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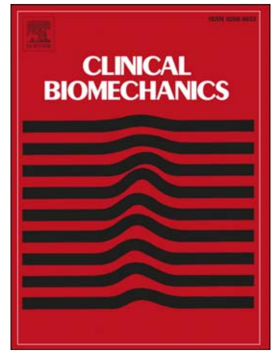

On the use of continuous relative phase: Review of current approaches and outline for a new standard

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On the use of continuous relative phase: Review of current approaches and outline for a new standard

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Abstract

In this paper we review applications of continuous relative phase and commonly reported methods for calculating the phase angle. Signals with known properties as well as empirical data were used to compare methods for calculating the phase angle. Our results suggest that the most valid, robust and intuitive results are obtained from the following steps: 1) centering the amplitude of the original signals around zero, 2) creating analytic signals from the original signals using the Hilbert transform, 3) calculating the phase angle using the analytic signal and 4) calculating the continuous relative phase. The resulting continuous relative phase values are free of frequency artifacts, a problem associated with most normalization techniques, and the interpretation remains intuitive. We propose these methods for future research using continuous relative phase in studies and analyses of human movement coordination.

Keywords: Phase angle, Continuous relative phase, Normalization, Gait data, Coordination, Movement variability

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1. Introduction

Within sports and health science, the biomechanical study of human movement has many purposes; these include, but are not limited to, rehabilitation, injury prevention and sports performance analysis. A common challenge for all of these domains is simplifying the high-dimensional information available from 2D video analysis, 3D motion capture systems or other modes of kinematic data collection. Dynamical systems theory approaches to movement analysis have gained support in recent years because it provides a theoretical framework for simplifying and working with complex systems (see, e.g. Kelso, 1995). Dynamical systems can be composed of many parts interacting and their behavior may often be described by a single low-dimensional term or measure. Most human movements involve a great number of moving parts, all coordinated together, explaining why so many researchers and clinicians have put such effort into modeling the human movement system as a dynamical system (e.g. Davids et al., 2003; Glazier & Davids, 2009; Stergiou, 2004). For example, in locomotion the lower extremity segments can be treated as a coupled system and the interaction of the segments acts to effectively displace the body's position during locomotion. By treating the musculoskeletal system as a system evolving over time, rather than focusing on particular events, a much richer description of the interaction of the individual and their environment can be achieved (Barela et al., 2000).

Rosen (1970) is often cited for suggesting that the behavior of a dynamical system can be described by plotting a variable versus its first derivative – these plots are commonly called phase portraits and provide qualitative utility in analyzing human movement (Bartlett & Bussey, 2012; Beek & Beek, 1988). According to

Clark et al. (1993), the phase portraits of the shank and thigh are similar to a limit cycle system – their coordination is cyclic and dissipative and therefore energy must be supplied to continue the behavior. Accordingly, their relation in phase space, or *relative phase*, can describe the dynamic coordination of these variables. Continuous relative phase is a measure, which describes the phase space relation between two segments (modeled as pendula) as it evolves throughout the movement, which makes continuous relative phase an attractive and popular collective variable for inter- and intra-limb coordination.

A central goal in dynamical systems theory is to identify the attractors, or stable states, of the system. Identifying stable states goes beyond simply identifying the common coordinative states for a particular movement; analysis of the variability of continuous relative phase allows one to investigate the stability of the system, or its resiliency to perturbation. Kelso (1995) noted that when coordination is perturbed beyond stability the relative phase pattern will fluctuate, indicated by an increase in variability, before settling on a new stable pattern. Analyses of the variability of continuous relative phase are insightful tools for understanding the dynamics of higher order coordination. Therefore, the importance of a valid, robust method for calculating phase angles, to be sure that the signal of interest is measured without contamination from frequency artifacts, should be clear and will be addressed in this paper.

Both the wide ranging applications of continuous relative phase as well as the varying methods used in its calculation warrant an in-depth overview and discussion of its application, calculation and interpretation. This paper provides an overview of the use of continuous relative phase in sport and health science before comparing the approaches that have been taken in the literature for its calculation.

We demonstrate the prominent procedures in the literature using synthetic and empirical data and outline what we suggest to be the new methodological standard for continuous relative phase in sports and health science.

2. Calculating Continuous Relative Phase

Continuous relative phase is a new signal generated representing the difference in phase angles of the two original signals. For the calculation of phase angles two different methods have commonly been used in studies of human movement. Firstly, continuous relative phase between two signals can be calculated based on phase portraits (Burgess-Limerick et al., 1993; Hamill et al., 1999) and, secondly, relative phase between two signals can be calculated using analytic signals generated by the Hilbert transform (Lamoth et al., 2009; Palut & Zanone, 2005). In the following two subsections we describe these methods in detail.

2.1. Phase Portraits

Studies of human movement coordination are often grounded in dynamical systems theory; therefore, system components can be assigned to a phase space in which each state of the dynamical system is described by certain properties. Pertaining to continuous relative phase analyses, the phase space usually consists of the measured (time dependent) signal $x(t)$ and its velocity $\dot{x}(t)$, the first derivative of the signal. The measured signal used in phase portraits is most often a segment or joint angle, although others have used higher derivatives to construct the phase space (Wagenaar & van Emmerik, 2000). To calculate the phase angle, frequency effects of the phase portrait on the phase angle are reduced by normalization methods.

Before introducing normalization methods we should first distinguish between analyzing sinusoidal signals and non-sinusoidal signals. Sinusoidal (harmonic) signals are signals which can mathematically be described by a sine wave, for example, the signal

$$x(t) = A\sin(\omega t + \psi) + d, \tag{1}$$

where ω denotes the frequency, ψ denotes a constant shift along the x-axis, A is a constant describing the magnitude of the amplitude, and d is a constant which describes a shift along the y-axis. Non-sinusoidal (non-harmonic) signals are those which cannot be mathematically described by only a sine wave (such as in equation 1). For each of these types of signals there are some commonly used normalization techniques.

In order to analyze a sinusoidal signal, Fuchs et al. (1996) showed that the phase portrait should be normalized so that the resulting trajectory in phase space is circular and centered around the origin of the phase space. To achieve the circularity they showed that the $\dot{x}(t)$ axis of the signals should be normalized by multiplying the $\dot{x}(t)$ axis by the factor $\frac{1}{\omega}$: the inverse of the signal's frequency. Furthermore, in case a sinusoidal oscillator is described by equation 1 with $d \neq 0$ the oscillator must be shifted by $-d$, so that the phase portrait is centered around the origin of the $x\dot{x}$ phase space. This ensures that phase portraits of different sinusoidal signals $x_1(t)$ and $x_2(t)$ are comparable and hence avoid artifacts caused by frequencies and/or different shifts d_1 and d_2. To calculate phase angles, the displacement of sinusoidal data does not need to be normalized because the phase angle ϕ of a sinusoidal oscillator (for simplicity we assume $d = 0$) does not influence the calculation of ϕ

$$\phi = \arctan\left(\frac{\dot{x}(t)}{x(t)}\right)$$
$$= \arctan\left(\frac{\omega\, A\, \cos(\omega\, t + \psi)}{A\, \sin(\omega\, t + \psi)}\right)$$
$$= \arctan\left(\frac{\omega\, \cos(\omega\, t + \psi)}{\sin(\omega\, t + \psi)}\right) \tag{2}$$

To analyze non-sinusoidal signals, different normalization methods have been used. The goal of normalizing the data has been to transform the phase portraits in such a way that both displacement of the signal and its first derivative are limited to the range between -1 and 1. In this paper we used the two most frequently used methods (similar to those reported by Kurz & Stergiou (2002)). First, normalization is accomplished for any input signal $y(t)$ by the function

$$f(y(t_i)) = \frac{y(t_i)}{\max(|y(t)|)}. \tag{3}$$

This technique limits the input signal of the function to either -1 or 1 depending on the maximum absolute value of $y(t)$. This method is often used for velocity normalization because the zero value has qualitative meaning and, arguably, should be preserved. In other words, after normalization the zero value represents the zero value in the original signal. A second normalization technique is based on the function

$$g(y(t_i)) = 2\left(\frac{y(t_i) - \min(y(t))}{\max(y(t)) - \min(y(t))}\right) - 1. \tag{4}$$

This function transforms the original values $y(t)$ in such a way that the minimum value of $g(y(t))$ equals -1 and the maximum value of $g(y(t))$ equals 1. Here the zero value is midway between the maximum and minimum and can, therefore, be arbitrary. Since angle definitions can be arbitrary, the method in equation 4 has often been used for normalizing joint or segment angles. We summarize the normalization methods found in the literature as follows:

- **Method A** uses equation 4 to normalize the joint angular displacement and equation 3 to normalize the angular velocities (Barela et al., 2000; Burgess-Limerick et al., 1993; Dierks & Davis, 2007; Hamill et al., 1999; Heiderscheit et al., 1999; Hein et al., 2012; Li et al., 1999; Miller et al., 2008, 2010; Stergiou et al., 2001a,b; Yen et al., 2009).

- **Method B** uses equation 4 for both angular displacement and angular velocity normalization (Figueiredo et al., 2012; Haddad et al., 2010; Kwakkel & Wagenaar, 2002; Lamoth et al., 2002; Meyns et al., 2013; Selles et al., 2001; van Emmerik & Wagenaar, 1996).

After normalization, the phase angle of the signal at time t_i is calculated based on the normalized phase portrait (Barela et al., 2000; Li et al., 1999; Peters et al., 2003)

$$\phi(t_i) = \arctan\left(\frac{\dot{x}_{\text{norm}}(t_i)}{x_{\text{norm}}(t_i)}\right). \tag{5}$$

Finally, the continuous relative phase, $\text{crp}(t_i)$, at time t_i between two signals $x_1(t)$ and $x_2(t)$ is calculated as

$$\begin{aligned}\text{crp}(t_i) &= \phi_1(t_i) - \phi_2(t_i)\\ &= \arctan\left(\frac{\dot{x}_{1,\text{norm}}(t_i)x_{2,\text{norm}}(t_i) - \dot{x}_{2,\text{norm}}(t_i)x_{1,\text{norm}}(t_i)}{x_{1,\text{norm}}(t_i)x_{2,\text{norm}}(t_i) + \dot{x}_{1,\text{norm}}(t_i)\dot{x}_{2,\text{norm}}(t_i)}\right).\end{aligned} \tag{6}$$

2.2. The Hilbert transform

Phase angles can also be calculated based on a measured signal $x(t)$ and its Hilbert transform $H(t) = H(x(t))$. The Hilbert transform allows the transformation of any real signal into a complex, analytic signal $\zeta(t)$ Gabor (1946) defined as

$$\zeta(t) = x(t) + iH(t) \tag{7}$$

where the Hilbert transform $H(t)$ of $x(t)$ serves as the imaginary part of the analytic signal[1]. Based on the complex signal the phase angle at time t_i can be calculated by

$$\phi(t_i) = \arctan\left(\frac{H(t_i)}{x(t_i)}\right). \tag{8}$$

The continuous relative phase $\text{crp}(t)$ between two signals $x_1(t)$ and $x_2(t)$ can be computed, first by transforming these signals into analytic signals using the Hilbert transform, then by subtracting the phase angles from each other. For example, the continuous relative phase for the two signals at time t_i is

$$\begin{aligned}\text{crp}(t_i) &= \phi_1(t_i) - \phi_2(t_i)\\ &= \arctan\left(\frac{H_1(t_i)x_2(t_i) - H_2(t_i)x_1(t_i)}{x_1(t_i)x_2(t_i) + H_1(t_i)H_2(t_i)}\right),\end{aligned} \tag{9}$$

where $H_1(t)$ and $H_2(t)$ denote the Hilbert transform of each signal, respectively.

In the next section we demonstrate with simulated data as well as kinematic

[1]In general, the Hilbert transform is considered a convolution of a function (signal) in the time domain. The Hilbert transform needs to be defined using the Cauchy principle value so that the integral converges and thus exists. As an integral, the Hilbert transform can be solved in the time domain. There are many methods for calculating the Hilbert transform; many software applications, such as MATLAB, calculate the Hilbert transform in the frequency domain using the (Fast) Fourier transform and its inverse.

data, the effect of the normalization methods A and B and the Hilbert transform on continuous relative phase values. To aid interpretation, whenever possible, we use modeled data which has been reported previously in the literature.

3. Modeled Data

3.1. Sinusoidal oscillators

In this section we begin with simple sinusoidal examples to demonstrate the effect of various normalization techniques. Therefore, we calculated the continuous relative phase for all testing cases using phase angles which were calculated based on: a) not normalizing the original data at all, b) normalizing velocity using the technique shown by Fuchs et al. (1996), and c) creating analytic signals using the Hilbert transform. These procedures are approximate reproductions of those shown in Peters et al. (2003), with the addition of the Hilbert transform method.

3.1.1. Example 1: two sinusoidal signals with the same frequency, shifted horizontally

Figure 1 illustrates a sinusoidal oscillator $x(t) = \sin(2t)$, $t \in [0, 2\pi]$, and the same sinusoidal oscillator shifted by 18°, the corresponding $x\dot{x}$ phase portraits, and a plot visualizing the continuous relative phase between these two oscillators calculated using different techniques. In this example the velocity of the two oscillators was normalized with respect to the frequency, $\omega = 2$, of the sinusoidal oscillator through the factor $\frac{1}{2}$. Note that in the right panel of Figure 1 the Hilbert transform is not shown because the transformed values lie in the complex plane rather than the phase plane in which the original and normalized values are located.

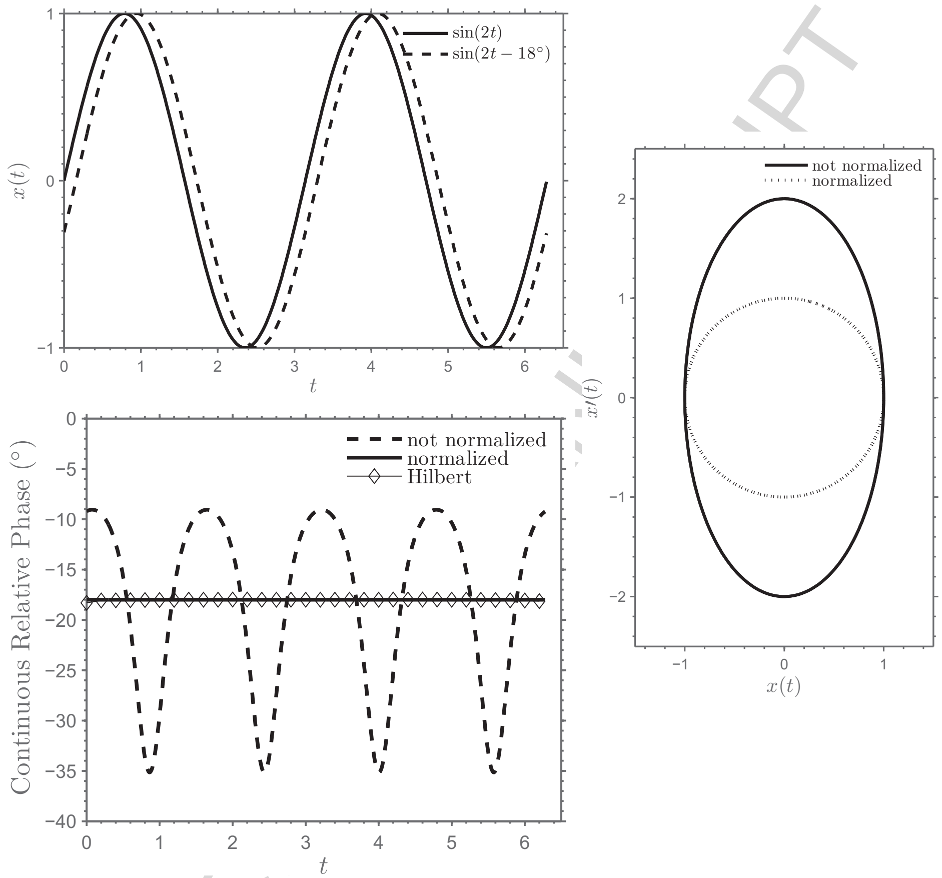

Figure 1: Two sinusoidal signals, one phase shifted by 18° (top left), the phase portraits for both signals (right) and the corresponding continuous relative phase calculated with: no normalization, with frequency normalization and the Hilbert transform (bottom left).

The continuous relative phase calculated based on non-normalized data (Fig. 1, bottom left, dashed line) shows oscillating behavior about a constant continuous relative phase, even though the two oscillators behave equally only phase shifted by 18°. One would expect the continuous relative phase of these two oscillators to be constant and equal to 18°; the oscillating behavior of the continuous relative phase of the non-normalized data represent frequency artifacts (Fuchs et al., 1996; Peters et al., 2003). This is made clear by the continuous relative phase values which were calculated based on frequency normalized velocities. The resulting continuous relative phase is constant and shows exactly the 18° difference between the two oscillators. Finally, we calculated continuous relative phase using the Hilbert transform based on the raw sinusoidal data. The resulting plot (Fig. 1, bottom left) also shows the expected constant difference of 18° between the two oscillators.

3.1.2. Example 2: two sinusoidal signals with different frequencies

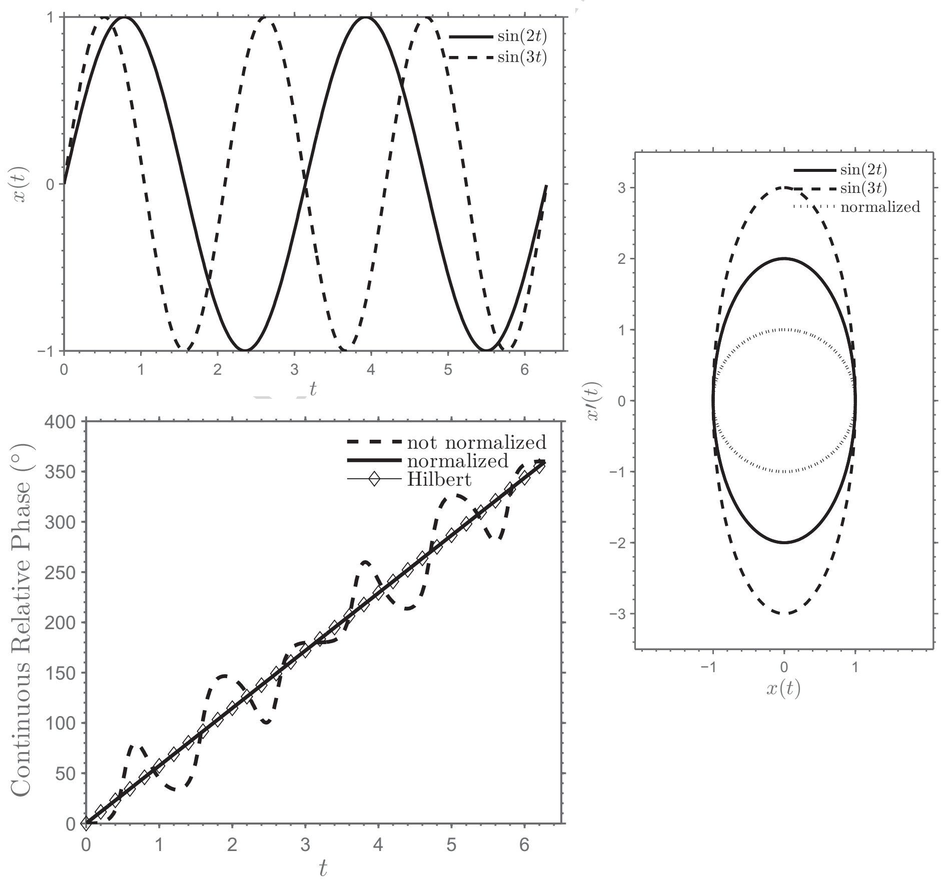

Figure 2: Two sinusoidal signals with different frequencies (top left), the phase portraits for both signals (right) and the corresponding continuous relative phase calculated with: no normalization, with frequency normalization and Hilbert transformation (bottom left).

Here we compare continuous relative phase calculations between two sinusoidal oscillators with different frequencies. The two oscillators are represented by $x_1(t) = \sin(2t)$ and $x_2(t) = \sin(3t)$, respectively. Figure 2 shows the two oscillators each within the interval $t \in [0, 2\pi]$, their respective phase portraits, and a plot containing continuous relative phase values. The velocities of the two oscillators were normalized each by the inverse of the respective frequency $\frac{1}{\omega}$ (as in the previous example (Fuchs et al., 1996)).

As already shown by Peters et al. (2003), continuous relative phase calculated based on non-normalized data shows a fluctuating pattern (Fig. 2, bottom left, dashed line); this can again be explained by frequency artifacts (Fuchs et al., 1996; Peters et al., 2003). After normalizing the velocities of the two oscillators, each with respect to its frequency, the continuous relative phase shows the expected pattern. The oscillators move linearly from in-phase to anti-phase and eventually back into in-phase during the respective time period $[0, 2\pi]$. Finally, we calculated continuous relative phase values using Hilbert transform based on the raw data. The resulting continuous relative phase values show the same linear pattern as the continuous relative phase values calculated based on normalized phase portraits.

3.2. Non-sinusoidal signals

In this section we compare the different methods for calculating continuous relative phase with respect to non-sinusoidal signals. The term non-sinusoidal can describe different kinds of data; thus we first distinguish between non-sinusoidal signals which are based on a mathematical description and empirical data. A mathematical description of a signal usually relies on a modeling process. The models are either the combination of basic functions like the signal in equation 10 or can be systems of differential equations (c.f. HKB model; Haken et al., 1985;

Kelso, 1984). Empirical data representing human movement is not mathematically described by functions, they are most often time series data, for example, kinematic joint angles (see section 4.1).

We compared continuous relative phase calculations using different techniques for both functional and experimental non-sinusoidal data. Continuous relative phase values were calculated and compared to each other based on phase angles calculated based on a) not normalizing the original data at all, b) normalizing data using the normalization methods A and B, and c) creating analytic signals using the Hilbert transform.

3.2.1. Example 4: two non-sinusoidal signals

This example is based on non-sinusoidal data which are represented by the function

$$x(t) = \frac{\cos(t - 0.25\pi)}{\sqrt{1 + 0.41418^2 - 2 \times 0.41418 \sin(t - 0.25\pi)}} \tag{10}$$

which is similar to the non-sinusoidal signal in Peters et al. (2003). In this section, continuous relative phase values between a signal modeled by equation 10 for $t \in [0, 2\pi]$ and the same signal shifted by 126° are compared. Figure 3 shows the two signals, their respective phase portraits, and continuous relative phase values calculated using the different techniques mentioned above.

Since the signals in Figure 3 are shifted but have the same frequency, neither signal will ever *catch up* to the other so that they are in-phase. In section 3.1.1 the two shifted sine waves had a constant continuous relative phase once the frequency artifacts were removed. Because the signals in Figure 3 are non-sinusoidal they are constantly increasing and decreasing their phase shift of 126°; therefore, the expected behavior of their continuous relative phase should fluctu-

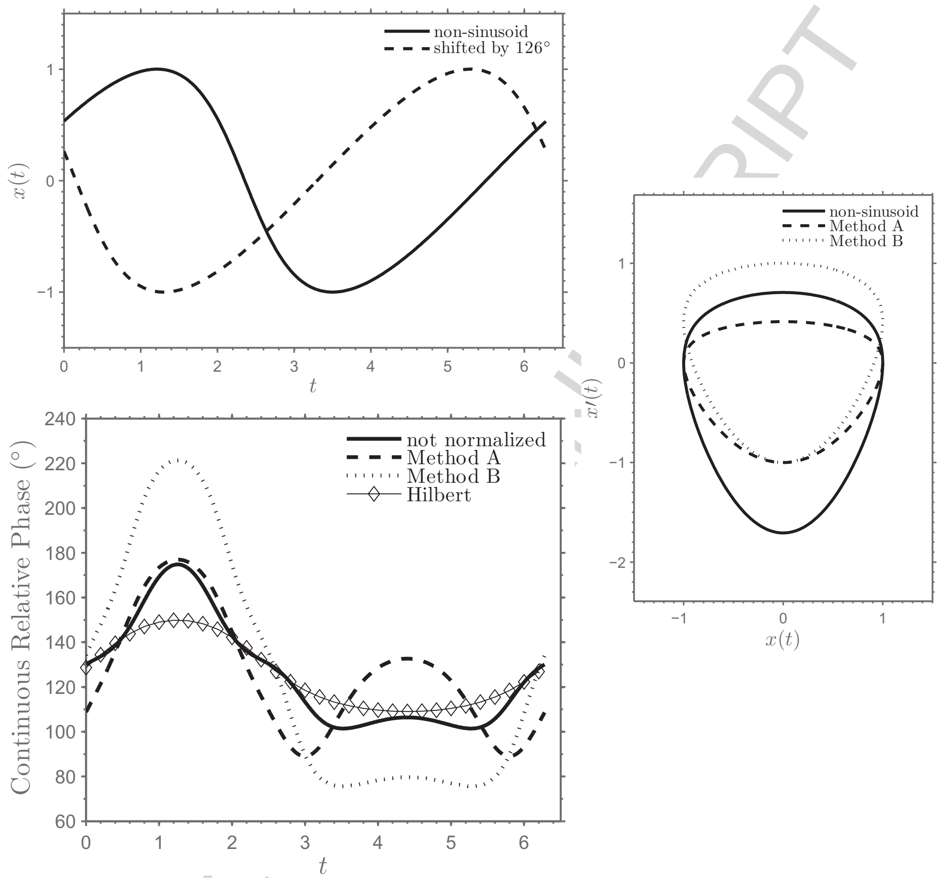

Figure 3: Two non-sinusoidal signals, one phase shifted by 126° (top left), the phase portraits for using different normalization methods (right) and the corresponding continuous relative phase diagrams (bottom left).

ate around 126°. The Hilbert transformed data show this behavior exactly, the non-normalized continuous relative phase values resemble those of the Hilbert transform most closely, although artifacts of the non-circular phase portrait are evident. The normalized continuous relative phase values show the greatest deviation from the Hilbert transformed values. This is because normalizing introduces artifacts when the original signal is non-sinusoidal (Kurz & Stergiou, 2002).

4. Empirical Data

4.1. Example 5: kinematic data

In this section the various methods for calculating phase angles are demonstrated using kinematic data representing hip-knee coupling during three strides of treadmill running.

The ranges of motion in Figure 4, on which the continuous relative phase calculations are based, were roughly between 152° and 195° for the hip and between 66° and 164° for the knee. Since the joint angles are located in the top right quadrant of the time domain plot (Fig. 4, top left), the analytic signals created by the Hilbert transform may only have positive real values. Hence, the two respective analytic signals are located in the right half of the complex plane. Consequently, the phase angles of these two signals are limited to the range $[-90°, 90°]$ at the most.

For this reason the trajectory of the signal should be transformed in such a way that it winds around the origin of the complex plane. Whereas Rosenblum et al. (2001) suggest transforming the signal by subtracting the mean value of the signal from the signal, we suggest centering the range of a signal's amplitude around zero by

$$x_{centered}(t_i) = x(t_i) - \min(x(t)) - (\max(x(t)) - \min(x(t)))/2, \qquad (11)$$

and eventually calculating the analytic signal using the Hilbert transform based on $x_{centered}(t)$.

The resulting analytic signal will have the same imaginary component, which

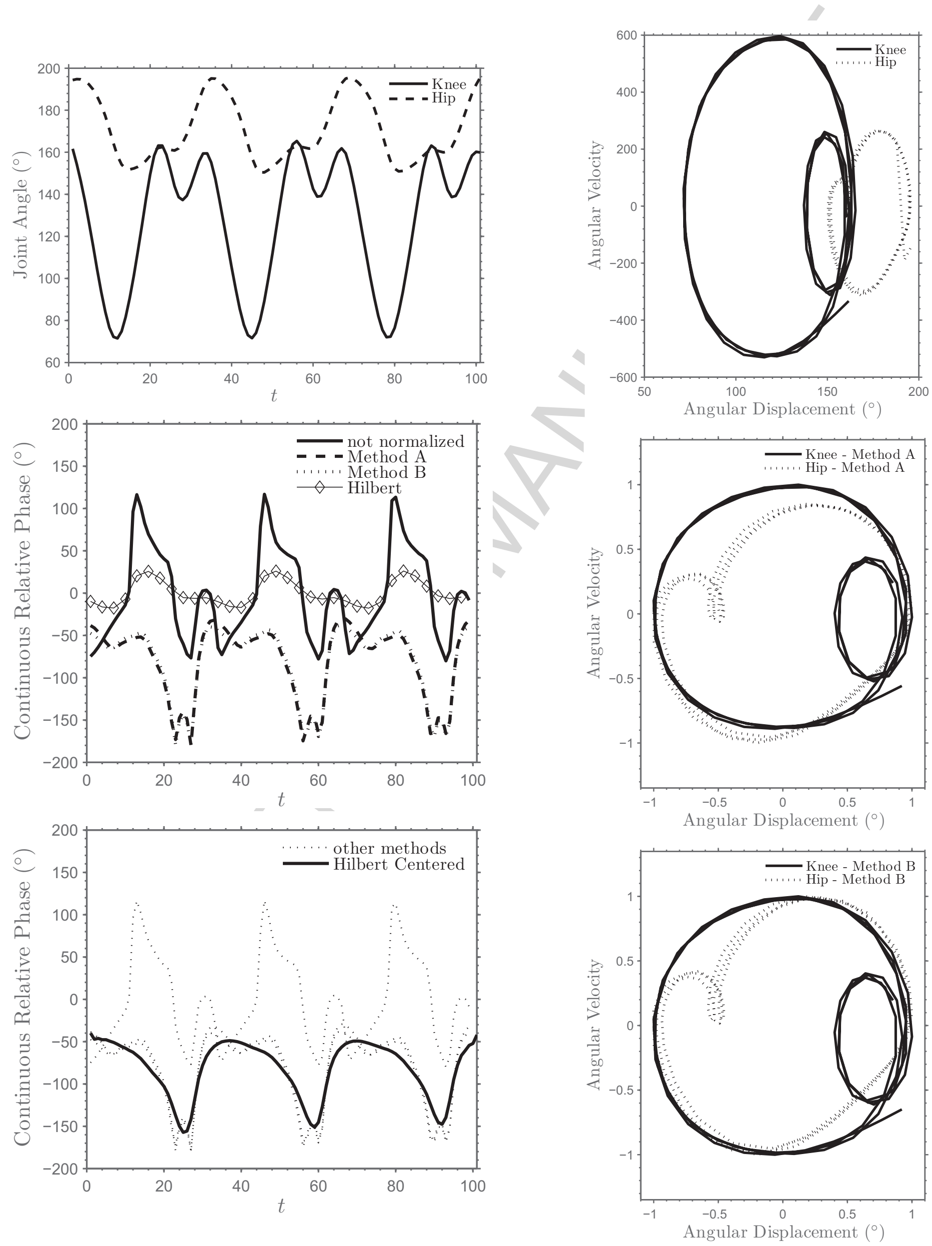

Figure 4: In the left panels, time domain plots of hip and knee joint angles for treadmill running (top), continuous relative phase values (middle) and the continuous relative phase values using the centered Hilbert transform (bottom; methods from the plot above are also included for reference). In the right panels, phase portraits for hip-knee coupling in treadmill running: without normalization (top), normalized according to Method A (middle) and Method B (bottom).

is determined by the Hilbert transform, as that of the raw data since

$$H(x(t)+c) = H(x(t)), \tag{12}$$

where c denotes a constant shift of the signal's amplitude (see appendix Appendix A). This approach allows the resulting phase angle to have values in the range $(-180°, 180°)$ (Fig. 4, bottom left).

5. Discussion

The purpose of this paper was to review applications of continuous relative phase and commonly used methods for calculating the phase angle, address important points which have been discussed in the relevant literature and, based on the results of our analyses, to propose a valid and robust method for calculating the phase angle applicable to most research questions in sports and health science. We have demonstrated the effect of different normalization techniques on the resulting continuous relative phase values. Several other issues pertaining to the interpretation of continuous relative phase are discussed in the following section.

5.1. Phase angle vs. continuous relative phase

Some debate has developed concerning the range used for continuous relative phase and the phase angle (Hamill et al., 1999; Kurz & Stergiou, 2002; Wheat et al., 2003). The arctan function outputs values in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$, or in degrees, $(-180°, 180°)$. In terms of relative phase, for the range $[-180°, 180°]$ a continuous relative phase value of $0°$ represents in-phase behavior and values of $-180°$ and $180°$ represent anti-phase behavior (Scholz & Kelso, 1989). Some authors have chosen to use the absolute value of continuous relative phase values

(Hamill et al., 1999; Heiderscheit et al., 1999; van Emmerik & Wagenaar, 1996), since the values -180° and 180° both indicate anti-phase behavior and by doing so, the necessity for using directional statistics is alleviated (Sparto & Schor, 2004). Conversely, others have suggested that the positive and negative values have qualitative meaning and should be preserved. If the phase angle of the proximal segment is subtracted from the phase angle of the distal segment, then positive continuous relative phase values indicate that the distal segment is ahead of the proximal segment in phase space (Barela et al., 2000; Clark & Phillips, 1993; Hamill et al., 2000; Kao et al., 2003; Kiefer et al., 2011; Kurz & Stergiou, 2002; Yen et al., 2009), or the complex plane, and vice versa.

This seems to have highlighted a point of misunderstanding between the terms *phase angle* and *continuous relative phase*. While continuous relative phase values may be manipulated into the range $[0^\circ, 180^\circ]$ for reasons mentioned above, this should not be confused with defining the phase angle in the range $(0^\circ, 180^\circ)$ (Hamill et al., 1999; van Emmerik & Wagenaar, 1996) or even $[-90^\circ, 90^\circ] \setminus \{0^\circ\}$ (Kurz & Stergiou, 2002).

Wheat et al. (2003) showed that by defining the phase angle in a 180° range, in their case $(0^\circ, 180^\circ)$, the subsequent continuous relative phase values are non-intuitive. Therefore, we suggest defining the phase angle as that which is naturally produced by the arctan function. For this reason we feel the need to emphasize that the phase angle and continuous relative phase cannot be used interchangeably. Phase angles should always be in the ranges $(-180^\circ, 180^\circ) \setminus \{0^\circ\}$ or $(0^\circ, 360^\circ) \setminus \{180^\circ\}$, while continuous relative phase may be expressed in the ranges $[0^\circ, 180^\circ]$ or $[-180^\circ, 180^\circ]$ or $[0^\circ, 360^\circ]$.

5.2. Joint vs. segment angles

Many studies which have used continuous relative phase have used joint angles as the original signals. The use of joint angles, however, is contradictory to modelling the segments as pendula. Consider, for example, adjacent joint relationships such as the coupling of the hip and knee. To calculate phase angles from the hip and knee joint angles, the thigh segment is included in both angles, and consequently influences the phase angles for each joint. Calculating phase angles in this way goes against the original interpretation under the dynamical systems framework, notably by Kugler et al. (1980) and by Clark & Phillips (1993) specifically to gait. Only segment angles measured relative to an external reference frame allow meaningful and interpretable results that can be used to describe phase relationships properly from a dynamical systems perspective.

5.3. Maximum and minimum values

Different methods for obtaining the maximum and minimum values used in the normalization procedures (equations 3 and 4) have also been reported. This pertains to whether the maximum value for each trial is used to normalize the respective trial or whether the maximum value among a group of trials (e.g. a single testing session) is used to normalize each trial (Hamill et al., 2000). Authors seldom report exactly how they obtain the maximum and minimum values. As Hamill et al. (2000) showed, when using phase space to calculate the phase angle, the method for determining maximum and minimum values affects the continuous relative phase calculation. One advantage of using the grouped approach (i.e. maximum or minimum value from a group of trials) for normalizing is that the trials being compared are scaled by a constant factor. However, rather than discussing this issue further, as we have shown thus far, the (centered) analytic signal based on the Hilbert transform provides the correct phase angle and, therefore, removes the need for normalization in order to fit the data into a unit phase space.

5.4. Inter- and intralimb couplings and normalization

Initially, continuous relative phase was used as a higher resolution form of discrete relative phase for assessing the coordination between two oscillating segments: often representing contralateral or interlimb coordination. For interlimb coordination one might expect that the limb being compared could oscillate in a near sinusoidal manner. For these situations the methods described by Fuchs et al. (1996) may satisfy the assumption that the phase space spanned by the two oscillators is circular and that the two oscillators are simply phase shifted. Furthermore Varlet & Richardson (2011) demonstrated a method for dealing with changes in frequency in interlimb coordination assumed to be sinusoidal (also based on the Hilbert transform). However, the current paper focuses on whole-body movements, for which continuous relative phase is most often used to represent intralimb coupling – or the coupling between adjacent joints. For questions of intralimb coordination, one can safely assume that the time-series of joint angles being compared are always non-sinusoidal (possibly with the exception of isokinetic exercises). To be clear, if two joints both oscillate sinusoidally, their continuous relative phase values throughout the measurement must be linear. If the two signals have the same frequency, the continuous relative phase values must be constant and equal to the phase shift (Fig. 1), and if the signals have different frequencies, the continuous relative phase values must be linearly increasing or decreasing depending on the frequency difference (Fig. 2). There can be no way for continuous relative phase to fluctuate throughout the movement if the joints oscillate sinusoidally and frequency artifacts have been removed. Therefore, for research into intralimb coordination using continuous relative phase we suggest using the amplitude centered Hilbert transform (as shown in Fig. 4) so that changes in coordination throughout the movement may be exposed.

5.5. Normalization

We have identified two main methods for normalization, which have been used to scale data to the unit phase space (Burgess-Limerick et al., 1993; Hamill et al., 1999). Others have argued for no normalization in favor of maintaining the original topology or aspect ratio of the data (Clark & Phillips, 1993; Kurz & Stergiou, 2002). While others have employed the Hilbert transform to create an analytic signal (Lamoth et al., 2009; Palut & Zanone, 2005). In Section 3.1 we showed that the scaling method of Fuchs et al. (1996) adequately transforms the data, thus removing frequency artifacts from the continuous relative phase calculation. However, since sinusoidal data does not arise from empirical measurements of human movement, the method will have limited use with such data. Understandably, many have used sinusoidal signals to demonstrate the effects of various normalization methods and phase angle definitions (Hamill et al., 1999; Kurz & Stergiou, 2002; Peters et al., 2003) – including the current paper – because of the simple characteristics of sine waves. However, the validity of transferring the demonstrated methods from sinusoidal to empirical data have not always been made clear.

Sinusoidal data often have their amplitude centered around zero, possibly for this reason the necessary shift of d when $d \neq 0$ has not been discussed. When dealing with empirical data, one should expect the data to be non-sinusoidal and have the amplitude not centered around zero. Therefore, we suggest that the data first be centered, so that zero represents the midpoint between the maximum and minimum values (Rosenblum et al., 2001). The amplitude centering is analogous to the shift of d for sinusoidal signals. However, for non-sinusoidal empirical data the Hilbert transform should be used to remove frequency effects.

5.6. Discrete and cyclic movements

In keeping with the resemblance of human movement to the limit cycle, most studies involving continuous relative phase as a measure of coordination have applied it to cyclic movements. Running (Dierks & Davis, 2007; Hamill et al., 1999; Hein et al., 2012; Miller et al., 2010, 2008; Kurz et al., 2005; Trezise et al., 2011) and walking (Barela et al., 2000; Clark & Phillips, 1993; Haddad et al., 2010; Kwakkel & Wagenaar, 2002; Lamoth et al., 2002; Li et al., 1999; Meyns et al., 2013; Wagenaar & van Emmerik, 2000; Wu et al., 2004), the transition between gait modes (Kao et al., 2003; Lamoth et al., 2009; Seay et al., 2006; van Emmerik & Wagenaar, 1996) and swimming (Figueiredo et al., 2012; Seifert et al., 2010, 2011) constitute the most common cyclic human activities studied (note that we only consider whole-body movements in this review). These types of movements closely correspond with the concept of phase analysis, which allows unique characteristics of the movement to be exposed qualitatively, because of the shape of the phase space trajectories. For example, a damped oscillator will show, in phase space, convergence to the origin as it loses energy. Accordingly, some have argued that studying cyclic movements (modeled as pendula) in terms of energy transfer with the environment can provide important insight into the changing state of the modeled system (Clark et al., 1993; Kurz & Stergiou, 2004).

However, central to dynamical systems theory is the continuous interaction between the many constraints (performer, the environment and the task (Newell, 1986)), which give rise to coordinated movement on the biomechanical level through self-organization. Furthermore, these interactions can influence performance of a task on different time scales (Schöllhorn et al., 2009). For example, fatigue can cause sprinters to make coordinative compensations for changing availability of energy resources (Trezise et al., 2011). On the other hand, for discrete tasks requiring precision, variability can also be managed throughout execution of the task to aid performance (Bootsma & van Wieringen, 1990). Therefore, although only a few studies have used continuous relative phase to study discrete movements (Burgess-Limerick et al., 1993; Robins et al., 2006), it seems reasonable to do so in order to reflect the changing constraints affecting the performance of the task, given a few caveats. The time scales between repetitions of discrete tasks are different from those of cyclic movements and should be acknowledged by authors using continuous relative phase for analyzing coordination variability in discrete tasks. Additionally, time continuous concepts such as relaxation time, the amount of time required after the system is perturbed to return to its original stable state (Scholz & Kelso, 1989), may not yet be meaningful for discrete tasks.

5.7. Interpretation

So far we have proposed that continuous relative phase should be calculated based on amplitude centered Hilbert transform values rather than phase angles obtained through plotting phase portraits when the original signals are non-sinusoidal. Yet to be discussed is the interpretation of the continuous relative phase using the Hilbert transform. As shown in Figure 4 (bottom left), the centered Hilbert transform gives similar continuous relative phase to those gained from normalizing the phase portraits; however, with the frequency artifacts removed. Therefore, the interpretation of the continuous relative phase values using the Hilbert transform should not change compared to the interpretation of continuous relative phase based on phase portraits (Hamill et al., 1999; Li et al., 1999). Furthermore, when using continuous relative phase for measures of variability such as an ensemble curve representing multiple trials or variability at each time point (Stergiou et al., 2001a,b; Yen et al., 2009) care should be taken to remove frequency artifacts as they could have significant influence on these measures.

Some have suggested, that continuous relative phase does not allow one to make inferences on the original signals (Miller et al., 2010; Peters et al., 2003). In-phase coordination simply means that the two joints occupy the same phase angle at the same time in the movement, whether the phase angle is measured in phase space or in the complex plane. Peters et al. (2003) stated that the non-intuitive result was generated when the original signals had the same slope in the time domain but were not *in-phase* according to continuous relative phase. It seems that the authors interpreted continuous relative phase with respect to direction (joint angle is increasing or decreasing) and velocity rather than displacement and velocity. Peters et al. (2003) highlighted two points on two lines with the same slope but obviously different displacement values and it is confusing that they suggest these should correspond with in-phase coordination according to continuous relative phase. That interpreting a joint angle's movement direction and velocity should predict its relative phase is a misinterpretation of relative phase, but may provide the basis for a new form of dynamic analysis of coordination – one which is more descriptive than discrete relative phase and simpler, or possibly more intuitive, than continuous relative phase.

5.8. Recommendations for future use

We have demonstrated the effects of normalization on various sinusoidal signals as well as on non-sinusoidal signals. Although normalizing sinusoidal signals adequately removes frequency effects, since sinusoidal signals will not be obtained from experimental data, we suggest that the normalization method provided by Fuchs et al. (1996) is irrelevant for studying multi-articular or whole body movements. Others have suggested different normalization methods, or in fact no normalization at all, to account for the frequency or amplitude of empirical data, but as we have shown, these methods either do not remove frequency artifacts from the calculated continuous relative phase values or do not allow the full range of phase angles on which continuous relative phase is based. In place of a) sinusoidal normalization, b) normalization methods A and B, or c) no normalization we propose the following steps:

1. centering the amplitude of the data around zero (equation 11)
2. transform each signal into an analytic signal using the Hilbert transform (equation 7)
3. calculate the phase angles for each signal (equation 8)
4. calculate the continuous relative phase (equation 9)

The Hilbert transform creates an analytic signal from non-sinusoidal signals, thereby removing frequency artifacts and making it appropriate for studying inter- and intralimb coordination in human movement. We should also mention that analytic signals can be created for any real signal but the phase angle only has a real physical meaning if the real signal is a narrow-band signal. Of course, kinematic data representing human movement satisfy this condition (Meng et al., 2006), but we bring attention to this in case researchers of human physiological or behavioral data encounter signals which do not have a narrow-band frequency spectrum (Boashash, 1992).

Applying the methods in this paper to other types of human movement data was out of the scope of this paper, but we will highlight one particular point of interest for researchers in other domains of human movement science seeking to use continuous relative phase. We have suggested the signal's amplitude be centered around zero; this is true for kinematic joint angles because the joint angle values are relatively arbitrary – they depend on how the joint angle is defined. However, if the values have qualitative meaning then another form of centering the data may be more appropriate. For example, Palut & Zanone (2005) looked at the lateral coordination of two tennis players on the court. The authors argued that the players could be modeled as a paired oscillator, which oscillates about the center line. In this case, the centerline (assigned as zero displacement) on the tennis court has qualitative meaning and should be preserved. For studies of player positional data, new methods for calculating the phase angle, such as those for tennis (Palut & Zanone, 2005), should be investigated.

6. Conclusions

In this paper we identified and compared commonly reported methods for calculating the phase angle for use in continuous relative phase analyses. Using synthetic and real data we compared the commonly reported normalization methods and showed that, after centering the signals' amplitudes around zero, the continuous relative phase values obtained from the analytic signal created using the Hilbert transform in all test cases gave the intuitive answer. We therefore suggest that future research adopt the amplitude centered Hilbert transform to remove frequency artifacts of the non-sinusoidal signals being studied.

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Appendix A. Invariance of the Hilbert transform with respect to a constant amplitude shift of the signal

According to Gabor (1946) the Hilbert transform of a real signal (time dependent) $x(t)$ is defined as

$$H(x(t)) = \frac{1}{\pi}\,\text{P.V.} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau, \tag{A.1}$$

where P.V. means that the integral is taken in the sense of the Cauchy principal value. The Hilbert transform of a signal $x(t)$ with respect to a constant shift c of the signal's amplitude is

$$\begin{aligned} H(x(t)+c) &= \frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{x(\tau)+c}{t-\tau}d\tau \\ &= \frac{1}{\pi}\,(\text{P.V.}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau + \underbrace{\text{P.V.}\int_{-\infty}^{\infty}\frac{c}{t-\tau}d\tau)}_{\overset{A.3}{=}0} \\ &= \frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau \\ &= H(x(t)) \end{aligned} \tag{A.2}$$

because

$$\begin{aligned} \text{P.V.}\int_{-\infty}^{\infty}\frac{c}{t-\tau}d\tau &= \lim_{a\to-\infty}\lim_{b\to\infty}\lim_{\varepsilon\to 0}(\text{P.V.}\int_{a}^{t-\varepsilon}\frac{c}{t-\tau}d\tau + \text{P.V.}\int_{t+\varepsilon}^{b}\frac{c}{t-\tau}d\tau) \\ &= \lim_{a\to-\infty}\lim_{b\to\infty}\lim_{\varepsilon\to 0}([-c\,\ln|t-\tau|]_{a}^{t-\varepsilon} + [-c\,\ln|t-\tau|]_{t+\varepsilon}^{b}) \\ &= \lim_{a\to-\infty}\lim_{b\to\infty}\lim_{\varepsilon\to 0}(c\,(-\ln|\varepsilon| + \ln|a| - \ln|b| + \ln|\varepsilon|) \\ &= \lim_{a\to-\infty}\lim_{b\to\infty}(c\,(\underbrace{\ln|a|}_{\to 0} - \underbrace{\ln|b|}_{\to 0})) \\ &= 0. \end{aligned} \tag{A.3}$$

676 Hence, the Hilbert transform of a signal $x(t)$ is invariant with respect to a constant
677 shift of the amplitude of $x(t)$.

Conflict of Interest Statement

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