sim \text{Ber}(\lambda) \). The parameters to be estimated are \( \alpha, \kappa, \theta, \sigma_V, \rho, \lambda, \mu_S, \sigma_S, \mu_V \). Based on the analysis in Table 3.13, the jumps in the stochastic volatility dominates the VRP. So, we are also interested in the stochastic volatility model with jumps in volatility (SVJ\( V \)) \( (\mu_S = 0 \) and \( \sigma_S = 0) \).
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Following Eraker et al. (2003), we run the MCMC algorithm for 100,000 iterations, discarding the first 10,000 as the burn-in period to achieve the convergence of the chain. For each parameter to be estimated, we use the same priors as in Eraker et al. (2003) and the sample period is from 2 January 1990 to 30 September 2016 (for summary statistics, see Table 3.1). Estimates are shown in Table 3.3.

Table 3.3: S&P 500 Parameter Estimates.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma_V$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu_S$</th>
<th>$\sigma_S$</th>
<th>$\mu_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>0.02158</td>
<td>1.191</td>
<td>0.1799</td>
<td>-0.7147</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.002676)</td>
<td>(0.101)</td>
<td>(0.009263)</td>
<td>(0.02662)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVJP</td>
<td>0.01915</td>
<td>1.194</td>
<td>0.1711</td>
<td>-0.7408</td>
<td>0.00938</td>
<td>-1.225</td>
<td>1.533</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0023)</td>
<td>(0.1051)</td>
<td>(0.00817)</td>
<td>(0.02623)</td>
<td>(0.004502)</td>
<td>(0.455)</td>
<td>(0.229)</td>
<td></td>
</tr>
<tr>
<td>SVJV</td>
<td>0.0288</td>
<td>0.8383</td>
<td>0.1603</td>
<td>-0.7536</td>
<td>0.004617</td>
<td>2.288</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.003065)</td>
<td>(0.07135)</td>
<td>(0.009288)</td>
<td>(0.02723)</td>
<td>(0.001512)</td>
<td></td>
<td>(0.3877)</td>
<td></td>
</tr>
<tr>
<td>SVCJ</td>
<td>0.02601</td>
<td>0.8337</td>
<td>0.1535</td>
<td>-0.7606</td>
<td>0.004876</td>
<td>-2.556</td>
<td>1.66</td>
<td>2.14</td>
</tr>
<tr>
<td></td>
<td>(0.003099)</td>
<td>(0.07424)</td>
<td>(0.008128)</td>
<td>(0.02758)</td>
<td>(0.00148)</td>
<td>(0.5633)</td>
<td>(0.2802)</td>
<td>(0.3571)</td>
</tr>
</tbody>
</table>

These reported parameters are quite informative. Table 3.3 shows that the values of the daily variance $\theta$ are 1.191, 1.194, 0.8383 and 0.8337, respectively, for the SV, SVJP, SVJV and SVCJ models, which are close to the unconditionally sampled standard deviation of the S&P500 return data, 1.1301. In the annualized view, the sampled standard deviation of data is 17.940. Using the jump-adjusted $\sqrt{2\kappa(\kappa \theta + \lambda \mu_V)/\kappa}$, the estimates of annualized volatility are 17.324, 17.381, 17.427 and 17.641, respectively, for the SV, SVJP and SVCJ models. We find that the SVCJ model is closest to the sampled standard deviation of data. Our MCMC estimates are quite accurate.

---

$^{27}$17.940 = 1.1301$\sqrt{252}$. 

Corresponding to S&P500 parameter estimates in Table III in Eraker et al. (2003), adding more jumps in the model leads to the lower estimated values of $\theta$, $\sigma_V$, but a higher value of $\rho$. In addition, we verify our MCMC codes with other literature in Appendix 3.6.2.

Figure 3.3: Latent volatility.

This figure displays the time series of the annualized latent instantaneous volatility estimated under the physical measure based on the SVCJ model. We express the volatility in percentage points per annum. The data are from 02 January 1990 to 30 September 2016.

[Graph showing the annualized latent instantaneous volatility over time from 1990 to 2016]

Following Neumann et al. (2016), we present the latent volatility implied by the MCMC based on the SVCJ model in Figure 3.3. During the 2008 financial crisis, the instantaneous volatility has rapid movements from September to October 2008. The means of the daily variance are 1.1563, 1.1213, 1.1975 and 1.1848, respectively,
for the SV, SVJP, SVJV and SVCJ models.\textsuperscript{28} The time series of the annualized latent instantaneous volatility is consistent with Neumann et al. (2016).

\subsection*{3.4.2 Equilibrium Model-implied ERP and VRP}

In order to get the risk-neutral parameters implied in our equilibrium model, we have to convert the daily percentage estimates into annualized values in Table 3.4.\textsuperscript{29} This is because the preference’s parameters, $\gamma$, $\psi$, and $\beta$, are annualized values in the existing literature (e.g., Bansal and Yaron (2004); Drechsler and Yaron (2011); Drechsler (2013); Jin (2015) and others). Here we set $\beta = -\ln(0.999)$, which is same as Drechsler and Yaron (2011); Drechsler (2013). The risk-free rate $r = 0.3507(\%)$ is the average three-month Treasury bill adjusted by the U.S. inflation rate over the same period, 02 January 1990 to 30 September 2016 (for summary statistics, see Table 3.1).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma_V$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu_S$</th>
<th>$\sigma_S$</th>
<th>$\mu_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>5.43816</td>
<td>0.03001</td>
<td>0.4533</td>
<td>-0.7147</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVJP</td>
<td>4.82580</td>
<td>0.03009</td>
<td>0.4312</td>
<td>-0.7408</td>
<td>2.36376</td>
<td>-0.01225</td>
<td>0.01533</td>
<td></td>
</tr>
<tr>
<td>SVJV</td>
<td>7.2576</td>
<td>0.02113</td>
<td>0.4040</td>
<td>-0.7536</td>
<td>1.1635</td>
<td></td>
<td></td>
<td>0.05766</td>
</tr>
<tr>
<td>SVCJ</td>
<td>6.55452</td>
<td>0.02101</td>
<td>0.3868</td>
<td>-0.7606</td>
<td>1.228752</td>
<td>-0.02556</td>
<td>0.0166</td>
<td>0.05393</td>
</tr>
</tbody>
</table>

We now adjust the values of RRA and EIS to fit the mean of ERP in Table 3.1, and we show the results in Figure 3.4. All results use the MCMC spot variance. Neumann et al. (2016) do a comparison analysis for the MCMC spot variance and

\textsuperscript{28}The means of the annualized variance are 0.02914, 0.02826, 0.03018 and 0.02986, respectively, for the SV, SVJP, SVJV and SVCJ models.

\textsuperscript{29}A detailed guide for the conversion of parameters can be found in Appendix 3.6.3 of Branger and Hansis (2015).
the spot variance estimated by the options data. They find that there is very little difference. We compare the results by using the MCMC spot variance and the 10,000 model simulations and find that there is very little to distinguish between the two sets. This is because the MCMC spot variance is the mean of 90,000 MCMC simulations, which is equivalent to or even better than the average variance obtained from 10,000 model Monte Carlo (MC) simulations. As the means of the annualized variance, $EV$, implied by the MCMC, in the equation system (3.29) are 0.02914, 0.02826, 0.03018 and 0.02986, respectively, for the SV, SVJP, SVJV and SVCJ models, given all parameters and $\gamma$, we are able to explicitly solve $a$ and $\psi$ from the equation system (3.29).
Figure 3.4: Equilibrium Model-implied Equity Premium.

We set the equilibrium model parameters as $\beta = -\ln(0.999)$ and $r = 0.3507\%$. The mean of the annualized ERP, 6.4299, is from 02 January 1990 to 30 September 2016. The risk-free rate $r = 0.3507$ is the average three-month Treasury bill over the same period. The means of the annualized variance, $EV$, implied by the MCMC, in the equation system (3.29) are 0.02914, 0.02826, 0.03018 and 0.02986, respectively, for the SV, SVJP, SVJV and SVCJ models.

Panel A: SV model

Panel B: SVJP model

Panel C: SVJV model

Panel D: SVCJ model
Two vertical dash lines in Figure 3.4 are the solutions of EIS for given RRA=1.008, in order to fit the market average ERP (which is the horizontal dashed line). Figure 3.4 shows that (i) the solutions of $a$ are all positive, which means that the equation system (3.29) is well-posed so that the DVRP is always negative (see Remark 4); (ii) higher RRA, higher $a$ and the average ERP; (iii) all of the above models can easily explain the equity premium puzzle in our cost-free production economy by using very low RRA.

We calculate the average VRP based on the equation system (3.29) by using the same parameters and then get Figure 3.5.
We set the equilibrium model parameters as $\beta = -\ln(0.999)$, $\gamma = 1.008$ and $r = 0.3507(\%)$.

The mean of the one-month VRP, $-10.7884$, is from 02 January 1990 to 30 September 2016. The risk-free rate $r = 0.3507$ is the average three-month Treasury bill over the same period. The means of the annualized variance, $EV$, implied by the MCMC, in the equation system (3.29) are 0.02914, 0.02826, 0.03018 and 0.02986, respectively, for the SV, SVJP, SVJV and SVCJ models.

From Table 3.1, the market average VRP is $-10.7884$. In order to fit the average VRP, we need large EIS, while, at same time, it leads to a large average ERP in Figure 3.4. We draw the solutions of EIS, which fit the market average ERP, in Figure 3.5 (i.e., the vertical dashed line) and then we get the explained VRP. We provide the exact solutions in Table 3.5.
Chapter 3. Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

Table 3.5: Model-implied Variance Risk Premium.

We set the equilibrium model parameters as $\beta = -\ln(0.999)$, $\gamma = 1.008$ and $r = 0.0730(\%)$. The mean of the one-month VRP, $-10.7884$, is from 02 January 1990 to 30 September 2016.

<table>
<thead>
<tr>
<th>Model-based $E[VRP_V^{\text{D}}]$</th>
<th>$VRP^{P,J}$</th>
<th>$VRP^{V,J}$</th>
<th>$E[VRP]$</th>
<th>$E[VRP]$</th>
<th>Explanation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV Model ($\psi = 0.04, a = 3.7036$)</td>
<td>-0.9829</td>
<td>0</td>
<td>0</td>
<td>-0.9829</td>
<td>-10.7884</td>
</tr>
<tr>
<td>SVJP Model ($\psi = 0.14, a = 3.8620$)</td>
<td>-0.9284</td>
<td>-0.0212</td>
<td>0</td>
<td>-0.9496</td>
<td>-10.7884</td>
</tr>
<tr>
<td>SVJV Model ($\psi = 0.44, a = 3.6721$)</td>
<td>-0.7583</td>
<td>-1.2433</td>
<td>-2.0016</td>
<td>-10.7884</td>
<td></td>
</tr>
<tr>
<td>SVCJ Model ($\psi = 0.17, a = 3.0432$)</td>
<td>-0.6350</td>
<td>-0.2346</td>
<td>-0.9613</td>
<td>-1.8308</td>
<td>-10.7884</td>
</tr>
</tbody>
</table>

Based on Table 3.5, the SVJV model works best among four affine models to fit the market average VRP.\(^{31}\) It reveals that jumps in the volatility process are the most important factor to explain the VRP. This is why the SVCJ and SVJV models work much better than the SV and SVJP models. However, compared with the period in Broadie et al. (2007) (1990–1999), in the large sample period with a low average ERP, our model works less well. This is a limitation of our cost-free production-based equilibrium model. Considering the property of the equilibrium, one way of improving the model’s explanation performance is to add more risks (i.e., jump and diffusive risks). For example, if we assume that the long-term mean level of $V_t$ and the jump intensity in SVJV model also follow the mean-reverting square root process, then the ERP in Equation (3.27) will not acquire any new risks, while the VRP in Equation (3.28) will acquire two more diffusive risks.

Combining Figure 3.4 and 3.5, given $\gamma = 1.008$, based on the SVJV model, we have $VRP = -8.4371, -9.8824, -12.1473$ and $-16.2199$, and $ERP = 11.3693, 11.8874$.

\(^{31}\) The explanation rate $= \frac{E[VRP]_{\text{Model}}}{E[VRP]_{\text{Data}}} \times 100\%$. 

12.5279 and 13.3505 when $\psi = 0.94, 0.95, 0.96$ and 0.97, respectively. The VRP is more sensitive than the ERP in a large EIS domain. Thus, only if the average ERP is at or larger than 11, the equation system (3.29) has available solutions with restrictions $\gamma > 1, \psi > 0$ and $0 < a < 1/\mu_V$.

For two subsamples with a large average ERP, 1990–1999 and 2010–2016, in Table 3.1, we just respectively choose $\psi = 0.965$ and $\psi = 0.95$ both with $\gamma = 1.008$ and then the SVJV model can well explain such a high ERP and VRP on average (i.e., for periods 1990–1999 and 2010–2016, data $VRP = -14.5477, -9.4441$, while the model $VRP = -13.8450, -8.4371$; data $ERP = 12.3619, 11.3884$, while the model $ERP = 12.9112, 11.3693$).

Overall, all models built in our cost-free production economy can perfectly explain the the equity premium puzzle with much low risk aversion. The SVCJ and SVJV models can explain both the larger ERP and VRP only if the average annualized ERP is at or larger than 11% (e.g., for the periods 1990–1999 and 2010–2016).

### 3.4.3 Other implements of the model

**Volatility market.**

Eraker and Wu (2017) use a consumption-based equilibrium to explain the negative returns of $VIX$ futures and other volatility claims. Our model also can explain it. Based on Equation (3.4), we can get the formula for CBOE VIX:

$$VIX^2 = A^Q \times V_t + B^Q,$$

where

$$A^Q = \frac{1 - e^{-\kappa^Q \tau}}{\kappa^Q \tau}, \quad B^Q = \left(\frac{\kappa^Q \theta^Q + \lambda^Q \mu_V^Q}{\kappa^Q \tau}\right) \left(1 - A^Q\right) + 2\lambda^Q \left(m^Q - \mu_S^Q\right),$$

As the SVCJ model works similarly to the SVJV model, we only discuss the SVJV model here.
and the risk-neutral parameters can be obtained by using the transition in (3.23).

For example, for the SVCJ model, we set $\gamma = 1.1$ and $\psi = 0.43$, and then we get $a = 10.0438, \kappa^Q = 0.01876, \lambda^Q = 0.01094, \mu^Q_S = -2.5863$ and $\mu^Q_V = 4.6688$. By using the MCMC spot volatility, we can generate the model-implied VIX in Figure 3.6 and give the summary statistic in Table 3.6.

**Figure 3.6: Equilibrium model-implied VIX in the SVCJ model.**

The VIX is from 02 January 1990 to 30 September 2016. We set the equilibrium model parameters as $\beta = -\ln(0.999), r = 0.0730(\%), \gamma = 1.1$ and $\psi = 0.78$. The physical parameters are $\kappa = 0.02601, \theta = 0.8337, \sigma_V = 0.1535, \rho = -0.7606, \lambda = 0.004876, \mu_S = -2.556, \sigma_S = 1.66, \mu_V = 2.14$ and the risk neutral parameters are $\kappa^Q = 0.01876, \lambda^Q = 0.01094, \mu^Q_S = -2.5863$ and $\mu^Q_V = 4.6688$. 
Table 3.6: Equilibrium model-implied VIX in the SVCJ model.

The VIX is from 02 January 1990 to 30 September 2016. We set the equilibrium model parameters as $\beta = -\ln(0.999)$, $r = 0.0730(\%)$, $\gamma = 1.1$ and $\psi = 0.43$. The physical parameters are $\kappa = 0.02601$, $\theta = 0.8337$, $\sigma_V = 0.1535$, $\rho = -0.7606$, $\lambda = 0.004876$, $\mu_S = -2.556$, $\sigma_S = 1.66$, $\mu_V = 2.14$ and the risk neutral parameters are $\kappa^Q = 0.01876$, $\lambda^Q = 0.01094$, $\mu^Q_S = -2.5863$ and $\mu^Q_V = 4.6688$.

<table>
<thead>
<tr>
<th></th>
<th>VIX (Data)</th>
<th>VIX (Model)</th>
<th>RV (Data)</th>
<th>RV (Model)</th>
<th>V RP (Data)</th>
<th>V RP (Model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std</td>
<td>7.8622</td>
<td>6.0230</td>
<td>47.6415</td>
<td>29.9255</td>
<td>34.2257</td>
<td>2.1751</td>
</tr>
<tr>
<td>Skewness</td>
<td>2.1053</td>
<td>3.6603</td>
<td>6.9652</td>
<td>6.4735</td>
<td>6.1047</td>
<td>-6.4735</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>10.7107</td>
<td>22.9600</td>
<td>64.9391</td>
<td>57.3278</td>
<td>79.8023</td>
<td>57.3278</td>
</tr>
</tbody>
</table>

Based on Figure 3.6 and Table 3.6, the SVCJ model built in our cost-free production economy can well capture the dynamics of the VIX index. By using the VIX future pricing formula in Zhu and Lian (2012), we are able to get the prices of VIX futures with different maturities and their returns, which are close to the market data. Thus, the SVCJ model (or SVJP model) built in a cost-free production economy can well explain the negative returns of VIX futures and other volatility claims, with low RRA and EIS.

Option market.

Based on our production-based equilibrium model, we are able to price the options (including index options, equity options, VIX options and others). Following Bates (1991, 1996, 2000); Liu et al. (2005); Zhang et al. (2012); Fu and Yang (2012), given the stock process in (3.3), once we have an explicit transition between the risk-neutral measure $Q$ and the physical probability measure $P$ in (3.23), we can price any options in the risk-neutral measure $Q$. Unlike Duffie et al. (2000), we provide a clear and explicit transition between the two measures, which are implied by a production-based equilibrium model.
3.5 Conclusion

This chapter discusses the equity and variance risk premiums in a cost-free production-based economy with one representative investor who has a recursive preference. After solving the equilibrium, we provide an explicit transition for model parameters between the risk-neutral measure and the physical probability measure. This transition documents that the diffusive volatility risk premium should be negative. Then, for the data period in Broadie et al. (2007), the SVCJ model built in our cost-free production economy can perfectly explain the equity premium puzzle and the large negative VRP. For the longer data period, the SVJ model works best in terms of explaining the VRP, and we find that the SVJ and SVCJ models are able to explain both large ERP and VRP when the ERP is large than 11% (e.g., the periods, 1990–1999 and 2010–2016).

In contrast to Bollerslev et al. (2009); Drechsler and Yaron (2011); Drechsler (2013) who use a long-run risks model, and Buraschi et al. (2014) who use a two-tree Lucas (1978) economy with two heterogeneous investors, we employ a simpler cost-free production-based equilibrium model to explain the large equity and variance risk premiums. Our analysis in this chapter can be extended to study the skewness risk premium (e.g., Neuberger (2012); Kozhan, Neuberger, and Schneider (2013)).

3.6 Appendix

3.6.1 Proof of Proposition 3.3.1

The value function satisfies the following HJB equation:

\[
0 = \max_{c,u} \left\{ f(c, J) + [r + (\phi - \lambda m)u - \frac{c}{W}]WJ_W + \frac{1}{2}u^2W^2VJ_{WW} + \kappa(\theta - V)J_V + \frac{1}{2}V^2J_{VV} + \sigma_V^2VuJ_{VV} + \sigma_V\rho uWVJ_{WV} + \lambda E[J(W + uW(e^x - 1), V + y) - J]\right\}. \tag{3.37}
\]
This leads to the two first-order conditions (FOCs):

\[ f_c(c, J) - J_W = 0, \]
\[ (\phi - \lambda m)WJ_W + uW^2VJ_WW + \sigma_V \rho VJ_WV, \]
\[ + \lambda E [J_W(W(1 + (e^x - 1)u), V + y)W(e^x - 1)] = 0. \]  
\[ (3.38) \]

Applying the market clearing condition \( u = 1 \) to (3.39), we can solve the equity premium as,

\[ \phi_t = \lambda m - \frac{1}{J_W} (WVJ_WW + \sigma_V \rho VJ_WV + \lambda E [J_W(We^x, V + y)(e^x - 1)]). \]
\[ (3.40) \]

In addition, from (3.38), we can solve the optimal consumption rate:

\[ c_t^* = \left( J_W [(1 - \gamma)J]^{\omega - 1} \beta^{-1} \right)^{-\psi}. \]
\[ (3.41) \]

Plugging (3.40) into (2.4) and using \( u = 1 \), we get the following partial differential equation (PDE),

\[ 0 = \frac{\beta(1 - \gamma)J}{1 - \psi^{-1}} \left[ \frac{e^{x - \psi - 1}}{((1 - \gamma)J)^{\omega}} - 1 \right] - c^* J_W + rWJ_W - \frac{1}{2} W^2VJ_WW + \kappa(\theta - V)J_V \\
\[ + \frac{1}{2} \sigma_V^2 VJ_{VV} + \lambda E [J(We^x, V + y) - J] - \lambda E [WJ_W(We^x, V + y)(e^x - 1)]. \]
\[ (3.42) \]

We conjecture the value function has the following form:

\[ J(W, V) = e^{aV + b} \frac{W^{1 - \gamma}}{1 - \gamma}, \]
\[ (3.43) \]

where \( a > 0 \) and \( \gamma > 1 \) (similar assumptions see Zhou and Zhu (2012); Wachter (2013)).
Substituting the conjectured value function in (3.43) into (2.5) and the PDE (3.42), the optimal consumption rate (2.5) can be rewritten as

$$c^*_t = \left( e^{-\psi(aV+b)\beta_V} \right) W_t,$$

and the PDE (3.42) becomes

$$0 = \beta(1 - \gamma) \left[ e^{-\psi(aV+b)\beta_V} - 1 \right] - (1 - \gamma) \left( e^{-\psi(aV+b)\beta_V} \right) + r (1 - \gamma) + \gamma (1 - \gamma) \frac{1}{2} \frac{\partial^2 V}{\partial V^2} + \kappa(\theta - V) a + \frac{1}{2} \sigma^2 V a^2 + \lambda E \left[ e^{ay+(1-\gamma)x} - 1 \right] - (1 - \gamma) \lambda E \left[ e^{ay-\gamma x} (e^x - 1) \right].$$

(3.45)

Using the affine approximation method (e.g., see Benzoni et al. (2011)), we expand the exponential term in $V$ near their long term mean level $\bar{V} = \frac{\kappa \theta + \lambda E[y]}{\alpha}$, $e^{-\psi(aV+b)} \approx e^{-\psi(a\bar{V}+b)} - \psi a e^{-\psi(a\bar{V}+b)} (V - \bar{V}) = (1 + \psi a \bar{V}) e^{-\psi(a\bar{V}+b)} - \psi a e^{-\psi(a\bar{V}+b)} V$ and collect the terms with the same power of $V$. Then $a$ and $b$ can be solved in the following equation.

$$\begin{cases} 
0 = \beta(1 - \gamma) \left[ e^{-\psi(a\bar{V}+b)\beta_V} - 1 \right] - (1 - \gamma) \left( e^{-\psi(a\bar{V}+b)\beta_V} \right) + r (1 - \gamma) + \gamma (1 - \gamma) \frac{1}{2} \frac{\partial^2 V}{\partial V^2} + \kappa(\theta - V) a + \frac{1}{2} \sigma^2 V a^2 + \lambda E \left[ e^{ay+(1-\gamma)x} - 1 \right] - (1 - \gamma) \lambda E \left[ e^{ay-\gamma x} (e^x - 1) \right], \\
0 = -ae^{-\psi(a\bar{V}+b)\beta_V} + \gamma (1 - \gamma) \frac{1}{2} - \kappa a + \frac{1}{2} \sigma^2 a^2. 
\end{cases}$$

(3.46)

Plugging (3.43) into (3.40), we get the equity premium as

$$\phi_t = (\gamma - \sigma_V \rho a) V_t + \lambda E \left[ (1 - e^{-\gamma x+ay}) (e^x - 1) \right],$$

(3.47)

and substituting (3.44) into (3.15), we obtain the optimal wealth process as follows,

$$\begin{cases} 
\frac{dW_t}{W_t} = \left[ (r + \phi_t - \lambda m) - \left( e^{-\psi(aV+b)\beta_V} \right) \right] dt + \sqrt{V_t} dB_{S,t} + (e^x - 1) dN_t, \\
\frac{dV_t}{V_t} = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dB_{V,t} + y dN_t. 
\end{cases}$$

(3.48)
Finally, we define the SDF as

$$\pi_t = \exp \left\{ \int_0^t f_J(c, J) ds \right\} f_c(c, J),$$

or

$$\frac{d\pi_t}{\pi_t} = f_J(c, J) dt + \frac{df_c(c, J)}{f_c(c, J)},$$

where

$$f_c(c, J) = J_W = e^{aV+b} W^{-\gamma},$$

and

$$f_J(c, J) = \frac{\beta(1-\gamma)}{1-\psi^{-1}} \left[ \frac{e^{1-\psi^{-1}(1-\omega)}}{((1-\gamma)J)^\psi - 1} \right] = \frac{\beta(1-\gamma)}{1-\psi^{-1}} \left[ e^{-\psi \omega (aV+b)} \beta \psi^{-1} (1 - \omega) - 1 \right].$$

Implying Itô’s Lemma to (3.51), we have

$$\frac{df_c(c, J)}{f_c(c, J)} = -\Gamma_t dt - \gamma \sqrt{V_t} dB_{S,t} + a \sigma \sqrt{V_t} dB_{V,t} + (e^{-\gamma x + ay} - 1) dN_t - \lambda E \left( e^{-\gamma x + ay} - 1 \right) dt,$$

where

$$\Gamma_t = \gamma \left( r + \phi_t - \lambda m - e^{-\psi \omega (aV+b)} \beta \psi \right) - \frac{1}{2} \gamma (\gamma + 1) V - a \kappa (\theta - V)$$

$$- \frac{1}{2} \sigma^2 a^2 V + \gamma a \rho \sigma V - \lambda E \left( e^{-\gamma x + ay} - 1 \right).$$

Finally, we plug (3.52) and (3.53) into (3.50) and then we get

$$\frac{d\pi_t}{\pi_t} = -r dt - \gamma \sqrt{V_t} dB_{S,t} + a \sigma \sqrt{V_t} dB_{V,t} + (e^{-\gamma x + ay} - 1) dN_t - \lambda E \left( e^{-\gamma x + ay} - 1 \right) dt,$$

(3.55)
where the constant $r$ satisfies

$$r = \Gamma_t - \frac{\beta(1 - \gamma)}{1 - \psi - 1} \left[ e^{-\psi \omega(aV + b)} \beta^{\psi - 1}(1 - \omega) - 1 \right].$$  \hspace{1cm} (3.56)

Plugging $\Gamma_t$ in Equation (3.54) with the equity premium $\phi_t$ in Equation (3.47) into Equation (3.56), we get

$$0 = \frac{\beta(1 - \gamma)}{1 - \psi - 1} \left[ e^{-\psi \omega(aV + b)} \beta^{\psi - 1} - 1 \right] - (1 - \gamma) \left[ e^{-\psi \omega(aV + b)} \beta^{\psi} \right] + r(1 - \gamma) + \gamma(1 - \gamma) \frac{1}{2} V$$

$$+ \kappa(\theta - V)a + \frac{1}{2} \sigma^2_V V a^2 + \lambda E \left[ e^{a y + (1 - \gamma) x} - 1 \right] - (1 - \gamma) \lambda E \left[ e^{a y - \gamma x} (e^x - 1) \right].$$  \hspace{1cm} (3.57)

We find that Equation (3.57) implied by the pricing kernel and Equation (3.45) implied by HJB equation are totally the same. According to Benzoni et al. (2011), this is because the pricing kernel we assume in (3.49) is identical to the HJB equation (2.4).

As $B_{S,t}$ and $B_{V,t}$ are a pair of correlated Brownian motions with correlation coefficient $\rho$, according to Girsanov’s theorem (see, e.g., Theorem 1.32 and Theorem 1.34, Øksendal and Sulem (2007)), we present the transition between the risk-neutral measure $\mathbb{Q}$ and the physical probability measure $\mathbb{P}$,

$$dB^{Q}_{S,t} = dB_{S,t} + (\gamma - a \rho \sigma_V) \sqrt{V_t} dt,$$

$$dB^{Q}_{V,t} = dB_{V,t} + \rho (\gamma - a \rho \sigma_V) \sqrt{V_t} dt - (1 - \rho^2) a \sigma_V \sqrt{V_t} dt = dB_{V,t} + (\rho \gamma - a \sigma_V) \sqrt{V_t} dt,$$

$$\mu^{Q}_S = \mu_S - \gamma \sigma^2_S,$$

$$\mu^{Q}_V = \frac{1}{1 - a \mu_V} \mu_V,$$

$$\lambda^{Q} = \lambda E \left[ e^{-\gamma x + ay} \right] = \lambda e^{\frac{1}{2} \sigma^2_S \gamma^2 - \mu_S \gamma} \frac{1}{1 - a \mu_V}.$$

As Equation (3.46) can yield multiple solutions to $a$ and $b$, we select $a$ with the restriction, $a < 1/\mu_V$, to make sure $\lambda^{Q} > 0$. The empirical results in Eraker (2004);
Broadie et al. (2007); Neumann et al. (2016), $\mu_Q > \mu_V$, suggest the restriction $a > 0$ as $\mu_V > 0$.

### 3.6.2 A Comparison of Our Codes

In order to verify the accuracy of our MCMC codes, we present the replicated estimates by using the same sample in Eraker et al. (2003) and Yun (2011) with our MCMC codes based on OpenBUGS, which are used in Section 3.4. We discard the first 10,000 runs as the “burn-in” period and use the last 100,000 iterations in MCMC simulations to estimate model parameters. Specifically, we take the mean of the posterior distribution as the parameter estimate and the standard deviation of the posterior as the standard error in parentheses.

As the jumps of the SVCJ model in Eraker et al. (2003) are correlated, we only present the estimates of the SV and SVJP models. As a complementary comparison, we compare our estimates of the SV, SVJP and SVCJ models with Yun’s (2011), which are estimated by using WinBUGS. From Table 3.7 and 3.8, we can confirm that our MCMC codes based on OpenBUGS can estimate similar model parameters with a very small difference to Eraker et al. (2003) and Yun (2011).

---

33 The reason we choose OpenBUGS instead of WinBUGS is that OpenBUGS running is faster and more functional. For changes between WinBUGS and OpenBUGS, see http://www.openbugs.net/w/OpenVsWin.
Chapter 3. Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

Table 3.7: A comparison with Eraker et al. (2003).

We present the replicated estimates by using the same MCMC methodology and the sample in Eraker et al. (2003) with our MCMC codes based on OpenBUGS, which will be used in Section 3.4. We discard the first 10,000 runs as the “burn-in” period and use the last 90,000 iterations in MCMC simulations to estimate model parameters. For each parameter to be estimated, we use the same priors as in Eraker et al. (2003). Specifically, we take the mean of the posterior distribution as the parameter estimate and the standard deviation of the posterior as the standard error in parentheses.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma_V$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu_\sigma$</th>
<th>$\sigma_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>Eraker et al. (2003)</td>
<td>0.0231</td>
<td>0.9052</td>
<td>0.1434</td>
<td>-0.3974</td>
<td>(0.0068)</td>
<td>(0.1077)</td>
</tr>
<tr>
<td></td>
<td>Our estimates</td>
<td>0.0289</td>
<td>0.9002</td>
<td>0.1544</td>
<td>-0.407</td>
<td>(0.00524)</td>
<td>(0.07645)</td>
</tr>
<tr>
<td>SVJP</td>
<td>Eraker et al. (2003)</td>
<td>0.0128</td>
<td>0.8136</td>
<td>0.0954</td>
<td>-0.4668</td>
<td>0.0060</td>
<td>-2.5862</td>
</tr>
<tr>
<td></td>
<td>(0.0039)</td>
<td>(0.1244)</td>
<td>(0.0104)</td>
<td>(0.057)</td>
<td>(0.0021)</td>
<td>(1.3034)</td>
<td>(1.7210)</td>
</tr>
<tr>
<td></td>
<td>Our estimates</td>
<td>0.01979</td>
<td>0.8534</td>
<td>0.1184</td>
<td>-0.479</td>
<td>0.00594</td>
<td>-2.978</td>
</tr>
<tr>
<td></td>
<td>(0.004218)</td>
<td>(0.08437)</td>
<td>(0.0106)</td>
<td>(0.05199)</td>
<td>(0.00203)</td>
<td>(1.304)</td>
<td>(0.7953)</td>
</tr>
</tbody>
</table>

Table 3.8: A comparison to Yun (2011).

We present the replicated estimates by using same MCMC methodology and the sample in Yun (2011) with our MCMC codes based on OpenBUGS, which will be used in Section 3.4. We discard the first 10,000 runs as “burn-in” period and use the last 90,000 iterations in MCMC simulations to estimate model parameters. For each parameter to be estimated, we use the same priors as in Eraker et al. (2003). Specifically, we take the mean of the posterior distribution as parameter estimate and the standard deviation of the posterior as standard error in parentheses.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma_V$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu_\sigma$</th>
<th>$\sigma_\sigma$</th>
<th>$\mu_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>Yun (2011)</td>
<td>0.0288</td>
<td>1.0210</td>
<td>0.1813</td>
<td>-0.4960</td>
<td>(0.0052)</td>
<td>(0.1133)</td>
<td>(0.0128)</td>
</tr>
<tr>
<td></td>
<td>(0.0528)</td>
<td>(0.0528)</td>
<td>(0.0528)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Our estimates</td>
<td>0.02796</td>
<td>1.051</td>
<td>0.1811</td>
<td>-0.4904</td>
<td>(0.005127)</td>
<td>(0.1168)</td>
<td>(0.01303)</td>
</tr>
<tr>
<td></td>
<td>(0.05238)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVJP</td>
<td>Yun (2011)</td>
<td>0.0204</td>
<td>0.9856</td>
<td>0.1458</td>
<td>-0.5912</td>
<td>0.0066</td>
<td>-3.3490</td>
<td>3.7940</td>
</tr>
<tr>
<td></td>
<td>(0.0048)</td>
<td>(0.1340)</td>
<td>(0.0132)</td>
<td>(0.0511)</td>
<td>(0.0025)</td>
<td>(1.4190)</td>
<td>(0.7329)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Our estimates</td>
<td>0.0184</td>
<td>1.035</td>
<td>0.1423</td>
<td>-0.5838</td>
<td>0.00439</td>
<td>-3.389</td>
<td>3.976</td>
</tr>
<tr>
<td></td>
<td>(0.034644)</td>
<td>(0.1548)</td>
<td>(0.01364)</td>
<td>(0.05226)</td>
<td>(0.02371)</td>
<td>(1.461)</td>
<td>(0.8397)</td>
<td></td>
</tr>
<tr>
<td>SVCJ</td>
<td>Yun (2011)</td>
<td>0.0362</td>
<td>0.7026</td>
<td>0.1459</td>
<td>-0.5712</td>
<td>0.0049</td>
<td>-4.4660</td>
<td>2.7100</td>
</tr>
<tr>
<td></td>
<td>(0.0006)</td>
<td>(0.0771)</td>
<td>(0.0131)</td>
<td>(0.0500)</td>
<td>(0.0017)</td>
<td>(1.1250)</td>
<td>(0.8013)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Our estimates</td>
<td>0.03495</td>
<td>0.7216</td>
<td>0.1451</td>
<td>-0.5611</td>
<td>0.005264</td>
<td>-4.149</td>
<td>2.514</td>
</tr>
<tr>
<td></td>
<td>(0.005097)</td>
<td>(0.07657)</td>
<td>(0.01199)</td>
<td>(0.0582)</td>
<td>(0.001824)</td>
<td>(1.18)</td>
<td>(0.6005)</td>
<td></td>
</tr>
</tbody>
</table>
3.6.3 ERP and VRP in Broadie et al. (2007)

Model-implied VRP

Table 3.9: Estimates in Broadie et al. (2007).


<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma_V$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu_S$</th>
<th>$\sigma_S$</th>
<th>$\mu_V$</th>
<th>$\kappa^Q$</th>
<th>$\mu_Q^0$</th>
<th>$\mu_V^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>0.023</td>
<td>0.90</td>
<td>0.14</td>
<td>-0.40</td>
<td>0.006</td>
<td>-2.59</td>
<td>4.07</td>
<td>0.028</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVJP</td>
<td>0.013</td>
<td>0.81</td>
<td>0.10</td>
<td>-0.47</td>
<td>0.066</td>
<td>-2.83</td>
<td>2.89</td>
<td>0.023</td>
<td>-2.59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVCJ</td>
<td>0.026</td>
<td>0.54</td>
<td>0.08</td>
<td>-0.48</td>
<td>0.006</td>
<td>-2.63</td>
<td>2.89</td>
<td>1.48</td>
<td>0.056</td>
<td>-6.58</td>
<td>10.81</td>
</tr>
</tbody>
</table>

We use the estimates in Broadie et al. (2007) (see Table 3.9) to calculate the model-implied VRP and use the returns of the S&P 500 index to calculate the real variance risk premium based on Definition 3.2.1 (for summary statistics, see Table 3.1). In Table 3.10, we find that the SVJP and SVCJ models with their estimates can well fit the real VRP (the mean in 1990–1999 is $-14.5477$). This gives us the motivation that the simple SVJP or SVCJ model built in our cost-free production economy may explain the high negative VRP.

Table 3.10: Model-implied Variance Risk Premium in Broadie et al. (2007).

We report simulated statistics based on 10,000 simulations, with each statistic calculated using a sample size equal to its VRP data counterpart. We report the mean of the VRP from 1990 to 1999, which is the overlapping period between Eraker et al. (2003) and Broadie et al. (2007) after VIX launched.

<table>
<thead>
<tr>
<th>Model</th>
<th>$E[\text{VRP}^{TV}]$</th>
<th>$V_{\text{VRP}}^{P,J}$</th>
<th>$V_{\text{VRP}}^{V,J}$</th>
<th>$E[\text{VRP}]$ (Model)</th>
<th>$E[\text{VRP}]$ (Data)</th>
<th>Explanation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>0.8241</td>
<td>0</td>
<td>0</td>
<td>0.8241</td>
<td>-14.5477</td>
<td>-</td>
</tr>
<tr>
<td>SVJP</td>
<td>1.5341</td>
<td>-11.6793</td>
<td>0</td>
<td>-10.1452</td>
<td>-14.5477</td>
<td>69.73%</td>
</tr>
</tbody>
</table>
Chapter 3. Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

Equilibrium Model-implied ERP and VRP

We set the equilibrium model parameters as $\beta = -\ln(0.999)$ and $\gamma = 1.1$, which is much lower than in the existing literature (e.g., 9.5 in Drechsler and Yaron (2011); 5 in Drechsler (2013)). The risk-free rate $r = 1.8421(\%)$ is the average three-month Treasury bill adjusted by the U.S. inflation rate over the same period, 02 January 1990 to 31 December 1999 (for summary statistics, see Table 3.1). The physical model parameters are given in Broadie et al. (2007) (i.e., Table 3.9). The mean of the annualized log S&P500 returns, 14.2040, is from 1990 to 1999 which is the overlapping period between Eraker et al. (2003) (1980–1999) and Broadie et al. (2007) (1987-2003) after VIX launched. Thus, the mean of the ERP is 12.3619. We report simulated statistics in Table 3.11 and 3.13 based on 10,000 simulations, with each statistic calculated using a sample size equal to its ERP and VRP data counterpart.

Table 3.11: Equilibrium Model-implied Equity Premium in Broadie et al. (2007).

<table>
<thead>
<tr>
<th>Model</th>
<th>$E[ERP^{PD}]$</th>
<th>$ERP^{PJ}$</th>
<th>$E[ERP]$ (Model)</th>
<th>$E[ERP]$ (Data)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV Model ($\gamma = 1.1, \psi = 0.70, a = 30.4081$)</td>
<td>12.2679</td>
<td>0</td>
<td>12.2679</td>
<td>12.3619</td>
</tr>
<tr>
<td>SVJJP Model ($\gamma = 1.1, \psi = 0.80, a = 39.5584$)</td>
<td>11.8854</td>
<td>0.3885</td>
<td>12.2740</td>
<td>12.3619</td>
</tr>
<tr>
<td>SVCJ Model ($\gamma = 1.1, \psi = 0.58, a = 16.1155$)</td>
<td>5.9040</td>
<td>6.4587</td>
<td>12.3627</td>
<td>12.3619</td>
</tr>
</tbody>
</table>
Chapter 3. *Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy*

Using the annualized estimates, we adjust $\psi$ to fit the mean of the ERP in Table 3.11 and then we get the best $\psi$ and the annualized risk-neutral parameters. We convert it into daily percentage estimates in Table 3.12.

**Table 3.12: Model-implied risk-neutral parameters in Broadie et al. (2007).**

The estimates correspond to daily percentage changes in the index value.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa^Q$</th>
<th>$\chi^Q$</th>
<th>$\mu_S^Q$</th>
<th>$\mu_V^Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>0.007365</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SVJP</td>
<td>0.002514</td>
<td>0.006180</td>
<td>-6.9860</td>
<td></td>
</tr>
<tr>
<td>SVCJ</td>
<td>0.02298</td>
<td>0.015489</td>
<td>-6.8591</td>
<td>3.7097</td>
</tr>
</tbody>
</table>

Here we discuss the parameters transition of the SVCJ model between the risk-neutral measure $Q$ and the physical probability measure $P$ in details. In the physical measure, $\kappa = 0.026$, $\lambda = 0.006$, $\mu_S = -2.63$ and $\mu_V = 1.48$ given in Table 3.9, while in the risk-neutral measure, $\kappa^Q = 0.02298$, $\chi^Q = 0.015489$, $\mu_S^Q = -6.8591$ and $\mu_V^Q = 3.7097$ shown in Table 3.12. Based on that, our production-based equilibrium reveals that the DVRP, $\kappa^Q - \kappa = -0.0038 < 0$, should be negative. From (3.23), we know that $\kappa^Q - \kappa = (\rho \gamma - a \sigma_V) \sigma_V < 0$ for $\gamma > 1$ and $0 < a < 1/\mu_V$. In addition, in the risk-neutral probability measure, we have a more negative mean of jump size in the price and a larger mean of jump size in the volatility. Furthermore, the transition suggests that the estimate of the jump intensity in the risk-neutral probability measure should be larger than in the physical probability measure. It will contribute part of the jump risk premium in the variance risk premium and the equality premium. Thus, here we suggest again that the risk-neutral parameter $\chi^Q$ should be estimated. Finally, we use the same annualized parameters to calculate the VRP in Table 3.13.
Chapter 3. Equilibrium Equity and Variance Risk Premiums in a Cost-free Production Economy

Table 3.13: Equilibrium Model-implied Variance Risk Premium in Broadie et al. (2007).

We set the equilibrium model parameters as \( \beta = -\ln(0.999) \). We report simulated statistics based on 10,000 simulations, with each statistic calculated using a sample size equal to its VRP data counterpart. We report the mean of the VRP from 1990 to 1999 which is the overlapping period between Eraker et al. (2003) and Broadie et al. (2007) after VIX launched. The risk-free rate \( r = 1.8421 \) is the average three-month Treasury bill adjusted by the U.S. inflation rate over the same period, 1990 to 1999.

<table>
<thead>
<tr>
<th>Model</th>
<th>VRP(^{P, D})</th>
<th>VRP(^{P, J})</th>
<th>VRP(^{V, J})</th>
<th>E[VRP] (Model)</th>
<th>E[VRP] (Data)</th>
<th>Explanation rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV Model (( \gamma = 1.1, \psi = 0.70, a = 30.4081 ))</td>
<td>-2.9572</td>
<td>0</td>
<td>0</td>
<td>-2.9572</td>
<td>-14.5477</td>
<td>20.33%</td>
</tr>
<tr>
<td>SVJP Model (( \gamma = 1.1, \psi = 0.80, a = 39.5584 ))</td>
<td>-1.8512</td>
<td>-0.2146</td>
<td>0</td>
<td>-2.0657</td>
<td>-14.5477</td>
<td>14.20%</td>
</tr>
<tr>
<td>SVCJ Model (( \gamma = 1.1, \psi = 0.58, a = 16.1155 ))</td>
<td>-0.4707</td>
<td>-3.2024</td>
<td>-9.2100</td>
<td>-12.8832</td>
<td>-14.5477</td>
<td>88.56%</td>
</tr>
</tbody>
</table>

From Table 3.11 and 3.13, we conclude that over the same period, 1990 to 1999, the SVCJ model built in our cost-free production economy can explain not only the high ERP but also the large VRP on average (the explanation rate is 88.56%). In other words, our model works well for extremely high ERP (e.g., \( E[ERP] > 11 \)). In addition, Table 3.13 examines the fraction of the VRP explained by the three components. We find that the jump VRP (especially the volatility jump VRP) explains around 71.45% of the total VRP. This is close to Li and Zinna (2017), who calculate the fraction at around 79%.
Chapter 4

A Demand-Based Equilibrium Model of Volatility Trading

This chapter is joint work with Jin E. Zhang. It has been accepted for presentation at 2017 Auckland Finance Meeting, 18-20 December 2017, AUT, Queenstown, New Zealand.

4.1 Introduction

This chapter is the first to provide a demand-based equilibrium model of volatility trading with three kinds of traders (i.e., dealers, asset managers and leveraged funds), which complements Eraker and Wu’s (2017) consumption-based equilibrium model. According to Mixon and Onur (2015), volatility derivatives are mainly traded by three kinds of traders: dealers, asset managers and leveraged funds. Dealers as market makers balance the orders of volatility derivatives. Asset managers as hedgers prefer to take long positions, while leveraged funds as speculators prefer to take short positions. Mixon and Onur (2015) collect the daily volatility derivatives transaction data from the Swap Data Repositories (SDR) reported by the Commodity Futures Trading Commission (CFTC) and find that the gross vega notional outstanding for variance swaps, in 2014, is over USD 2 billion, with USD 1.5 billion in S&P 500
variance swaps. From Bollen, O’Neill, and Whaley (2017), the dollar value of open interest of the CBOE Market Volatility Index (VIX) futures in 2013 is around USD 7 billion, and the dollar market value of VIX Exchange Traded Products (ETPs) linked to the short-term S&P 500 VIX futures index is around USD 2 billion. The market for volatility trading has become an important new avenue of financial markets in addition to equity and fixed income securities over last decade.

However, since 2009, according to Whaley (2013), there has been a huge loss of investing in positive multiplier VIX Exchange Traded Notes (ETNs) (e.g., iPath S&P 500 VIX Short-Term Futures ETN (VXX)). This is because of the negative return of VIX futures (Eraker and Wu (2017)). In this chapter, we mainly investigate the question: How do volatility trading activities affect the VRP and the VIX futures’ price and return? By using an equilibrium model of volatility trading, we provide an economic theory to explain that the lower positions (more net short) of dealers, the lower short positions of leverage funds and the higher long positions of asset managers lead to a higher VIX futures price and a more negative VIX futures return, so that the positive multiplier VIX ETNs produce a huge loss. This is different from the explanation in Eraker and Wu (2017), which states that the negative futures return is only determined by the investors risk aversion. Besides considering the risk aversion, adding the trading behaviour of the three main traders complement their consumption-based equilibrium model. The novel model proposed in this chapter is our main contribution. Furthermore, empirically we use the weekly Traders in Financial Futures (TFF) reports data to test the impact of trading on the VIX futures return. These results are new.

In terms of demand-based equilibrium models, Garleanu et al. (2009) propose a demand-based equilibrium model for option pricing, with two kinds of agents: dealers and end users. Dong (2016) extends their model to explain how the demand for ETPs demand affects the VIX futures price. In contrast to the previous literature,
our model considers three kinds of traders (i.e., dealers, asset managers and leveraged funds) and analyses how traders demand for VIX-linked futures influences the volatility market. The daily trading data in Mixon and Onur (2015) and the weekly data used in this chapter strongly support that we should use a three-trader model instead of the two-trader model.

There are a huge number of papers studying the VRP and its predictive power. For example, Carr and Wu (2009) find that there exists a large and negative mean of the VRP on five stock indexes and 35 individual stocks. Recently, González-Urteaga and Rubio (2016) have discussed and tested the volatility risk premium at the individual and portfolio level. Barras and Malkhozov (2016) formally compare the market VRP inferred from equity and option markets.1 However, there is a paucity of research on predicting the VRP empirically. Konstantinidi and Skiadopoulos (2016) compare four predicting models and find that the trading activity model is the best performing. Fan, Imerman, and Dai (2016) claim that the magnitude of VRP is significantly affected by investors’ demand for hedging tail risk. Two papers do not use the explicit trading positions of dealers, asset managers and leveraged funds as predictive variables. For example, Konstantinidi and Skiadopoulos (2016) use the trading volume of all S&P 500 futures contracts and the TED spread to explain the high negative VRP. In this chapter, we test the impacts of the trading positions of the three main traders and find that the high negative VRP is driven by the higher short positions of dealers, the lower short positions of leverage funds and the higher long positions of asset managers in variance swaps. This is new empirical evidence. In addition, our theoretical model also gives a very neat economic theory to explain the

1Furthermore, Todorov (2010); Bollerslev and Todorov (2011) use the rare events to account for the large average VRP. Aït-Sahalia et al. (2015); Li and Zinna (2017) examine the term structures of the VRP. Choi, Mueller, and Vedolin (2017) study variance risk premiums in the bond market. Bollerslev et al. (2015); Jin (2015) and others study the predictive power of the VRP for the stock return, while Londono and Zhou (2017) recently provide evidence of the predictive power of the VRP for the currency return.
This chapter makes at least two contributions. First, it is the first paper to provide a demand-based equilibrium model of volatility trading with three kinds of traders (i.e., dealers, asset managers and leveraged funds) that fully supports the existing empirical results. Second, due to our novel model, this chapter is the first to test the impact of the three main traders net positions on the VRP and the VIX futures return.

The remainder of this article is organized as follows. Section 4.2 presents the model and results, and Section 4.3 provides two endogenous cases. Section 4.4 gives the empirical analysis. Section 4.5 concludes. Appendix 4.6.1 collects all proofs, and Appendix 4.6.2 gives solutions for the endogenous cases.

4.2 Models and results

4.2.1 Heston model

We set up our demand-based equilibrium model starting from the Heston (1993) model, which is the most popular stochastic volatility model in the literature. We adopt it to describe the dynamics of the stock price (i.e., S&P 500 Index (SPX)) in the physical measure $\mathbb{P}$ as follows:

$$\left\{\begin{array}{l}
\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dB_{S,t}, \\
v_t = \kappa (\theta - v_t) dt + \sigma_v \sqrt{v_t} dB_{v,t},
\end{array}\right. \quad (4.1)$$

where $v_t$ is the instantaneous variance; $B_{S,t}$ and $B_{v,t}$ are a pair of correlated Brownian motions with correlation coefficient $\rho$. Empirical evidence documents that $\rho$.

---

2Bollerslev et al. (2009); Drechsler and Yaron (2011); Bollerslev et al. (2012); Drechsler (2013); Jin (2015) adopt the long-run risks model (i.e., long-run risks and investor preferences) to explain the negative VRP and Buraschi et al. (2014) use a two-tree Lucas (1978) economy with two heterogeneous investors (i.e., disagreement). In contrast to them, we use the trading positions of dealers, asset managers and leveraged funds to explain the high VRP.
is negative for SPX, so that here we assume \(-1 < \rho < 0\) (e.g., Eraker et al. (2003); Eraker (2004) and Broadie et al. (2007)).

Developing Heston (1993), we assume the stock price in the risk-neutral measure \(Q\) as follows:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rdt + \sqrt{v_t}dB^Q_{S,t}, \\
\frac{dv_t}{v_t} &= \left[\kappa(\theta - v_t) - \lambda(S,v,\Theta,t)\right]dt + \sigma_v\sqrt{v_t}dB^Q_{v,t},
\end{align*}
\] (4.2)

where \(\lambda(S,v,\Theta,t)\) represents the market price of the volatility risk and \(\Theta\) captures the volatility trading activities, and

\[
\frac{dB^Q_{S,t}}{S_t} = dB_{S,t} + \frac{\mu - r}{\sqrt{v_t}}dt; \quad dB^Q_{v,t} = dB_{v,t} + \frac{\lambda(S,v,\Theta,t)}{\sigma_v\sqrt{v_t}}dt.
\] (4.3)

The above transformation between the physical measure \(P\) and the risk-neutral measure \(Q\) indicates that the pricing kernel (or state-price density) \(\pi_t\) must satisfy,

\[
\frac{d\pi_t}{\pi_t} = -rdt - \frac{\mu - r}{\sqrt{v_t}}dB_{S,t} - \frac{\lambda(S,v,\Theta,t)}{\sigma_v\sqrt{v_t}}dB_{v,t}.
\] (4.4)

In Heston (1993), he assumes \(\lambda(S,v,\Theta,t)\) as \(\lambda(S,v,t)\).\(^5\) It means that the market price of the volatility risk is determined only by the stock price and its volatility. In our setting, besides the stock price and its volatility, the volatility trading activities contributes to the market price of the volatility risk as well.

We simplify the Heston (1993) model based on the following assumptions.

\(^3\)To be clear, all notions (e.g., Brownian motion, conditional expectation, conditional variance and conditional covariance) with superscript \(\cdot\) throughout the chapter are in the risk-neutral measure \(Q\), while all notions without superscript \(\cdot\) are in the physical measure \(P\).

\(^4\)Strictly speaking, the market price of the volatility risk is \(\frac{\lambda(S,v,\Theta,t)}{\sigma_v\sqrt{v_t}}\). As \(\sigma_v\sqrt{v_t}\) is fixed, then \(\lambda(S,v,\Theta,t)\) is able to measure the magnitude of the market price of the volatility risk.

\(^5\)Furthermore, Heston (1993) just assumes \(\lambda(S,v,t) = \lambda v_t\), where \(\lambda\) is a constant.
**Assumption 4.2.1** To simplify the model, we set

\[ r = 0, \quad \kappa = 0. \]  

(4.5)

In addition,

\[ -1 < \rho < 0, \quad 0 < \sigma_v < 1, \]  

(4.6)

which leads to \( \text{Var}_t(R_T) > |\text{Cov}_t(R_T, v_T)| \) where \( R_T = \ln \frac{S_T}{S_t} \) and the conditional variance and covariance are given in Appendix 4.6.1.

We note that Assumption 4.2.1 is made for notational simplicity only, and is unimportant for the conclusions we get below. The results can be extended if we relax the above assumption.

Under Assumption 4.2.1, we rewrite the dynamics of the stock price in the physical measure \( \mathbb{P} \) as

\[
\begin{cases}
\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dB_{S,t}, \\
\frac{dv_t}{v_t} = \sigma_v \sqrt{v_t} dB_{v,t},
\end{cases}
\]  

(4.7)

and in the risk-neutral measure \( \mathbb{Q} \) as

\[
\begin{cases}
\frac{dS_t}{S_t} = \sqrt{v_t} dB_{S,t}^Q, \\
\frac{dv_t}{v_t} = -\lambda(S, v, \Theta, t) dt + \sigma_v \sqrt{v_t} dB_{v,t}^Q.
\end{cases}
\]  

(4.8)

### 4.2.2 Variance risk premium and the CBOE VIX Index

Given SPX in (4.7), we are able to obtain the annualized realized variance (RV) at time \( t \) during period \([t, t + \tau]\) as

\[
RV_t = E_t \left[ \frac{1}{\tau} \int_t^{t+\tau} (d \ln S_u)^2 du \right] = E_t \left[ \frac{1}{\tau} \int_t^{t+\tau} v_u du \right] = v_t. \]  

(4.9)
Similarly, the annualized implied variance (IV) at time $t$ during period $[t, t + \tau]$ can be written as

$$IV_t = E_t^Q \left[ \frac{1}{\tau} \int_t^{t+\tau} (d \ln S_u)^2 du \right] = E_t^Q \left[ \frac{1}{\tau} \int_t^{t+\tau} v_u du \right] = v_t - \frac{1}{\tau} \int_t^{t+\tau} E_t^Q[\lambda(S, v, \Theta, u)]du. \tag{4.10}$$

According to Carr and Wu’s (2009) definition for the variance risk premium (VRP),

$$VRP_t = RV_t - IV_t,$$

we have the following theorem.

**Theorem 4.2.2 (Variance risk premium)** The VRP at time $t$ during period $[t, t + \tau]$ can be defined as

$$VRP_t \equiv RV_t - IV_t = \frac{1}{\tau} \int_t^{t+\tau} E_t^Q[\lambda(S, v, \Theta, u)]du. \tag{4.11}$$

Based on the above theorem, the sign of $VRP_t$ depends on the sign of $\lambda$ the more negative the $\lambda$, the more negative the $VRP_t$. The VRP on average is empirically negative, so that $\lambda$ should have a negative mean (e.g., Carr and Wu (2009)).

Following the definition of the CBOE VIX Index,\(^6\) the VIX Index at time $t$ can be defined as the product of the square root of the implied variance during period $[t, t + 21/252]$ and the notional amount, 100, i.e.,

$$VIX_t = \sqrt{IV_t} \times 100 = 100 \times \sqrt{v_t - \frac{1}{\tau} \int_t^{t+\tau} E_t^Q[\lambda(S, v, \Theta, u)]du}, \tag{4.12}$$

where $\tau = 21/252$.

**Lemma 4.2.3** Based on simplified Heston (1993), we have the following:

(i) The implied volatility and the CBOE VIX increase with the more negative $\lambda$.

(ii) The negative VRP is caused by the negative $\lambda$.

(iii) In addition, more negative $\lambda$ leads to more negative VRP.

---

\(^6\)The CBOE VIX white paper can be found at https://www.cboe.com/micro/vix/vixwhite.pdf.
In Heston’s (1993) setup, the VRP is determined by the average market price of the volatility risk $\lambda$ during a future period $(t, t+\tau)$. In addition, the implied volatility and CBOE VIX are contributed by $\lambda$. This is the key to link the VRP and the existing risks in the financial market. In this chapter, the risks are from the traders’ holdings in volatility products, i.e., $\Theta$. However, for different purposes, we could set is so that $\lambda$ is related to other market risks.

### 4.2.3 Volatility market

As Heston (1993) mentioned, any derivatives with the particular payoff function $U_T$ must satisfy the following partial differential equation (PDE):

$$
\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma_v v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma_v^2 \frac{v^2}{v^2} \frac{\partial^2 U}{\partial v^2} - \lambda \frac{\partial U}{\partial v} + \frac{\partial U}{\partial t} = 0. \tag{4.13}
$$

The value of the derivative is determined by the market price of the volatility risk $\lambda$. Here we assume that there is a volatility derivative, i.e., $VIX^2$ futures, written on the square of the CBOE VIX Index with the maturity date $T$. Then, its fair price at time $t$ is given by

$$
F_{t,T}^{VIX^2} = E_t^Q [VIX^2_T] = E_t^Q \left[ v_T - \frac{1}{\tau} \int_T^{T+\tau} E_t^Q [\lambda(S, v, \Theta, u)] du \right] \times 100^2
$$

$$
= \left( v_t - \int_T^t E_t^Q [\lambda(S, v, \Theta, u)] du - \frac{1}{\tau} \int_T^{T+\tau} E_t^Q [\lambda(S, v, \Theta, u)] du \right) \times 100^2. \tag{4.14}
$$

Applying Itô’s Lemma to $F_{t,T}^{VIX^2}$, we get

$$
d \frac{F_{t,T}^{VIX^2}}{100^2} = dv_t + \lambda(S, v, \Theta, t) dt = \sigma_v \sqrt{v_t} dB_{v,t}^Q. \tag{4.15}
$$

$VIX^2$ futures can be regarded as a proxy of VIX futures because they have similar properties. The reason we use $VIX^2$ futures instead of VIX futures is that the pricing formula of $VIX^2$ futures is more tractable. The tractable formula produces a lot of intuitions in Section 4.2.5. It can be extended into VIX futures without affecting the main results in the chapter.
which is a martingale in the risk-neutral measure $Q$. It can be also rewritten as

$$
\frac{d F_{VIX}^2}{100^2} = \frac{1}{F_{VIX}^2} \sigma_v \sqrt{v_t} dB_{v,t}^Q = \frac{1}{F_{VIX}^2} \left[ \lambda(S, v, \Theta, t) dt + \sigma_v \sqrt{v_t} dB_{v,t}^Q \right]. \tag{4.16}
$$

From (4.16), we can get the expected return of $VIX^2$ futures as follows.

**Theorem 4.2.4 ($VIX^2$ futures return)** The expected return of $VIX^2$ futures is

$$
R_{VIX}^t \equiv \frac{1}{dt} \left[ \frac{d F_{VIX}^2}{100^2} \right] = \frac{\lambda(S, v, \Theta, t)}{v_t - \int_t^T E_Q^{\Theta}(\lambda(S, v, \Theta, u)) du - \frac{1}{2} \int_T^{t+\tau} E_Q^{\Theta}(\lambda(S, v, \Theta, u)) du}.
$$

This indicates that the return of $VIX^2$ futures depends on the average $\lambda$ during future periods $(t, t+\tau)$ and $(T, T+\tau)$; the negative $\lambda$ leads to a negative return of $VIX^2$ futures. In addition, the value of $VIX^2$ futures basis at time $t$ can be solved in the following.

**Theorem 4.2.5 ($VIX^2$ futures basis)** The value of $VIX^2$ futures basis at time $t$ is

$$
\text{Basis}_{VIX}^t \equiv F_{VIX}^t - VIX_t^2 = \left[ - \int_t^T E_Q(\lambda(S, v, \Theta, u)) du + \frac{1}{\tau} \left( \int_t^{t+\tau} E_Q(\lambda(S, v, \Theta, u)) du - \int_T^{T+\tau} E_Q(\lambda(S, v, \Theta, u)) du \right) \right] \times 100^2. \tag{4.18}
$$

**Lemma 4.2.6** Based on simplified Heston (1993), we have the following:

(i) The price of the $VIX^2$ futures increases with more negative $\lambda$.

(ii) The negative return of the $VIX^2$ futures is caused by the negative $\lambda$.

(iv) More negative $\lambda$ leads to more negative return of $VIX^2$ futures.

(v) In addition, more negative $\lambda$ leads to more positive $VIX^2$ futures basis.
In Heston’s (1993) framework, the variables related to the volatility of the underlying are affected by the market price of the volatility risk $\lambda$. In other words, once we determine the value of $\lambda$, we are able to price any volatility derivatives in Heston’s (1993) model.

### 4.2.4 The market price of the volatility risk and traders

In the economy, there are three kinds of traders: dealers (market makers), asset managers (hedgers) and leveraged funds (speculators). The trading data of the three main traders are reported by the Commitments of Traders (COT) reports and the TFF reports published by CFTC. We consider only a single-period model. There are 2 dates, $t$ and $T$, where $0 \leq t < T + \tau$. All traders make their decisions at time $t$ and hold it until time $T$. In detail, the optimal futures positions in volatility markets (i.e., $VIX^2$ futures market) of dealers, asset managers and leveraged funds are $x_t, y_t$ and $z_t$.

**Assumption 4.2.7** We assume the market price of the volatility risk is only related to the volatility trading activities,

$$\lambda(S, v, \Theta, t) = \lambda(\Theta, t) := \lambda_t. \quad (4.19)$$

In addition, the market price of the volatility risk is related to the trading strategies of dealers, $x_t$, i.e.,

$$\Theta = \{x\}. \quad (4.20)$$

Based on the single-period model, the trading strategies $x_t$ will never change until date $T$, so that we assume $\lambda_t$ is time-homogeneous.\(^8\)

---

\(^8\)In reality, the net position of dealers is the sum of net positions of the asset managers and leveraged funds. This is why we set $\Theta = \{x\}$ rather than $\Theta = \{x, y, z\}$.

\(^9\)In other words, $x_t$ is time-homogeneous. That is, $x_u = x_t$ and $\lambda_u = \lambda_t$ for any $u$, where $t \leq u \leq T$, so that $E^Q_t[\lambda_u] = E^Q_t[\lambda_t] = \lambda_t$ where $t \leq u \leq T$. As the formulas of futures price and return are involved in the term $\int^{T+\tau}_T E^Q_t[\lambda_u]du$, we further assume $\lambda_u = \lambda_t$ for any $u$ where $T < u \leq T + \tau$. 
Under Assumption 4.2.7, for any time $u$ where $t \leq u \leq T$, the implied variance becomes $IV_u = v_u - \lambda_t$, and then the VRP is

$$VRP_u = \lambda_t; \quad (4.21)$$

the price of the $VIX^2$ futures becomes

$$F_{VIX^2}^{u,T} = \frac{v_u - \lambda_t(T - u + 1)}{100^2}; \quad (4.22)$$

the expected return of $VIX^2$ futures,

$$R_{VIX^2}^{t,T} = \lambda_t F_{VIX^2}^{u,T}; \quad (4.23)$$

and the value of $VIX^2$ futures basis,

$$Basis_{VIX^2}^{u,T} = -\lambda_t(T - u) \times 100^2. \quad (4.24)$$

We consider the following special case in which the market price of volatility $\lambda_t$ is in proportion to dealers’ positions.

**Assumption 4.2.8** *In particular, we assume*

$$\lambda_t = ax_t + b, \quad (4.25)$$

where $a$ and $b$ will be solved in equilibrium.$^{10}$

**Assumption 4.2.9** *The trading strategies of dealers $x_t$ are exogenous.$^{11}$

$^{10}$For the multiple-period model, $a$ can be recursively solved; see Garleanu et al. (2009).

$^{11}$Here we assume the optimal trading orders of dealers $x_t$ are exogenous (actually, $x_t$, $y_t$, and $z_t$ are all exogenous in this case). As similar idea can be found in, for example, Bansal and Yaron (2004). The optimal consumption in Bansal and Yaron (2004) is solved from an equilibrium, but it is exogenously given.
Leveraged funds (speculators). To take the advantage of the negative return of \( VIX^2 \) futures, leveraged funds (speculators) prefer to short \( VIX^2 \) futures.\(^{12}\) Thus, we assume their demand of \( VIX^2 \) futures is \( y_t < 0 \) at time \( t \).

Asset managers (hedgers). In order to hedge their long positions in the stock market, asset managers (hedgers) prefer to long \( VIX^2 \) futures. Thus, we assume their demand of \( VIX^2 \) futures is \( z_t > 0 \) at time \( t \).

Dealers (market makers). Dealers are risk-averse. They choose the optimal order \( \phi_t \) in stocks and \( x_t \) in \( VIX^2 \) futures to maximize the mean-variance preferences with the risk aversion coefficient \( \gamma_D \), i.e.,

\[
\max_{\phi, x} E_t[W_{D,T}] - \frac{\gamma_D}{2} Var_t(W_{D,T}), \tag{4.26}
\]

with terminal wealth process \( W_{D,T} \) given by\(^{13}\)

\[
W_{D,T} = W_{D,t} + \phi_t S_t R_T + x_t \left( F_{T,t}^{VIX^2} - F_{t,T}^{VIX^2} \right), \tag{4.27}
\]

where \( W_{D,t} \) is their initial wealth and the stock return \( R_T = \ln \frac{S_T}{S_t} \).

Using the \( VIX^2 \) futures price formula (4.22), the terminal wealth processes of dealers can be rewritten as

\[
W_{D,T} = W_{D,t} + \phi_t S_t R_T + x_t \left[ v_T - v_t + (ax_t + b)(T - t) \right] \times 100^2. \tag{4.28}
\]

\(^{12}\)As Mixon and Onur (2015) mentions, leveraged funds always short negative-return VIX futures. Our empirical results also support this assumption.

\(^{13}\)In reality, \( W_{D,T} = W_{D,t} + \phi_t (S_T - S_t) + x_t \left( F_{T,t}^{VIX^2} - F_{t,T}^{VIX^2} \right) \). Here we use the continuously compounded return \( \ln \frac{S_T}{S_t} \) to approximate the simple return \( \frac{S_T - S_t}{S_t} \), then we get the wealth process (2.3). This approximation is due to the simplicity of calculating \( E_t[R_T], Var_t[R_T] \) and \( Cov_t[R_T, v_T] \). However, whether we use this approximation or not does not change the main results in the chapter.
Thus, their optimization problem becomes

$$\max_{\phi, x} \quad W_{D,t} + \phi_t S_t E_t[R_T] + x_t [(a x_t + b)(T - t)] \times 100^2$$

$$- \frac{\gamma D}{2} [\phi_t^2 S_t^2 Var_t[R_T] + x_t^2 Var_t[v_T] \times 100^4 + 2\phi_t x_t S_t Cov_t[R_T, v_T] \times 100^2].$$

The first-order conditions (FOCs) lead to

$$x_t = \frac{b(T - t) - \gamma \phi_t S_t Cov_t[R_T, v_T]}{\gamma Var_t[v_T] \times 100^2 - 2a(T - t)},$$

(4.29)

and

$$\phi_t = \frac{E_t[R_T] - x_t \gamma Cov_t[R_T, v_T] \times 100^2}{\gamma S_t Var_t[R_T]},$$

(4.30)

where $Cov_t[R_T, v_T], Var_t[v_T], E_t[R_T]$ and $Var_t[R_T]$ are shown in Appendix 4.6.1.

We rearrange Equation (4.29) and get

$$a x_t = \frac{\gamma Var_t[v_T] \times 100^2}{2(T - t)} x_t - \frac{b}{2(T - t)} + \frac{\gamma \phi_t S_t Cov_t[R_T, v_T]}{2(T - t)}.$$  \tag{4.31}

Plugging Equation (3.40) into (4.31),

$$0 = \left( a - \frac{\gamma (Var_t^2[R_T] - Cov_t^2[R_T, v_T]) \times 100^2}{2(T - t) Var_t[R_T]} \right) x_t + \frac{E_t[R_T] Cov_t[R_T, v_T] - b Var_t[R_T]}{2(T - t) Var_t[R_T]}.$$  \tag{4.32}

Under Assumption 4.2.9, $x_t$ is an exogenously given random variable. Thus, the coefficient of $x_t$ should be zero, and then we get

$$a = \frac{\gamma (Var_t^2[R_T] - Cov_t^2[R_T, v_T]) \times 100^2}{2(T - t) Var_t[R_T]} > 0,$$  \tag{4.33}

and

$$b = \frac{E_t[R_T] Cov_t[R_T, v_T]}{Var_t[R_T]} < 0.$$  \tag{4.34}
4.2.5 Equilibrium

Definition 4.2.1 (Volatility market equilibrium I) Equilibrium in our economy is defined in a standard way: The equilibrium VIX² futures order \( x_t \) of dealers is such that dealers maximize their mean-variance preferences, and the VIX² futures market is clear, i.e., \( x_t + y_t + z_t = 0 \).

In equilibrium, we summarize all trading activities in futures market as the following theorem.

Theorem 4.2.10 (Benchmark) Under Assumption 4.2.1-4.2.9, the equilibrium solutions are solved as

\[
\begin{align*}
\lambda_t &= ax_t + b, \\
a &= \frac{\gamma(\text{Var}^2[R_T] - \text{Cov}^2[R_T, v_T]) \times 100^2}{2(T-t)\text{Var}[R_T]} > 0, \\
b &= \frac{\text{E}[R_T] \text{Var}_{R_T, v_T}}{\text{Var}_{R_T}} < 0, \\
x_t &= -y_t - z_t, \\
y_t &< 0, z_t > 0,
\end{align*}
\]

(4.35)

where \( x_t, y_t \) and \( z_t \) are all exogenous.

By using Theorem 4.2.10, we get the following lemma.

Lemma 4.2.11 (Benchmark) In equilibrium, under Assumption 4.2.1-4.2.9,

(i) The larger short position of dealers leads to more negative \( \lambda_t \).

(ii) The larger short position of leverage funds, \( y_t \), leads to lower negative \( \lambda_t \).

(iii) The larger long position of asset managers, \( z_t \), leads to more negative \( \lambda_t \).

(iv) Higher risk aversion, \( \gamma \), more negative \( \lambda_t \), if \( x_t < 0 \), and vice versa.
Lemma 4.2.11 shows how the volatility trading activities influence the market price of the volatility risk $\lambda_t$. We plug $\lambda_t$ into Equation (4.21)-(4.24) and summarize as

$$
\begin{align*}
VRP_t &= ax_t + b = ay_t - az_t + b, \\
\frac{F_t^{VIX^2}}{100^2} &= - (ax_t + b) (T - t + 1) + vt = (ay_t + az_t - b) (T - t + 1) + vt, \\
\frac{Basis_t^{VIX^2}}{100^2} &= - (ax_t + b) (T - t) = (ay_t + az_t - b) (T - t), \\
\frac{R_t^{VIX^2}}{100^2} &= ax_t + b = \frac{-az_t + b}{F_t^{VIX^2}}.
\end{align*}
$$

(4.36)

By using Equation (4.36) and Lemma 4.2.3-4.2.11, we get the following proposition.

**Proposition 4.2.1 (Benchmark)** In equilibrium, under Assumption 4.2.1-4.2.9

(i) The larger short positions of dealers lead to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price (equivalently, higher $VIX^2$ futures basis).

(ii) The larger short positions of leverage funds lead to less negative VRP and $VIX^2$ futures return, and a lower $VIX^2$ futures price.

(iii) The larger long positions of asset managers leads to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price.

(iv) The higher risk aversion of dealers leads to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price if dealers are in a short position.

Mixon and Onur (2015) empirically test part of the results (i)-(iii) in Proposition 4.2.1; i.e., the futures price is negatively (positively) related to the level of positioning by dealers (asset managers and leveraged funds). Furthermore, dealers as market makers balance the futures positions. If buyers (i.e., asset managers) need more hedging demand than sellers (i.e., leveraged funds) supply, dealers will issue new futures for asset managers. In this case, the futures price will be higher and the return will be negatively lower. The higher risk aversion of dealers leads to more negative futures return and higher futures price only if dealers are in a short position.
The economic mechanism is very clear and intuitive. Asset managers as hedgers have a high demand for volatility derivatives, which raises the volatility derivatives prices. At the same time, due to their high demand, asset managers are willing to pay the high risk premium to hold volatility derivatives. On the sellers side, if leveraged funds short more volatility derivatives (i.e., higher supply), it leads to lower prices of volatility products. Facing the higher supply and lower prices, buyers, of course, are willing to pay a lower risk premium to hold these risky products. As the market makers, dealers positions indicate the balance of the supply and demand of volatility derivatives. If the demand is higher than the supply, dealers will issue new volatility contracts for buyers. The higher net short positions of dealers imply the higher demand for volatility derivatives, so that their prices will be higher. In the same scenario, dealers need to short more volatility contracts in order to cater to the needs of buyers. If dealers are more risk-averse, however, they will prefer to short less. Then the supply will be lower and consequently the price will be higher. Finally, higher prices always lead to a lower return.

If we treat the $VIX^2$ futures as variance swaps, from (i)-(iii) in Proposition 4.2.1, the model well explains the empirical results in Konstantinidi and Skiadopoulos (2016); i.e., volatility trading activities strongly predict the market VRP. In contrast to Konstantinidi and Skiadopoulos (2016), who claim that this is because for dealers holding a short position in index options based on Garleanu et al. (2009), the model suggests that the high negative VRP is driven by the larger short positions of dealers, the lower short positions of leverage funds and the larger long positions of asset managers in variance swaps.

To summarize, the volatility market is affected contemporaneously by three types of traders: asset managers, leveraged funds and dealers.
4.3 **Endogenous trading strategies**

In this section, we endogenize the trading strategies and find that the main conclusions are not changed.

4.3.1 **Case I: One-market equilibrium with endogenous trading strategies**

In contrast to Assumption 4.2.9, we give a new assumption as follows.

**Assumption 4.3.1** The trading strategies of dealers $x_t$ are endogenous.$^{14}$

As $x_t$ is endogenous and there is only one equilibrium in the volatility market, we cannot determine the parameters $a$ and $b$ in Assumption 4.2.8. Thus, there are two solutions to overcome this issue: (i) reducing parameters (e.g., $b$) and (ii) defining additional equilibrium in the stock market. We analyse the former in this subsection and discuss the latter in the next subsection. Then, in order to decrease the number of unknowns, we change Assumption 4.2.8 as follows.

**Assumption 4.3.2** We assume

$$\lambda_t = ax_t, \quad (4.37)$$

where $a$ will be solved in equilibrium.

In addition, we relax the assumption that the trading strategies of leveraged funds and asset managers are endogenous as well. We show the details of their trading behaviours as follows.

*Leveraged funds (speculators).* By taking advantage of the negative return of $VIX^2$ futures, leveraged funds (speculators) prefer to short $VIX^2$ futures and only speculate on the $VIX^2$ futures market. They choose the optimal order $y_t$ in $VIX^2$ futures

---

$^{14}$Actually, $x_t, y_t$ and $z_t$ are endogenous, while $\psi_t$ is exogenous in this case.
to maximize the mean-variance preferences with the risk-aversion coefficient $\gamma_L$, i.e.,

$$\max_y E_t[W_{L,T}] - \frac{\gamma_L}{2} Var_t(W_{L,T}),$$  \hspace{1cm} (4.38)$$

with terminal wealth process $W_{L,T}$ given by

$$W_{L,T} = W_{L,t} + y_t \left( F^{VIX^2}_{T,T} - F^{VIX^2}_{L,T} \right),$$  \hspace{1cm} (4.39)$$

where $W_{L,t}$ is their initial wealth. The terminal wealth processes of dealers can be rewritten as

$$W_{L,T} = W_{L,t} + y_t \left[ v_T - v_t + \lambda_t (T - t) \right] \times 100^2.$$  \hspace{1cm} (4.40)$$

Thus their optimization problem becomes

$$\max_y W_{L,t} + y_t \left[ \lambda_t (T - t) \right] \times 100^2 - \frac{\gamma_L}{2} y_t^2 Var_t[v_t] \times 100^4.$$  

The FOC leads to

$$y_t = \frac{\lambda_t (T - t)}{\gamma_L Var_t[v_T] \times 100^2} < 0 \text{ if } \lambda_t = ax_t < 0.$$  \hspace{1cm} (4.41)$$

The position of leveraged funds in the futures market depends on the sign of $\lambda_t$, which indicates the sign of the $VIX^2$ futures returns. Thus, the short positions in $VIX^2$ futures of leveraged funds are due to the negative return of futures (i.e., $\lambda_t < 0$).

Asset managers (hedgers). In order to hedge their long positions $\psi_t > 0$ in the stock market, asset managers (hedgers) prefer to long $VIX^2$ futures, due to the negative correlation between stock return and volatility derivatives. Thus, given $\psi_t > 0$
in stocks, they choose optimal $z_t$ in $VIX^2$ futures in order to maximize the mean-variance preferences with the risk aversion coefficient $\gamma_A$, i.e.,

$$\max_z E_t[W_{A,T}] - \frac{\gamma_A}{2} Var_t(W_{A,T}), \quad (4.42)$$

with terminal wealth process $W_{A,T}$ given by

$$W_{A,T} = W_{A,t} + \psi_t S_t R_T + z_t \left( F_{T,T}^{VIX^2} - F_{t,T}^{VIX^2} \right), \quad (4.43)$$

where $W_{A,t}$ is their initial wealth. Similarly, the terminal wealth processes of dealers can be rewritten as

$$W_{A,T} = W_{A,t} + \psi_t S_t R_T + z_t [v_T - v_t + \lambda_t (T - t)] \times 100^2. \quad (4.44)$$

Thus their optimization problem becomes

$$\max_z W_{A,t} + \psi_t S_t E_t[R_T] + z_t [\lambda_t (T - t)] \times 100^2 - \frac{\gamma_A}{2} \left[ \psi_t^2 S_t^2 Var_t[R_T] + z_t^2 Var_t[v_t] \times 100^4 + 2\psi_t z_t Cov_t[R_T, v_T] \times 100^2 \right].$$

The FOC leads to

$$z_t = \frac{\lambda_t (T - t) - \gamma_A \psi_t S_t Cov_t[R_T, v_T]}{\gamma_A Var_t[v_T] \times 100^2} = \frac{\lambda_t (T - t)}{\gamma_A Var_t[v_T] \times 100^2} + \frac{-\psi_t S_t Cov_t[R_T, v_T]}{Var_t[v_T] \times 100^2}. \quad (4.45)$$

The positions of asset managers in $VIX^2$ futures are contributed from two components. Due to the negative return of $VIX^2$ futures, they try to take short positions. However, as they hold a bunch of stocks, they have to long $VIX^2$ futures to hedge their long positions in stocks by using the negative correlation between the stock
return and the futures price (i.e., \( \text{Cov}_t[R_T, v_T] < 0 \)). Finally, their positions are determined by the size of the initial wealth in stocks. If \( \psi_t S_t \) is large, then \( z_t > 0 \), and vice versa. This explains why sometimes asset managers have net short positions in the volatility market. It is because they may reduce their position in the stock market and the return of volatility derivatives is deeply negative.

**Dealers (market makers).** Dealers (market makers) are the same here as in Section 4.2.4. Thus, their optional endogenous trading strategies are

\[
x_t = \frac{-\gamma_D \phi_t S_t \text{Cov}_t[R_T, v_T]}{\gamma_D \text{Var}_t[v_T] \times 100^2 - 2a(T - t)},
\]

(4.46)

and

\[
\phi_t = \frac{E_t[R_T] - x_t \gamma_D \text{Cov}_t[R_T, v_T] \times 100^2}{\gamma_D S_t \text{Var}_t[R_T]}.
\]

(4.47)

**Definition 4.3.1 (Volatility market equilibrium II)** Equilibrium in our economy is defined in a standard way: The equilibrium \( \text{VIX}^2 \) futures orders of dealers, \( x_t \), and the stock orders of dealers, \( \phi_t \), are such that they maximize their mean-variance preferences; the equilibrium \( \text{VIX}^2 \) futures orders of leveraged funds, \( y_t \), and the equilibrium \( \text{VIX}^2 \) futures orders of asset managers, \( z_t \), are such that they maximize their mean-variance preferences, and the \( \text{VIX}^2 \) futures market is clear, i.e., \( x_t + y_t + z_t = 0 \).

In this case, the competition among the three traders is close to in a von Stackelberg game with the dealers as the leader. Dealers propose an “a” in the market and then asset managers and leveraged funds submit their trading positions of the \( \text{VIX}^2 \) futures. Finally, dealers as market makers maximize their expected utilities with the market clearing constriction, \( x_t = -y_t - z_t \), so that an equilibrium “a” is determined.
Based on Definition 4.3.1, we summarize the equilibrium as the following theorem.\textsuperscript{15}

**Theorem 4.3.3 (Case I)** Under Assumption 4.2.1-4.2.7 and 4.3.1-4.3.2, the equilibrium solutions can be solved from the following system,

\[
\begin{align*}
\lambda_t &= ax_t, \\
x_t &= \frac{-\gamma_D \psi_t S_t Cov_t[R_T, v_T]}{\gamma_D Var_t[R_T]} \\
\phi_t S_t &= \frac{E_t[R_T] - x_t \gamma_D Cov_t[R_T, v_T]}{\gamma_D Var_t[R_T]} \\
y_t &= \frac{\lambda_t (T-t)}{\gamma_L Var_t[v_T] \times 100^2}, \\
z_t &= \frac{\lambda_t (T-t) - \gamma_A \psi_t S_t Cov_t[R_T, v_T]}{\gamma_A Var_t[v_T] \times 100^2}, \\
x_t + y_t + z_t &= 0.
\end{align*}
\]

(4.48)

The analytical solutions are provided in Appendix 4.6.2.

The equilibrium system (4.48) has six equations with six unknowns, \(a, \lambda_t, x_t, y_t, z_t, \phi_t S_t\). Given parameters \(v_t, T-t, \mu, \rho, \sigma_v, \psi_t S_t, \gamma_D, \gamma_L, \gamma_A\), we can easily solve them by most solvers. All solutions are provided in Appendix 4.6.2. For example, the key parameter \(a\) is solved as

\[
a = \frac{\gamma_A \gamma_L \left( \gamma_D \psi_t S_t Cov_t^2[R_T, v_T] - \gamma_D \psi_t S_t Var_t[R_T] Var_t[v_t] - E_t[R_T] Var_t[v_t] \right) \times 100^2}{(T-t) ((\gamma_L + \gamma_A) E_t[R_T] - 2 \gamma_A \gamma_L \psi_t S_t Var_t[R_T])}.
\]

(4.49)

We assume their risk aversion as \(\gamma_L < \gamma_D < \gamma_A\). This is because asset managers are the most risk-averse and want to hedge their risks as much as they can, while leveraged funds, as speculators, are risk takers, who prefer to sell \(VIX\) futures to gamble on more profits. The risk aversion of dealers lies somewhere between them.

\textsuperscript{15}Even though, in the equilibrium, there are three traders. Asset managers and leveraged funds submit their orders of volatility products and deals as market makers clear the volatility market. Our model is essential different to Kyle (1985)-type model which derives equilibrium security prices when traders have asymmetric information. All traders in our model have symmetric information, while they have different hedging needs of volatility products.
To simplify, we fix $\gamma_D = \gamma$, and then we set $\gamma_L = \gamma/\delta, \gamma_A = \gamma\delta$, where $\delta > 1$ can be a measure of risk-averse heterogeneity. Following Aït-Sahalia and Kimmel (2007), we set $\rho = -0.75, \sigma_v = 0.5$. In addition, we set $\mu = 0.06, \nu_t = 0.2^2, T - t = 1, \gamma = 2, \delta = 2$ and $\psi_t S_t = 1$. Then we have $a = 161.61 > 0, \lambda_t = -0.0094 < 0, x_t = -0.58 \times 10^{-4} < 0, y_t = -0.94 \times 10^{-4} < 0, z_t = 1.51 \times 10^{-4} > 0, \phi_t S_t = 0.29$.

We analyse the sensitivity of $a$ to other parameters, e.g., the risk-averse heterogeneity $\delta$, investment horizon $T - t$ and the initial wealth of asset managers in stocks $\psi_t S_t$ (equivalently, the demands of asset managers in $VIX^2$ futures). In Figure 4.1, we find the value of $a$ is always positive. Now we can conclude that, under reasonable parameter settings, the value of $a$ can be always positive.
Figure 4.1: The value of parameter $a$ in Case I.

This figure shows the value of $a$. The benchmark model parameters are as follows: $\rho = -0.75, \sigma_v = 0.5, \mu = 0.06, v_t = 0.2^2, T - t = 1, \gamma = 2, \delta = 2$ and $\psi_tS_t = 1$. 

Panel A: Sensitivity to risk-averse heterogeneity

Panel B: Sensitivity to investment horizon

Panel C: Sensitivity to the initial wealth of asset managers in stocks
In this case, the sign of \( \lambda \) is purely determined by the net positions of dealers. By using the positive \( a \) and the market clearing condition \( x_t = -y_t - z_t \), we can get the following lemma, which is similar to Lemma 4.2.11.

**Lemma 4.3.4 (Case I)** In equilibrium, under Assumption 4.2.1-4.2.7 and 4.3.1-4.3.2,

(i) The larger short position of dealers leads to more negative \( \lambda_t \).

(ii) The larger short position of leverage funds, \( y_t \), leads to lower negative \( \lambda_t \).

(iii) The larger long position of asset managers, \( z_t \), leads to more negative \( \lambda_t \).

Lemma 4.3.4 shows that under Assumption 4.2.1-4.2.7 and 4.3.1-4.3.2, the conclusions in Lemma 4.3.4 are not changed, compared with the results (i)-(iii) in Lemma 4.2.11. Similarly, we plug \( \lambda_t \) into Equation (4.21)-(4.24) and summarize as

\[
\begin{align*}
V R P_t &= a x_t = -a y_t - a z_t , \\
F_{V I X}^{T,t} & = -a x_t (T - t + 1) + v_t = (a y_t + a z_t) (T - t + 1) + v_t , \\
B_{basis}^{V I X} & = -a x_t (T - t) = (a y_t + a z_t) (T - t) , \\
R_{V I X}^{T,t} & = \frac{a x_t}{F_{V I X}^{T,t}} = -\frac{a y_t - a z_t}{F_{V I X}^{T,t}} .
\end{align*}
\]  
\tag{4.50}

By using Equation (4.50) and Lemma 4.2.3-4.2.6 and 4.3.4, we get the following proposition.

**Proposition 4.3.1 (Case I)** In equilibrium, under Assumption 4.2.1-4.2.7 and 4.3.1-4.3.2,

(i) The larger short positions of dealers lead to more negative VRP and \( V I X^2 \) futures return, and a higher \( V I X^2 \) futures price (equivalently, higher \( V I X^2 \) futures basis).

(ii) The larger short positions of leveraged funds lead to less negative VRP and \( V I X^2 \) futures return, and a lower \( V I X^2 \) futures price.

(iii) The larger long positions of asset managers leads to more negative VRP and \( V I X^2 \) futures return, and a higher \( V I X^2 \) futures price.
The results in Proposition 4.3.1 are same as the results (i)-(iii) in Proposition 4.2.1. Thus, the economic mechanism is the same as in Section 4.2.5. Here, we are interested in how risk aversion, risk-averse heterogeneity, investment horizon and the hedging demand of asset managers affect the VRP, the $VIX^2$ futures return and the level of the $VIX^2$ futures price.

**Figure 4.2: The value of $\lambda$ in Case I.**

This figure shows the value of $\lambda$. The benchmark model parameters are as follows: $\rho = -0.75$, $\sigma_v = 0.5$, $\mu = 0.06$, $v_t = 0.2^2$, $T - t = 1$, $\gamma = 2$, $\delta = 2$ and $\psi_t S_t = 1$.

Panel A in Figure 4.2 shows higher risk-averse heterogeneity leads to less negative $\lambda$, which means the less negative VRP and $VIX^2$ futures return, and lower $VIX^2$ futures price, while Panel B-D in Figure 4.2 shows that the higher risk aversion of dealers (or total social risk aversion), shorter horizon and larger hedging demand of asset managers generate more negative $\lambda$. We summarize these results.
in the following proportion.

**Proposition 4.3.2 (Case I)** In equilibrium, under Assumption 4.2.1-4.2.7 and 4.3.1-4.3.2,

(i) Higher risk-averse heterogeneity leads to a less negative VRP and \( VIX^2 \) futures return, and a lower \( VIX^2 \) futures price.

(ii) The larger risk aversion of dealers leads to more negative VRP and \( VIX^2 \) futures return, and a higher \( VIX^2 \) futures price (equivalently, higher \( VIX^2 \) futures basis).

(iii) The shorter investment horizon leads to more negative VRP and \( VIX^2 \) futures return, and a higher \( VIX^2 \) futures price.

(iv) The larger hedging demand of asset managers leads to more negative VRP and \( VIX^2 \) futures return, and a higher \( VIX^2 \) futures price.

### 4.3.2 Case II: Two-market equilibrium with endogenous trading strategies

In this case, we take Assumption 4.2.1-4.2.8 and 4.3.1 and then we have one more unknown, i.e., \( b \), so that the equilibrium in the stock market has to be defined, in order to determine the additional unknown \( b \).\(^\text{16}\) The behaviours of traders are same as in Case I, except for asset managers’ trading activities in the stock market. Thus, the optimal positions in \( VIX^2 \) futures of leveraged funds is same as (4.41), and the optimal portfolios of dealers in \( VIX^2 \) futures and stocks are same as (4.29) and (3.40). Asset managers choose both the optimal order \( \psi_t \) in stocks and \( z_t \) in \( VIX^2 \) futures to maximize the mean-variance preferences, i.e.,

\[
\max_{\psi, z} E_t[W_{A,T}] - \frac{\gamma_A^2}{2} Var_t(W_{A,T}). \tag{4.51}
\]

\(^{16}\)Actually, \( x_t, y_t, z_t \) and \( \psi_t \) are all endogenous in this case.
Here, in the volatility market, asset managers are hedgers and dealers are the market makers, while, in the stock market, asset managers are the market makers (clearing the stock market) and dealers are hedgers (hedging their long or short position in $VIX^2$ futures). Leveraged funds only speculate in the volatility market.

Similarly, we can solve the optimal order $\psi_t$ in stocks and $z_t$ in $VIX^2$ futures as

$$z_t = \frac{\lambda_t(T-t) - \gamma_A \psi_t S_tCov_t[R_T, v_T]}{\gamma_A Var_t[v_T] \times 100^2},$$

(4.52)

and

$$\psi_t = \frac{E_t[R_T] - z_t \gamma_A Cov_t[R_T, v_T] \times 100^2}{\gamma_A S_t Var_t[R_T]}.$$ (4.53)

The definition of the equilibrium in volatility and stock markets is given as follows.

**Definition 4.3.2 (Two-market equilibrium)** Equilibrium in our economy is defined in a standard way: Equilibrium $VIX^2$ futures orders $x_t, y_t, z_t$ and stock orders $\phi_t$ and $\psi_t$ maximize all traders’ mean-variance preferences, and $VIX^2$ futures and stock markets clear, i.e., $x_t + y_t + z_t = 0$ and $\phi_t + \psi_t = Z$, where $Z$ is the total amount of stocks.\(^{17}\)

Based on Definition 4.3.2, we summarize the equilibrium as the following system.

\(^{17}\)In a more general case, $Z$ can be regarded as the remainder of the total supply traded by purely-stock-trading traders.
Theorem 4.3.5 (Case II) Under Assumption 4.2.1-4.2.8 and 4.3.1, the equilibrium solutions can be solved from the following system,

\[
\begin{align*}
\lambda_t &= ax_t + b, \\
x_t &= \frac{b(T-t) - \gamma_D \phi_t S_t \text{Cov}(R_T,v_T)}{\gamma_D \text{Var}(v_T) \times 100^2}, \\
\phi_t S_t &= \frac{E_t[R_T] - x_t \gamma_D \text{Cov}(R_T,v_T) \times 100^2}{\gamma_D \text{Var}(R_T)}; \\
y_t &= \frac{\lambda_t(T-t)}{\gamma_L \text{Var}(v_T) \times 100^2}, \\
z_t &= \frac{\lambda_t(T-t) - \gamma_A \psi_t S_t \text{Cov}(R_T,v_T)}{\gamma_A \text{Var}(v_T) \times 100^2}, \\
\psi_t S_t &= \frac{E_t[R_T] - z_t \gamma_A \text{Cov}(R_T,v_T) \times 100^2}{\gamma_A \text{Var}(R_T)}, \\
x_t + y_t + z_t &= 0, \\
\phi_t S_t + \psi_t S_t &= ZS_t.
\end{align*}
\]

The analytical solutions are provided in Appendix 4.6.2.

The equilibrium system (3.10) has eight equations with six unknowns \(a, b, \lambda_t, x_t, y_t, z_t, \phi_t S_t\) and \(\psi_t S_t\). Given parameters \(v_t, T - t, \mu, \rho, \sigma_v, ZS_t, \gamma_D, \gamma_L\) and \(\gamma_A\), we can easily solve them (see Appendix 4.6.2). Similarly, we set \(\rho = -0.75, \sigma_v = 0.5, \mu = 0.06, v_t = 0.2^2, T - t = 1, \gamma = 2, \delta = 2\) and \(ZS_t = 1\). Then we have \(a = 21.53 > 0, b = -0.012 < 0, \lambda_t = -0.010 < 0, x_t = 0.78 \times 10^{-4} > 0, y_t = -1.05 \times 10^{-4} < 0, z_t = 0.027 \times 10^{-4} > 0, \phi_t S_t = 0.69\) and \(\psi_t S_t = 0.31\). We find that, in this model, the positive \(x_t = 1.12 \times 10^{-4} > 0\) can produce the negative \(\lambda_t = -0.012 < 0\). This is consistent with the position data in our chapter (i.e., Table 4.1) and Mixon and Onur (2015).

We analyse the sensitivities of \(a\) and \(b\) against other parameters in Figure 4.3, which shows that the value of \(a\) is always positive and the value of \(b\) is always negative. So, under reasonable parameter settings, the value of \(a\) can be always positive, while \(b\) can be always negative. The signs of them are same as the signs in the exogenous case (i.e., Equation (4.33) and (4.34)).
Figure 4.3: The value of parameter $a$ in Case II.

This figure shows the value of $a$. The benchmark model parameters are as follows: $\rho = -0.75, \sigma_v = 0.5, \mu = 0.06, v_t = 0.2^2, T - t = 1, \gamma = 2, \delta = 2$ and $ZS_t = 1$.

In this model, the sign of $\lambda$ is determined by not only the positions of dealers in $VIX^2$ futures, but also the value of $b$, which measures the impacts of the position in stocks. This explains why the positive $x_t$ can produce the negative $\lambda_t$. 
Figure 4.4: The value of $\lambda$ in Case II.

This figure shows the value of $\lambda$. The benchmark model parameters are as follows: $\rho = -0.75$, $\sigma_v = 0.5$, $\mu = 0.06$, $v_t = 0.2^2$, $T - t = 1$, $\gamma = 2$, $\delta = 2$ and $ZS_t = 1$.

The effects in Figure 4.4 are similar to Figure 4.3, except the investment horizon. In Panel C, Figure 4.4, a longer horizon leads to more negative $\lambda_t$. We get the following proposition, which is the same as Proposition 4.3.2.

**Proposition 4.3.3 (Case II)** In equilibrium, under Assumption 4.2.1-4.2.8 and 4.3.1,

(i) The higher risk-averse heterogeneity most likely leads to less negative VRP and $VIX^2$ futures return, and a lower $VIX^2$ futures price.

(ii) The larger the risk aversion of dealers leads to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price (equivalently, higher $VIX^2$ futures basis).

(iii) The longer investment horizon leads to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price.
(iv) The larger hedging demand of asset managers leads to more negative VRP and $VIX^2$ futures return, and a higher $VIX^2$ futures price.

4.4 Empirical analysis

In this section, first, Konstantinidi and Skiadopoulos (2016) test the trading activity model by using the trading volume of all S&P 500 futures contracts and the TED spread (which measures traders funding liquidity), while we extend the model with the net positions of dealers, asset managers and leveraged funds, which reveals the relation between the VRP and the net positions of three main traders. Second, inspired by Eraker and Wu (2017), we newly investigate the impact of the net positions on the VIX futures return. Finally, we empirically study the results in Mixon and Onur (2015) by using obtainable data.\footnote{The daily trading data used in Mixon and Onur (2015) is not available to the public. Thus, we have to use weekly data to test our model. Our empirical results are consistent with Mixon and Onur (2015).}

Based on Proposition 4.2.1, we propose the following six hypotheses:

(i-a) VRP is positively related to the level of positioning by dealers.

(i-b) VRP is negatively related to the level of positioning by asset managers and leveraged funds.

(ii-a) VIX futures return is positively related to the level of positioning by dealers.

(ii-b) VIX futures return is negatively related to the level of positioning by asset managers and leveraged funds.

(iii-a) VIX futures basis is negatively related to the level of positioning by dealers.

(iii-b) VIX futures basis is positively related to the level of positioning by asset managers and leveraged funds.
Hypothesis (i-a) and (i-b) are designed to explicitly explain the observation in Constantinidi and Skiadopoulos (2016); hypothesis (ii-a) and (ii-b) are extended from Eraker and Wu (2017); and hypothesis (i-a) and (i-b) correspond to the empirical findings in Mixon and Onur (2015).

4.4.1 Data

CFTC began to publish weekly TFF reports on 4 September 2009 to add further transparency to the financial futures markets, together with the disaggregated data in the CFTC’s weekly COT reports. Supporting the legacy COT reports, the TFF reports provide a breakdown of each Tuesday’s open interest for markets in which 20 or more traders hold positions equal to or above the reporting levels established by the CFTC and separates large traders in the financial markets into the following four categories: dealers, asset managers, leveraged funds and other reportables. We download TFF Futures Only Reports weekly data from the CFTC website. The available time period is from 13 June 2006 to 25 Oct 2016.

The TFF reports disclose the long and short open interest for four categories. We are interested in their net positions in VIX futures. So, we convert the long and short open interest variables into net positions, which are defined as the long open interest minus the short open interest. Here the net positions are the total aggregated open interest in VIX futures across different maturities for each type of trader. The statistics are given in Table 4.1.

---

Table 4.1: Summary statistics on net positions.

We report the summary statistics of net positions of the different types of traders. $NP_i$ where $i = d, am, lf, or$ represents the net positions of dealers, asset managers, leveraged funds and other reportables. The time period is from 13 June 2006 to 25 October 2016.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NP_d$</td>
<td>26,031.73</td>
<td>46,864.47</td>
<td>-58,735</td>
<td>144,190</td>
</tr>
<tr>
<td>$NP_{am}$</td>
<td>16,056.04</td>
<td>19,204.95</td>
<td>-13,396</td>
<td>90,706</td>
</tr>
<tr>
<td>$NP_{lf}$</td>
<td>-40,255.67</td>
<td>755,660.03</td>
<td>-195,486</td>
<td>63,753</td>
</tr>
<tr>
<td>$NP_{or}$</td>
<td>-1,235.41</td>
<td>7,186.36</td>
<td>-38,950</td>
<td>33,621</td>
</tr>
</tbody>
</table>

Table 4.1 shows that the sum of the average net positions of asset manager and leveraged funds is $24,199.63$, which is close to the average net position of dealers, 26,031.73. The average net position of other reportables is very small, around 3% of leveraged funds. Thus, contributions from other reportables can be omitted. In addition, Table 4.1 documents that we have to consider at least three main traders in the equilibrium instead of using the two-trader equilibrium model in Garleanu et al. (2009); Dong (2016). Our equilibrium model is more realistic than others.
Figure 4.5: Net positioning in VIX futures by asset managers, leveraged funds, and dealers.

The figure displays the net positions, aggregated within each of the dealers, asset managers and leveraged funds. \( NP_i \) where \( i = d, am, lf, or \) represents the net positions of dealers, asset managers, leveraged funds and other reportables. The time period is from 13 June 2006 to 25 October 2016.

Furthermore, from Figure 4.5, the trading in VIX futures is not very active before 2012. In order to make an easy comparison with the empirical results in Mixon and Onur (2015), we consider the last four years trading data, i.e., from 23 October 2012 to 25 October 2016.

We download VIX, S&P 500 index and VIX futures daily data from Bloomberg. As the TFF reports provide only the total open interest of VIX futures markets held by the three main traders across different maturity contracts, we examine only the
futures basis for the first and second nearest futures contracts, which are the most active. Table 4.2 also supports our treatment. The average open interest of the first and second nearest VIX futures is two times more than the rest of contracts.

**Table 4.2: Open interest of VIX futures across different maturities.**

We report the summary statistics of open interest of VIX futures across different maturities (first six maturities). The time period is from 23 October 2012 to 25 October 2016.

<table>
<thead>
<tr>
<th>Contract Expiry</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>142,654.0</td>
<td>46,600.07</td>
<td>44,360</td>
<td>295,871</td>
</tr>
<tr>
<td>2</td>
<td>111,317.4</td>
<td>45,966.74</td>
<td>32,357</td>
<td>295,930</td>
</tr>
<tr>
<td>3</td>
<td>43,304.3</td>
<td>12,214.04</td>
<td>16,890</td>
<td>95,476</td>
</tr>
<tr>
<td>4</td>
<td>31,663.4</td>
<td>7,533.20</td>
<td>14,389</td>
<td>56,336</td>
</tr>
<tr>
<td>5</td>
<td>25,618.2</td>
<td>7,035.65</td>
<td>9,940</td>
<td>47,991</td>
</tr>
<tr>
<td>6</td>
<td>18,398.0</td>
<td>5,298.13</td>
<td>7,609</td>
<td>35,450</td>
</tr>
</tbody>
</table>

Following Mixon and Onur (2015), we calculate the daily futures basis as

\[
Basis_i^t = Price\_future_i^t - VIX_t
\]  

(4.55)

where \( Price\_future_i^t \) is VIX futures last price at date \( t \) for the first nearest \( (i = 1) \) and the second nearest \( (i = 2) \).\(^{21}\) In addition, according to Bollerslev et al. (2009) and González-Urteaga and Rubio (2016), the daily VRP is defined as

\[
VRP_i^{t \rightarrow t+21} = RV_i^{t \rightarrow t+21} - VIX_t^2
\]  

(4.56)

where \( RV_i^{t \rightarrow t+21} \) is calculated as the variance of the daily percentage returns of the S&P 500 over 21-day windows at day \( t \); \( VIX_t^2 \) is the daily squared VIX index divided

\[^{21}\text{We only consider the last price because the settlement price is the same as the last price in the VIX futures data downloaded from Bloomberg.}\]
by 12 as one-month horizon at time $t$.

Now we merge the daily $Basis_i^t$ and $VRP_i^t$ with weekly net position data and then we calculate the weekly returns of VIX futures $i$,

$$Ret_{\text{future}}_i = \ln(Price_{\text{future}}_i^t) - \ln(Price_{\text{future}}_{i-1}^t).$$  \hspace{1cm} (4.57)

Finally, we provide summary statistics for all of the variables in Table 4.3.

**Table 4.3: Summary statistics on all variables.**

We report the summary statistics of variables in weekly frequency. The time period is from 23 October 2012 to 25 October 2016.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis$^1$</td>
<td>0.58</td>
<td>1.13</td>
<td>-10.70</td>
<td>2.66</td>
</tr>
<tr>
<td>Basis$^2$</td>
<td>1.46</td>
<td>1.83</td>
<td>-13.47</td>
<td>4.50</td>
</tr>
<tr>
<td>VIX</td>
<td>15.43</td>
<td>3.54</td>
<td>10.99</td>
<td>36.02</td>
</tr>
<tr>
<td>$VRP_{i\rightarrow t+21}$</td>
<td>-6.64</td>
<td>13.33</td>
<td>-56.54</td>
<td>64.91</td>
</tr>
<tr>
<td>$NP_{lf}$</td>
<td>-74,160.72</td>
<td>61,307.91</td>
<td>-195,486</td>
<td>63,753</td>
</tr>
<tr>
<td>$NP_{am}$</td>
<td>29,727.25</td>
<td>20,625.31</td>
<td>-13,396</td>
<td>90,706</td>
</tr>
<tr>
<td>$NP_d$</td>
<td>48,697.36</td>
<td>52,010.26</td>
<td>-58,735</td>
<td>144,190</td>
</tr>
<tr>
<td>$Ret_{\text{future}}^1$</td>
<td>-0.0011152</td>
<td>0.11429</td>
<td>-0.37194</td>
<td>0.57328</td>
</tr>
<tr>
<td>$Ret_{\text{future}}^2$</td>
<td>-0.0008592</td>
<td>0.07553</td>
<td>-0.18382</td>
<td>0.39609</td>
</tr>
</tbody>
</table>

In Table 4.3, the average future basis is positive and the means of the VRP and the futures return are negative, which implies that the market price of the volatility risk $\lambda$ in our model should be negative in the real world. The mean of leveraged funds’ net positions is negative and that of asset managers’ net positions is positive. This is consistent with our model assumptions in Section 4.2.4. Using the weekly data, we are able to analyse the impact of trading on futures basis, the VRP and the
VIX futures return.

### 4.4.2 Empirical results: the impact of trading on VRP

Konstantinidi and Skiadopoulos (2016) compare the predictive ability of four models and conclude that the trading activity model is the best to predict the VRP. They claim that the greater VRP is due to dealers’ greater short positions in index options. However, their trading activity variables are the trading volume of all S&P 500 futures contracts and the TED spread. They do not explicitly test the impacts of the net positions of dealers, asset managers and leveraged funds on the VRP. To fill this gap, we further develop their trading activity model into a VIX futures trading model by using the net positions of the three main traders. Then we are able to investigate the impacts of volatility trading activities on the VRP. Corresponding to Konstantinidi and Skiadopoulos (2016), we run the following regressions:\(^22\)

\[
VRP_t^{t \rightarrow t+21} = \alpha + \beta_1 NP_d t + \varepsilon_t, \quad \text{Hypothesis (i-a)}
\]  

Hypothesis (4.58)

\[
VRP_t^{t \rightarrow t+21} = \alpha + \beta_1 NP_{am} t + \beta_2 NP_{lf} t + \varepsilon_t, \quad \text{Hypothesis (i-b)}
\]  

Hypothesis (4.59)

In contrast to the impact of trading on futures basis, based on (4.36), the sign of the coefficient \(\beta\) signifies whether the VRP is positively (negatively) related to the level of positioning by dealers (asset managers and leveraged funds). The results are given in Table 4.4.

\(^{22}\)The better way to examine the impact of volatility trading on VRP is to use S&P 500 variance swaps trading data for the three major traders and to calculate the VRP based on variance swaps rates. However, the data are unobtainable.
Table 4.4: Results from regressing VRP on dealer and non-dealer VIX futures positions.

The table displays estimation results for the regressions in Equation (4.58) shown in Panel A and in Equation (4.59) shown in Panel B. T-statistics are based on Newey-West (1987) standard errors with 3 Newey-West lags. The regressions are estimated on weekly data spanning the period 23 October 2012 to 25 October 2016. ***, ** and * represent statistical significance at the 1, 5 and 10% level.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Cons.</th>
<th>Adj. $R^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NP_{dl}$ ($\times 10^{-5}$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.92**</td>
<td>-9.00***</td>
<td>3.24</td>
</tr>
<tr>
<td>(2.04)</td>
<td>(-4.83)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Cons.</th>
<th>Adj. $R^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NP_{am}$ ($\times 10^{-5}$)</td>
<td>$NP_{lf}$ ($\times 10^{-5}$)</td>
<td></td>
</tr>
<tr>
<td>-8.86</td>
<td>-5.19*</td>
<td>-7.84***</td>
</tr>
<tr>
<td>(-1.07)</td>
<td>(-1.83)</td>
<td>(-2.89)</td>
</tr>
</tbody>
</table>

Panel A of Table 4.4 shows that the coefficient is significantly positive. In other words, the VRP is positively related to the level of positioning by dealers. The larger the short position of dealers, the more negative the VRP. Panel B shows that both the coefficients are negative, where the coefficient of leveraged funds’ net positions is significantly negative. Similarly, we can empirically conclude that the VRP is negatively related to the level of positioning by asset managers and leveraged funds. The larger long positions of asset managers in VIX futures lead to more negative VRP, while the larger short positions of leveraged funds lead to the more negative VRP. All results correspond to Proposition 4.2.1 in this chapter.

Our demand-based equilibrium model provides a channel to explain the high negative VRP. Based on our empirical and theoretical results, the high negative VRP is driven by the large hedging demand of asset managers in VIX futures to hedge the underlyings they hold. It is very intuitive that the buyers of volatility derivatives
are willing to pay some risk premium (i.e., $-VRP$) to protect their long positions in underlyings. On the other hand, if leveraged funds increase (decrease) their short position, the supply of volatility derivatives will increase (decrease), so that volatility derivatives buyers are willing to pay less (more) to buy these derivatives. This is why the VRP decreases with a short position of leveraged funds.

The large negative mean of the VRP is mainly captured by the constant term $\alpha$ in Regression (4.58) and (4.59), which essentially is the solution $b$ in our benchmark model. Table 4.4 shows that the constants are significantly negative with a large magnitude. Based on Equation (4.34), we know that the negative sign of $b$ comes from the negative correlation between the stock return and its volatility. Therefore, we observe that the large negative VRP is caused by the volatility trading activities and the negative correlation between the stock return and its volatility. Actually, due to the negative correlation, volatility derivatives provide a channel for investors to hedge the position in the stock market or speculate the position in the volatility market.

4.4.3 Empirical results: the impact of trading on VIX futures return

In order to investigate the impact of trading on the VIX futures return, we run the following regressions,

$$\text{Ret}_{\text{future}}_i = \alpha_i + \beta_{1i} NP_{d} + \varepsilon_t, \quad \text{Hypothesis (ii-a)}$$  \hspace{1cm} (4.60)

and

$$\text{Ret}_{\text{future}}_i = \alpha_i + \beta_{1i} NP_{am} + \beta_{2i} NP_{lf} + \varepsilon_t, \quad \text{Hypothesis (ii-b)}.$$  \hspace{1cm} (4.61)

Consistent with Eraker and Wu (2017), the negative VIX futures return is contemporaneously related to the negative VRP. The results in Table 4.5 are similar to
the results in Table 4.4. In other words, the VIX futures return and the VRP are positively (negatively) related to the level of positioning by dealers (asset managers and leveraged funds); see Proposition 4.2.1-4.3.1. The larger the short position of dealers, the more negative the return of VIX futures (see Panel A of Table 4.5). In addition, the larger long positions of asset managers in VIX futures lead to more negative return, while the large short positions of leveraged funds lead to the more negative return (see Panel B of Table 4.5).

Table 4.5: Results from regressing VIX futures return on VIX and dealer and non-dealer VIX futures positions.

The table displays estimation results for the regressions in Equation (4.60) shown in Panel A and in Equation (4.61) shown in Panel B. Regressions are estimated separately for each contract (i.e., 1 and 2). T-statistics are based on Newey-West (1987) standard errors with 3 Newey-West lags. The regressions are estimated on weekly data spanning the period 23 October 2012 to 25 October 2016. ***, ** and * represent statistical significance at the 1, 5 and 10% level.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Contract Expiry</th>
<th>( NP_{dl} \times 10^{-7} )</th>
<th>Cons. ((\times 10^{-2}))</th>
<th>Adj. ( R^2 ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.50**</td>
<td>1.32</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.32)</td>
<td>(-1.43)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.06</td>
<td>0.60</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.34)</td>
<td>(-0.91)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Contract Expiry</th>
<th>( NP_{am} \times 10^{-7} )</th>
<th>( NP_{lf} \times 10^{-7} )</th>
<th>Cons. ((\times 10^{-3}))</th>
<th>Adj. ( R^2 ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-7.57*</td>
<td>-3.54***</td>
<td>4.75</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.88)</td>
<td>(-2.80)</td>
<td>(-0.39)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-3.51</td>
<td>-1.55*</td>
<td>1.87</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.36)</td>
<td>(-1.75)</td>
<td>(-0.22)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Our demand-based equilibrium model very intuitively explains the mechanism.
As hedgers, the high demand of asset managers in volatility derivatives acts to increase the prices of volatility derivatives and make the returns more negative. On the other hand, the leveraged funds, as speculators, selling (shorting) more volatility derivatives brings a high supply, so that the prices of volatility derivatives decrease and their returns become less negative. The balance of the supply and demand determines the prices of volatility derivatives and their returns.

4.4.4 Empirical results: the impact of trading on futures basis

In order to test whether the VIX futures (i.e., VIX futures basis) varies according to the level of different types of traders’ net positions in VIX futures, following Mixon and Onur (2015), we run the following regressions:

\[
Basis_i = \alpha_i + \eta_i VIX + \beta_{i1} NP_d + \varepsilon_i, \quad \text{Hypothesis (iii-a)} \quad (4.62)
\]

and

\[
Basis_i = \alpha_i + \eta_i VIX + \beta_{i1} NP_{am} + \beta_{i2} NP_{lf} + \varepsilon_i, \quad \text{Hypothesis (iii-b)} \quad (4.63)
\]

The signs of the coefficient \(\beta_i\) signifies whether the futures price (equivalently, the futures price basis) is negatively (positively) related to the level of positioning by dealers (asset managers and leveraged funds). The results are given in Table 4.6.
Table 4.6: Results from regressing VIX futures basis on VIX and dealer and non-dealer VIX futures positions.

The table displays estimation results for the regressions in Equation (4.62) shown in Panel A and in Equation (4.63) shown in Panel B. Regressions are estimated separately for each contract \((i = 1\) and \(2\)). T-statistics are based on Newey-West (1987) standard errors with 3 Newey-West lags. The regressions are estimated on weekly data spanning the period 23 October 2012 to 25 October 2016. \(*\), \(**\) and \(*\) represent statistical significance at the 1, 5 and 10% level.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Contract Expiry</th>
<th>VIX</th>
<th>(NP_d (\times 10^{-6}))</th>
<th>Cons.</th>
<th>Adj. (R^2) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.24(<em>)</em>**</td>
<td>-5.50(<em>)</em>**</td>
<td>4.61(<em>)</em>**</td>
<td>47.52</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-4.30)</td>
<td>(-2.82)</td>
<td>(5.04)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.44(<em>)</em>**</td>
<td>-1.66</td>
<td>8.34(<em>)</em>**</td>
<td>69.28</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-7.36)</td>
<td>(-0.61)</td>
<td>(8.53)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Contract Expiry</th>
<th>VIX</th>
<th>(NP_{am} (\times 10^{-5}))</th>
<th>(NP_{lf} (\times 10^{-6}))</th>
<th>Cons.</th>
<th>Adj. (R^2) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.23(<em>)</em>**</td>
<td>1.91(<em>)</em>**</td>
<td>7.56(<em>)</em>**</td>
<td>4.15(<em>)</em>**</td>
<td>50.97</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-4.09)</td>
<td>(4.28)</td>
<td>(3.89)</td>
<td>(4.43)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.42(<em>)</em>**</td>
<td>3.49(<em>)</em>**</td>
<td>6.54(<em>)</em>**</td>
<td>7.35(<em>)</em>**</td>
<td>77.45</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-7.20)</td>
<td>(6.39)</td>
<td>(2.73)</td>
<td>(7.55)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results in both Panel A and Panel B of Table 4.6 are consistent with Mixon and Onur (2015), even though we use weekly data. Panel A shows that the coefficient is significantly negative for the first nearest futures contracts. Even though the coefficient of the second nearest contracts is not significant, the sign is negative. Thus, we can see the futures basis is negatively related to the level of positioning by dealers. In Panel B, the coefficients of asset managers and leveraged funds net positions for the first and second nearest contracts are all significantly positive. This means that the futures basis is positively related to the level of positioning by asset...
managers and leveraged funds. As on average, the asset managers net positions are positive and leveraged funds net positions are negative, the larger long positions of asset managers act to bring about higher VIX futures prices, while the larger short positions of leveraged funds act to lower VIX futures prices. Our empirical results are consistent with Proposition 4.2.1-4.3.1 and Mixon and Onur (2015).

4.5 Conclusion

We provide a very neat demand-based equilibrium model of volatility trading, which reveals an intuitive economic mechanism of how asset managers, leveraged funds and dealers’ volatility trading activities affect the volatility market. Our model complements Eraker and Wu’s (2017) consumption-based equilibrium model. After solving the equilibrium, we get several theoretical results which are consistent with the empirical tests in Mixon and Onur (2015) and the observation in Konstantinidi and Skiadopoulos (2016). Our empirical tests significantly support the theoretical results implied by our equilibrium model. As this is the first paper to model the volatility trading flows, our model can be easily extended into more complicated settings.

4.6 Appendix

4.6.1 Proof

Denoting \( R_T = \ln \frac{S_T}{S_t} = \int_t^T (\mu - \frac{1}{2} \sigma_u) \, du + \int_t^T \sigma_u dB_{S,u} \), we have

\[
E_t[R_T] = \mu(T - t) - \frac{1}{2} \int_t^T E_t[\sigma_u] du. \tag{4.64}
\]
As \( v_t \) is a martingale (see Equation (4.7)), i.e., \( E_t[v_u] = v_t \) for \( u > t \), Equation (4.64) therefore can be simplified as

\[
E_t[R_T] = \mu(T - t) - \frac{1}{2} v_t(T - t). \tag{4.65}
\]

By using \( v_u = v_t + \int_t^u \sigma_v \sqrt{v_s} dB_{V,s} \) for \( u > t \), we have

\[
Var_t[R_T] = E_t[R_T - E_t[R_T]]^2 =
E_t \left[ \int_t^T \left( \mu - \frac{1}{2} v_u \right) du + \int_t^T \sqrt{v_u} dB_{S,u} - \int_t^T \left( \mu - \frac{1}{2} E_t[v_u] \right) du \right]^2
= E_t \left[ \int_t^T \sqrt{v_u} dB_{S,u} - \frac{1}{2} \int_t^T (v_u - E_t[v_u]) du \right]^2
= E_t \left[ \int_t^T \sqrt{v_u} dB_{S,u} - \frac{1}{2} \int_t^T \int_t^u \sigma_v \sqrt{v_s} dB_{V,s} du \right]^2
= E_t \left[ \int_t^T \sqrt{v_u} dB_{S,u} - \frac{1}{2} \int_t^T \int_t^T \sigma_v \sqrt{v_u} dB_{V,u} du \right]^2
= E_t \left[ \int_t^T \sqrt{v_u} dB_{S,u} - \frac{1}{2} \int_t^T \sigma_v \sqrt{v_u} (T - u) dB_{V,u} \right]^2
= E_t \left[ \int_t^T v_u du \right] - E_t \left[ \int_t^T \sqrt{v_u} dB_{S,u} \left( \int_t^T \sigma_v \sqrt{v_u} (T - u) dB_{V,u} \right) \right]
+ \frac{1}{4} E_t \left[ \left( \int_t^T \sigma_v \sqrt{v_u} (T - u) dB_{V,u} \right)^2 \right]
= v_t(T - t) - \rho \sigma_v v_t \int_t^T (T - u) du + \frac{1}{4} \sigma_v^2 v_t \int_t^T (T - u)^2 du
= v_t(T - t) - \frac{1}{2} \rho \sigma_v v_t (T - t)^2 + \frac{1}{12} \sigma_v^2 v_t (T - t)^3.
\]

In addition,

\[
E_t[v_T] = v_t, \tag{4.66}
\]

and

\[
Var_t[v_T] = E_t \left[ \sigma_v^2 \int_t^T v_u du \right] = \sigma_v^2 v_t (T - t). \tag{4.67}
\]
Furthermore, we calculate the conditional covariance between $R_T$ and $v_T$ as

\[
\text{Cov}_t[R_T, v_T] = E_t[(R_T - E_t[R_T]) (v_T - E_t[v_T])]
\]

\[
= E_t \left[ \left( \int_t^T \sqrt{v_u} dB_{S,u} - \frac{1}{2} \int_t^T \sigma_v \sqrt{v_u} \sqrt{T - u} dB_{V,u} \right) \left( \int_t^T \sigma_v \sqrt{v_u} dB_{V,u} \right) \right]
\]

\[
= \rho \sigma_v E_t \left[ \int_t^T v_u du \right] - \frac{1}{2} \sigma_v^2 E_t \left[ \int_t^T v_u (T - u) du \right]
\]

\[
= \rho \sigma_v v_t (T - t) - \frac{1}{4} \sigma_v^2 v_t (T - t)^2.
\]

By using the conditions, $-1 < \rho < 0$ and $0 < \sigma_v < 1$, under mild conditions (e.g., $T - t$ is relatively small), we have

\[
|\text{Cov}_t[R_T, v_T]| < v_t (T - t) < \text{Var}_t[R_T]. \tag{4.68}
\]
4.6.2 The solution of the equilibrium

Given the equilibrium system (4.48), we solve the equilibrium as

\[ a = \gamma_A \gamma_L \left( \text{Cov}^2[R_T, v_T](\psi_S \gamma_D - \text{Var}[R_T] \text{Var}_t[\psi_S \gamma_D - E_t[R_T] \text{Var}_t[\psi_T]]) \times 100^2 \right) \frac{(T-1)}{(-2 \text{Var}_t[R_T]\psi_S \gamma_A \gamma_L + E_t[R_T](\gamma_A + \gamma_L))}, \]

\[ \lambda_t = \gamma_A \gamma_L \left( \text{Cov}^2[R_T, v_T] \psi_S \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_S \gamma_D - E_t[R_T] \text{Var}_t[\psi_T]] \right) \text{Cov}_t[R_T, v_T] \frac{(T-1)}{(-2 \text{Var}_t[R_T]\psi_S \gamma_A \gamma_L + E_t[R_T](\gamma_A + \gamma_L))}, \]

\[ \phi_S = \frac{2 \text{Cov}^2[R_T, v_T] \psi_S \gamma_A \gamma_D \gamma_L - E_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2E_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - E_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L}{\text{Cov}_t[R_T, v_T] \gamma_A \gamma_D + \text{Cov}^2[R_T, v_T] \gamma_D \gamma_L - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L}. \]

\[ x_t = \frac{1}{100^2 \text{Cov}^2[R_T, v_T] \gamma_A \gamma_D + \text{Cov}^2[R_T, v_T] \gamma_D \gamma_L - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L}, \]

\[ y_t = \frac{\text{Cov}_t[R_T, v_T] \gamma_A \gamma_D + \text{Cov}^2[R_T, v_T] \gamma_D \gamma_L - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L}}{100^2 \text{Var}_t[\psi_T] (\text{Cov}^2[R_T, v_T] \gamma_A \gamma_D + \text{Cov}^2[R_T, v_T] \gamma_D \gamma_L - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L)}, \]

\[ z_t = \frac{-\text{Cov}_t[R_T, v_T] \left( \text{Cov}^2[R_T, v_T](\psi_S \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_S \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \psi_S \gamma_A \gamma_D + E_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L) \right)}{-100^2 \text{Var}_t[\psi_T] (\text{Cov}^2[R_T, v_T] \gamma_A \gamma_D + \text{Cov}^2[R_T, v_T] \gamma_D \gamma_L - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - 2 \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_A \gamma_D - \text{Var}_t[R_T] \text{Var}_t[\psi_T] \gamma_D \gamma_L)}}. \]
Similarly, given the equilibrium system (3.10), we solve the equilibrium as

\[ a = \frac{(Cov[R_T, \varphi] - Var[R_T]Var[\varphi]) \times 10^2}{T - t} \]

\[ b = \frac{-2Var[R_T]Var[\varphi]ZS_1 \gamma_{D}^2} {T - t \cdot Cov[R_T, \varphi] \gamma_{D}^2} \]

\[ \lambda_t = \frac{Var[\varphi] \gamma_A \left(-Var[R_T]ZS_1 \gamma_{D}^2 + E_t[\gamma_A]D + E_t[\gamma_D] \gamma_D\right)} {\gamma_A (T - t \cdot Cov[R_T, \varphi] \gamma_D)} \]

\[ \phi_t = \frac{Var[\varphi] \left(-Var[R_T]ZS_1 \gamma_{A,D} \gamma_L + E_t[\gamma_A]D + E_t[\gamma_D] \gamma_A\right)} {Cov[R_T, \varphi] \gamma_D \gamma_A} \]

\[ \psi_t = \frac{Var[\varphi] \left(-Var[R_T]ZS_1 \gamma_{A,D} \gamma_L + E_t[\gamma_A]D + E_t[\gamma_D] \gamma_A\right)} {Cov[R_T, \varphi] \gamma_D \gamma_A} \]

\[ \chi_t = \frac{-2Var[R_T]Var[\varphi]ZS_1 \gamma_{A,D} \gamma_L + Var[R_T]Var[\varphi]ZS_1 \gamma_{A,D} \gamma_L + Var[R_T]Var[\varphi](\gamma_A \gamma_L) - E_t[R_T]Var[\varphi]Var[R_T] \gamma_A \gamma_D} {100 \cdot Cov[R_T, \varphi] \gamma_D \gamma_A (Cov[R_T, \varphi] - Var[R_T]Var[\varphi])} \]

\[ \phi_t = \frac{-Var[R_T]ZS_1 \gamma_{A,D} \gamma_L + E_t[\gamma_A]D + E_t[\gamma_D] \gamma_A} {100 \cdot Cov[R_T, \varphi] \gamma_D \gamma_A (Cov[R_T, \varphi] - Var[R_T]Var[\varphi])} \]

\[ \phi_t = \frac{-Var[R_T]ZS_1 \gamma_{A,D} \gamma_L - Var[R_T]Var[\varphi] \gamma_A \gamma_D + E_t[R_T]Var[\varphi] \gamma_A \gamma_D + E_t[R_T]Var[\varphi] \gamma_D \gamma_A} {100 \cdot Cov[R_T, \varphi] \gamma_D \gamma_A (Cov[R_T, \varphi] - Var[R_T]Var[\varphi])} \]
Chapter 5

Conclusion

This thesis mainly studies three classes of equilibrium models and their applications. In Chapter 2, we study a consumption-based equilibrium model built in a pure exchange economy in which there are two types of investors, with lower and higher risk aversion, each with CRRA utility. The equilibrium is completely solved by using perturbation methods. Taking advantage of the tractability of our solutions, we analyse the effect of the size of investors and the effect of the risk-aversion heterogeneity and discuss the economic mechanism implied by our solutions.

In Chapter 3, in contrast to Bollerslev et al. (2009); Drechsler and Yaron (2011); Drechsler (2013), who use a long-run risks model, and Buraschi et al. (2014) who use a two-tree Lucas (1978) economy with two heterogeneous investors, we employ a simpler cost-free production-based equilibrium model to explain the large equity and variance risk premiums. For the data period in Broadie et al. (2007), the SVCJ model built in our cost-free production economy can perfectly explain the equity premium puzzle and the large negative VRP. For the longer data period, the SVJ model works best in terms of explaining the VRP, and we find that the SVJ model explains both large ERP and VRP when the ERP is larger than 11% (e.g., the periods, 1990–1999 and 2010–2016).

In Chapter 4, we provide an equilibrium model, which is the first demand-based
equilibrium model of volatility trading with three kinds of traders (i.e., dealers, asset managers and leveraged funds). It reveals an intuitive economic mechanism of how asset managers, leveraged funds and dealers' volatility trading activities affect the volatility market and fully supports the existing empirical results. The empirical results significantly support the theoretical results, which are implied by our equilibrium model. Finally, this chapter newly studies the impact of the three main traders net positions on the VRP and the VIX futures return based on our novel equilibrium model.
Appendix A

A list of research output during the PhD period

In addition to the three essays included in this thesis, I have also published two papers and completely finished seven working papers during the PhD period from 2014 to 2017 as follows.


Paper [3] is the first studying Kyle’s (1985) model with heterogeneous beliefs on the proportion of insiders. As the proportion of insiders in the market is unknown, the extension is more realistic. Paper [3] therefore investigates how heterogeneous beliefs on the proportion of insiders affect the equilibrium.
Appendix A. A list of research output during the PhD period


Paper [4] extends Pindyck and Wang (2013) into a production economy with shocks in the volatility of capital stock. In the last decade, the variance risk premium (VRP) in the US financial markets has generated strong interest among academic researchers. As an application, we employ the production-based asset pricing model to successfully explain the higher negative VRP.


Paper [5] studies the price of liquidity beta in China. The conventional, risk-based view on liquidity beta is a dismal story for China: High liquidity beta stocks underperform low liquidity beta stocks by 1.17% per month in China. We propose a competing, sentiment-based explanation on the reversed pricing pattern and find that liquidity beta is a negative return predictor at the firm level and the return differential between high and low liquidity beta stocks is more dramatic following high market liquidity periods.


Paper [6] considers an asset pricing model with a multiple-priors recursive utility incorporating decision makers’ concern with ambiguity on the drift and the jumps of driving process. In the last 30 years, the equity premium puzzle in the US financial markets has generated strong interest among academic researchers. The model in this paper can well explain the equity premium puzzle, since the ambiguity aversion, as a complementary aversion of risk aversion, can increase
the equity premium and decrease the risk-free rate, which documents that ambiguity on uncertainty is a resolution of the equity premium puzzle.


Paper [7] comprehensively studies the risk-neutral moments and cumulants in the crude oil market. The crude oil market has become the most active energy market in terms of trading volume and the variety of derivative products. Paper [7] calculates the innovations in the risk-neutral moments (i.e., volatility, skewness and kurtosis) and the third and fourth order cumulants and investigates the predictability of stock and option returns in time series and cross-sectional levels.


Paper [8] newly investigates the predictability of market moment spreads in the cross section of expected returns, taking the case of the energy stock market, which is highly correlated to energy commodity markets and essentially important for the development of a country. Paper [8] is the first to study the predictability of market moment spreads in the energy stock sector.


Paper [9] estimates the skew risk premia for the individual stocks and indexes in the US financial markets and analyzes the determinants of the cross-sectional variation of skew risk premia. Due to the prosperity of the variance swaps market and the needs of skew swap to speculate or hedge the skewness risk, studying skew swaps becomes much more important. Paper [9] is the first to provide
a comprehensive study of the cross-sectional variation of skew risk premia and their determinants.

For more details, see my homepage: sites.google.com/site/ruanxinf.
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