Asymptotics of solutions in evolutionary formulations of the Einstein constraint equations

Author: Joshua Ritchie
Supervisor: Florian Beyer

Department of Mathematics and Statistics
University of Otago
New Zealand
Acknowledgements

This research has been my first introduction to General Relativity and I would like to express gratitude towards everyone who has helped me learn about the theory. So, thank you Jörg Frauendiener for the idea of the project and for advice throughout its duration. I would also like to thank Vee-Liem Saw (Veefessional) for letting me blabber on about my project to him. Most of all, I’d like to thank my supervisor Florian Beyer, for his patience, guidance, and for answering all of my questions. Even the foolish ones. I would also like to thank my brother and mother for all of their support. Finally, I’d like to thank my flat-mate Ashley, chur.
## Contents

1 Introduction ......................................................... 1  
1.1 History and motivation ........................................ 1  
1.2 Notation and conventions ..................................... 2  
1.3 Thesis summary .................................................. 4  

2 PDE summary .......................................................... 7  
2.1 Types of PDE ....................................................... 7  
2.2 Spin-weighted-spherical-harmonics (SWSH) .................. 11  
2.3 Linearisation ...................................................... 15  

3 The geometry of the $(n+1)$-decomposition .................. 17  
3.1 Hypersurfaces ...................................................... 17  
3.2 Foliating space-time with Cauchy surfaces ................. 19  
3.3 The $(n+1)$-decomposition of the Riemann tensor ........ 21  
  3.3.1 The Gauss, Codazzi, and Ricci equations ............... 21  

4 The Einstein constraint equations ............................... 25  
4.1 Derivation .......................................................... 25  
4.2 The ADM equations ............................................... 27  
4.3 Well-posedness .................................................... 29  

5 The elliptic form of the constraints ............................ 31  
5.1 Conformal transformations ..................................... 31  
5.2 (Multiple) Black hole solution ................................. 34  

6 An evolutionary formulation of the constraint equations ... 37  
6.1 Derivation .......................................................... 37  
  6.1.1 Momentum constraint .................................... 38  
  6.1.2 Hamiltonian constraint ................................... 39  
6.2 Form of the equations ........................................... 42  
  6.2.1 Strongly hyperbolic ....................................... 42  
  6.2.2 Parabolic-hyperbolic ...................................... 44  
6.3 Linearisation of the evolutionary form of the constraint equations ... 45  
  6.3.1 Linearisation: strongly hyperbolic ................. 45  
  6.3.2 Linearisation: parabolic-hyperbolic .................. 46  

7 Schwarzschild decomposition ...................................... 47  
7.1 The $(3+1)$-decomposition ..................................... 47  
7.2 The $(2+1)$-decomposition ..................................... 49
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>Asymptotically hyperboloidal foliations</td>
<td>50</td>
</tr>
<tr>
<td>8.1</td>
<td>The strongly hyperbolic formulation</td>
<td>50</td>
</tr>
<tr>
<td>8.1.1</td>
<td>Non-linear perturbations in the Minkowski space-time</td>
<td>51</td>
</tr>
<tr>
<td>8.1.2</td>
<td>Linear perturbations in the Schwarzschild space-time</td>
<td>54</td>
</tr>
<tr>
<td>8.1.3</td>
<td>Linear perturbations as a second order PDE</td>
<td>63</td>
</tr>
<tr>
<td>8.1.4</td>
<td>Linear perturbations as a boundary value problem</td>
<td>68</td>
</tr>
<tr>
<td>8.2</td>
<td>The parabolic-hyperbolic formulation</td>
<td>73</td>
</tr>
<tr>
<td>8.2.1</td>
<td>Linear perturbations</td>
<td>76</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Linear perturbations as a second order system</td>
<td>82</td>
</tr>
<tr>
<td>8.2.3</td>
<td>Non-linear perturbations</td>
<td>83</td>
</tr>
<tr>
<td>9</td>
<td>Asymptotically flat black hole(s) in the parabolic-hyperbolic formulation</td>
<td>87</td>
</tr>
<tr>
<td>9.1</td>
<td>Preparations</td>
<td>87</td>
</tr>
<tr>
<td>9.1.1</td>
<td>Kerr-Schild metrics</td>
<td>87</td>
</tr>
<tr>
<td>9.1.2</td>
<td>Non-linear spherically symmetric perturbations of the Schwarzschild Kerr-Schild data</td>
<td>88</td>
</tr>
<tr>
<td>9.2</td>
<td>Construction of binary black hole initial data sets</td>
<td>92</td>
</tr>
<tr>
<td>9.2.1</td>
<td>Model outline</td>
<td>93</td>
</tr>
<tr>
<td>9.2.2</td>
<td>Constructing the free data</td>
<td>94</td>
</tr>
<tr>
<td>9.2.3</td>
<td>Choosing adapted coordinates</td>
<td>97</td>
</tr>
<tr>
<td>9.2.4</td>
<td>Expected results</td>
<td>99</td>
</tr>
<tr>
<td>9.3</td>
<td>Asymptotics of binary black holes</td>
<td>102</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Convergence tests: shifted Kerr-Schild Schwarzschild initial data</td>
<td>102</td>
</tr>
<tr>
<td>9.3.2</td>
<td>Convergence tests: binary black holes</td>
<td>103</td>
</tr>
<tr>
<td>9.3.3</td>
<td>Binary black holes with a vanishing curvature parameter</td>
<td>107</td>
</tr>
<tr>
<td>9.3.4</td>
<td>Iteratively constructing asymptotically flat binary black hole data</td>
<td>111</td>
</tr>
<tr>
<td>10</td>
<td>Summary and conclusions</td>
<td>116</td>
</tr>
<tr>
<td>Appendices</td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>A</td>
<td>Lie derivatives</td>
<td>I</td>
</tr>
<tr>
<td>B</td>
<td>Black hole model: calculations</td>
<td>III</td>
</tr>
<tr>
<td>B.1</td>
<td>Coordinate free</td>
<td>III</td>
</tr>
<tr>
<td>B.1.1</td>
<td>Extrinsic 3-curvature</td>
<td>III</td>
</tr>
<tr>
<td>B.1.2</td>
<td>Extrinsic 2-curvature</td>
<td>V</td>
</tr>
<tr>
<td>B.2</td>
<td>Coordinates: $(r, \theta, \phi)$</td>
<td>VI</td>
</tr>
<tr>
<td>B.2.1</td>
<td>Derivatives of $\rho$</td>
<td>VI</td>
</tr>
</tbody>
</table>
B.2.2 Derivatives of $l_a$ ........................................ VII
B.3 Coordinates: $(\rho, \vartheta, \varphi)$ ................................ VIII

C Black hole model: code .......................................................... IX

References ................................................................. XXVIII
1 Introduction

1.1 History and motivation

In his 1905 paper ‘On the electrodynamics of moving bodies’ [1], Albert Einstein revolutionised our understanding of the relationship between space and time. In particular, he showed that observers moving in reference frames relative to one another experience different measures of time. This paper only dealt with inertial reference frames (i.e. frames in which Newton’s laws hold), and Einstein would not generalise his theory to include accelerating observers for another 10 years. With the support of numerous experimental results, Einstein’s General Theory of Relativity (GR) is widely accepted as the correct extension of Newtonian gravity. Here, we interpret the effect of gravity as the intrinsic curvature of a space-time with pseudo-Riemannian geometry. Every massive object inside of the space-time is governed by the Einstein equations, which form a set of ten, coupled, non-linear, second order partial differential equations (PDE) for the metric components, which describe how distance behaves in the space-time. Only a handful of exact solutions are known as the PDE system is notoriously difficult to solve. Moreover, if one is able to find an exact solution, it is difficult to establish whether or not the solution has physical significance; Einstein himself famously disregarded the Schwarzschild space-time as being unphysical. Nevertheless, we now know how to interpret this solution as the simplest description of a black hole.

Initially, the field equations were solved by imposing high degrees of symmetry. This is not the only way that it could be done, and in 1952 Choquet-Bruhat showed that the Einstein equations were well-posed, and hence could be solved as an initial value problem (IVP) [2]. Geometrically, this involves splitting the four-dimensional space-time into a set of three-dimensional submanifolds. Initial data is specified on one of these submanifolds and the Einstein equations are then used to evolve the data through space-time. Mathematically, this allowed researchers to split the Einstein equations into two sets of PDE, namely the constraint and evolution equations, which could be solved independently of one another. More specifically, to solve the full system one would first solve the constraints and then use their solution as initial data in the evolution equations. Furthermore, it was shown that if one were to choose an initial data set such that the constraints are satisfied then, there exists a corresponding unique solution to the full Einstein equations; this is a fact that prompts researchers to study the constraint equations.

The constraints themselves are a set of 4 equations for 12 unknowns. It follows that in order to solve the constraints one must first specify 8 of the components. It is unclear how these should be chosen, one of the most successful methods that has been employed is the conformal transformation. More recently, it has been suggested that the constraints are solved as an IVP. In a series of papers [3–6], Rácz uses a set of two-
dimensional surfaces in order to cast the constraints as an IVP. In his work, two such formulation were derived, namely the strongly hyperbolic and the parabolic-hyperbolic formulations. This is not the first exploitation of lower dimensional geometries to be proposed. In [7] an adapted set of 2-surfaces are used to construct an approximate form of the solution to the constraints, and in [8] the 3-dimensional geometry is constructed via surfaces of revolution. Such constructions are appealing as they allow the researcher more ‘control’ over the geometry. The properties of Rácz’s formulations are as of yet unclear. For example, if a solution is perturbed from its equilibrium state then we should expect this deviation to decay during our evolutions. This is the behaviour that we are interested in, and is what we will be studying in this thesis.

1.2 Notation and conventions

In this text we will adopt the abstract index notation for the representation of tensor quantities. In particular, a variable with indices will not describe a single tensorial component but rather will represent the abstract coordinate-free tensor itself.

We will be considering manifolds with various dimensions and hence find it prudent to distinguish between these. Three types of indices will be used, namely Greek, lower-case Latin, and upper-case Latin. These will run from 0-3, 1-3, and 2-3 respectively, the reason for this will become clear in the following work.

Greek indexed objects can have their indices raised and lowered in the standard way, using the space-time metric denoted by \( g_{\alpha\beta} \). In a similar way lower-case Latin indexed objects can have their indices lowered using the three-dimensional metric, typically represented as \( \gamma_{ab} \). The same follows for upper-case Latin indexed objects.

The 2-dimensional metric is typically written as \( h^{AB} \).

For a vector \( v^\alpha \) we have,

\[
\begin{align*}
 v^\alpha v_\alpha < 0 &\quad \Rightarrow \quad v^\alpha \text{ is time-like,} \\
 v^\alpha v_\alpha > 0 &\quad \Rightarrow \quad v^\alpha \text{ is space-like,} \\
 v^\alpha v_\alpha = 0 &\quad \Rightarrow \quad v^\alpha \text{ is null or light-like.}
\end{align*}
\]

There are two types of curvatures that will be introduced, namely the extrinsic and the intrinsic curvatures. Both of these will be dealt with respect to manifolds of different dimensionalities, sometimes simultaneously. To help differentiate between these, the second fundamental form associated with an \( n \)-dimensional manifold will be called the ‘extrinsic \( n \)-curvature’. The corresponding intrinsic curvature will be denoted by \( (n)R \).

Some metrics will occur frequently and as such have particular symbols associated with them. These are listed as follows.
1. The flat four-dimensional metric in standard polar coordinates:

\[ \eta_{\alpha\beta} = -dt_\alpha dt_\beta + dr_\alpha dr_\beta + r^2 d\theta_\alpha d\theta_\beta + r^2 \sin^2(\theta) d\phi_\alpha d\phi_\beta. \quad (1.4) \]

2. The three-dimensional Euclidean metric in standard polar coordinates:

\[ \delta_{ab} = dr_a dr_b + r^2 d\theta_a d\theta_b + r^2 \sin^2(\theta) d\phi_a d\phi_b. \quad (1.5) \]

The associated covariant derivative is \((\delta)\nabla_a\).

3. The two-dimensional 2-sphere metric in standard polar coordinates:

\[ \sigma_{AB} = d\theta_A d\theta_B + \sin^2(\theta) d\phi_A d\phi_B. \quad (1.6) \]

The corresponding covariant derivative is \((\sigma)\nabla_A\).

If \(g_{\alpha\beta}\) is a metric and \(\nabla_\alpha\) is the associated covariant derivative, we will write

\[ \nabla^2 = \nabla^\alpha \nabla_\alpha = g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \quad (1.7) \]

Finally, due to the nature of this work a large number of equations are present, all of which are important to develop an understanding of the work. However, some hold more importance than others. To help highlight the more prominent equations coloured boxes have been used. Red boxes are used to emphasize equations that the section is focused on. In a similar manner, green boxes are used to draw attention to equations that are fundamental in understanding the content of the red boxes. The symbol ‘\(\dot{=}\)’ will be used to mean that the relation only holds in a particular coordinate system.
1.3 Thesis summary

We end the introduction by presenting a paragraph for each section. The first part of this thesis is dedicated to the mathematical theory required for our research.

2. PDE Summary:

A brief summary of the theory of partial differential equations (PDEs) is given. We focus primarily on the relationship between the type of PDE system and what this tells us about the equations with regards to the initial value problem (IVP). We also introduce spin-weighted spherical harmonics (SWSH) and briefly describe how they may be used numerically. Finally, we discuss the mathematical process of linearisation.

3. The geometry of the \((n + 1)\)-decomposition:

The core concept of a hypersurface is introduced. We discuss the first and second fundamental forms of a hypersurface, and derive the Gauss, Codazzi, and Ricci equations.

4. The Einstein constraint equations:

The \((3+1)\)-form of the Einstein equations is derived, and split into the constraint and evolution equations. The ADM equations are also derived, and are used to demonstrate why it is non-trivial to show that the IVP of Einstein’s equations is well-posed. Finally, an outline of the proof that the IVP of Einstein’s equations is well-posed is presented.

5. The elliptic form of the constraints:

Focusing on how one may solve the constraint equations, we introduce the conformal method. A general conformal transformation is performed and used to show how one may pick gauge freedoms to simplify the resulting equations. A simple example of how to solve the constraints is then used as a demonstration of the usefulness of this formulation.

6. An evolutionary formulation of the constraint equations:

A \((2+1)\)-decomposition is performed on the constraints so that they may be cast as a well-posed IVP. Due to the present freedoms, two evolutionary systems are derived namely, the ‘strongly hyperbolic’ and the ‘parabolic-hyperbolic’ systems. We prove the type of equations that each formulation presents, and give their linearisations.

The second part of the thesis is focused on our original work. These paragraphs will not only outline the contents of each section but will also summarise the key results.
7. Schwarzschild decomposition:

We take a known solution of Einstein’s equations and perform the $(3 + 1)$ and $(2 + 1)$-decompositions, which will be used throughout the remainder of the work.

8. Asymptotically hyperbolic foliations:

We begin by defining what is meant by ‘asymptotically hyperboloidal’.

8.1 The strongly hyperbolic formulation.

An appropriate foliation of the Minkowski space-time is performed and the constraints are solved to produce a family of spherically symmetric solutions. We show that these can be embedded into a Schwarzschild space-time, and proceed to study linear perturbations of these data. Despite violating the hyperbolicity condition the equations are stable to perturbations, a fact that is explained via the consideration of an ‘equivalent’ second-order PDE. We end by showing how one may solve the linearised equations as a boundary value problem (BVP).

8.2 The parabolic-hyperbolic formulation.

We directly consider foliations of the Schwarzschild space-time and show that the resulting $(2 + 1)$-system is stable to both linear and non-linear perturbations. We also show that the linearised equations are ‘equivalent’ to a second order PDE.

9. Asymptotically flat black hole(s) in the parabolic-hyperbolic formulation:

We begin with a summary of metrics that take a Kerr-Schild form.

9.1 Non-linear perturbations of the Schwarzschild solution.

An asymptotically flat foliation of the Schwarzschild solution is performed and the corresponding constraints are solved for a general family of spherically symmetric solutions. We show that these are asymptotically flat only if certain conditions are met.

9.2 A model for multiple black holes.

We introduce a model that can be used to choose free data that describe multiple black holes. A more in-depth discussion is presented for binary black holes. We predict that the asymptotics of the produced solution can be matched to the previously found spherically symmetric solutions. We use this model to show that there exists a set of cases that are asymptotically flat. For the ones that are not inherently flat, we introduce an
iterative construction to choose the initial data such that they are asymptotically flat.
2 PDE summary

We begin by summarising needed information on the theory of partial differential equations. We will first introduce different types of PDE and some of the associated results. We will then discuss methods for solving and studying the equations. These discussions will be based on [9–16]. In this section, indices will not always be consistent with what was outlined in Section 1.2 as our discussion is not restricted by dimensionality. Furthermore, we will be considering open subsets of $\mathbb{R}^n$, this is not a restriction as non-Euclidean geometries are locally $\mathbb{R}^n$.

2.1 Types of PDE

General relativity is a geometric theory, primarily centred around solving the Einstein equations. Whilst these equations are hyperbolic in nature, the constraint equations (which will be introduced in the following sections) may not be, this is due to the number of freedoms present. As a simplified illustration of this, one may consider the following PDE

$$\partial_x^2 f + \partial_y^2 f + \partial_x^2 g - \partial_y^2 g + \partial_x^2 h - \partial_y h = 0,$$

(2.1)

which forms a single PDE for three unknown functions $f = f(x, y)$, $g = g(x, y)$, and $h = h(x, y)$. Thus, to find a solution one must specify two of the functions, depending on the choices made the equation will take a different form. In particular, solving for $f$, $g$, and $h$ will give an elliptic, hyperbolic, and parabolic equation, respectively.

Whilst there are several advantages in being able to identify the type of PDE that one is dealing with, we will primarily be interested in the relationship between the form of the PDE and what this may tell us about it having a well-posed IVP.

**Definition 2.1.** A system of partial differential equations with solution $u = u(t, \vec{x})$ (where $\vec{x} = (x_1, \ldots, x_{n-1})$ is the vector of variables) is said to be well-posed provided that there exists a unique solution that depends continuously on the initial data.

In other words, a system of PDE is well-posed if small changes in the initial data correspond to small changes in the solution itself [11].

To understand the relationship between the form of the equation and being well-posed, we introduce the principal symbol [9].

**Definition 2.2.** A general linear system of homogeneous partial differential equations takes the form

$$\sum_{|\alpha| \leq s} A^\alpha(\vec{x}) D^\alpha u = 0,$$

(2.2)
where $j$ is a multi-index with maximum order $s$, $D^j u$ is the collection of all partial derivatives of the unknown function $u = u(t, \vec{x})$, and $\vec{x} = (x^1, ..., x^{n-1})$ is the collection of variables. Then the principal symbol is the map from $\mathbb{R}^{2n}$ to $k \times k$ real matrices defined by the relation

$$
\sigma(t, \vec{x}, \xi^i) = \sum_{|j|=s} A^j(t, \vec{x}) \xi^j,
$$

(2.3)

and the characteristic polynomial of the system is the determinant of the principal symbol.

The characteristic polynomial associated with any PDE system allows for the classification of the PDE type \[9\].

**Definition 2.3.** A set of PDE is called:

1. ‘Hyperbolic’ provided that all roots of the characteristic polynomial are real.
2. ‘Elliptic’ if the set $\{ (\vec{x}, \xi^i) : \det |\sigma(\vec{x}, \xi^i)| = 0 \}$ is empty.

There are several notions of both hyperbolicity and ellipticity present within the literature. To explore these, we restrict ourselves to second order systems.

$$
A^{(0,0)} \partial^2_t u + B^{\alpha} \partial_\alpha u + L[u] = s(u),
$$

$$
L[u] = \sum_{i,j=1}^n A^{(i,j)} \partial_{x^i} \partial_{x^j} u + \sum_{i=1}^n B^i \partial_{x^i} u,
$$

(2.4)

where the coefficients $A^{(\alpha,\beta)}$ and $B^\alpha$ are $n \times n$ matrices, $u = u(t, \vec{x})$ is an $n$-dimensional vector of unknowns, and $s(u)$ is a source term that may depend on $u$ but not its derivatives. This is a system of first order evolution equations if $A^{(\alpha,\beta)} = 0$.

The concept of hyperbolicity plays a fundamental role in General Relativity. To help develop a more intuitive understanding of hyperbolicity we consider a special case of hyperbolic PDE.

**Special case.** Suppose that $A^{(\alpha,\beta)}$ is positive definite. Then the operator $A^{(0,0)} \partial^2_t - L$ is ‘hyperbolic’ if there exists a positive constant $\nu \in \mathbb{R}^+$ such that,

$$
A^{(i,j)} \xi_i \xi_j \geq \nu \delta^{ij} \xi_i \xi_j,
$$

(2.5)

where $\delta^{ij}$ is the $n$-dimensional Euclidean metric and $\xi_i \in \mathbb{R}^n$.

From a heuristic perspective, hyperbolic equations can be thought of as admitting wave-like solutions \[10\].
Example 1. The wave-equation\(^1\),

\[
\partial_t^2 u(t, \vec{x}) - c(t)^2 \delta^{ab} \partial_a \partial_b u(t, \vec{x}) = 0,
\]

(2.6)
is hyperbolic where \(c(t)^2\) is a positive, real-valued function with \(t \in I\) where \(I\) is the interval of interest. We will require that \(c(t)^2\) is bounded away from zero for all \(t \in I\). Here we have that \(A^{(a,b)} = c(t)^2 \delta^{ab}\). Choosing \(\nu = \min_{t \in I} c(t)^2\) gives

\[
c(t)^2 \delta^{ab} \xi_a \xi_b \geq \nu \delta^{ab} \xi_a \xi_b.
\]

which proves the claim.

This example lends support to the heuristic interpretation. A particularly important property of hyperbolic equations is that they are well-posed from the perspective of the IVP.

Definition 2.4. A system of evolution equations,

\[
B^0 \partial_t u + \sum_{i,j=1}^n A^{(i,j)} \partial_{x^i} \partial_{x^j} u + \sum_{i=1}^n B^i \partial_{x^i} u = s(u),
\]

(2.7)
is called ‘symmetric hyperbolic’ if

1. \(B^0\) is positive definite and each \(B^i\) is symmetric,

2. or there exists a matrix \(H\) such that \(HB^0\) is positive definite and each \(HB^i\) is symmetric. In this case, \(H\) is called a ‘symmetrizer’.

In each of the above definitions we see that the PDE types are (partially\(^2\)) classified by the relationship between the \(t\) and \(x^i\) variables. This trend continues into the notion of ellipticity.

Definition 2.5. Suppose the matrices \(A^{(0,0)} = B^0 = 0\) and each \(A^{(a,b)}\) are negative definite. Then the operator \(L\) is called ‘uniformly elliptic’ if there exists a positive constant \(\nu \in \mathbb{R}\) such that,

\[
\sum_{\alpha, \beta = 0}^n A^{(\alpha, \beta)} \xi_\alpha \xi_\beta \geq \nu \delta^{\alpha \beta} \xi_\alpha \xi_\beta.
\]

(2.8)

\(^1\)For the standard wave equation we typically have \(c(t)^2 = \text{constant}\). This has been generalised to a function, as it will be more relevant in the coming sections.

\(^2\)The geometry of the problem will also have an effect on the relationship between the \(t\) and \(x_i\) coordinates.
Intuitively, one may view elliptic equations as giving rise to solutions with exponential-like behaviour. It follows that the Laplace equation is an example of an elliptic system.

**Example 2.** The two-dimensional Laplace equation,

$$\partial_t^2 u = -\partial_x^2 u,$$  \hspace{1cm} (2.9)

is not a well-posed IVP [11].

To demonstrate this we define,

$$u^{(1)} = \partial_t u, \quad u^{(2)} = \partial_x u,$$  \hspace{1cm} (2.10)

and consider their time derivatives.

$$\partial_t u^{(1)} = \partial_t^2 u = -\partial_x^2 u = \partial_x u^{(2)},
\partial_t u^{(2)} = \partial_t \partial_x u = \partial_x \partial_t u = \partial_x u^{(1)}.$$  \hspace{1cm} (2.11)

Choosing the Fourier mode $u = u_0 e^{k(t+ix)}$ as our initial datum gives,

$$u^{(1)} = u_0 k e^{k(t+ix)}, \quad u^{(2)} = u_0 k i e^{k(t+ix)}.$$  \hspace{1cm} (2.12)

Here we see that the solution will grow with a rate dependent on the exact value of $k$ and hence will possess a strong dependence on the initial data.

The above example suggest that elliptic systems may not be well-posed with respect to the IVP. Indeed this is the case. To further justify this we express $u$ as an infinite sum of Fourier modes:

$$u = \sum_{k=0}^{\infty} u_k e^{k(t+ix)}.$$  \hspace{1cm} (2.13)

By making the assumption that $u$ is sufficiently smooth we are able to calculate the following:

$$u^{(1)} = \sum_{k=0}^{\infty} u_k k e^{k(t+ix)}, \quad u^{(2)} = i \sum_{k=0}^{\infty} k u_k e^{k(t+ix)}.$$  \hspace{1cm} (2.14)

If we were to solve these equations numerically, then we would not be able to calculate the complete sum. To discuss this, we split each sum into two parts

$$u^{(1)} = \sum_{k=0}^{L} u_k k e^{k(t+ix)} + \sum_{k=L+1}^{\infty} u_k k e^{k(t+ix)},$$  \hspace{1cm} (2.15)

$$u^{(2)} = i \sum_{k=0}^{L} k u_k e^{k(t+ix)} + i \sum_{k=L+1}^{\infty} k u_k e^{k(t+ix)}.$$  \hspace{1cm} (2.16)
where $L = \text{constant}$ is the band-limit (i.e. the largest $k$ value considered). We assume that the first sum in each of the above equations approximates the overall solution, and interpret the second summation as the error in our approximation. For this to be a reasonable estimate of the solution we would expect that the error goes to zero as the band-limit goes to infinity. However, this will not be the case here as the error term is unbounded. This helps demonstrate that elliptic PDE do not have a well-posed IVP. For a more in depth discussion on error we refer the interested reader to [16].

The concept of ellipticity allows us to define the parabolic PDE.

**Definition 2.6.** The equation,
\[
\partial_t u = L[u], \tag{2.17}
\]
is ‘uniformly parabolic’ if the operator $L$ is uniformly elliptic [10].

The typical example of a parabolic PDE is the heat equation $\partial_t u = \partial^2_x u$. One immediately sees that it is of the form given in the above definition. It follows that we may think of as such equations as giving rise to diffusive solutions. Parabolic PDEs are well-posed with respect to the IVP only in the direction of increasing $t$. Under the transformation $t \mapsto -t$ the equations are no longer well-posed.

### 2.2 Spin-weighted-spherical-harmonics (SWSH)

SWSH provide a tool for solving a PDE on the 2-sphere by allowing one to write the original system as a set of ODE. We will only give a brief summary of the theory as many of the precise details go beyond the scope of this thesis. The discussion we present will be based on [12,15].

To begin with we consider a unit sphere embedded into a three-dimensional vector space equipped with the standard Euclidean norm. For each point $p$ on the sphere, one may define the unit normal $n^j$ and construct an orthonormal frame $(n^j, a^j, b^j)^4$, where its orientation can be fixed by imposing the right-hand rule [15]. Using this frame, one may introduce a complex triad $(n^j, m^j, \bar{m}^j)$

\[
m^j = \frac{1}{\sqrt{2}} (a^j - ib^j), \quad \bar{m}^j = \frac{1}{\sqrt{2}} (a^j + ib^j). \tag{2.18}
\]

---

3Here we have assumed that $t \in [t_0, t_f]$ with $t_f > t_0$. If $t_0 > t_f$ then the word ‘increasing’ should be replaced with ‘decreasing’.

4As a consequence of the no hair theorem we note that such a frame cannot be smoothly constructed at all points on the 2-sphere. However, we will not discuss this problem further as it will not cause us any issues. For more details, we refer the reader to [15].
Even though the orientation of the frame has been fixed, the exact choice of \( a^i \) and \( b^j \) is not uniquely determined. In fact, a frame \((n^j, \tilde{m}^j, \bar{m}^j)\) can be related to the frame \((m^j, \tilde{m}^j)\) by a rotation through an angle \( \rho \), i.e.

\[
(\tilde{m}^j, \bar{m}^j) = (e^{i\rho} m^j, e^{-i\rho} \tilde{m}^j).
\] (2.19)

Suppose that a frame \((m^j, \tilde{m}^j)\) has been picked and we let the corresponding dual frame be represented as \((w_k, \bar{w}_k)\). Since the duality relation must be satisfied, we will have the transformation rule

\[
(\tilde{w}_k, \bar{w}_k) = (e^{-i\rho} w_k, e^{i\rho} \bar{w}_k).
\] (2.20)

Consider an arbitrary smooth tensor field \( T \) of type \((g + h, p + q)\), defined on the unit sphere with \( g, h, p, q \geq 0 \). Then the components of the tensor may be written as functions of the form \([12]\),

\[
\tilde{T}^{a_1 \ldots a_s b_1 \ldots b_h c_1 \ldots c_p, d_1 \ldots d_q} := T(\tilde{w}_{a_1}, \ldots, \tilde{w}_{a_s}, \tilde{w}_{b_1}, \ldots, \tilde{w}_{b_h}, \tilde{m}^{c_1}, \ldots, \tilde{m}^{c_p}, \bar{m}^{d_1}, \ldots, \bar{m}^{d_q}) \quad (2.21)
\]

Making use of the transformation rules of Eqs. (2.19) and (2.20) we are able express the tensor in terms of \((m^j, \tilde{m}^j)\) and \((w_k, \bar{w}_k)\)

\[
\tilde{T}^{a_1 \ldots a_s b_1 \ldots b_h c_1 \ldots c_p, d_1 \ldots d_q} = e^{i(h+p-q-g)\rho} T(\bar{w}_{a_1}, \ldots, \bar{w}_{a_s}, \bar{w}_{b_1}, \ldots, \bar{w}_{b_h}, \bar{m}^{c_1}, \ldots, \bar{m}^{c_p}, \bar{m}^{d_1}, \ldots, \bar{m}^{d_q})
\]

\[
= e^{i(-g+h+p-q)\rho} \tilde{T}^{a_1 \ldots a_s b_1 \ldots b_h c_1 \ldots c_p, d_1 \ldots d_q}. \quad (2.22)
\]

This allows one to see that for an arbitrary tensor field as given above the spin-weight \( s \) of the tensor components is \( s = -h + g + p - q \).

Hitherto the vectors \((a^i, b^j)\) have been arbitrary. However, we will proceed by fixing the initial basis in polar coordinates with the standard restriction \( \theta \in (0, \pi) \) and \( \phi \in (0, 2\pi) \). Then \( a^i = \partial \theta^i \) and \( b^j = \csc(\theta) \partial \phi^j \).

Consider now the space of square integrable functions\(^5\) on the 2-sphere with inner product,

\[
<f, g> = \int_{S^2} f \bar{g} \sin(\theta) d\theta d\phi, \quad (2.23)
\]

where \( \bar{g} \) is the complex conjugate of \( g \), for all \( f, g \in S^2 \). The SWSH, \( Y_{l,m}(\theta, \phi) \), can be thought of as the generalisation of the standard spherical harmonics, \( Y_{l,m}(\theta, \phi) \) and form a complete orthonormal basis for all square integrable functions on the 2-sphere with spin-weight \( s \). From the orthogonality and completeness of the SWSH, it follows that

\[
\int_{S^2} Y_{l_1,m_1}(\theta, \phi) \bar{Y}_{l_2,m_2}(\theta, \phi) \sin(\theta) d\theta d\phi = \delta_{l_1,l_2} \delta_{m_1,m_2}. \quad (2.24)
\]

\(^5\)For \( s = 0 \) these are the standard square integrable functions defined on the 2-sphere (i.e. \( f, g \in L^2(S^2) \)). For \( s \neq 0 \) we refer the reader to [12] for a more precise definition.
Furthermore, any square-integrable function with spin-weight $s$ defined on the 2-

sphere can be written as a sum of SWSHs.

$$s f = \sum_{l=s}^{\infty} \sum_{m=-|l|}^{|l|} f_{l,m} Y_{l,m}(\theta, \phi),$$  \hspace{1cm} (2.25)

where $f_{l,m}$ are the complex coefficients associated with $s f$. In particular, these coef-
ficients do not (by definition) possess a dependence on $\theta$ or $\phi$. Each spin-weighted
function with spin $s$ can be related to a $s+1$ (or $s-1$) function via the application
of the $\eth$ operators

$$\eth(\cdot) : s f \rightarrow s+1 f, \quad s f \mapsto \eth(s f) = \sqrt{2} m^a \partial_a s f - s f s \cot(\theta),$$  \hspace{1cm} (2.26)

$$\bar{\eth}(\cdot) : s f \rightarrow s-1 f, \quad s f \mapsto \bar{\eth}(s f) = \sqrt{2} \bar{m}^a \partial_a s f + s f s \cot(\theta).$$  \hspace{1cm} (2.27)

Furthermore, any of the SWSH can be obtained through the use of these operators

$$\eth(\cdot) : s Y_{l,m}(\theta, \phi) \rightarrow s+1 Y_{l,m}(\theta, \phi),$$
$$s Y_{l,m}(\theta, \phi) \mapsto \eth(s Y_{l,m}(\theta, \phi)) = -\sqrt{(l-s)(l+s+1)} s+1 Y_{l,m}(\theta, \phi),$$  \hspace{1cm} (2.28)

$$\bar{\eth}(\cdot) : s Y_{l,m}(\theta, \phi) \rightarrow s-1 Y_{l,m}(\theta, \phi),$$
$$s Y_{l,m}(\theta, \phi) \mapsto \bar{\eth}(s Y_{l,m}(\theta, \phi)) = \sqrt{(l+s)(l-s+1)} s-1 Y_{l,m}(\theta, \phi).$$  \hspace{1cm} (2.29)

Application of the $\eth$ operators onto the SWSH has the effect of raising and lowering
the spin-weight. Thus, one may view the $\eth$ operators as ‘ladder operators’ for the
SWSH. Moreover, it should be noted that for $s = 0$ the standard spherical harmonics
are returned.

To demonstrate the power of this formalism we consider an example.

**Example 3.** The wave equation\(^6\) on the 2-sphere is,

$$\partial_r^2 u - c(r)^2 \left( \partial_\theta^2 u + \csc(\theta) \partial_\phi^2 u + \cot(\theta) \partial_\theta u \right) = 0,$$  \hspace{1cm} (2.30)

for an unknown scalar function $u = u(r, \theta, \phi)$. The goal here is to express this equation
in terms of the SWSH. We begin by noting that,

$$\partial_\theta^a = \frac{1}{\sqrt{2}} (m^a + \bar{m}^a), \quad \partial_\phi^a = \frac{i}{\sqrt{2}} (m^a - \bar{m}^a) \sin(\theta).$$  \hspace{1cm} (2.31)

\(^6\)Once again the wave equation we present here differs from its standard form. Not only is $c(r)$
non-constant, we also have $r$ playing the role of the *time coordinate*. The reason for this will become
clear in the following sections.
which allows us to write,

\[0 = \partial^2_r u - c^2(r) \left( \frac{1}{\sqrt{2}} (\partial_b ((m^a + \bar{m}^a) \partial_a u + i \csc(\theta) \partial_a (m^a - \bar{m}^a) \partial_a u)) + \cot(\theta) \partial_b u \right) = \partial^2_r u - c^2(r) \left( m^a \partial_a (\bar{m}^b \partial_b u) + \bar{m}^a \partial_a (m^b \partial_b u) + \cot(\theta) \partial_b u \right). \tag{2.32} \]

In order to express this equation in terms of the \( \eth \) operators we must make the assertion that \( u \) is a smooth function. Using that \( u \) has zero spin allows us to write

\[\bar{m}^a \partial_a (m^b \partial_b u) = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \eth(u) + m^a \partial_a u \cot(\theta) \right), \tag{2.33}\]

\[m^a \partial_a (\bar{m}^b \partial_b u) = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \eth(u) + m^a \partial_a u \cot(\theta) \right), \tag{2.34}\]

which we put into Eq. (2.32) to get

\[0 = \partial^2_r u - c^2(r) \left( \eth(u) + \eth(u) \right). \tag{2.35} \]

Using the expansion,

\[u = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} u_{l,m} Y_{l,m}(\theta,\phi), \tag{2.36}\]

we get\(^7\),

\[0 = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} (\partial^2_r u_{l,m} - c^2(r) l(l+1) u_{l,m}) Y_{l,m}(\theta,\phi). \tag{2.37} \]

It follows from the orthogonality of the SWSH that there can be no cancellation and hence we have that for each \( l = 0, \ldots, \infty \) and \( m = -l, \ldots, l \),

\[\partial^2_r u_{l,m} - c^2(r) l(l+1) u_{l,m} = 0. \tag{2.38} \]

Thus, use of the SWSH has allowed us to express this PDE as a system of ODE. Moreover, to develop an understanding of the \( r \)-behaviour of the equation, it is enough to study the behaviour of the coefficients.

\(^7\)In general, the relation \( \partial^2_r (\sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} u_{l,m} Y_{l,m}(\theta,\phi)) = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} \partial^2_r (u_{l,m}) Y_{l,m}(\theta,\phi) \) is not true. However, under the assumption that \( u \) is sufficiently smooth the equation will hold.
To the best of the author’s knowledge the numerical treatment of SWSH was first developed in [17]. Here spin-weighted quantities are written as a band-limited expression of the form,

$$s f = \lim_{L \to \infty} \sum_{l=0}^{L} \sum_{m=-l}^{l} f_{l,m} s Y_{l,m} (\theta, \phi).$$

(2.39)

It is assumed that the function in question may be completely decomposed into a finite linear combination of basis functions. If the function $s f$ is to be smooth, then we require that the coefficients decay exponentially. i.e.

$$f_{l,m} \sim \alpha e^{-|\kappa|l}.$$

(2.40)

where $\alpha, \kappa \in \mathbb{R}$ are constants. Our initial data should, in principle, have infinitely many modes, this is the set of all smooth functions. However, the restriction to finitely many modes must be made for the numerical implementation.

### 2.3 Linearisation

Linearisation is a process that approximates small changes in a neighbourhood of a known solution to some equation [13]. This is done by adding a linear perturbation to a known solution and examining the behaviour in the neighbourhood of the solution [13]. Such techniques can be employed to study the stability of a solution by examining the long-term behaviour after such a perturbation.

**Definition 2.7.** A solution $u$ is said to be ‘asymptotically stable’ if the perturbations behave in a controlled manner.

The above is a working definition. A requirement on the decay rate must also be imposed. The exact rate of decay changes depending on the geometry of the system. This will be discussed more explicitly in Section 8. For now, we will outline the process of linearisation.

Suppose one has a known exact solution to an equation that takes the following form,

$$F(g_0) = 0.$$

(2.41)

Then, perturbations of $g_0$ can be written in the form of a family of one-parameter (or multi-parameter) exact solutions $g = g(\lambda)$ where $\lambda$ measures the size of the perturbation from the initial solution [13]. Then,

$$F(g(\lambda)) = 0.$$

(2.42)
The statement that \( \lambda \) measures the size of the perturbation from the initial solution means that firstly, \( g \) depends differentiably on \( \lambda \) near \( \lambda = 0 \); and secondly, that \( g(0) = g_0 \). Although it would be ideal to find an explicit formula for \( g(\lambda) \) as a solution of Eq. (2.42) it may in general, be impossible [13]. However, Eq. (2.42) can still be used to find a linear approximation of small perturbations. This can be done by considering the derivative of Eq. (2.42) at \( \lambda = 0 \),

\[
\left. \frac{dF(g(\lambda))}{d\lambda} \right|_{\lambda=0} = DF(g(\lambda)) \Big|_{g(\lambda)=g(0)} \cdot \left. \frac{dg(\lambda)}{d\lambda} \right|_{\lambda=0} = 0.
\]

(2.43)

Here \( DF(g(\lambda)) \) is meant as the Jacobian of \( F \). Eq. (2.43) is a linear equation of the form,

\[
L(u) = 0,
\]

(2.44)

where

\[
u = \left. \frac{dg(\lambda)}{d\lambda} \right|_{\lambda=0},
\]

(2.45)

is the unknown of Eq. (2.44) and \( L \) is a linear operator [13]. Then, Eq. (2.44) may be easier to solve than Eq. (2.42). Applying Taylor’s Theorem, \( g(\lambda) \) can be expressed as

\[
g(\lambda) = g_0 + u\lambda + \mathcal{O}(\lambda^2),
\]

(2.46)

and hence, linear perturbations can be expressed as

\[
g(\lambda) \approx g_0 + u\lambda,
\]

(2.47)

for sufficiently small \( \lambda \). However, it should be said that it can be very challenging to estimate the error involved in using this approximation. Furthermore, while the existence of a family of solutions to Eq. (2.42) implies the existence of a solution to Eq. (2.44), the converse statement is not true [13].

Eq (2.44) is the linearisation of Eq. (2.42) with background solution \( g_0 \). Through the use of Eq. (2.47), we interpret the function \( u \) as a linear perturbation of \( g_0 \).

---

8The statement ‘for sufficiently small \( \lambda \)...’ means that \( \lambda \) should be chosen such that the term \( \lambda u \) is much smaller than \( g_0 \). Under the scaling \( u \mapsto \alpha u \) for \( \alpha \in \mathbb{R} \) we will require that \( \lambda \alpha u \) is much smaller than \( g_0 \).
3 The geometry of the \((n + 1)\)-decomposition

3.1 Hypersurfaces

We begin by considering an \((n + 1)\)-dimensional manifold \(M\) equipped with a Lorentzian metric \(g_{\alpha\beta}\) (i.e. metrics that have signature \((-1, 1, \ldots, 1)\)).

**Definition 3.1.** A submanifold of \(M\) is the embedding of an \(m\)-dimensional manifold, \(\tilde{\Sigma}\), into \(M\). An embedding \(\Phi\) is a diffeomorphism\(^9\) [19].

\[
\Phi : \tilde{\Sigma} \rightarrow \Phi(\tilde{\Sigma}) := \Sigma(\iota) \subset M. \tag{3.1}
\]

If \((n + 1) - m = 1\) the submanifold is referred to as a hypersurface.

The diffeomorphic quality of \(\Phi\) ensures that \(\Sigma(\iota)\) is not self-intersecting. The manifold \(M\) acts as an ambient space of \(\Sigma(\iota)\) [20].

**Example 4.** Consider the manifold \(\mathbb{R}^3\). An embedding, using the standard Cartesian coordinates, is

\[
\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^4, \quad (x, y, z) \mapsto (0, x, y, z) \tag{3.2}
\]

Depending on its orientation within the space-time, a hypersurface may be space-like, time-like, or null. The treatment of null hypersurfaces is rather different from that of the space-like and time-like hypersurfaces and as such will not be discussed here.

The causal character of the orthogonal covector field \(n_\sigma \in T^*_p \Sigma(\iota)\) can be used to identify the structure of the hypersurfaces. If \(\varepsilon := n_\sigma n^\sigma\) is positive then, \(n_\sigma\) is space-like and \(\Sigma(\iota)\) is time-like. Similarly if \(\varepsilon\) is negative, then \(n_\sigma\) is time-like and \(\Sigma(\iota)\) is space-like.

Given the unit normal \(n_\sigma\) we can define the projection operator,

\[
\gamma^\alpha_\beta := g^\alpha_\beta - \varepsilon n^\alpha n_\beta, \tag{3.3}
\]

which permits us to calculate the induced metric

\[
\gamma_{\alpha\beta} = \gamma^\gamma_\alpha \gamma^\iota_\beta g_{\gamma\iota} = g_{\alpha\beta} - \varepsilon n_\alpha n_\beta. \tag{3.4}
\]

Restricting to intrinsic coordinates, we can consider the signature of \(\gamma_{\alpha\beta}\). If the metric is positive definite, then \(\Sigma(\iota)\) is space-like. Similarly if it is Lorentzian, then the hypersurface is time-like.

For the covariant derivative, we have the following.

\(^9\)For a more detailed discussion on diffeomorphisms and embeddings see [18].
Theorem 3.1. Let \((n+1)\nabla_\alpha\) and \((n)\nabla_\alpha\) be the covariant derivatives associated with \(g_{\alpha\beta}\) and \(\gamma_{\alpha\beta}\) respectively. Then for \(u_\alpha \in T_p^*\Sigma^{(i)}\) we have
\[
(n)\nabla_\alpha u_\beta = \gamma'_\alpha \gamma'_{\beta} (n+1)\nabla_\sigma u_\delta.
\] (3.5)

Proof. [20].

In the full \((n + 1)\)-dimensional space-time quantities such as curvature can only be defined intrinsically. This stems from the lack of an ambient space. For the \(n\)-dimensional hypersurface, on the other hand, the space-time acts as the ambient space, which allows us to define the ‘extrinsic curvature’ as a measure how \(\Sigma^{(i)}\) bends inside of \(M\),
\[
K_{\alpha\beta} := -\gamma'_\alpha \gamma'_{\beta} (n+1)\nabla_\sigma n_\delta = -\gamma'_\alpha (n+1)\nabla_\sigma n_\beta.
\] (3.6)

Mathematicians commonly refer to this curvature as the second fundamental form. Since this is projected fully onto the hypersurface it follows that \(K_{\alpha\beta}\) is a purely spatial quantity.

Proposition 3.1. \(K_{\alpha\beta}\) is a symmetric tensor [11].

Proof. Let \(\xi_\beta\) be a unit covector field tangent to a congruence of time-like geodesics such that \(w^\beta \xi_\beta = 0\) for all \(w^\beta \in T_p\Sigma^{(i)}\). Then we consider a separate foliation \(\Sigma := \{\Sigma_T \mid \dot{T} = \text{constant}\}\) with \(\xi_\beta := (n+1)\nabla_\beta \dot{T}\). Since the components \(\xi_\beta\) are the gradient of a scalar it follows that \((n+1)\nabla_\beta \xi_\delta = (n+1)\nabla_\delta \xi_\beta\). Furthermore, since \(\xi_\beta\) is a unit vector that satisfies the geodesic equation, it follows that \((n+1)\nabla_\beta \xi_\delta \in T_p\Sigma^{(i)} \times T_p\Sigma^{(i)}\). \(\xi_\beta\) will coincide with \(n_\beta\) in \(\Sigma^{(i)}\) but not necessarily outside of it. Thus the derivatives of \(\xi_\beta\) and \(n_\beta\) in the directions tangential to \(\Sigma^{(i)}\) will be equal. i.e.
\[
\gamma'_\alpha (n+1)\nabla_\sigma n_\beta = \gamma'_\alpha (n+1)\nabla_\sigma \xi_\beta = (n+1)\nabla_\sigma \xi_\beta = (n+1)\nabla_\beta \xi_\alpha = \gamma'_\beta (n+1)\nabla_\sigma \xi_\alpha
\] (3.7)

This proves the claim. □

The eigenvalues of the second fundamental form are the principal curvatures of the hypersurface.

The induced acceleration is,
\[
a_\alpha := n^\sigma (n+1)\nabla_\sigma n_\alpha,
\] (3.8)
and is related to the extrinsic curvature by [22],
\[
K_{\alpha\beta} = -\gamma'_\alpha (n+1)\nabla_\sigma n_\beta = -(g^\sigma_\alpha - \varepsilon n^\sigma n_\alpha) (n+1)\nabla_\sigma n_\beta = -(n+1)\nabla_\alpha n_\beta + \varepsilon n_\alpha a_\beta.
\] (3.9)
Figure 1: A visual representation of the extrinsic curvature in a space-time. The dashed arrow represents the parallel transport of normal vector, $n^\sigma$, at $p$ along some geodesic that connects $p$ to $p'$. The difference between this transported vector and the exact normal vector $n^\alpha$ at $p'$ is the result of the bending of $\Sigma^{(i)}$ in the space-time $(M, g_{\alpha\beta})$. The projection of this difference directly measures the amount by which $\Sigma^{(i)}$ bends.

### 3.2 Foliating space-time with Cauchy surfaces

**Definition 3.2.** A Cauchy surface is a space-like hypersurface $\Sigma^{(i)}$ embedded into a manifold $M$ such that any inextendable time-like or null curve intersects $\Sigma^{(i)}$ exactly once.

Not all space-times admit such a surface; the interior solution of a Kerr black hole is one example. If a Cauchy surface can be defined, the space-time is said to be ‘globally hyperbolic’ [21]. We will restrict ourselves to such a space-time.

**Definition 3.3.** For a given function $T = T(x^\alpha)$, a foliation is a set of Cauchy surfaces $\Sigma := \{\Sigma^{(i)} \mid T = \text{constant}\}$ such that for all $p \in M$, there exists a unique $\Sigma^{(i)} \in \Sigma$ with $p \in \Sigma^{(i)}$. The function $T$ is called the ‘time-function’, and its level sets define the foliation. Each $\Sigma^{(i)} \in \Sigma$ is called a ‘leaf’ or a ‘slice’ of the foliation.

Since all points of $M$ must also be contained within $\Sigma$, we see that $M = \bigcup_{i \in \mathbb{R}} \Sigma^{(i)}$. Further, for any pair $\Sigma^{(i)}, \Sigma^{(j)} \in \Sigma$ with $i \neq j$, we have $\Sigma^{(i)} \cap \Sigma^{(j)} = \emptyset$. The time-function gives the interpretation that each hypersurface is a level surface of $M$, which is to say that for $p \in M$ and $\Sigma^{(i)} \in \Sigma$, we have $p \in \Sigma^{(i)}$ if and only if $T(p) - \text{constant} = 0$ and $dT(p) \neq 0$.

The gradient of the time-function can be used to calculate normal co-vector field

$$n_\beta = \pm \alpha^{(n+1)} \nabla_\beta T, \quad (3.10)$$

where $\alpha$ is the lapse function which we introduce as a normalisation factor of $n_\alpha$. The $\pm$ is chosen to ensure $n_\alpha$ is future-pointing.
We now seek coordinates that are adapted to the foliation. Fix a well-defined (local) coordinate system \( \{ x^a \}_{a \in [1,n]} \) on each \( \Sigma_{(i)} \in \Sigma \). In general, on each surface, the coordinates will not be the same\(^{10} \). However, if we assume that they vary smoothly between each hypersurface, then we may define a coordinate system on the full space-time \( \{ x^\alpha \}_{\alpha \in [0,n]} = \{ t, x^a \}_{a \in [1,n]} \), where \( t = T \). The corresponding coordinate basis is

\[
\{ \partial_t^\alpha, \partial_a^\alpha \}_{\alpha \in [1, n]} = \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x^a} \right\}_{a \in [1, n]}.
\]

The vector \( \partial_t^\alpha \) is tangent to lines of constant spatial coordinates and as such will have a component in the direction of \( n^\beta \). The amount of proper-time elapsed between each leaf is the lapse function \( \alpha \) and can be calculated as

\[
\pm \frac{1}{\alpha^2} = g^{\sigma \delta} (n+1) \nabla_\sigma T (n+1) \nabla_\delta T, \tag{3.12}
\]

where the sign is chosen to ensure that \( \alpha \) is real-valued. It can be related to the acceleration via the equation

\[
a_\beta = -n^{\sigma} (n+1) \nabla_\sigma \left( \alpha (n+1) \nabla_\beta T \right)
= \pm \left( n^{\sigma} (n+1) \nabla_\sigma \alpha (n+1) \nabla_\beta T + \alpha n^{\sigma} (n+1) \nabla_\sigma (n+1) \nabla_\beta T \right)
= \frac{1}{\alpha} \left[ n_\beta n^\sigma (n+1) \nabla_\sigma \alpha + \alpha n^{\sigma} (n+1) \nabla_\sigma \left( \frac{\nabla_\beta}{\alpha} \right) \right]
= \frac{1}{\alpha} \left( n^{\sigma} n_\beta (n+1) \nabla_\sigma \alpha - (n+1) \nabla_\beta \alpha \right) = -\epsilon^{\gamma} \nabla_\beta \alpha. \tag{3.13}
\]

The shift vector is defined as the difference between \( \alpha n^\beta \) and \( \partial_\beta T \)

\[
\partial_\beta^\gamma = \alpha n^\gamma + \beta^\gamma. \tag{3.14}
\]

Calculating the magnitude,

\[
g_{\sigma \delta} \partial_\sigma^\gamma \partial_\delta^\gamma = g_{\sigma \delta} \beta^\sigma \beta^\delta - \alpha^2. \tag{3.15}
\]

Thus if \( g_{\sigma \delta} \beta^\sigma \beta^\delta \) is less than, equal to, or greater than \( \alpha^2 \), then \( \partial_\beta^\gamma \) will be time-like, null, or space-like, respectively. Despite not always being time-like, \( \partial_\beta^\gamma \) is called the ‘time-vector’ [21]. A visualisation of the time-vector is given in Fig. 2.

\(^{10}\text{i.e. The coordinates will not flow in the direction of the norm } n^\alpha.\)
The line of \( x^\alpha = \text{constant} \) intersects each slice of the foliation exactly once and can be used to define the time-vector \( \partial_t^\alpha \). This line can be further used to define the shift vector associated with the coordinate system, \( \{ x^i \}_{i \in [1,n]} \), on each of the hypersurface, \( \Sigma(i) \in \Sigma \).

### 3.3 The \((n+1)\)-decomposition of the Riemann tensor

The \(n\)-dimensional Riemann tensor associated with \( \gamma_{ab} \) is given by the following Ricci identity

\[
2 (n) \nabla_{[\alpha} (n) \nabla_{\beta]} v_\sigma = (n) R^\delta_{\sigma \beta \alpha \nu} v_\delta,
\]

where \( v_\delta \) is any purely spatial vector.

The aim of this section is to find relations between the \((n+1)\)-dimensional Riemann tensor and the hypersurface quantities that have been introduced above.

The four-dimensional Riemann tensor can be expressed in terms of \(3 + 1\) quantities by considering the varying projections onto the hypersurface and in the normal direction. The derivation presented here is based on [21,22]

### 3.3.1 The Gauss, Codazzi, and Ricci equations

For this, instead of directly considering the full projection, we will first consider the Ricci identity given in Eq. (3.16). This we will be done by considering the spatial derivative of a purely spatial vector \( V^\sigma \),

\[
(n) \nabla_\beta V^\sigma = \gamma^\rho_{\beta \gamma^\sigma \lambda} \nabla_\rho V^\lambda = \gamma^\rho_{\beta} (\delta^\sigma_\lambda - \varepsilon n^\sigma n_\lambda) (n+1) \nabla_\rho V^\lambda
\]

\[
= \gamma^\rho_{\beta} (n+1) \nabla_\rho V^\sigma + \varepsilon \gamma^\rho_{\beta n^\sigma V^\lambda (n+1) \nabla_\rho n_\lambda}
= \gamma^\rho_{\beta} (n+1) \nabla_\rho V^\sigma - \varepsilon n^\sigma V^\nu K_{\beta \nu}.
\]
We now consider the second spatial derivative of a purely spatial vector $V^\sigma$,

\[(n)\nabla_\alpha (n)\nabla_\beta V^\sigma = \gamma^\psi_\alpha \gamma^\mu_\beta \gamma^\pi_\sigma (n+1)\nabla_\psi ((n)\nabla_\mu V^\pi)
\]

\[= \gamma^\psi_\alpha \gamma^\mu_\beta \gamma^\pi_\sigma (n+1)\nabla_\psi (\gamma^\rho_\mu (n+1)\nabla_\rho V^\sigma - \varepsilon n^\sigma V^\nu K_{\mu\nu})
\]

\[= \gamma^\psi_\alpha \gamma^\rho_\beta (n+1)\nabla_\psi (n+1)\nabla_\rho V^\sigma + \varepsilon (V^\lambda K_{\beta\lambda} K_{\alpha}^\sigma + n^\rho K_{\alpha\beta} (n+1)\nabla_\rho V^\sigma).
\]

(3.18)

Anti-symmetrising allows us to calculate the $n$-dimensional Riemann tensor as,

\[(n)R^\sigma_{\beta\alpha} = 2\gamma^\psi_\alpha \gamma^\rho_\beta \gamma^\pi_\sigma (n+1)\nabla_\psi (n+1)\nabla_\rho V^\pi + 2\varepsilon K_{[\alpha}^\sigma K_{\beta]\lambda} V^\lambda
\]

(3.19)

where we have used that $K_{[\alpha\beta]} = 0$. Then, one may use the definition of the $(n + 1)$-dimensional intrinsic curvature to finally get

\[\frac{(n)R^\nu_{\sigma\beta\alpha} - \varepsilon (K_{\alpha\sigma} K_{\nu}^\beta - K_{\beta\sigma} K_{\nu}^\alpha)}{(n)R^\delta_{\rho\pi\psi}} = \gamma^\psi_\alpha \gamma^\rho_\beta \gamma^\pi_\sigma \gamma^\nu_{\delta (n+1)R^\delta_{\rho\pi\psi}}.
\]

(3.20)

This is the so-called Gauss equation, which relates the full spatial projection of the four-dimensional Riemann tensor to the intrinsic curvature defined on the hypersurfaces.

We now consider the spatial projection of three of the indices and one index projected in the normal direction

\[\gamma^\rho_\alpha \gamma^\xi_\beta \gamma^\pi_\sigma n^{\psi(n+1)} R_{\rho\xi\pi\psi} = \gamma^\rho_\alpha \gamma^\xi_\beta \gamma^\pi_\sigma ((n+1)\nabla_\rho (n+1)\nabla_\xi (n+1)\nabla_\xi (n+1)\nabla_\rho) n_{\pi}
\]

\[= \gamma^\rho_\alpha \gamma^\xi_\beta \gamma^\pi_\sigma ((n+1)\nabla_\rho (-K_{\xi\pi} + \varepsilon n_{\xi\alpha}) - (n+1)\nabla_\xi (-K_{\rho\pi} + \varepsilon n_{\rho\alpha}))
\]

\[= \gamma^\rho_\alpha \gamma^\xi_\beta \gamma^\pi_\sigma (- (n+1)\nabla_\rho K_{\xi\pi} + (n+1)\nabla_\xi K_{\rho\pi}),
\]

(3.21)

and thus we have,

\[\gamma^\rho_\alpha \gamma^\xi_\beta \gamma^\pi_\sigma n^{\psi(n+1)} R_{\rho\xi\pi\psi} = (n)\nabla_\beta K_{\alpha\sigma} - (n)\nabla_\alpha K_{\beta\sigma}.
\]

(3.22)

This is the so-called Codazzi equation.

Next, project $(n+1)R_{\alpha\beta\sigma\delta}$ twice in the normal direction, on non-consecutive indices, and twice onto the hypersurface

\[n^{\alpha\gamma_\eta} n^{\sigma\gamma_\delta} (n+1) R_{\alpha\beta\sigma\delta} = n^{\alpha\gamma_\eta} n^{\sigma\gamma_\delta} ((n+1)\nabla_\beta (n+1)\nabla_\alpha n^{\delta} - (n+1)\nabla_\alpha (n+1)\nabla_\beta n^{\delta})
\]

\[= \gamma^\beta_\eta \gamma^\delta_\alpha \left( K^{\delta}_{\alpha} (n+1)\nabla_\beta n^{\alpha} + (n+1)\nabla_\beta a^{\delta} + n^{\alpha} (n+1)\nabla_\alpha K^{\beta}_{\delta} - \varepsilon a^{\delta} a_{\beta} \right).
\]

(3.23)

The relation between the acceleration and the lapse function then allows us to write,

\[n^{\alpha\gamma_\eta} n^{\sigma\gamma_\delta} (n+1) R_{\alpha\beta\sigma\delta} = -K^\gamma_{\alpha} K^{\alpha}_{\eta} + n^{\alpha} (n+1)\nabla_\alpha K^{\gamma}_{\eta} - \varepsilon \alpha^{-1} (n)\nabla_\eta (n)\nabla_\alpha.
\]

(3.24)
At this point, it is convenient to consider the projection of the Lie derivative of the extrinsic curvature (a review of the Lie derivative is summarised in Appendix A).

\[ \mathcal{L}_n K_\eta = \gamma^\alpha_\eta \gamma^\beta_\eta \mathcal{L}_n K_{\alpha\beta} = \gamma^\alpha_\eta \gamma^\beta_\eta n^{(n+1)} \nabla_\sigma K_{\alpha\beta} - 2 K_{\eta\alpha} K^{\sigma}_{\eta} . \]  

(3.25)

This result can then be used to rewrite Eq. (3.24) as

\[ n^\alpha \gamma^\beta_\eta n^{(n+1)} R_{\alpha\beta\sigma\delta} = \mathcal{L}_n K_\eta + K_{\eta\alpha} K^{\alpha}_{\eta} - \varepsilon^{(n)} \nabla_\eta (n) \nabla_\alpha , \]  

(3.26)

which is the Ricci equation.

There are two other projections that we wish to consider. The first is the full projection of the space-time intrinsic curvature along the direction of the normal. Both of these projections are identically zero. For completeness this is shown in the following calculations. First, we make use of the Ricci identity,

\[ n^\nu n^\sigma n^\rho n^\mu (n+1) R^\mu_{\nu\rho\sigma} = n^\nu n^\sigma n^\rho (n+1) \nabla_\sigma (n+1) \nabla_\rho - (n+1) \nabla_\rho n^{(n+1)} \nabla_\sigma n_\nu . \]  

(3.27)

Before continuing with calculation, consider the following

\[ n^\nu (n+1) \nabla_\sigma n_\nu = 0 \implies (n+1) \nabla_\rho (n^\nu (n+1) \nabla_\sigma n_\nu) \]

\[ = n^\nu (n+1) \nabla_\rho (n+1) \nabla_\sigma n_\nu + (n+1) \nabla_\rho n^{(n+1)} \nabla_\sigma n_\nu = 0 , \]  

(3.28)

and thus we have

\[ n^\nu n^\sigma n^\rho n_\mu (n+1) R^\mu_{\nu\rho\sigma} = n^\nu n^\rho (n+1) \nabla_\sigma (n+1) \nabla_\rho - (n+1) \nabla_\rho n^{(n+1)} \nabla_\sigma n_\nu \]

\[ = n^\nu n^\rho \left( (n+1) \nabla_\sigma n^{(n+1)} \nabla_\rho n_\nu - (n+1) \nabla_\sigma n^{(n+1)} \nabla_\rho n_\nu \right) \]

\[ = 0 . \]  

(3.29)

Similarly,

\[ \gamma^\xi_\beta n^\rho n^\psi (n+1) R^\pi_{\psi\rho\xi} = \gamma^\xi_\beta n^\rho (n^\psi (n+1) \nabla_\xi (n+1) \nabla_\rho - (n+1) \nabla_\rho (n+1) \nabla_\xi) n_\psi = 0 . \]  

(3.30)

Together the Gauss, Codazzi, and Ricci equations provide all the information needed to perform a (3 + 1)-decomposition of Einstein’s equations. However, it will first be convenient to consider contracted versions of these equations.

We take a contraction of the Gauss equation to get

\[ \gamma^\psi_\alpha \gamma^\pi_\sigma (n+1) R_{\psi\pi} + \gamma^\psi_\alpha \gamma^\pi_\sigma n^\alpha n^\beta (n+1) R_{\alpha\psi\beta\pi} = (n) R_{\sigma\alpha} + \varepsilon (K_{\sigma\alpha} K - K_{\psi\alpha} K^{\nu}_{\psi} . \]  

(3.31)

Further contraction gives

\[ (n+1) R = (n) R + \varepsilon (K^2 - K_{\psi\alpha} K^{\nu}_{\psi}) . \]  

(3.32)
This is the scalar Gauss relation, which is the generalisation of his famed Theorem Egremium [21].

Returning to the Codazzi equation, we consider its contraction

\[ n^\psi (n+1) R_{\beta \psi} = (n+1) \nabla_\beta K - (n+1) \nabla_\sigma K^\sigma_{\beta \psi}. \quad (3.33) \]

This is the contracted Codazzi relation.

Finally, we consider the Ricci equation. However, this time we will not take a contraction. Instead, we note that Eq. (3.31) can be rearranged as

\[ \gamma^\psi_\alpha \pi_\sigma n^\alpha n^\beta (n+1) R_{\alpha \psi \beta \pi} = (n) R_{\sigma \alpha} + \varepsilon (K_{\sigma \alpha} K - K_{\nu \sigma} K^\nu_{\alpha}) - \gamma^\psi_\alpha \pi_\sigma (n+1) R_{\psi \pi}. \quad (3.34) \]

Equating this to the Ricci equation then gives

\[ \gamma^\beta_\eta \gamma^\alpha_\iota (n+1) R_{\alpha \beta} = (n) R_{\eta \iota} + \varepsilon K K_{\eta \iota} - \zeta_n K_{\iota \eta} - (1 - \varepsilon) K_{\iota \delta} K^\delta_\eta + \varepsilon n^{-1} (n) \nabla_\epsilon (n) \nabla_\eta \alpha. \quad (3.35) \]

For \( n + 1 = 4 \), the Weyl tensor \( C_{\alpha \beta \delta \sigma} \) can be decomposed into two independent symmetric tensors [23]

\[ E_{\beta \delta} := n^\alpha n^\sigma C_{\alpha \beta \delta \sigma}, \quad H_{\alpha \beta} := \frac{1}{2} \epsilon_{\psi \alpha \sigma \pi} n^\psi C^{\pi \sigma}_{\beta \psi}, \quad (3.36) \]

where \( \epsilon_{\psi \alpha \sigma \pi} \) is the Levi-Civita symbol, \( E_{\beta \delta} \) is called the ‘electric part’ and \( H_{\alpha \beta} \) the ‘magnetic part’ of the Weyl tensor. Both of these tensors are purely intrinsic to \( \Sigma_{(i)} \), a fact that follows from Eqs. (3.29) and (3.30). If the Einstein vacuum equations are satisfied, then Eq. (3.31) allows us to write the electric part of the Weyl tensor as

\[ E_{\alpha \beta} = - (3) R_{\alpha \beta} + \varepsilon (K_{\nu \alpha} K^\nu_{\alpha} - K_{\alpha \sigma} K^\sigma_{\alpha}). \quad (3.37) \]

This equation gives us a relationship between the extrinsic geometry of the hypersurface and the intrinsic geometry of the space-time it is embedded in. For example, if \( E_{\alpha \beta} \neq 0 \), then the triple \( (\Sigma_{(i)}, \gamma_{ab}, K_{ab}) \) cannot be embedded into Minkowski space-time.

For the remainder of this thesis we will restrict ourselves to \( n + 1 = 4 \).
4 The Einstein constraint equations

4.1 Derivation

Armed with an understanding of hypersurfaces, we turn our attention to the Einstein equations. In particular we will use the contracted Gauss, Codazzi, and Ricci relations to derive an evolutionary formulation of the equations.

In a four-dimensional space-time, the Einstein field equations take the form

\[ G_{\alpha\beta} := (4)^{R_{ab}} - \frac{1}{2} (4)^{Rg_{\alpha\beta}} - \Lambda g_{ab} = 8\pi T_{\alpha\beta}, \]  

where \( T_{\alpha\beta} \) is the energy-momentum tensor, \( G_{\alpha\beta} \) is the Einstein tensor, \( \Lambda \) is the cosmological constant, \( (4)^R \) and \( (4) \) are the Ricci tensor and scalar associated with the space-time metric \( g_{\alpha\beta} \), respectively. This gives a relationship between the matter content of the universe, given by the energy-momentum tensor, and part of its curvature, given by the Einstein tensor.

To express these equations as an IVP, we begin by foliating the space-time with a set of space-like Cauchy surfaces \( \Sigma := \{ \Sigma_T \mid T(x^\alpha) = \text{constant} \} \), which allows for the construction of a normal vector field \( n^\alpha \) on each \( \Sigma \) such that \( n^\alpha n_\alpha = -1 \). Then, we decompose the metric and energy momentum tensor in the standard way \[21\]

\[ g_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta, \]  

\[ T_{\alpha\beta} = E n_\alpha n_\beta + 2 n_{(\alpha} j_{\beta)} + S_{\alpha\beta}, \]  

where \( S_{\alpha\beta} = \gamma^\sigma g_\sigma^\alpha g_\delta^\beta T_{\sigma\delta}, \) \( j_\alpha = -\gamma^\sigma n^\alpha T_{\sigma\beta}, \) \( E = n^\alpha n^\beta T_{\alpha\beta} \) and \( \gamma_{\alpha\beta} \) is the induced metric on \( \Sigma \). This decomposition can only make sense if \( j_\alpha, S_{\alpha\beta} \) are purely spatial quantities, and hence we will also have the following orthogonality properties

\[ S_{\alpha\beta} n^\alpha = \gamma_{\alpha\beta} n^\alpha = 0, \]  

\[ j_\alpha n^\alpha = 0. \]  

Similarly, the Einstein tensor may be decomposed as

\[ G_{\alpha\beta} = \frac{1}{2} \mathcal{H} n_\alpha n_\beta + 2 n_{(\alpha} M_{\beta)} + E_{\alpha\beta}, \]  

where \( E_{\alpha\beta} = \gamma^\sigma a^\gamma^\delta G_{\sigma\delta}, \) \( M_\alpha = -\gamma^\sigma a^\nu T_{\sigma\beta} \) and \( \mathcal{H} = 2 n^\alpha n^\beta G_{\alpha\beta} \). Each of these quantities must satisfy orthogonality relations similar to the ones given above

\[ E_{\alpha\beta} n^\alpha = 0, \]  

\[ M_\alpha n^\alpha = 0. \]  

In particular this allows one to write,

\[ \mathcal{H} = 16\pi E, \]  

\[ M_\alpha = 8\pi j_\alpha, \]  

\[ E_{\alpha\beta} = 8\pi S_{\alpha\beta}. \]  

25
To obtain explicit formulas we consider the various projections of $G_{\alpha\beta}$, beginning with the fully normal projection

$$\mathcal{H} = 2^{(4)} R_{\alpha\beta} n^\alpha n^\beta - (4^{(4)} R g_{\alpha\beta} n^\alpha n^\beta + 2\Lambda$$

$$= (3) R + (K^2 - K_{\alpha\beta} K^{\alpha\beta} - 2\Lambda)$$

(4.8)

where the contracted Gauss equation has been used with $\varepsilon = -1$. We now make use of the Codazzi equation in the mixed projection of $G_{\alpha\beta}$.

$$\mathcal{M}_\alpha = \gamma^\gamma_{\alpha n} R^{(4)} \sigma_{\delta} - \frac{1}{2} (4^{(4)} R \gamma^\sigma_{\delta} + \Lambda \gamma^\sigma_{\delta}) n^\delta$$

$$= (3) \nabla_\alpha K - (3) \nabla_\sigma K^\sigma_{\alpha}.$$  (4.9)

where $^{(3)}\nabla$ is the covariant derivative associated with $\gamma_{\alpha\beta}$. Before considering the fully spatial projection, we will calculate the trace of Einstein’s equations

$$G^\alpha_{\alpha} = (4) R + 4\Lambda = 8\pi (E - S),$$  (4.10)

where $S = S^\alpha_{\alpha}$. Then,

$$\mathcal{E}_{\alpha\beta} = \gamma^\gamma_{\alpha n} (4^{(4)} R_{\sigma\delta} - 4\pi (E - S) \gamma_{\alpha\beta} + \Lambda \gamma_{\alpha\beta}$$

$$= (3) R + K K_{\alpha\beta} - 2 K_{\alpha\sigma} K^\sigma_{\beta} - \frac{1}{\alpha} (3) \nabla_\alpha \gamma_{\beta\alpha} - \mathcal{L}_n K_{\alpha\beta} + \Lambda \gamma_{\alpha\beta} - 4\pi (E - S) \gamma_{\alpha\beta}$$

$$\Leftrightarrow (3) R + K K_{\alpha\beta} - 2 K_{\alpha\sigma} K^\sigma_{\beta} - \frac{1}{\alpha} (3) \nabla_\alpha \gamma_{\beta\alpha} - \mathcal{L}_n K_{\alpha\beta} + \Lambda \gamma_{\alpha\beta}$$

$$= 8\pi \left( S_{\alpha\beta} - \frac{1}{2} (S - E) \right).$$  (4.11)

This concludes the $(3 + 1)$-decomposition of the Einstein field equations. However, this does not constitute the full evolutionary system as an evolution equation for the 3-metric is still needed. Such an equation may be derived as follows:

$$K_{\alpha\beta} = -\frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta} = \frac{1}{\alpha} (\mathcal{L}_\alpha \gamma_{\alpha\beta} + \mathcal{L}_\beta \gamma_{\alpha\beta})$$

$$\Leftrightarrow \mathcal{L}_\alpha \gamma_{\alpha\beta} = \mathcal{L}_\beta \gamma_{\alpha\beta} - 2\alpha K_{\alpha\beta}.$$  (4.12)

This allows us to write Einstein’s field equations as the following PDE system [21].

$$16\pi E = (3) R \left( K^2 - K_{\alpha\beta} K^{\alpha\beta} - 2\Lambda \right),$$  (4.13)

$$8\pi j_\alpha = (3) \nabla_\sigma K^\sigma_{\alpha} - (3) \nabla_\alpha K,$$  (4.14)

$$\mathcal{L}_\alpha \gamma_{\alpha\beta} = \mathcal{L}_\beta \gamma_{\alpha\beta} - 2\alpha K_{\alpha\beta}$$  (4.15)

$$(3) R_{\alpha\beta} + K K_{\alpha\beta} - 2 K_{\alpha\sigma} K^{\sigma}_{\beta} - \frac{1}{\alpha} (3) \nabla_\alpha \gamma_{\beta\alpha} - \mathcal{L}_n K_{\alpha\beta} + \Lambda \gamma_{\alpha\beta}$$

$$= 8\pi \left( S_{\alpha\beta} - \frac{1}{2} (S - E) \right).$$  (4.16)

26
Eqs. (4.13) and (4.14) form what is known as the constraint equations. There is no normal derivative present within these equations and hence they may be solved independently of Eqs. (4.15) and (4.16) (known as the evolution equations). The constraints may be thought of as the condition that the triple \((\Sigma_i, \gamma_{ab}, K_{ab})\) must satisfy if it is to be embedded into a space-time with solution \((M, g_{\alpha\beta})\) [22,24]. Given one such triple, the evolution equations describe how the field data change between each leaf of the foliation.

Notice that there are no evolution equations for the lapse function and shift vector, a consequence of which is that we may interpret these quantities as \textit{gauge freedoms} [11,25].

For the remainder of this work we will restrict ourselves to \(\Lambda = 0\).

### 4.2 The ADM equations

With the goal of simplifying the above PDE system, we introduce a coordinate basis that is adapted to the foliation. For this we use that

\[
\partial_t^\delta = (1, 0, 0, 0). \tag{4.17}
\]

Since \((^4\nabla_\delta T)\) is constant along the \(x^\delta = \text{constant}\) lines the contraction \(\partial_{x^i} (^4\nabla_\delta T)\) must vanish. This implies that the spatial components of the normal co-vector field \(n_\delta\) must be identically zero, i.e. \(n_a = 0\). Coupled with the fact that the time-component of the shift vector must also be zero, we may construct the following

\[
n^\delta = \frac{1}{\alpha}(1, -\beta^a), \quad n_\delta = -\alpha(1, 0, 0, 0), \tag{4.18}
\]

and a simple calculation allows one to check that \(n^\delta n_\delta = -1\).

In this coordinate system, we have \(\gamma_{ab} = g_{ab}\). More specifically, components of the 3-metric induced on the hypersurface \(\Sigma_i\) are simply the spatial components of the full space-time metric. In matrix form,

\[
g^{\delta\sigma} = \begin{pmatrix}
-\alpha^2 + \beta^a\beta_a \\
\beta_a \\
\beta_b \\
\gamma_{ab}
\end{pmatrix}, \quad g^{\delta\sigma} = \begin{pmatrix}
-\alpha^{-2} \\
-\alpha^{-2}\beta^b \\
-\alpha^{-2}\beta^b \\
-\alpha^{-2}\beta^a\beta^b
\end{pmatrix}, \tag{4.19}
\]

with the associated line element

\[
ds^2 = g^{\delta\sigma} dx^\delta dx^\sigma = (-\alpha^2 dt^2 + \beta_a\beta^a) + 2\beta_a dx^a dt + \gamma_{ab} dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ab}(dx^a + \beta^a dt)(dx^b + \beta^b dt), \tag{4.20}
\]

where we have introduced the label \(t = x^0\).
Making use of Eq. (A.9) we re-express the evolution equations:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \gamma_{\delta\sigma} &= -2\alpha K_{\delta\sigma}, \\
\left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) K_{\delta\sigma} &= \alpha \left( (3) R + KK_{\delta\sigma} - 2K_{\delta\psi}K^{\psi}_{\sigma} \right) \\
&\quad + 4\pi\alpha \left( (S - E) \gamma_{\delta\sigma} - 2S_{\delta\sigma} \right)
\end{align*}
\] (4.21)

Together with the constraint equations, Eqs. (4.21) and (4.22) form the ADM equations.

To explore the properties of these equations, we note that the lapse function and shift vector are gauge functions and as such they may be specified freely. For simplicity, we may pick the following:\n
\[\alpha = 1, \quad \beta = 0.\] (4.23)

In this setting, the world-lines of all Eulerian observers become geodesics. Owing to this, picking the lapse and shift in this way is known as geodesic slicing [25]. Further, we only consider the equations in a source-free region. Since the matter content is a freedom of the equations any results will also hold for regions with a non-vanishing energy-momentum tensor. Inserting these choices into the ADM equations gives

\[
\begin{align*}
(3) \nabla_d K^d_a - (3) \nabla_a K &= 0, \\
(3) R + K^2 - K_{ab}K^{ab} &= 0, \\
\frac{\partial}{\partial t} \gamma_{ab} + 2K_{ab} &= 0, \\
\frac{\partial}{\partial t} K_{ab} - \left( (3) R_{ab} + KK_{ab} - 2K_{ad}K^d_b \right) &= 0.
\end{align*}
\] (4.24)

Since all quantities present are well defined on each \(\Sigma(i) \in \Sigma\), we have made the restriction to spatial coordinates. Making use of Eq. (4.26), one may ‘remove’ the extrinsic curvature [21]:

\[
\begin{align*}
(3) \nabla_d \left( \frac{\partial}{\partial t} \gamma_{da} \right) - (3) \nabla_a \left( \gamma^{cd} \frac{\partial}{\partial t} \gamma_{cd} \right) &= 0, \\
4 (3) R + \left( \gamma^{cd} \frac{\partial}{\partial t} \gamma_{cd} \right)^2 - \gamma^{ac} \gamma^{bd} \left( \frac{\partial}{\partial t} \gamma_{ab} \right) \left( \frac{\partial}{\partial t} \gamma_{cd} \right) &= 0, \\
\frac{\partial^2}{\partial t^2} \gamma_{ab} - \left( 2 (3) R_{ab} + \frac{1}{2} \left( \gamma^{cd} \frac{\partial}{\partial t} \gamma_{cd} \right) \left( \frac{\partial}{\partial t} \gamma_{ab} \right) - \gamma^{cd} \left( \frac{\partial}{\partial t} \gamma_{ad} \right) \left( \frac{\partial}{\partial t} \gamma_{cb} \right) \right) &= 0.
\end{align*}
\] (4.28)

\[\text{It should be emphasized that this choice has only been made for illustrative purposes and, in general, we will not be fixing the lapse and shift in this way.}\]
We now have a set of ten equations for six unknowns, namely the components of $(\gamma_{ab}, K_{ab})$, and hence the system appears over-determined.

### 4.3 Well-posedness

Even though the $(3 + 1)$-system appears to be over-determined this is not the case, a fact which was first demonstrated by Choquet Bruhat in [2]. The details of the proof are more technical than we wish to discuss here. We will nevertheless present an outline of the argument which is based on [26, 27].

We do not consider the ADM equations directly but instead note that they are equivalent to the full Einstein equations. In vacuum these are

\[
^{(4)}R_{\alpha\beta} = -\frac{1}{2} \partial^\sigma g_{\alpha\beta} + \partial_{(\alpha} \Gamma_{\beta)} + \Gamma_{\sigma} \Gamma^{\sigma}_{\alpha\beta} + 2g^{\delta\sigma} \Gamma^{(\alpha}_{\delta} \Gamma_{\beta)} + g^{\delta\sigma} \Gamma^{\delta\sigma}_{\alpha\beta} = 0, \quad (4.31)
\]

where,

\[
\Gamma_{\alpha\beta\sigma} = \frac{1}{2} (2\partial_{(\alpha} g_{\beta)\sigma} - \partial_{\beta} g_{\alpha\sigma}), \quad \Gamma_{\beta} = g^{\sigma\beta} \Gamma_{\alpha\beta\sigma}. \quad (4.32)
\]

It should be emphasized that this argument need not be restricted to a source-free region, this is a choice we have made to simplify the discussion. To gain an understanding of the behaviour of the PDE system, it is enough to only examine the principal part:

\[
-\frac{1}{2} g^{\mu\nu} \partial_{\mu} \partial_{\nu} g_{\alpha\beta} + ^{(4)}\nabla_{(\alpha} \Gamma_{\beta)}. \quad (4.33)
\]

This is a quasi-linear system and standard results tell us that it forms a well-posed system provided that a coordinate system can be introduced such that $^{(4)}\nabla_{(\alpha} \Gamma_{\beta)} = 0$ [26, 27]. It what follows, we aim to find such a coordinate system. To do this, we consider the abridged PDE

\[
^{(4)}\tilde{R}_{\alpha\beta} = ^{(4)}R_{\alpha\beta} - ^{(4)}\nabla_{(\alpha} \Gamma_{\beta)} + ^{(4)}\nabla_{(\alpha} F_{\beta)}, \quad (4.34)
\]

where $F_{\nu}$ is a gauge source function chosen such that $D_{\nu} := F_{\nu} - \Gamma_{\nu} = 0$ initially. If the function depends at most on the metric but not its derivatives, then this augmented system is well-posed. Suppose such a solution is known. Then,

\[
^{(4)}R_{\alpha\beta} = -^{(4)}\nabla_{(\alpha} D_{\beta)} \Rightarrow G_{\alpha\beta} = -^{(4)}\nabla_{(\alpha} D_{\beta)} + \frac{1}{2} g_{\alpha\beta} ^{(4)}\nabla_{\sigma} D^{\sigma} = 0. \quad (4.35)
\]

Using the Bianchi identities allows the above to be re-expressed as

\[
^{(4)}\nabla_{\nu} ^{(4)}\nabla^{\nu} D_{\mu} + ^{(4)}R_{\mu}^{\nu} D_{\nu} = 0. \quad (4.36)
\]
Clearly, $\mathcal{D}_\nu$ satisfies a homogeneous wave equation and hence if both $\mathcal{D}_\nu$ and its normal derivatives vanish initially, then it will remain zero in the domain of dependence of the hypersurface. This motivates us to choose an initial hypersurface $\Sigma_{(i)}$ such that $F_\nu = \Gamma_\nu$. We then solve the auxiliary system on $\Sigma_{(i)}$. Here, the constraints must hold and hence the normal-normal and normal-tangential derivatives of $\mathcal{D}_\nu$ must be zero. Then Eq. (4.36) implies that $\mathcal{D}_\nu$ will remain zero. Thus, we have that a solution of $^{(4)}\mathring{R}_{\alpha\beta} = 0$ implies a local solution of $^{(4)}R_{\alpha\beta} = 0$. Patching together several local solutions gives a development.

From this we are able to conclude that, given an initial data set $(\Sigma, \gamma_{ab}, K_{ab})$ to Eq. (4.31) we are able to set up a corresponding initial data set to Eq. (4.34) such that $\mathcal{D}_\mu$ and its derivatives vanish initially. By solving this abridged system, we are able to construct a solution to Eq. (4.31), and hence to the ADM equations [27].
5 The elliptic form of the constraints

5.1 Conformal transformations

Thus far we have found that the Einstein equations give rise to a well-posed IVP provided that the constraint equations are satisfied on some initial space-like hypersurface $\Sigma_{(i)}$. The data set that satisfies the constraints consists of the first and second fundamental forms of $\Sigma_{(i)}$. Together, the metric and extrinsic curvature give a total of 12 unknowns but they only need to satisfy 4 equations. This means that if we are to find a solution, we must first specify 8 of the components \[24, 25\]. Whilst there is no physically distinguished or geometrically preferable way to choose the free data, one of the most successful methods is to perform a conformal decomposition.

Definition 5.1. Two tensors $T^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m}$ and $S^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m}$ are said to be conformally related if there exists a positive function $\psi$ and a constant $k \in \mathbb{R}$ such that

$$T^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} = \psi^k S^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m}.$$  \hspace{1cm} (5.1)

We call $\psi$ the ‘conformal factor\(^\text{12}\)’.

Under the transformation

$$\psi \mapsto \psi^n, \quad k \mapsto k - n,$$  \hspace{1cm} (5.2)

for $n = \text{constant}$, we see that the pair $(k, \psi)$ is not uniquely determined, this is a gauge freedom, once $k$ has been chosen then the function $\psi$ is fixed. In addition, $\psi$ should be bounded away from zero.

Before performing the transformations we express $K_{ab}$ in terms of its trace $\bar{K}$ and trace-free $A_{ab}$ components. The full extrinsic curvature is reconstructed by the sum

$$K_{ab} = A_{ab} + \frac{1}{3} \bar{K} \gamma_{ab}.$$  \hspace{1cm} (5.3)

We now suppose that the data $(\gamma_{ab}, A_{ab}, K)$ is conformally related to the known fields $(\bar{\gamma}_{ab}, \bar{A}_{ab}, \bar{K})$ via the transformations

$$\gamma_{ab} = \psi^k \bar{\gamma}_{ab}, \quad A^{ab} = \psi^s \bar{A}^{ab}, \quad K = \psi^w \bar{K},$$  \hspace{1cm} (5.4)

for known constants $k, s, w \in \mathbb{R}$. We will call $(\bar{\gamma}_{ab}, \bar{A}_{ab}, \bar{K})$ the ‘background data’. These fields need not be related to one another. As was mentioned above, each of

\(^\text{12}\)Conformal transformations are typically restricted to the metric and the quantities derived from it. Nevertheless, we will use this definition as it is convenient for this section.
the exponents are gauge freedoms. It is a priori unclear how to choose these. We will pick them such that the resulting constraint equations take the simplest form.

We begin with $\gamma_{ab}$. A metric multiplied by its inverse must return the identity; as such $\bar{\gamma}_{ab}$ will transform with power $-k$.

$$\gamma_{ab} = \psi^k \bar{\gamma}_{ab}, \quad \gamma^{ab} = \psi^{-k} \bar{\gamma}^{ab}. \quad (5.5)$$

The value of $k$ will be fixed by considering the conformal transformation of the Ricci scalar.

To do this, we seek a relationship between connections associated with $\gamma_{ab}$ and $\bar{\gamma}_{ab}$:

$$\Gamma_{ab}^{\ c} = \frac{1}{2} \chi^{cd} \left( \partial_c \gamma_{ab} + \partial_b \gamma_{ca} - \partial_a \gamma_{bc} \right)$$

$$= \bar{\Gamma}_{ab}^{\ c} + k \left( \frac{1}{2} \psi^{-1} \left( \delta^a_b \partial_c \psi + \delta^a_c \partial_b \psi - \bar{\gamma}_{bc} \bar{\gamma}^{ad} \partial_d \psi \right) \right)$$

$$= \bar{\Gamma}_{ab}^{\ c} + \kappa \left( \delta^a_b \bar{\nabla} \ln \psi + \delta^a_c \bar{\nabla} \ln \psi - \bar{\gamma}_{bc} \bar{\gamma}^{ad} \bar{\nabla} \ln \psi \right), \quad (5.6)$$

where it is understood that barred quantities are associated with $\bar{\gamma}_{ab}$. This formula allows us to express $\bar{\nabla} \ln \psi$ in terms of $\nabla \ln \psi$ in the standard way:

$$\bar{\nabla} \ln \psi = \nabla \ln \psi + \frac{\kappa}{2} \left( \bar{\gamma}_{ab} \bar{\nabla} \ln \psi - \bar{\gamma}_{bc} \bar{\gamma}^{ad} \bar{\nabla} \ln \psi \right), \quad (5.7)$$

We are now in a position to find a formula relating the two Ricci tensors:

$$(\nabla \ln \psi \cdot T^{ab}) = (\bar{\nabla} \ln \psi \cdot T^{ab}) - \frac{\kappa}{2} \left( \bar{\gamma}_{ab} \bar{\nabla} \ln \psi - \bar{\gamma}_{bc} \bar{\gamma}^{ad} \bar{\nabla} \ln \psi \right)$$

where

$$C_{bc} := \Gamma_{bc} - \bar{\Gamma}_{bc} = \frac{k}{2} \left( \delta^a_b \bar{\nabla} \ln \psi + \delta^a_c \bar{\nabla} \ln \psi - \bar{\gamma}_{bc} \bar{\gamma}^{ad} \bar{\nabla} \ln \psi \right). \quad (5.8)$$

We use Eq. (5.8) we get

$$C_{ba} = \frac{3k}{2} (\bar{\nabla} \ln \psi, \quad \bar{\nabla} \ln \psi C_{ab} = k (\bar{\nabla} \ln \psi - \frac{k}{2} \bar{\gamma}_{ab} \bar{\nabla} \ln \psi). \quad (5.10)$$

Then the Ricci tensor is

$$(\nabla \ln \psi \cdot R^{ab}) = (\bar{\nabla} \ln \psi \cdot R^{ab}) - \frac{\kappa}{2} \left( \bar{\gamma}_{ab} \bar{\nabla} \ln \psi + \bar{\gamma}_{ab} \bar{\nabla} \ln \psi \right)$$

$$+ \frac{k^2}{4} \left( \bar{\nabla} \ln \psi \cdot \bar{\nabla} \ln \psi - \bar{\gamma}_{ab} \bar{\nabla} \ln \psi \cdot \bar{\nabla} \ln \psi \right). \quad (5.11)$$
From this we are able to calculate the corresponding Ricci scalar

\[
(3) R = \gamma^{ab}(3) R_{ab} = \psi^{-k(3)} R - \psi^{-k}(2k^{(3)} \nabla^2 \ln \psi + \frac{k^2}{2} (3) \nabla^a \ln \psi (3) \nabla_a \ln \psi).
\] (5.12)

This equation may be simplified further by using that

\[
(3) \nabla^2 \ln \psi = \psi^{-1(3)} \nabla^2 \psi - (3) \nabla_a \ln \psi (3) \nabla_a \ln \psi,
\] (5.13)

which allows us to write,

\[
(3) R = \psi^{-k(3)} R - 2k \psi^{-k-1(3)} \nabla^2 \psi + \psi^{-k(3)} \nabla^a \ln \psi (3) \nabla_a \ln \psi \left(2k - \frac{k^2}{2}\right).
\] (5.14)

A desire to remove the lower order derivatives of \(\psi\) prompts the choice \(k = 4\).

We now perform a conformal transformation of \(A^{ab}\):

\[
A^{ab} = \psi^s \tilde{A}^{ab},
\] (5.15)

To find the best choice of \(s\), we will consider its derivative, as it appears in the momentum constraint:

\[
(3) \nabla_b A^{ab} = (3) \nabla_b A^{ab} + C^{be}_{ac} A^{eb} + C^{b}_{ac} A^{ae} = \psi^{-10(3)} \nabla_b (\psi^{10} A^{ab}) = \psi^{-10(3)} \nabla_b (\psi^{10+s} \tilde{A}^{ab}),
\] (5.16)

so this would suggest that the optimal choice is \(s = -10\).

The last transformation we wish to consider involves \(K\):

\[
K = \psi^w \tilde{K}.
\] (5.17)

This exponent is fixed by considering the full form of \(K_{ij}\) as it appears in the momentum constraint

\[
(3) \nabla_b K^{ab} - (3) \nabla^a K = (3) \nabla_b (A^{ab} + \frac{1}{3} \gamma^{ab} K) - (3) \nabla^a K
= \psi^{-10(3)} \nabla_b \tilde{A}^{ab} + 2 \psi^{-4(3)} \nabla^a \tilde{K} + 2 \psi^{4(3)} \tilde{K}^w (3) \nabla^a \psi.
\] (5.18)

Since we wish to remove lower order terms, we pick \(w = 0\).

Thus, we express our unknowns as

\[
\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}, \quad A^{ab} = \psi^{-10} \tilde{A}^{ab}, \quad K = \tilde{K}.
\] (5.19)
The conformal transformation of the extrinsic curvature takes the form
\[ A_{ab} = \psi^{-10} \tilde{A}_{ab}, \quad (5.20) \]
\[ K = \tilde{K}, \quad (5.21) \]
and,
\[ A_{ab} = \gamma_{ac} \gamma_{bd} A^{cd} = \psi^{-2} \gamma_{ac} \gamma_{bd} \tilde{A}^{cd} = \psi^{-2} \tilde{A}_{ab}. \quad (5.22) \]

We are now in a position to perform a full conformal transformation\(^{13}\) of the constraint equations. In order to make a choice on the value of \( w \), the momentum constraint has already been transformed. Then, with \( w = 0 \), the momentum constraint takes the form
\[ (3) \; \bar{\nabla}_b \bar{A}^{ab} - \frac{2}{3} \psi^6 (3) \bar{\nabla}^a \bar{K} = 8\pi \psi^{10} p^a. \quad (5.23) \]

For the Hamiltonian constraint, we consider the following:
\[ K_{ab} K^{ab} = (A_{ab} + \frac{1}{3} \gamma_{ab} K)(A^{ab} + \frac{1}{3} \gamma^{ab} K) = A_{ab} A^{ab} + \frac{1}{9} \gamma^{ab} \gamma_{ab} K^2 \]
\[ = \psi^{-12} \tilde{A}_{ab} \tilde{A}^{ab} + \frac{1}{3} \bar{K}^2. \quad (5.24) \]
This gives us the conformal constraints:
\[ 8 (3) \; \bar{\nabla}^2 \psi - \psi \bar{R} - \frac{2}{3} \psi^5 \bar{K}^2 + \psi^{-7} \tilde{A}^{ab} \tilde{A}_{ab} = -16\pi \psi^5 E, \quad (5.25) \]
\[ \psi^{-10} (3) \; \bar{\nabla}_b \bar{A}^{ab} - \frac{2}{3} \psi^{-4} (3) \bar{\nabla}^a \bar{K} = 8\pi j_a. \quad (5.26) \]

5.2 (Multiple) Black hole solution

The formulation presented above is the ‘standard’ method that is used in most relativity works, and it is useful in demonstrating the power of this formulation of the constraint equations, which we will now endeavour to solve on a spherically symmetric space-time. We will make use of the conformal method presented above, basing our discussion on [11, 22]. For simplicity, we restrict ourselves to considering a time-symmetric hypersurface \( \Sigma \), or a ‘moment of time symmetry’, embedded into a source-free space-time, i.e.
\[ E = j_i = 0. \quad (5.27) \]

\(^{13}\)Again, we emphasize that this is not a conformal transformation in the standard sense.
A consequence of time symmetry is that all of the time derivatives of the metric must be zero and the 4-dimensional line element must be invariant under time reversal, i.e.

\[ t \mapsto -t \implies ds^2 \mapsto ds^2. \quad (5.28) \]

This invariance implies that the shift vector will also vanish, a fact that follows from Eq. (4.20). Then, Eq. (4.15) implies that the extrinsic curvature must also vanish on the hypersurface:

\[ K_{ij} = 0, \quad K = 0. \quad (5.29) \]

In this case the momentum constraint is trivially satisfied. The Hamiltonian constraint of Eq. (5.25) is now given as

\[ (\sqrt{\gamma}) \nabla^2 \psi = \frac{1}{8} \psi \sqrt{\gamma} \mathcal{R}. \quad (5.30) \]

We make the choice of the background metric as being the Euclidean metric. In such situations, the conformal metric is referred to as being conformally flat. At this point, one could note that any spherically symmetric spatial metric is conformally flat. Hence we can always assume conformal flatness without loss of generality [22]. This assumption of conformal flatness simplifies the equation in that the intrinsic curvature must vanish, as must the Ricci scalar. Then the Hamiltonian constraint simply becomes a Laplace equation:

\[ (\sqrt{\gamma}) \nabla^2 \psi = 0. \quad (5.31) \]

A spherically symmetric solution can be given as

\[ \psi = C + \frac{M}{2r}, \quad (5.32) \]

where \( C \) and \( M \) are constants of integration. However, it can be shown that \( M \), unlike \( C \), has physical meaning.

In order to fix the value of \( C \) we must pick a boundary condition. In this section, we will require that the space-time is asymptotically flat.

**Definition 5.2.** An initial data set \( (\Sigma, \gamma_{ab}, K_{ab}) \) is called ‘asymptotically flat’ if

1. \( \Sigma \) is diffeomorphic to \( \mathbb{R}^3 \) and,

2. there exists coordinates \( \{x^i\} \) on \( \Sigma \) such that the components of \( \gamma_{ab} \) and \( K_{ab} \) satisfy

\[ \gamma_{ab} = \left( 1 + \frac{2M}{\rho} \right) \delta_{ab} + \mathcal{O}(\rho^{-2}), \quad (5.33) \]

\[ K_{ab} = \mathcal{O}(\rho^{-2}), \quad (5.34) \]
in the limit

\[ \rho := \sum_{i=1}^{3} (x^i)^2 \to \infty. \]  

(5.35)

[28, 29].

This can translate into a requirement on \( \psi \) by making the association \( \rho = 4r \), which gives

\[ \psi \to 1 + \mathcal{O}(r^{-1}) \text{ as } r \to \infty. \]  

(5.36)

Under this condition this solution reduces to

\[ \psi = 1 + \frac{M}{2r}. \]  

(5.37)

This is simply a slice of the Schwarzschild metric in isotropic coordinates:

\[ ds^2 = \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2). \]  

(5.38)

One can then see that \( M \) is in fact the mass of the black hole. This is the ‘physical meaning’ mentioned previously.

One example of why it is beneficial to use this method in solving the constraint equations is that it simplifies the task of finding solutions for an arbitrary number of black holes. To see this, consider the solution found previously. A more general solution to Eq. (5.31) can be given as the superposition of solutions, each of the form given in Eq. (5.32). This takes the following form:

\[ \psi = \sum_{i=1}^{N} C_i + \frac{M_i}{2r_i}, \]  

(5.39)

where \( N \) is the number of black holes. Under the requirement that the space-time is asymptotically flat, this becomes

\[ \psi = 1 + \sum_{i=1}^{N} \frac{M_i}{2r_i}, \]  

(5.40)

where \( M_i \) is the mass of the \( i \)-th black hole and \( r_i \) is the distance of the \( i \)-th black hole from the origin. Then, the metric takes the form

\[ ds^2 = \left(1 + \sum_{i=1}^{N} \frac{M_i}{2r_i}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2). \]  

(5.41)
6 An Evolutionary Formulation of the Constraint Equations

The conformal method allows us to cast the constraints as a BVP, a benefit of which is that it allows us control over the asymptotic geometry of the solution [24]. It is not necessary, and possibly not physically advantageous, to formulate the constraints in this way and one may instead choose to formulate them as an IVP. In [3–6] Rácz introduces two such formulations: One that gives a set of two equations that form a strongly-hyperbolic system, and another that produces two hyperbolic (when coupled) equations and one parabolic equation.

6.1 Derivation

We start with a four-dimensional space-time equipped with a Lorentzian metric $g_{\alpha\beta}$ and assume that this space-time can be foliated with a set of smooth Cauchy surfaces $\Sigma = \{\Sigma_T \mid T = \text{constant}\}$. The vacuum constraint equations must hold on some initial hypersurface $\Sigma(i) \in \Sigma$ with data $(\gamma_{ab}, K_{ab})$:

\begin{align}
(3) R - K_{ab}K^{ab} + K^2 &= 0, \\
\nabla_a K^a_c - \nabla_c K &= 0,
\end{align}

where $\nabla_a$ is the covariant derivative associated with the metric $\gamma_{ab}$. We assume that this initial hypersurface can be foliated by a set of topological 2-spheres $S = \{S_\rho \mid \rho = \text{constant}\}$. The corresponding unit normal to each of the leaves will be denoted as $N^a$ such that $N^a N_a = 1$. This decomposition is in full analogy to the $(n + 1)$-decomposition presented in Section 3, with $\varepsilon = 1$ and $n = 2$. The metric and extrinsic curvature can be decomposed into normal and tangential components in the standard way:

\begin{align}
\gamma_{ab} &= h_{ab} + N_a N_b, \\
K_{ab} &= q_{ab} + 2p_a N_b + \kappa N_a N_b,
\end{align}

where $q_{ab} = h^c_a h^d_b K_{cd}$, $p_a = h^c_a N^b K_{cb}$, $\kappa = N^a N^b K_{ab}$ and $h_{ab}$ is the induced metric on $S_\rho$. As in Section 4, these decompositions are only unique (and hence ‘meaningful’) if we also have the following orthogonal properties

\begin{align}
q_{ab} N^a &= h_{ab} N^a = 0, \quad p_a N^a = 0,
\end{align}

i.e. $q_{ab}$ and $p_a$ are entirely intrinsic to $S_\rho$. 

37
6.1.1 Momentum constraint

To decompose the constraint equations, we begin by decomposing the covariant derivative. Fix a \((0, 1)\) tensor \(u_a\) over \(T_p S^c\):

\[
\nabla_a u_b = (h^c_a + N^e N_a)(h^d_b + N^d N_b) \nabla_c u_d
\]

\[
= D_a u_b + N_a N^e h^d_b \nabla_c u_d - h^c_a N_b u_d \nabla_c N^d - u_d N^e N_a N_b \nabla_c N^d
\]

\[
= D_a u_b + N_a D u_b + N_b u_d k^d_a - u_d N_a N_b v^d,
\]

(6.6)

where \(D_a u_b = h^c_a h^d_b \nabla_c u_d\) is the (intrinsic) covariant derivative associated with the leaf \(S^c\), \(v_a = N^d \nabla_d N_a\) is the acceleration on \(S^c\) and \(k_{ab} = -h^c_a h^d_b \nabla_c N_d = -h^c_a \nabla_c N_b\) is the extrinsic curvature of \(S^c\). Furthermore, it should be noted that the following operator has been defined:

\[
D u^{a_1 \ldots a_n}_{b_1 \ldots b_m} = N^e \prod_{i=1}^n h^{a_i}_{b_i} \prod_{j=1}^m h^{c_j}_{b_j} \nabla_{d_1 \ldots d_m} e_{c_1 \ldots c_m},
\]

(6.7)

We now fix \((0, 2)\) tensor \(u_{ab}\) over \(T_p S^c\). Then,

\[
\nabla_{ab} u_{cd} = (h^e_{ab} + N^f N_{ab})(h^g_{cd} + N^g N_{cd}) \nabla_e u_f
\]

\[
= D_{ab} u_{cd} + N_{ab} D u_{cd} + h^e_{ab} h^f_{cd} N_{ef} \nabla_e u_f + h^e_{cd} N_{ab} N_{ef} \nabla_e u_f
\]

\[
+ N^g N_{ab} h^f_{cd} \nabla_e u_f + N^g N_{cd} N_{ab} h^f_{ef} \nabla_e u_f
\]

\[
= D_{ab} u_{cd} + N_{ab} D u_{cd} + u_{fc} k^f_{de} N_{ab} - N_{ab} u_{fc} k^f_{de} + k^g_{ab} u_{bg} N_{cd} - u_{bg} N_{cd} v^g.
\]

(6.8)

The last decomposition we need is of the derivative of the co-normal:

\[
\nabla_a N_b = (h^c_a + N^e N_a) \nabla_c N_b = -k_{ab} + N_a v_b.
\]

(6.9)

We now have all the tools we need to decompose the momentum constraint given by Eq. (6.2). We will begin by considering the derivative of the extrinsic constraint curvature \(K_{ab}\):

\[
\nabla_a K_{bc} = \nabla_a q_{bc} + 2 N_{(b} \nabla_{[a} p_{c]} + 2 p_{(b} \nabla_{[a} N_{c]} + N_{bc} N_{ab} N_{c} + 2 \kappa N_{(b} \nabla_{[a} N_{c]}).\]

(10)

We proceed to use Eq. (6.9) to replace the derivatives of the co-normal:

\[
\nabla_a K_{bc} = \nabla_a q_{bc} + 2 N_{(b} \nabla_{[a} p_{c]} + p_{(c} (N_a v_b - k_{ab}) + p_{b} (N_a v_c - k_{ac}) + N_{ab} N_{cd} \nabla_{[a} N_{c]} + \kappa N_{b} (N_a v_c - k_{ac}) + \kappa N_{c} (N_b v_a - k_{ba}).
\]

(111)

Using Eq. (6.6) allows us rewrite the second term in Eq. (6.11).

\[
N^a \nabla_a p_c = D p_c - N_c p q v^d,
\]

(12)

\[
N_c \nabla_a p^c = N_c (D a p^c - v^c p_c).
\]

(13)
We now use Eq. (6.8) to rewrite the first term in Eq. (6.11).

\[ \nabla_a q^c = D_a q^c - q_{ac} v^a + k^{ab} q_{ab} N_c. \]  

(6.14)

Raising the \( b \) index in Eq. (6.11) and contracting with the \( a \) index gives

\[ \nabla_a K^a_b = (D_a q^c - q_{ac} v^a - p_k k + \kappa v + D p_c - p_k k^b_c) + N_c (k^{ab} q_{ab} + D \kappa - k \kappa - 2 p_{d} v^d + D a p^a), \]  

(6.15)

where \( k = k^a_a \).

Next we consider the second term in the momentum constraint (Eq. (6.2)):

\[ \nabla_c K = \nabla_c (q + \kappa) = D_c (q + \kappa) + N_c (q + \kappa), \]  

(6.16)

where \( q = q^a_a \).

It follows that we can split the momentum constraint into the following equations:

\[ 0 = D_a q^c - q_{ac} v^a - p_k k + \kappa v + D p_c - p_k k^b_c - D_c (q + \kappa), \]  

(6.17)

\[ 0 = -k^{ab} q_{ab} + \kappa + 2 p_{d} v^d - D_a p^a + D q. \]  

(6.18)

These equations form a PDE system for the variables\(^{14} \) \( q \) and \( p_a \). This can be made explicit by decomposing \( q_{ab} \) into its trace and trace-free components:

\[ q_{ab} = Q_{ab} + \frac{1}{2} q h_{ab}, \quad Q^a_a = 0. \]  

(6.19)

This allows us write the momentum constraint as

\[
\begin{align*}
D p_c - \frac{1}{2} D_c q &= D_c \kappa - D_a Q^a_c + Q_{ac} v^a + \frac{1}{2} q v + p_k k - \kappa v_c + p_k k^c, \\
D q - D_a p^a &= k^{ab} Q_{ab} + \frac{1}{2} q k - k \kappa - 2 p_{d} v^d.
\end{align*}

(6.20)

(6.21)

6.1.2 Hamiltonian constraint

There are two ways one may view the Hamiltonian constraint (Eq. (6.1)). It may be treated as an algebraic equation for \( \kappa \), or as a PDE for the lapse function \( A \).

First we consider the algebraic approach. The Hamiltonian constraint, given by Eq. (6.1), can be rewritten by making use of Eq. (6.4):

\[ -(3)^R + K_{ab} k^{ab} - K^2 = -(3)^R - q^2 - 2 \kappa q + 2 p^a p_a + q_{ab} q^{ab} = 0. \]

(6.22)

\(^{14}\)These may not be the only variables as the Hamiltonian constraint may give rise to more unknowns.
Using Eq. (6.19) and solving for $\kappa$ gives

$$\kappa = (2q)^{-1}[-(3)R + Q^{ab}Q_{ab} - \frac{1}{2}q^2 + 2p_ap^a].$$  \hspace{1cm} (6.23)

Taking the derivative,

$$D_c\kappa = -\frac{\kappa}{q}D_cq + \frac{1}{2q}D_c(-(3)R + Q_{ab}Q^{ab}) - \frac{1}{2}D_cq + \frac{2}{q}p_aD_cp^a.$$  \hspace{1cm} (6.24)

Inserting this derivative into Eq. (6.20) gives our final system of equations:

$$Dp_c + \frac{\kappa}{q}D_cq - \frac{2}{q}p_aD_c p^a = -D_aQ^a_c + Q_{ac}v^a + \frac{1}{2}qv_c + p_ck - \kappa v_c + p^b k_{bc}$$

$$+ \frac{1}{2q}D_c(-(3)R + Q_{ab}Q^{ab}),$$  \hspace{1cm} (6.25)

$$Dq - D_ap^a = k^{ab}Q_{ab} + \frac{1}{2}qk - k\kappa - 2p_d v^d.$$  \hspace{1cm} (6.26)

Upon examination, one may note that this forms a PDE system for the unknowns $(p_A, q)$ with freely specifiable data $(Q_{ab}, h_{ab}, v_a, k_{ab}, (3)R)$, where the dependence on $\kappa$ can be removed algebraically through the use of Eq. (6.23).

Next, we consider the Hamiltonian as a PDE for the lapse function. For this, we need the following relation

$$N^a = A^{-1}(\rho^a - B^a),$$  \hspace{1cm} (6.27)

where $\rho^a$\textsuperscript{15} is the ‘time vector’ and $B^a$ is the shift vector associated with the foliation of $\Sigma(i)$. Furthermore, we note that the intrinsic curvatures of $\Sigma(i)$ and $S_j \in S$ can be related via the following formula

$$(3)R = (2)R - (k^2 + k_{ab}k^{ab} + 2A^{-1}D^aD_aA - 2\mathcal{L}_Nk).$$  \hspace{1cm} (6.28)

Then, the Hamiltonian constraint becomes,

$$-(2)R + (k^2 + k_{ab}k^{ab} + 2A^{-1}D^aD_aA - 2\mathcal{L}_Nk) - 2\kappa q + 2p^a p_a + Q_{ab}Q^{ab} - \frac{1}{2}q^2 = 0.$$  \hspace{1cm} (6.29)

\textsuperscript{15}In what follows we will use the notation $\rho^a \nabla_a = \nabla_\rho$. Here, $\rho$ should not be mistaken as a space-time index.
The extrinsic curvature can be written as
\[ k_{ab} = -\frac{1}{2} \mathcal{L}_N h_{ab} = -A^{-1} \left( \frac{1}{2} \mathcal{L}_\rho h_{ab} - D_{(a} B_{b)} \right) = A^{-1} k_{\cdot ab}, \] (6.30)
\[ k = \gamma^{ab} k_{ab} = -A^{-1} \left( \frac{1}{2} \gamma^{ab} \mathcal{L}_\rho h_{ab} - \gamma^{ab} D_{a} B_{b} \right) = A^{-1} \hat{k}, \] (6.31)
where we have defined \( \hat{k}_{ab} \) as
\[ \hat{k}_{ab} := \frac{1}{2} \mathcal{L}_\rho h_{ab} - D_{(a} B_{b)} = \mathcal{L}_N k_{ab}. \] (6.32)
Moreover,
\[ \mathcal{L}_N k = A^{-1} \mathcal{L}_N \hat{k} - A^{-2} \hat{k} \mathcal{L}_N A = A^{-2} (\partial_{\rho} \hat{k} - B^a D_a \hat{k}) - A^{-3} \hat{k} (\partial_{\rho} A - B^a D_a A). \] (6.33)
Substituting this into Eq. (6.29),
\[ 0 = -(^2 R + A^{-2} \hat{k}^2) + A^{-2} \hat{k}_{ab} \hat{k}^{ab} + 2A^{-1} D^a D_a A + Q_{ab} Q^{ab} - \frac{1}{2} q^2 \]
\[ - 2A^{-2} (\partial_{\rho} \hat{k} - B^a D_a \hat{k}) + 2A^{-3} \hat{k} (\partial_{\rho} A - B^a D_a A). \] (6.34)
For convenience, we introduce the following notation:
\[ E = (^2 R + 2\kappa q - 2p^a p_a - Q_{ab} Q^{ab} + \frac{1}{2} q^2, \] (6.35)
\[ F = 2(\partial_{\rho} \hat{k} - B^a D_a \hat{k}) - \hat{k}_{ab} \hat{k} - (\hat{k})^2. \] (6.36)
Substituting this notation into Eq. (6.34) and multiplying by \( A^{-3} \),
\[ 2\hat{k} (\partial_{\rho} A - B^a D_a A) + 2A^2 D^a D_a A = A^3 E + AF \] (6.37)
Noting that we can write
\[ Dq = N^a \partial_a q = A^{-1} (D_{\rho} q - B^a D_a q), \] (6.38)
\[ D_{pc} = N^a h^b_c \nabla_a p_b = A^{-1} \rho^a h^b_c \nabla_a p_b - B^a h^b_c \nabla_a p_b = A^{-1} (h^b_c \nabla_{\rho} p_b - B^a D_a p_c), \] (6.39)
lets us write the system as
\[ (\partial_{\rho} q - B^a \partial_a q) - AD_{a} p^a = k_{ab} Q_{ab} + \frac{1}{2} qk - k\kappa - 2A p_d v^d, \] (6.40)
\[ 2k \partial_{\rho} A - 2A^2 D^a D_a A = A^3 E + AF + 2kB^a D_a A, \] (6.41)
\[ (h^b_c \nabla_{\rho} p_b - B^a D_a p_c) - \frac{1}{2} AD_{c} q = AD_{c} \kappa - AD_{a} Q^a_c + AQ_{ac} v^a + \frac{1}{2} A v_c \]
\[ + k p_c - \kappa A v_c + p^b k_{bc}, \] (6.42)
One may now note that this forms a PDE system for the unknowns \((p_A, q, A)\) with freely specifiable data \((B^A, Q_{ab}, h_{ab}, \kappa, v_A, k, (2) R)\). It should also be emphasized that 
\(^* k\) is purely intrinsic to the spheres and hence is determined by the metric \(h_{ab}\).

### 6.2 Form of the equations

#### 6.2.1 Strongly hyperbolic

We first consider Eqs. (6.25) and (6.26). We aim to find the condition under which these equations are hyperbolic. To do this, we must construct the principal symbol of the equations. According to the discussion in Section 2.1, this can be done by performing the following transformation \((D, D_C) \rightarrow (\xi, \xi_C)\). Then the principal symbol may be written as

\[
\sigma = \begin{pmatrix} \xi \delta^A_C - \frac{2}{q} p^A \xi_C & \frac{q}{q} \xi C \\ -\xi^A_C & \xi \end{pmatrix}.
\]  

(6.43)

We now wish to show under which condition this matrix is not invertible. This is equivalent to saying that there exists no \(v = (u_A, u)^T \neq (0, 0, 0)^T\) such that \(\sigma v = 0\), i.e.

\[
\begin{pmatrix} \xi \delta^A_C - \frac{2}{q} p^A \xi_C & \frac{q}{q} \xi C \\ -\xi^A_C & \xi \end{pmatrix} \begin{pmatrix} u_A \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]  

(6.44)

we combine\(^{16}\) these equations to get

\[
\xi^2 u_C - \frac{2}{q} p^A \xi_C u_A + \frac{\kappa}{q} \xi C \xi_A u^A = 0.
\]  

(6.45)

Upon examination of this equation, we see that since \(u_A \neq 0\) we must have \(u_A = c \xi_A\) for some function \(c : \Sigma(u) \rightarrow \mathbb{R}\):

\[
c \left( \xi^2 - \frac{2}{q} \xi p^A \xi_A + \frac{\kappa}{q} \xi_A \xi^A \right) \xi_C = 0 \Rightarrow \xi^2 - \frac{2}{q} \xi p^A \xi_A + \frac{\kappa}{q} \xi_A \xi^A = 0.
\]  

(6.46)

Application of the quadratic formula then gives,

\[
\xi_\pm = \frac{p^A \xi_A \pm \sqrt{(p^A \xi_A)^2 - \kappa q \xi A \xi^A}}{q}.
\]  

(6.47)

\(^{16}\)Explicitly, we take the equation \(\xi^A u_A = \xi u\) and use it to replace \(u\) in the remaining equation. Doing this leads us to make the assumption that \(\xi \neq 0\).
In order to have non-complex solutions, we must have

\[(p^A \xi_A)^2 - \kappa q \xi_A \xi^A > 0.\]  

(6.48)

This equation always holds, in particular, it must hold when \(\xi_A\) and \(p_A\) are perpendicular i.e. \(\xi_A p^A = 0\). Using \(\xi_A \xi^A \geq 0\) gives,

\[\kappa q < 0.\]  

(6.49)

It follows that if this inequality is satisfied then Eqs. (6.25) and (6.26) are hyperbolic in the sense of Def. 1, for this reason we call Eq. (6.49) the ‘hyperbolicity condition’.

By making use of Eq. (6.27), we note that the equations can be written in the following matrix form:

\[
\begin{pmatrix}
\delta^m_c \\
0 \\
1
\end{pmatrix}
\partial_{\rho}
\begin{pmatrix}
p_m \\
q
\end{pmatrix}
+
\begin{pmatrix}
-B^k \delta^m_c - \frac{2A}{q} p^n \delta^k_c \\
-h^{km} A \\
-B^k
\end{pmatrix}
\partial_k
\begin{pmatrix}
p_m \\
q
\end{pmatrix}
=
\begin{pmatrix}
C_c \\
C
\end{pmatrix},
\]

(6.50)

where

\[C_c = A \left( -D_a Q_a^c + Q_{ac} v^a + \frac{1}{2} q v_c + p_c k - \kappa v_c + p^b k_{bc} + \frac{2}{q} p_a h^b c \Gamma^c_{ab} p_e \right) + A \left( N^c h^b c \Gamma^c_{ab} p_e + \frac{1}{2} q D_c (\mathcal{R} + Q_{ab} Q^{ab}) \right), \]

(6.51)

\[C = A \left( k^{ab} Q_{ab} + \frac{1}{2} q k - k \kappa - 2 p_d v^d h^{ab} \Gamma^c_{ab} p_e \right). \]

(6.52)

Then, provided that the hyperbolicity condition holds, the matrix,

\[
\begin{pmatrix}
h^{cn} \\
0 \\
-\frac{\kappa}{q}
\end{pmatrix}
\]

(6.53)

is a symmetrizer of Eq. (6.50), and produces the following set of equations:

\[
\begin{pmatrix}
h^{mn} \\
0 \\
1
\end{pmatrix}
\partial_{\rho}
\begin{pmatrix}
p_m \\
q
\end{pmatrix}
+
\begin{pmatrix}
-B^k h^{mn} - \frac{2A}{q} p^n h^{km} \\
-\frac{\kappa}{q} h^{km} A \\
-\frac{\kappa}{q} B^k
\end{pmatrix}
\partial_k
\begin{pmatrix}
p_m \\
q
\end{pmatrix}
=
\begin{pmatrix}
h^{cn} C_c \\
-\frac{\kappa}{q} C
\end{pmatrix}.
\]

(6.54)

Since the system can be symmetrized it follows that the equations are symmetric hyperbolic. We will refer to this system as the “strongly hyperbolic formulation”.

As we will see shortly, this approach offers increased control over the geometry owing to the fact that the 3-metric is freely specifiable, which is not true for Eqs. (6.40)-(6.42). However, the hyperbolicity condition depends on the unknowns themselves. As such, it is, in a general situation, unclear if the condition will remain satisfied throughout the evolution.
6.2.2 Parabolic-hyperbolic

We now turn our attention to Eqs. (6.40), (6.41) and (6.42). Of particular interest is Eq. (6.41):

\[ 2 \partial_{\rho} A + 2 A^2 h^{AB} D_A D_B A = A^3 E + AF + 2 k B^A D_A A. \]  

(6.55)

Note that if the inequality

\[ k < 0 \]  

(6.56)

holds then Eq. (6.55) is reminiscent of a heat equation, with the term \( k B^A D_A A \) serving as an advection term\(^{17}\).

We proceed to show that Eq. (6.55) is uniformly parabolic in the sense of Def. 2.6. To do so, we note that we are dealing with a compact differentiable manifold with the topology of a 2-sphere, and hence any pair of metrics that have been equipped onto the manifold are conformally related via a conformal factor \( \psi \). i.e \( h_{AB} = \psi \sigma_{AB} \).

This allows us to write

\[ 2 A^2 h^{AB} \xi_A \xi_B = 2 A^2 \psi^{-1} \sigma^{AB} \xi_A \xi_B \geq C \sigma^{AB} \xi_A \xi_B, \]  

(6.57)

which holds if one picks \( C = \min 2 A^2 \psi^{-1} \), where the minimum should be taken over the domain of interest. Thus if Eq. (6.56) holds then Eq. (6.41) is a uniformly parabolic.

We now wish to show that Eqs. (6.40) and (6.42) form a hyperbolic system. This will be done by considering their equivalent form, given by Eqs. (6.20) and (6.21). Proceeding as above, the principal symbol is constructed with the transformation \((D,D) \rightarrow (\xi,\xi_C)\).

\[ \sigma = \begin{pmatrix} 2 \xi & -\xi_C \\ -\xi_C & \xi \end{pmatrix}, \]  

(6.58)

It is clear here that the equations are symmetric hyperbolic\(^{18}\). Nevertheless, to illustrate the differences between these equations (Eqs. (6.20) and (6.21)) and the previous ones (Eqs. (6.25) and (6.26)), we will show this explicitly. As before, we aim to demonstrate under which conditions the above matrix is not invertible. This is equivalent to saying that there exists \( v = (u_A, u)^T \neq (0, 0, 0)^T \) such that \( \sigma v = 0 \), i.e.

\[ \sigma = \begin{pmatrix} 2 \xi & -\xi_C \\ -\xi_C & \xi \end{pmatrix} \begin{pmatrix} u_C \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]  

(6.59)

\(^{17}\)The presence of such term means that Eq. (6.55) takes the form of a reaction diffusion equation, see [10,30] for more details.

\(^{18}\)This is because symmetric matrices always have real eigenvalues.
so we may use $\xi^C u_C = \xi u$ to get

$$2\xi^2 u_C - \xi C \xi^A u_A = 0. \quad (6.60)$$

Upon examination of this equation we see that since $u_A \neq 0$ we must $u_A = c \xi_A$ for some function $c : \Sigma_i \to \mathbb{R}$:

$$c (2\xi^2 - \xi^A \xi_A) \xi_C = 0 \Rightarrow 2\xi^2 - \xi^A \xi_A = 0. \quad (6.61)$$

Application of the quadratic formula then gives

$$\xi = \pm \sqrt{2\xi^A \xi_A}. \quad (6.62)$$

Thus, there are no complex eigenvalues and hence the equations are symmetric-hyperbolic. The symmetry can be seen from Eq. (6.59).

Then, the PDE system found by solving the Hamiltonian constraint as a PDE for the lapse function (Eqs. (6.40)-(6.42)) is parabolic-hyperbolic.

### 6.3 Linearisation of the evolutionary form of the constraint equations

By following the process outlined in Section 2.3, we will write down the linearisation of the two evolutionary formulations of the constraints, introduced above.

#### 6.3.1 Linearisation: strongly hyperbolic

We begin by fixing our free data, $\{A, B^A, h_{ab}, Q_{ij}, v_A\}$, and supposing that there exists a one-parameter family of solutions $\{p_A(\lambda), q(\lambda)\}$ that depend, differentably on an affine parameter $\lambda$ and satisfies the property $\{p_A(0), q(0)\} = \{\hat{p}_A, \hat{q}\}$, where $\{\hat{p}_A, \hat{q}\}$ is a known solution of the equations. Then the solutions may be written as a formal series

$$p_A(\lambda) = \hat{p}_A + \lambda \delta p_A + \sum_{j=2}^{\infty} \lambda^j \frac{\partial^j p_A(\lambda)}{\partial \lambda^j} \bigg|_{\lambda=0}, \quad (6.63)$$

$$q(\lambda) = \hat{q} + \lambda \delta q + \sum_{j=2}^{\infty} \lambda^j \frac{\partial^j q(\lambda)}{\partial \lambda^j} \bigg|_{\lambda=0}. \quad (6.64)$$
where \( \{ \delta p_A, \delta q \} = \left\{ \frac{\partial p_A(\lambda)}{\partial \lambda} \bigg|_{\lambda=0}, \frac{\partial q(\lambda)}{\partial \lambda} \bigg|_{\lambda=0} \right\} \). We now wish to find \( \{ \delta p_A, \delta q \} \) as a solution to Eqs. (6.25) and (6.26) by solving the equations

\[
\delta \kappa = \left( \frac{(3) R - Q^{ab} Q_{ab} - 2 \dot{p}_a \dot{q}_a - 1}{4} \right) \delta q + 2 \dot{p}_a \delta p_a \quad (6.65)
\]

\[
D \delta q - D_c \delta p^c = \frac{1}{2} k \delta q - k \delta \kappa - 2 \nu^a \delta p_a \quad (6.66)
\]

\[
D \delta p_c + \frac{\kappa}{q} D_c \delta q + \frac{\delta k q}{q^2} D_c \dot{q} - 2 \dot{p}_a D_c \delta p^a - 2 \left( \frac{\delta p_c \dot{q} - \delta q \dot{p}_c}{q} \right) D_c \dot{p}_a
\]

\[
= \frac{1}{2} \delta q v_c + \delta p_c k - \delta k v_c + k_{bc} \delta p^b - \frac{\delta q}{2q^2} D_c (- (3) R + Q^{ab} Q_{ab}). \quad (6.68)
\]

### 6.3.2 Linearisation: parabolic-hyperbolic

As above, we once again fix our free data \( \{ k, B^A, h_{ab}, Q_{ij}, v_a \} \), and supposing that there exists a one parameter family of solutions \( \{ p_A(\lambda), q(\lambda), A(\lambda) \} \) that depend, differentially, on an affine parameter \( \lambda \) and satisfies the property \( \{ p_A(0), q(0), A(0) \} = \{ \dot{p}_A, \dot{q}, \dot{A} \} \), where \( \{ \dot{p}_A, \dot{q}, \dot{A} \} \) is a known solution of the equations. We then proceed to expand the family of solutions as a formal series about \( \lambda = 0 \):

\[
p_A(\lambda) = \dot{p}_A + \lambda \delta p_A + \sum_{j=2}^{\infty} \chi^j \frac{\partial^j p_A(\lambda)}{\partial \lambda^j} \bigg|_{\lambda=0}, \quad (6.69)
\]

\[
q(\lambda) = \dot{q} + \lambda \delta q + \sum_{j=2}^{\infty} \chi^j \frac{\partial^j q(\lambda)}{\partial \lambda^j} \bigg|_{\lambda=0}, \quad (6.70)
\]

\[
A(\lambda) = \dot{A} + \lambda \delta A + \sum_{j=2}^{\infty} \chi^j \frac{\partial^j A(\lambda)}{\partial \lambda^j} \bigg|_{\lambda=0}, \quad (6.71)
\]

where \( \{ \delta p_A, \delta q, \delta A \} = \left\{ \frac{\partial p_A(\lambda)}{\partial \lambda} \bigg|_{\lambda=0}, \frac{\partial q(\lambda)}{\partial \lambda} \bigg|_{\lambda=0}, \frac{\partial A(\lambda)}{\partial \lambda} \bigg|_{\lambda=0} \right\} \). To find \( \{ \delta p_A, \delta q, \delta A \} \) as a solution to Eqs. (6.42), (6.40) and (6.41), this is done by solving the linear system

\[
(\partial_\rho \delta q - B^I D_I \delta q) - \delta A D^I \delta p_I - \dot{A} D^I \delta p_I = \frac{1}{2} k \delta q - 2 \left( \delta A \dot{p}_I + \dot{A} \delta p_I \right) v^I \quad (6.72)
\]

\[
2k \partial_\rho \delta A + 4 A \dot{A} D^2 \dot{A} + 2 A^2 D^2 \delta A = (3 \dot{A}^2 \dot{E} + F) \delta A + \dot{A}^3 \left( (2 \kappa + \dot{q}) \delta q - 4 \dot{p}_I \delta p^I \right) + 2k B^I D_I \delta A \quad (6.73)
\]

\[
\partial_\rho \delta p_I - D^2 \delta p_I = (Q_{IJ} v^J - D^J Q^J_I) \delta A + \frac{1}{2} \left( \dot{A} \delta q + \dot{q} \delta A - 2 \kappa \right) v_I + \dot{k} \delta p_I, \quad (6.74)
\]
7 Schwarzschild decomposition

For later use, it is convenient to take a known solution of the Einstein equations and perform the \((3 + 1)\)-and \((2 + 1)\)-decompositions.

### 7.1 The \((3 + 1)\)-decomposition

We will consider the Schwarzschild space-time in the standard polar coordinates \((t, r, \theta, \phi)\),

\[
g_{\alpha\beta} = -\left(1 - \frac{2M}{r}\right) dt_\alpha dt_\beta + \left(1 - \frac{2M}{r}\right)^{-1} dr_\alpha dr_\beta + r^2 \sigma_{\alpha\beta}, \tag{7.1}
\]

and perform a foliation by the set of Cauchy surfaces \(\Sigma := \{\Sigma_T \mid T(t, r) = t - H(r)\}\), for some known function \(H(r) \neq \text{constant}\). We have chosen \(H(r)\) to only depend on \(r\) as this will lead a spherically symmetric foliation, which is what we wish to consider. The corresponding normal vector is

\[
n_\beta = \alpha (dt_\beta - \partial_r H(r) dr_\beta), \quad \alpha = \left(\frac{2M}{r} - 1\right) \left(\partial_r H(r)\right)^2 - \frac{r}{2M - r} \right)^{-\frac{1}{2}}. \tag{7.2}
\]

This allows for the calculation of the induced 3-metric:

\[
\gamma_{\alpha\beta} = \frac{\left(r - 2M\right)\partial_r H(r)}{r(r - 2M)^2 \left(\partial_r H(r)\right)^2 - r^2 \left(-\left(2M - r\right)^2 \left(\partial_r H(r)\right) dt_\alpha dt_\beta + 2r dt_\alpha dr_\beta\right)}
\]

\[
+ \frac{r^3}{\left(r - 2M\right) \left(r^2 - (r - 2M)^2 \left(\partial_r H(r)\right)^2\right)} + r^2 \sigma_{\alpha\beta}. \tag{7.3}
\]

The corresponding extrinsic curvature is

\[
K_{\alpha\beta} = \hat{K} \alpha \left(-\frac{(2M - r)^3}{r^3} dt_\alpha dt_\beta + \frac{2(2M - r)}{r} dt_\alpha dr_\alpha + \frac{1}{(r - 2M)} dr_\alpha dr_\beta\right)
\]

\[
+ \alpha(2M - r)\partial_r H(r))\sigma_{\alpha\beta}, \tag{7.4}
\]

with

\[
\hat{K} = \frac{3Mr^2 \partial_r H(r) + (2M - r)(r^3 \partial_r^2 H(r) - M(2M - r)(\partial_r H(r))^3)}{(r^2 - (r - 2M)^2 (\partial_r H(r))^2)^2}. \tag{7.5}
\]
To remove any explicit dependence on $dt^\alpha$, we introduce adapted 3-dimensional coordinates $(T, R, \vartheta, \phi)$. The basis vectors of the new coordinate system are

\[
\partial_T^\alpha = \frac{\partial t}{\partial T} \partial_t^\alpha + \frac{\partial r}{\partial T} \partial_r^\alpha + \frac{\partial \vartheta}{\partial T} \partial_\vartheta^\alpha + \frac{\partial \phi}{\partial T} \partial_\phi^\alpha, \quad (7.6)
\]

\[
\partial_R^\alpha = \frac{\partial t}{\partial R} \partial_t^\alpha + \frac{\partial r}{\partial R} \partial_r^\alpha + \frac{\partial \vartheta}{\partial R} \partial_\vartheta^\alpha + \frac{\partial \phi}{\partial R} \partial_\phi^\alpha, \quad (7.7)
\]

\[
\partial_\vartheta^\alpha = \frac{\partial t}{\partial \vartheta} \partial_t^\alpha + \frac{\partial r}{\partial \vartheta} \partial_r^\alpha + \frac{\partial \vartheta}{\partial \vartheta} \partial_\theta^\alpha + \frac{\partial \phi}{\partial \vartheta} \partial_\phi^\alpha, \quad (7.8)
\]

\[
\partial_\phi^\alpha = \frac{\partial t}{\partial \phi} \partial_t^\alpha + \frac{\partial r}{\partial \phi} \partial_r^\alpha + \frac{\partial \vartheta}{\partial \phi} \partial_\theta^\alpha + \frac{\partial \phi}{\partial \phi} \partial_\phi^\alpha. \quad (7.9)
\]

The above are the general transformation rules. For explicit formulas, we make the associations $R = r, \vartheta = \theta$ and $\phi = \phi$ to get

\[
\partial_R^\alpha = \frac{\partial H(r)}{\partial r} \partial_t^\alpha + \partial_r^\alpha, \quad (7.10)
\]

\[
\partial_\vartheta^\alpha = \partial_\theta^\alpha, \quad \partial_\phi^\alpha = \partial_\phi^\alpha. \quad (7.11)
\]

Projecting the first and second fundamental forms onto this basis gives

\[
\gamma_{ab} = \left( \left( \frac{2M}{r} - 1 \right) \left( \partial_r H(r) \right)^2 + \frac{r}{r - 2M} \right) dr_a dr_b + r^2 d\sigma_{ab}, \quad (7.12)
\]

and

\[
K_{ab} = \frac{1}{K} \left( \frac{\hat{K}}{(r - 2M)r^3} dr_a dr_b + \left( (2M - r) \partial_r H(r) \right) d\sigma_{ab} \right), \quad (7.13)
\]

the trace of which is,

\[
K = \frac{r^2(2r - M)\partial_r H(r) + (3M - 2r)(r - 2M)^3(\partial_r H(r))^3 + r^3(r - 2M)\partial_r^2 H(r)}{r^2((r - 2M)^2(\partial_r H(r))^2 - r^2)K}. \quad (7.14)
\]

Finally, the intrinsic curvature is

\[
R = \frac{2(r - 2M)^2\partial_r H(r)((r - 2M)\partial_r^2 H(r) - (r - 2M)(\partial_r H(r))^3))}{(r^3 - r(r - 2M)^2(\partial_r H(r))^2)^2}
\]

\[
+ \frac{2(r - 2M)^2\partial_r H(r)(r(4M + r)\partial_r H(r))}{(r^3 - r(r - 2M)^2(\partial_r H(r))^2).} \quad (7.15)
\]

This completes our $(3 + 1)$-decomposition.
7.2 The \((2 + 1)\)-decomposition

We further foliate with a set of 2-spheres \(S := \{S_r | r = \text{constant}\}\). The corresponding normal co-vector is

\[
    N_a = A dr_a, \quad A = \sqrt{\frac{r^2 - (r - 2M)^2(\partial_r H(r))^2}{r(r - 2M)}}. \tag{7.16}
\]

The induced 2-metric and extrinsic 2-curvature are,

\[
    h_{ab} = r^2 \sigma_{ab}, \quad k_{ab} = -r A \sigma_{ab}, \tag{7.17}
\]

it follows that,

\[
    k = \frac{-2}{r} A^{-1}, \quad \kappa = -\frac{2}{r}. \tag{7.18}
\]

Decomposing the extrinsic 3-curvature,

\[
    K_{ab} = \frac{\hat{K}}{K} N_a N_b + \frac{2M - r}{r^2 K} \partial_r h(r) h_{ab}, \tag{7.19}
\]

where

\[
    Q_{ab} = 0, \quad v_a = p_a = 0, \tag{7.20}
\]

and

\[
    \kappa = \frac{\hat{K}}{K}, \quad q = \frac{2(2M - r)}{r^2 K} \partial_r h(r). \tag{7.21}
\]

The corresponding Ricci scalar is

\[
    (^{(2)}R) = \frac{2}{r^2}. \tag{7.22}
\]
8 Asymptotically hyperboloidal foliations

Our main interest in the study of these formulations will be the asymptotic behaviour of perturbations, and whether or not an initial data set is stable to perturbations.

**Definition 8.1.** A data set \((\Sigma, \gamma_{ab}, K_{ab})\) is called ‘asymptotically stable’ to perturbations if the perturbations are stable in the sense of Def. 2.7 and the asymptotic decay rates are unchanged.

It follows that in order to study this behaviour, we must first decide on the asymptotic geometry of the triple \((\Sigma, \gamma_{ab}, K_{ab})\). In [29], perturbations of asymptotically flat data were considered within the strongly hyperbolic formulation. In this section, we will consider geometries that are asymptotically hyperboloidal in the sense presented in [31].

**Definition 8.2.** Consider a space-like hypersurface, \(\Sigma(i)\), equipped with a Riemannian metric \(\gamma_{ab}\). We call the pair \((\Sigma(i), \gamma_{ab})\) ‘asymptotically hyperboloidal’ if and only if there exists a triple \((\Lambda, \varrho, \psi)\) where

1. \(\Lambda\) is a smooth manifold with boundary.
2. \(\varrho : \Lambda \rightarrow \mathbb{R}\) is a smooth non-negative function, with \(\varrho(x^a) = 0\) if and only if \(x^a \in \partial \Lambda\) and with \(d\varrho \neq 0\) for \(x^a \in \partial \Lambda\).
3. \(\psi : \Lambda \setminus \partial \Lambda \rightarrow \Sigma(i)\) is a smooth diffeomorphism, with \(\varrho^2 \psi^*(\gamma_{ab})\) a smooth Riemannian metric on \(\Lambda \setminus \partial \Lambda\) which extends smoothly to \(\Lambda\).

This is not the only definition we will need. A requirement on the behaviour of the extrinsic 3-curvature must also be imposed.

**Definition 8.3.** A set of initial data \((\Sigma, \gamma_{ab}, K_{ab})\) is ‘asymptotically hyperboloidal’ if

1. \((\Sigma, \gamma_{ab})\) is asymptotically hyperboloidal.
2. \(K\) is bounded away from zero asymptotically.
3. The trace-free part of \(K_{ab} = \gamma^{ac} \gamma^{bd} K_{cd}\) is order \(\varrho^3\) asymptotically.

If a data set is asymptotically stable, we simply call it stable.

8.1 The strongly hyperbolic formulation

We begin our discussion by considering the strongly hyperbolic formulation. Unless specified otherwise, all \((3 + 1)\)-and \((2 + 1)\)-decompositions will be carried out by following the process outlined in section 7.

50
8.1.1 Non-linear perturbations in the Minkowski space-time

We will begin by considering perturbations within the Minkowski solution \( (M = 0) \), and calculate the hyperbolicity condition for an arbitrary \( H(r) \) (as in Section 7).

\[
\kappa q \leq 0 \iff \frac{\partial^2 H(r)}{\partial r^2} \frac{\partial H(r)}{\partial r} \leq 0 \iff \frac{\partial}{\partial r} \left( \frac{\partial H(r)}{\partial r} \right)^2 \leq 0.
\] (8.1)

If we impose that our embedded manifold is smooth at the origin, by which we mean \( \left. \frac{\partial H(r)}{\partial r} \right|_{r=0} = 0 \), then the above inequality cannot be satisfied in some region sufficiently close to the origin [32]. However, this result offers no insight into understanding the behaviour as \( r \to \infty \), which is what we will be focusing on.

In this space-time, asymptotically hyperboloidal leaves can be produced by looking at hyperboloids. These are defined by the relation \( H(r) = \sqrt{r^2 - c^2} \), for some constant \( c \in \mathbb{R} \). These slices give rise to free data

\[
^{(3)}R = -\frac{6}{c^2}, \quad v_a = 0, \quad Q_{ab} = 0,
\] (8.2)

with metric

\[
\gamma_{ab} = \frac{c^2}{r^2 + c^2} dr_a dr_b + r^2 \sigma_{ab}.
\] (8.3)

This is the ‘hyperbolic metric’.

As an intermediate step, we will assume that the solutions (of the strongly hyperbolic constraints) will remain spherically symmetric. This motivates us to fix \( p_a = 0 \), which is in agreement with the exact Minkowski and Schwarzschild values. This assumption, reduces the PDE system to

\[
q(r) \frac{\partial q(r)}{\partial r} = \frac{3}{2r} \left( q(r)^2 - \frac{4}{c^2} \right),
\] (8.4)

which has general solution

\[
q(r) = \pm \sqrt{\frac{\tilde{c}}{r^3} + \frac{4}{c^2}} \implies \kappa = \pm \frac{\tilde{c}c^2 - 8c^3}{4c^2r^3} \left( \frac{4}{c^2} + \frac{\tilde{c}}{r^3} \right)^{-1/2},
\] (8.5)

where \( \tilde{c} \in \mathbb{R} \) is an integration constant. If \( \tilde{c} = 0 \), then the Minkowski solution is returned. This means that we have found a family of spherically symmetric solutions

\footnote{It should be noted that this is not the only way we could choose \( H(r) \). It is, however, the simplest.}
with \( \tilde{c} \) acting as a measure of how the solution differs from the precise Minkowski data. Due to this, one may interpret this exact solution as a non-linear perturbation of the Minkowski data.

The hyperbolicity condition is

\[
\kappa q = \frac{8r^3 - \tilde{c}c^2}{4c^2 r^3} < 0, \tag{8.6}
\]

which is only satisfied if

\[
r < r_\star = \frac{3}{\sqrt{\tilde{c}c^2}}. \tag{8.7}
\]

Therefore, the use of these initial data allows one to study the behaviour of the perturbed solutions as \( r \) approaches, and goes beyond, \( r_\star \). Due to this \( r_\star \) will be referred to as the transition point. More specifically, it is the point in which the PDE system transitions from strongly-hyperbolic to a non-hyperbolic. The surface defined at \( r_\star \) is a characteristic surface of the PDE. Moreover, we note that one of the principal curvatures vanishes before changing sign, at this point. As such, the hypersurfaces change convexity at \( r_\star \) [20]. It should be noted that currently, there is no standard theory that may be employed in the study of PDE’s that experience such transitional behaviour.

Of further note, the electric part of the Weyl tensor (constructed from our spherically symmetric solutions) only vanishes if \( \tilde{c} = 0 \). Since this data set is spherically symmetric, Birkhoff’s theorem tells us that we must be able to isometrically embed the data into a Schwarzschild space-time\(^{20}\), with some mass parameter \( M \). By following the process outlined in Section 7 for an arbitrary \( H(r) \), we find that if a foliation of the Schwarzschild solution is to produce the hyperbolic metric, then we must have

\[
H(r) = H_0 \pm \int_1^r \sqrt{\frac{2Mc^2 s + s^4}{(s - 2M)^2(s^2 + c^2)}} ds. \tag{8.8}
\]

Here we will interpret \( H_0 \in \mathbb{R} \) as the constant such that \( t - H(r) = H_0 \). A calculation of the extrinsic 3-curvature for this embedding allows us to make the identification \( \tilde{c} = 8M \). Some surfaces defined by Eq (8.8) are shown in Fig. 3. The transition point now becomes \( r_\star^3 = Mc^2 \) which suggests that it may be related to some kind of geometrically significant volume.

The metric is asymptotically hyperboloidal by construction and has remained unchanged from this perturbation. This is not true for the extrinsic 3-curvature

\(^{20}\)For a more in-depth discussion of Birkhoff’s theorem see [33].
Figure 3: Here we see isometric embeddings of initial data slices within the Schwarzschild space-time with mass $M = 1$. The grey curves represent surfaces of constant proper time. The blue lines are surfaces with $c = 1$ and $H_0 = -7, -8$. The red dashed curve is the surface with $c = 10$ and $H_0 = -8$. This prompts one to interpret $c$ as being related to some kind of measure of how the hypersurface ‘bends’ within the space-time. We label the event horizon, null and space-like infinity as $H^\pm$, $J^\pm$, and $i^0$, respectively.
which now has the form
\[
K_{ab} = -\frac{1}{2r^2} \sqrt{\frac{8M}{r^3} + \frac{4}{c^2} \left( \frac{r^3(c^2 + r^2)(r_\star^3 - r^3)}{c^2(2r_\star^3 + r^3)} \partial_r a \partial_r b + \partial_\theta a \partial_\theta b + \csc^2(\theta) \partial_\phi a \partial_\phi b \right)}.
\] (8.9)

The trace is,
\[
K = -\frac{6(Mc^2 + r^3)}{c^2r^3} \left( \frac{4}{c^2} + \frac{8M}{r^3} \right)^{-\frac{1}{2}},
\] (8.10)

with limit \( K \to -\frac{3}{2} \) as \( r \to \infty \), which is clearly bounded away from zero asymptotically. Expressing the trace-free components of \( K_{ab} = \gamma_{ac} \gamma_{bd} K_{cd} \) gives
\[
K_{ab} - \frac{1}{3} K \gamma_{ab} = \begin{pmatrix}
\mathcal{O}(\varrho^3) & 0 & 0 \\
0 & \mathcal{O}(\varrho^5) & 0 \\
0 & 0 & \mathcal{O}(\varrho^5)
\end{pmatrix},
\] (8.11)

where it has been used that one may approximate \( \varrho \) as \( \varrho = \varrho_0 + \mathcal{O}(r^{-2}) \), for some function \( \varrho_0 = \varrho_0(x^A) \), and \( \varrho \) is the conformal factor, as in Def. 8.2. Then, all conditions that are required for this data set to be asymptotically hyperboloidal are satisfied.

### 8.1.2 Linear perturbations in the Schwarzschild space-time

We will now use the above data set to study its linear perturbations.

\[
\dot{p}_A = 0, \quad \dot{q} = -\sqrt{\frac{8M}{r^3} + \frac{4}{c^2}}.
\] (8.12)

The linearised equations for these data set are
\[
0 = \sqrt{\frac{c^2}{c^2 + r^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta p_A - \frac{r^3 - r_\star^3}{4r^3 + 2r^3} \frac{\partial}{\partial x^A} \delta q,
\] (8.13)
\[
0 = \sqrt{\frac{c^2}{c^2 + r^2}} \left( \frac{\partial}{\partial r} + \frac{3(r^3 + r_\star^3)}{2r^2r_\star^r + r^4} \right) \delta q - \frac{1}{r^2} \sigma^{AB} (\sigma) \nabla_A \delta p_B.
\] (8.14)

With the intent of expressing the linearised equations in terms of the \( \mathcal{E} \)-\( \mathcal{H} \) operators introduced in Section 2.2, we will make use of the following:
\[
m^a \partial_a (\bar{\rho}) = \frac{1}{\sqrt{2}} \bar{\partial} (\bar{\rho}) - \frac{1}{\sqrt{2}} \bar{\rho} \cot(\theta), \quad \bar{m}^a \partial_a (\rho) = \frac{1}{\sqrt{2}} \bar{\delta} (\rho) - \frac{1}{\sqrt{2}} \rho \cot(\theta),
\] (8.15)

54
where the notation $\bar{p} = m^A\delta p_A$ and $p = m^A\delta p_A$ has been introduced. Then Eq. (8.14) becomes

$$\sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r_*^2r + r^4} \right) \delta q - \frac{1}{\sqrt{2r^2}}(\bar{\delta}(\bar{p}) + \bar{\delta}(p)) = 0.$$  

We now expand each variable in terms of SWSH, to get

$$\delta q = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_{l,m} Y_{l,m}(\theta, \phi),$$  

$$\bar{p} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \bar{p}_{l,m} Y_{l,m}(\theta, \phi),$$  

$$p = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} p_{l,m} Y_{l,m}(\theta, \phi),$$

Then, we have

$$0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r_*^2r + r^4} \right) \delta q_{l,m} \right) Y_{l,m}(\theta, \phi) + \frac{1}{\sqrt{2r^2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \bar{p}_{m,l} \sqrt{l(l+1)} - p_{m,l} \sqrt{l(l+1)} \right) Y_{l,m}(\theta, \phi).$$  

This equation must hold for each $l = 0, ..., \infty$ and $m = -l, ..., +l$ and hence

$$\sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r_*^2r + r^4} \right) \delta q_{l,m} + \frac{2\sqrt{l(l+1)}}{\sqrt{2r^2}} (p_{l,m} - \bar{p}_{l,m}) = 0.$$  

We now turn our attention to Eq. (8.13), and consider the projections onto the frame vectors $m^A$ and $\bar{m}^A$:

$$0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p} - \frac{r_*^3 - r^3}{4r_*^3 + 2r^3} m^a \partial_a \delta q,$$  

and

$$0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p - \frac{r_*^3 - r^3}{4r_*^3 + 2r^3} m^a \partial_a \delta q.$$
The eth operators acting on a spin zero function are

\[ m^a \partial_a \delta q = \frac{1}{\sqrt{2}} \bar{\delta}(\delta q), \quad \bar{m}^a \partial_a \delta q = \frac{1}{\sqrt{2}} \bar{\delta}(\delta q). \]  

(8.24)

Then

\[
\sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p} - \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} \bar{\delta}(\delta q) = 0,
\]

(8.25)

\[
\sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p - \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} \delta(\delta q) = 0.
\]

(8.26)

Using the expansions given by Eqs. (8.17)-(8.19) gives

\[
0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p}_{l,m} + \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)} \right) -1 Y_{l,m}(\theta, \phi),
\]

(8.27)

\[
0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_{l,m} - \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)} \right) +1 Y_{l,m}(\theta, \phi).
\]

(8.28)

As before, these equations must hold for all \( l = 0, \ldots, \infty \) and \( m = -l, \ldots, +l \), so

\[
0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p}_{l,m} + \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)},
\]

(8.29)

\[
0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_{l,m} - \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)}.
\]

(8.30)

Finally, we are left with the following system of equations:

\[
0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p}_{l,m} + \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)},
\]

(8.31)

\[
0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_{l,m} - \frac{1}{\sqrt{2}} \frac{r^3 - r^3}{4r_*^3 + 2r^3} q_{l,m} \sqrt{l(l+1)},
\]

(8.32)

\[
0 = \sqrt{\frac{r^2 + c^2}{c^2}} \left( \frac{\partial}{\partial r} + \frac{3(r^2 + r^3)}{2r_*^3 r + r^4} \right) q_{l,m} + \frac{2\sqrt{l(l+1)}}{\sqrt{2}r^2}(p_{l,m} - \bar{p}_{l,m}).
\]

(8.33)

For each \( l = 0, \ldots, \infty \) and \( m = -l, \ldots, +l \), this is a set of three coupled ODE for three variables, namely \( p_{l,m}, \bar{p}_{l,m} \) and \( q_{l,m} \). Notice here that the transitional behaviour (from
hyperbolic to non-hyperbolic) of the equations is ‘irrelevant’ from the point of view of these ODE. Moreover, the ODE system for each mode can be solved as an IVP irrespective of whether we are in the hyperbolic or non-hyperbolic regime.

To study the asymptotic behaviour, it is enough to only consider the leading order terms in a Taylor expansion with respect to $r$ at infinity, a fact that is justified through the use of Fuchsian analysis; see [34,35] for more details.

The resulting ODE system is

$$
\begin{align*}
& r \frac{\partial}{\partial r} \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix} + \begin{pmatrix} 2 & 0 & -\frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & 2 & \frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix} = 0. \\
& (8.34)
\end{align*}
$$

Under the coordinate change

$$
\begin{align*}
& u = -\ln \left( \frac{1}{r} \right) \implies r \frac{\partial}{\partial r} = r \frac{\partial u}{\partial r} = \frac{\partial}{\partial u}, \\
& (8.35)
\end{align*}
$$

we get

$$
\begin{align*}
& \frac{\partial}{\partial u} \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix} + \begin{pmatrix} 2 & 0 & -\frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & 2 & \frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix} = 0. \\
& (8.36)
\end{align*}
$$

which is an ODE of form $\frac{d}{du} \vec{z}(u) = W \vec{z}(u)$ with

$$
W = \begin{pmatrix} -2 & 0 & -\frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & -2 & \frac{c}{2} \frac{\sqrt{l(l+1)}}{\sqrt{2}} \\ 0 & 0 & -3 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix}. \\
(8.37)
$$

Equations of this form have the solution

$$
\vec{z}(u) = \sum_{j=1}^{3} A_j \exp(\lambda_j u) \vec{\lambda}_j, \\
(8.38)
$$

where the $\lambda_j$ are the eigenvalues of $W_{\alpha\beta}$, $\vec{\lambda}_j$ are the corresponding eigenvectors, and each $A_j \in \mathbb{R}$ is a constant. Solving the eigenvalue equation gives

$$
0 = \det|\eta_{ij} - \lambda \delta_{ij}| = (2 + \lambda)^2(3 + \lambda) \Rightarrow \lambda \in \{-2, -2, -3\}. \\
(8.39)
$$

57
The corresponding eigenvectors are

\[ \vec{\lambda} \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \sqrt{\frac{1}{2} (l+1)} \\ \frac{\sqrt{2}}{2} \\ -c \sqrt{\frac{1}{2} (l+1)} \end{pmatrix} \right\} . \]  

(8.40)

It follows that the general solution is

\[ \begin{pmatrix} p_{l,m} \\ \bar{p}_{l,m} \\ q_{l,m} \end{pmatrix} = e^{-2u} \begin{pmatrix} A_p \\ A_{\bar{p}} \\ 0 \end{pmatrix} + A_q e^{-3u} \begin{pmatrix} c \sqrt{\frac{1}{2} (l+1)} \\ -c \sqrt{\frac{1}{2} (l+1)} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \]  

(8.41)

\[ = \frac{1}{r^2} \begin{pmatrix} A_p \\ A_{\bar{p}} \\ 0 \end{pmatrix} + \frac{A_q}{r^3} \begin{pmatrix} c \sqrt{\frac{1}{2} (l+1)} \\ -c \sqrt{\frac{1}{2} (l+1)} \\ \frac{\sqrt{2}}{2} \end{pmatrix} , \]

for integration constants \( A_p, A_{\bar{p}}, A_q \in \mathbb{C} \). Thus, to leading order, we have

\[ (\delta q, p, \bar{p}) = \left( \frac{1}{r^3} \hat{q}, \frac{1}{r^2} \hat{p}, \frac{1}{r^2} \hat{\bar{p}} \right) , \]  

(8.42)

for functions \( \hat{p} = \hat{p} (\theta, \phi) \), \( \hat{\bar{p}} = \hat{\bar{p}} (\theta, \phi) \) and \( \hat{q} = \hat{q} (\theta, \phi) \).

Using these values we are able to reconstruct the extrinsic curvature and, once again, check that the resulting initial data is still asymptotically hyperboloidal. This is done by following the same steps outlined previously. As it turns out, this is indeed the case, and hence, at least asymptotically, perturbations of the initial data are stable. We will see shortly that these equations can be viewed as an ‘elliptic’ set of equations, and hence this result is unexpected, since ‘elliptic’ equations are typically do not have a well-posed IVP. Another reason that this result is noteworthy is because in [29] perturbations to asymptotically flat data were found to have the ‘wrong’ fall-off rates.

This long term behaviour can be confirmed by considering the numerical solutions of Eqs. (8.32) and (8.33). The numerical analysis presented here will be restricted to the case with axial symmetry i.e. \( m = 0 \). The numerical solutions corresponding to \( l = 15 \) are shown in Fig. 4, where we see that \( p_{l,m} = -\bar{p}_{l,m} \). This relation holds for all modes with \( l \neq 0 \). To check the fall-off rates, a linear relation is fitted to the logarithm of the solution. This is shown in Fig 5. The asymptotic behaviours of the
Figure 4: The solutions $p_{15,m}, \bar{p}_{15,m}$ and $q_{15,m}$ for the $l = 15$ mode with $M = c = 1$ and $r_\star = 1$. Here we see that $p_{15,m} = -\bar{p}_{15,m}$.

solutions are $q_{15,m} \propto r^{-2.95}$ and $p_{15,m}, \bar{p}_{15,m} \propto r^{-1.95}$. These decays are incorrect by $-0.05$. The accuracy can be improved by considering a larger range of $r$ values.\footnote{This statement has the following justification: Initially, we considered a smaller range of $r$, and ultimately increased the maximum radius to obtain better accuracy. The case presented here was the largest range of $r$ values we considered and had the closest accuracy.}

Turning our attention to the transition point, we note that wave-like dynamics should be expected for $r \leq r_\star$ and, for some short region, we expect exponential growth dynamics for $r > r_\star$. A plot of the solutions for an interval around $r_\star$ is shown in Fig. 6 for the $l = 15$ mode.

So far we have seen that Eqs. (8.31)-(8.33) have a well-posed IVP, which implies that the full system (Eqs. (8.13) and (8.14) is also well-posed, within the class of initial data with finitely many modes. To get evidence that the same conclusion can be drawn when the initial data has infinitely modes we must check numerical convergence. This is done as follows:

1. **Choose two spatial discretisations**: The discretisation is the set of discrete $\theta$ values $^1\Theta := \{\theta_j \mid j \in I\}$, where $I$ is an indexing set. The total number of grid points is the cardinality and will be represented by the notation $N(\Theta)$. For comparative purposes two discretisations will be defined, namely $(1)\Theta$ and $(2)\Theta$. Here, it will be assumed that $(1)\Theta \subset (2)\Theta$.

2. **Solve the PDE system for each discretisation**: One now solves the PDE system for both discretisations\footnote{To numerically solve these equations we will make use of the axial symmetric spin-weighted}, it should be emphasised here that we must
Figure 5: Logarithmic plots of the numerical solutions for the $l = 15$ mode with $M = c = 1$. $q_{15,m}$ falls off at a rate $r^{-2.95}$ and $p_{15,m}, \tilde{p}_{15,m}$ decays at $r^{-1.95}$.

Figure 6: Plot of the numerical solutions of $p_{15,m}$ and $q_{15,m}$. 9000 grid points are considered for $r \in [0.01, 2]$. Once again the choices $c = M = 1$ have been made which gives $r_* = 1$. 
choose the same initial data for both discretisations. The solution corresponding to \( i \Theta \) will be represented by the notation \( \phi \in \{ p^{(i)}, \bar{p}^{(i)}, q^{(i)} \} \).

3. **Compare the two sets at each shared grid point**: To compare the solutions, we define the norm

\[
\| \phi \|_{\Theta} := \max_{\theta_1, \theta_2} \| \phi^{(2)} - \phi^{(1)} \|,
\]

where \( \phi \) is defined as in the previous step. Obviously, it only makes sense to compare solutions at shared grid points. In order to ensure that such situations occur, we will impose the relationship \( N^{(2)} = 2N^{(1)} - 1 \).

4. **Complete steps 1 – 3 at least twice**: On an intuitive level, one may view Eq. (8.43) as a measure of error. Provided that the PDE has a well-posed IVP, one would expect the (numerically) measured error to decrease as the set \( N^{(1)} \) becomes finer. However, if no such convergence is seen, then the equations may not possess a well-posed IVP.

The final conclusion of the above sentence assumes that there is no error in the code. We state that this is the case as all expected results have been produced.

Since we require our initial data to have infinitely many modes we will specify the function in coefficient space. Functions such as \( e^{\cos(\theta)} \) have infinitely many modes and can be specified in physical space. Nevertheless, we have instead chosen to specify the function in coefficient space as it minimise the numerical error involved. The chosen initial data takes the form

\[
\begin{align*}
\delta q^{(0)} &= \sum_{l=0}^{\infty} \sum_{m=-|l|}^{|l|} e^{-\frac{1}{10} l} Y_{l,m}(\theta, \phi), \\
p^{(0)} &= \bar{p}^{(0)} = 0.
\end{align*}
\]

Since the coefficients decay exponentially, this is a smooth function. It should be noted however, that is the decay is slow. We have chosen this as it ensures that the coefficients do not become noisy too soon. A plot of \( \delta q^{(0)} \) is given in Fig. (7).

Three of plots of the numerical error are presented in Fig. 8 for \( N^{(1)} = 30, 40, \) and 60. Comparing these one sees that as the cardinality of \( \Theta \) increases the associated error does so as well. This shows that the growth within this region is unbounded and hence the initial value problem is not well-posed. Note that this result only applies to data with infinitely many modes, data with finitely many modes is well-posed. This phenomena appears surprising as the geometry is asymptotically stable to perturbations.

---

**functions** Python module, presented in [36].
Figure 7: These are the plots of the initial function. The graph on the left is in physical space, and demonstrates exponential decay with increasing values of $\theta$. The graph on the right is shown in the coefficient space and shows how the coefficients decay with increasing $l$.

Figure 8: A semi-log plot of the compared solutions of the linearised equations. Notice that as the $N_{(4)}$ becomes finer, we see an increase in logarithm of the error.
8.1.3 Linear perturbations as a second order PDE

To further investigate the behaviour identified in the previous subsection, we consider the PDE:

\[ 0 = \frac{\partial^2 u}{\partial r^2} - \mathcal{F}(r) \frac{\partial u}{\partial r} - \mathcal{C}^2(r) \nabla^2 u \]

\[ \mathcal{F}(r) = \frac{1}{2r} - \frac{r}{c^2 + r^2} - \frac{9r^2}{2(2r_*^2 + r^3)} + \frac{3r^2}{r^3 - r_*^3}, \]

\[ \mathcal{C}^2(r) = \frac{r^3(r_*^3 - r^3)}{2r^2(c^2 + r^2)(2r_*^2 + r^3)^3}, \]

for some unknown function \( u = u(x^a) \). The functions \( \mathcal{F}(r) \) and \( \mathcal{C}^2(r) \) will be referred to as the ‘friction’ and ‘speed’ terms, respectively.

**Proposition 8.1.** Suppose there exists a sufficiently smooth solution \( u(x^a) \) of Eq. (8.46). Then, the functions

\[ \delta q = \frac{4}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \frac{16r^3 + r^3 \partial u(x^a)}{16r_*^3 - r^3}, \quad \delta p_A = \frac{1}{r^2} \frac{\partial u(x^a)}{\partial x^A}, \]

satisfy the linearised equations (Eqs. (8.13) and (8.14)).

Moreover, suppose \( \delta p_A \) and \( \delta q \) are smooth solutions of the linearised equations (Eqs. (8.13) and (8.14)) such that

\[ \omega := \frac{\partial \delta p_A}{\partial \phi} - \frac{\partial \delta p_A}{\partial \theta} = 0, \]

initially. Then, there exists a uniquely determined family of solutions \( U := \{ u(x^i) + \lambda | \lambda = \text{constant} \} \) such that all elements \( u \in U \) satisfy Eqs. (8.46) and (8.47).

**Proof.** Let \( u = u(x^a) \) be a smooth solution of Eq. (8.46) and define \( \delta q \) and \( \delta p_A \) as in Eq. (8.47). The \( r \)-derivatives are,

\[ \frac{\partial \delta p_A}{\partial r} = -\frac{2}{r^3} \frac{\partial u}{\partial x^A} + \frac{1}{r^2} \frac{\partial u}{\partial x^A} = -\frac{2}{r} \delta p_A + \sqrt{\frac{c^2 + r^2}{c^2}} \frac{r_*^3 - r^3}{4r_*^3 + 2r^3} \frac{\partial \delta q}{\partial x^A} \]

\[ \Leftrightarrow 0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta p_A - \sqrt{\frac{c^2 + r^2}{c^2}} \frac{r_*^3 - r^3}{4r_*^3 + 2r^3} \frac{\partial \delta q}{\partial x^A}. \]
and
\[
\frac{\partial \delta q}{\partial r} = \left( -\frac{2}{r} + \frac{r}{c^2 + r^2} + \frac{3r^2}{2r_*^4 + r^3} + \frac{3r^2}{r_*^4 - r^3} \right) \delta q + \frac{4}{r^2} \sqrt{\frac{c^2 + r^2}{c^2} 16r_*^3 + r^3} \frac{\partial u^2}{\partial r} \\
= -\frac{3(r_*^3 + r^3)}{2r_*^3 r + r^4} \delta q + \frac{1}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \left( \frac{\partial^2}{\partial \theta^2} + \csc^2(\theta) \frac{\partial^2}{\partial \phi^2} + \cot(\theta) \frac{\partial}{\partial \theta} \right) u \\
\iff 0 = \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r_*^3 r + r^4} \right) \delta q - \frac{1}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \left( \frac{\partial \delta p_\theta}{\partial \theta} + \csc^2(\theta) \frac{\partial \delta p_\phi}{\partial \phi} + \cot(\theta) \delta p_\theta \right). 
\]
\( (8.50) \)

These are the linearised equations. Thus, \( \delta q \) and \( \delta p_A \) defined in this way produce a solution to the linearised system.

Suppose now that \( \delta q \) and \( \delta p_A \) are solutions of the linearised equations such that \( \omega = 0 \) initially and define the 1-form
\[
\Omega := \frac{r^2}{4} \sqrt{\frac{c^2 + r^2}{c^2} 16r_*^3 - r^3} \delta q dr + r^2 \delta p_\theta d\theta + r^2 \delta p_\phi \\
= \Omega_r dr + \Omega_\theta d\theta + \Omega_\phi d\phi. 
\]
\( (8.51) \)

We wish to show the existence of a solution to the PDE system described by Eq. (8.47). A necessary condition for this is that the exterior derivative of \( \Omega \) vanishes, and hence \( \Omega \) is closed, i.e.
\[
d\Omega = (\partial_r \Omega_\theta - \partial_\theta \Omega_r) dr \land d\theta + (\partial_r \Omega_\phi - \partial_\phi \Omega_r) dr \land d\phi + r^2 \omega d\theta \land d\phi = 0. 
\]
\( (8.52) \)

We begin by showing that \( \partial_r \Omega_A - \partial_A \Omega_r = 0 \).
\[
\partial_r \Omega_A - \partial_A \Omega_r = \frac{\partial}{\partial r} \left( r^2 \delta p_A \right) - r^2 \sqrt{\frac{c^2 + r^2}{c^2}} \frac{r_*^3 - r^3}{4r_*^4 + 2r^3} \partial x^A \\
= r^2 \left( \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta p_A - \sqrt{\frac{c^2 + r^2}{c^2}} \frac{r_*^3 - r^3}{4r_*^4 + 2r^3} \partial x^A \right), 
\]
\( (8.53) \)

which is zero by assumption.

We know that \( \omega \) vanishes at the initial radius \( r_0 \). To show that this remains true for all \( r \geq r_0 \), we consider the \( r \)-derivative:
\[
\frac{\partial \omega}{\partial r} = \frac{\partial}{\partial \phi} \frac{\partial \delta p_\theta}{\partial r} - \frac{\partial}{\partial \theta} \frac{\partial \delta p_\phi}{\partial r} \\
= \sqrt{\frac{c^2}{c^2 + r^2}} \left( \frac{r_*^3 - r^3}{4r_*^4 + 2r^3} \left( \frac{\partial}{\partial \theta} \frac{\partial \delta p_\theta}{\partial \phi} - \frac{\partial}{\partial \phi} \frac{\partial \delta p_\theta}{\partial \theta} \right) - \frac{2}{r} \left( \frac{\partial \delta p_\theta}{\partial \phi} - \frac{\partial \delta p_\phi}{\partial \theta} \right) \right) \\
= -\frac{2}{r} \sqrt{\frac{c^2}{c^2 + r^2}} \omega \iff \omega = \frac{\omega_0 r}{c(c + \sqrt{c^2 + r^2})}, 
\]
\( (8.54) \)
where \( \omega_0 \in \mathbb{R} \) is a constant. Since \( r > 0 \) and \( c \) is finite we have that if \( \omega = 0 \) on the initial surface then \( \omega_0 = 0 \) and hence \( \omega \) will remain zero. Thus \( d\Omega = 0 \) and hence there exists a solution \( u = u(x^a) \) of Eq. (8.47).

Finally we show that \( u \) also satisfies Eq. (8.46):

\[
0 = \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r^3r + r^4} \right) \delta q - \frac{1}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \left( \frac{\partial \delta p_\theta}{\partial \theta} + \csc^2(\theta) \frac{\partial \delta p_\phi}{\partial \phi} + \cot(\theta) \delta p_\theta \right)
\]

\[
= \left( \frac{\partial}{\partial r} + \frac{3(r_*^3 + r^3)}{2r^3r + r^4} \right) \left( 4 \frac{r^2}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \frac{2r^3 + r^3}{2(r_*^3 - r^3)} \frac{\partial u}{\partial r} \right)
\]

\[
- \sqrt{\frac{c^2 + r^2}{c^2}} \left( \frac{\partial^2 u}{\partial \theta^2} + \csc^2(\theta) \frac{\partial^2 u}{\partial \phi^2} + \cot(\theta) \frac{\partial u}{\partial \theta} \right)
\]

\[
= - \frac{2r^6(c^2 + 2r^2) + r^3c^2(c^2 - r^2) - r^3(11c^2r^3 + 10r^5)}{cr^3 \sqrt{c^2 + r^2(r_*^3 - r^3)^2}} \frac{\partial u}{\partial r}
\]

\[
+ \frac{4}{r^2} \sqrt{\frac{c^2 + r^2}{c^2}} \frac{2r^3 + r^3}{2r_*^3 - 2r^3} \frac{\partial^2 u}{\partial r^2} - \sqrt{\frac{c^2 + r^2}{c^2}} \left( \frac{\partial^2 u}{\partial \theta^2} + \csc^2(\theta) \frac{\partial^2 u}{\partial \phi^2} + \cot(\theta) \frac{\partial u}{\partial \theta} \right)
\]

\[
\Leftrightarrow 0 = \frac{\partial^2 u}{\partial r^2} - \left( \frac{1}{2r} - \frac{r}{c^2 + r^2} - \frac{18r^2}{\tilde{c}c^2 + 4r^3} + \frac{24r^2}{8r^3 - \tilde{c}c^2} \right) \frac{\partial u}{\partial r}
\]

\[
- \frac{r^3c^2 - c^2r^3}{2r^2(c^2 + r^2)(2r_*^3 + r^3)} \left( \frac{\partial^2 u}{\partial \theta^2} + \csc^2(\theta) \frac{\partial^2 u}{\partial \phi^2} + \cot(\theta) \frac{\partial u}{\partial \theta} \right) u.
\]

(8.55)

Thus, the solution \( u \) also satisfies Eq. (8.46).

This proposition allows us to interpret the two systems as being equivalent, with \( u(x^a) \) acting as a potential for \( \delta p_A \) and \( \delta q \). Moreover, the second order system elucidates the transition from hyperbolic to elliptic.

The principal part of this second order system changes its form depending on the sign of the ’speed’ term

\[
C^2(r) = \frac{c^2(r_*^3 - r^3)}{2r^2(c^2 + r^2)(2r_*^3 + r^3)}.
\]

(8.56)

The system takes the form of a wave equation for \( r \leq r_* \) and a Laplace equation for \( r \geq r_* \). Further, we note that as \( r \to \infty \) we have \( C^2(r) \to 0 \) with leading order \( r^{-4} \). This behaviour can be observed in Fig. 9. To help justify this, we consider the equation

\[
\frac{\partial^2 u}{\partial r^2} = \left( \frac{c^2}{r_*^3} - \frac{2}{r} \right) \frac{\partial u}{\partial r},
\]

(8.57)
Figure 9: Graph of $C^2(r)$ vs. $r$, with $8M = c = 1$ and hence $r_\ast = 0.5$. Here we see the predicted behaviour: The function is positive for $r < r_\ast$. For $r > r_\ast$ the function begins to grow negativity but then quickly tends to zero.

which is Eq. (8.46) where all terms that fall off at $r^{-4}$ or faster have been ignored. This equation has a solution of the form

$$u(x^a) = c_0(x^A) - c_1(x^A) \sqrt{\frac{\pi}{2c^2}} \int_0^{\sqrt{2r}} e^{-s^2} ds = c_0(x^A) - \frac{c_1(x^A)}{r} + \mathcal{O}(r^{-3}),$$ (8.58)

where $c_0(x^A)$ and $c_1(x^A)$ are unknown functions of $x^A = (\theta, \phi)$ and the Taylor expansion has been performed at $r = \infty$. The transformation rules at the beginning of this section give,

$$\delta p_A = \frac{1}{r^2} \left( \frac{\partial c_0(x^I)}{\partial x^A} - \frac{1}{r} \frac{\partial c_1(x^I)}{\partial x^A} \right), \quad \delta q = \frac{c_1(x^I)}{r^3}. \quad (8.59)$$

This is in agreement with the previously predicted leading order behaviour, as would be expected.

We now consider the ‘friction/dampening’ term:

$$\mathcal{F}(r) = \frac{1}{2r} - \frac{r}{c^2 + r^2} - \frac{9r^2}{4r^3 + 2r^3} + \frac{3r^2}{r^3 - r_\ast^3}. \quad (8.60)$$

We note here that $\mathcal{F}(r)$ has a singularity at $r = r_\ast$. However, the first order system experiences no such singularity, which implies that this behaviour has been introduced by us. With this in mind we claim that the solution $u$ will not experience such singular behaviour. This can be shown by considering the mode relations between the first and second order systems. As we have done previously, we will expand $u$ as

$$u = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} u_{l,m} y_{l,m}(\theta, \phi). \quad (8.61)$$
This allows us to write Eq. (8.46) in coefficient space as
\[ \partial_r^2 u_{l,m} - F(r) \partial_r u_{l,m} - C^2(r) l(l+1) u_{l,m} = 0. \]  
(8.62)

The modes of each system are related as follows:
\[ \delta p_A = r^{-2} \delta_A u \Rightarrow m^A \delta p_A = r^{-2} m^A \partial_A u = -\frac{\partial(u)}{r^2 \sqrt{2}} \]
\[ \Leftrightarrow m^B \delta p_B = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} p_{l,m} + 1 Y_{l,m}(\theta, \phi) = \frac{1}{r^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} u_{l,m} \partial(g Y_{l,m}(\theta, \phi)) \]
\[ = \frac{1}{r^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} -\frac{u_{l,m} \sqrt{l(l+1)}}{\sqrt{2}} + 1 Y_{l,m}(\theta, \phi). \]
(8.63)

Since this must hold for all \( l = 1, \ldots, \infty \) and \( m = -l, \ldots, l \) we find that
\[ p_{l,m} = -\frac{u_{l,m} \sqrt{l(l+1)}}{r^2 \sqrt{2}}. \]
(8.64)

Similarly,
\[ \tilde{p}_{l,m} = \frac{\sqrt{l(l+1)}}{r^2 \sqrt{2}} u_{l,m}, \quad u_{l,m} = u_0 + \int_0^r \frac{s^2}{2} \sqrt{\frac{c^2}{c^2 + s^2}} r^3 - s^3 \delta q_{l,m}(s) ds, \]
(8.65)

where \( u_0 \) is a constant. Of interest here is the following:
\[ \tilde{p}_{l,m} = -p_{l,m}, \text{ for } l \neq 0. \]
(8.66)

This result was stated previously. Since there is no singular behaviour in Eqs. (8.64) and (8.65), these relations can be used to find the value of \( u_{l,m} \) for each \( l = 0, \ldots, \infty \).

Fig. 10 makes use of these relations for the \( l = 15 \) mode.

The singularity present in the friction term makes numerical studies of this system difficult. Nevertheless, it does allow for a heuristic interpretation of the observed behaviour of the first order system. For the range \( 0 < r < r_* \), the linearised equations show wavelike behaviour, as is expected due to the hyperbolic nature of the equations in this regime. The frequency of these oscillations diminishes as the solutions approach \( r_* \) and in particular, the first derivatives of \( u \) goes to zero at the transition point. This serves to counter the singular behaviour of the friction term at \( r_* \). As \( r \) goes beyond \( r_* \), the speed term grows quickly for a finite range. In this region, the solution experiences exponential like behaviour and hence is not stable with respect to the IVP. After a sufficiency large evolution, the speed term tends to zero and the friction term becomes relevant, serving to dampen the remaining growth of the solution. Because of this, the second order system tends to a constant. Moreover, the first order derivatives of \( u \) (which are, loosely speaking, the solutions to the first order system) tend to zero.
Figure 10: Plot of $u_{15,m}$ and $\partial_r u_{15,m}$, which are calculated from the $p_{15,m}$ and $q_{15,m}$ modes. Notice that neither the solution nor its first derivative experiences singular behaviour at $r_\star$. Moreover at $r_\star$ we have that $\partial_r u_{15,m} = 0$, which serves to counter the singular behaviour of the friction term.

8.1.4 Linear perturbations as a boundary value problem

Thus far we have seen that the linearised equations are only stable from the perspective of an IVP if the initial data has a finite number of modes. This is owing to the fact that the hyperbolicity condition is violated after a finite evolution.

The standard approach in dealing with elliptic equations is to solve them as a BVP. In what follows we will consider such an approach. It should be noted that the primary focus of this work is to solve the constraints as an IVP and as such we will only briefly discuss this approach; strictly within the context of the linearised equations.

To simplify the problem, the restriction to a setup with axial symmetry will be made. This will allow us to deal with the second order system, which we aim to solve as a two-point BVP. Dealing with this equation gives rise to two problems. Firstly, due to the singular behaviour of the friction-term, we must pick data such that $\partial_r u_{l,m} = 0$ at $r = r_\star$. Secondly, a way to specify the boundary data at infinity must be found. To overcome these issues, we make the ansatz that in coefficient space, the solution to the BVP will have the same form as the solution to the IVP; in the following sense:

Let $u^{(0)}_{l,m}$ and $u^{(f)}_{l,m}$ represent the initial and final values of $u_{l,m}$, respectively. Then
we solve the following systems of ODE:

\[
\begin{align*}
\frac{d^2 u_{l,m}}{dr^2} - F(r) \frac{du_{l,m}}{dr} - C^2(r) l(l+1) u_{l,m} &= 0, \\
r \in [r_0, r_*], \quad r_0 < r_*, \quad (u_{l,m}^{(0)}, u_{l,m}^{(f)}) &= (u_{l,m}^{(s)}, u_{l,m}^{(s)}). \\
\end{align*}
\]

(8.67)

where \( u_{l,m}^{(s)} \in \mathbb{C} \) and \( \hat{r} \in \mathbb{R} \) are constants. The idea is as follows: Fix the choice of initial and final data, as well as an initial point \( r_0 < r_* \). The ODE is now solved in the range \([r_0, r_*]\) with boundary data \((u_{l,m}^{(0)}, u_{l,m}^{(f)})\), where \( u_{l,m}^{(s)} \) is some freely chosen constant. At \( r_* \) the solution is \( u_{l,m}^{(s)} \). The ODE is now solved in the range \((r_*, \hat{r}]\) with boundary data \((u_{l,m}^{(s)}, u_{l,m}^{(f)})\), where \( \hat{r} \) is some sufficiently large radius. For \( r \geq \hat{r} \), the solution is assumed to be constant. This is the behaviour that was observed previously. The quantity \( \hat{r} \) will be referred to as the “cut-off point”.

The arbitrary choice of \( u_{l,m}^{(s)} \) is a freedom that arises from the fact that \( u \) acts as a potential for the curvature quantities. We will make the choice \( u_{l,m}^{(s)} = 0 \). Searching for solutions of this form does not guarantee that \( \frac{du_{l,m}}{dr} \bigg|_{r=r_*} = 0 \) and hence we must check that this condition is satisfied. This will be demonstrated numerically for the \( l = 15 \) mode with boundary data \((u_{15,m}^{(0)}, u_{15,m}^{(f)}) = (0, 5)\).

Finite differencing is employed here with grid points \( N = 5000 \). For the region \( r \in (0, r_*) \) we see oscillatory behaviour, as would be expected. In the region \( r \in (r_*, \hat{r}] \) we see that solution quickly grows to \( u_{15,m}^{(f)} = 5 \). In particular we note that the solution appears to become constant after a sufficiently large evolution. This lends credence to the ansatz made about the form of the solution. For further analysis of this approximation, we consider two solutions with the same boundary data allowing one to have a larger cut-off point than the other. We then consider the difference of the two solutions at the shared grid points. The solution \( u_{l,m} \) will have a cut-off point \( \hat{r}_1 \) and the solution \( \tilde{u}_{l,m} \) will have a cut-off point \( \hat{r}_2 \), where \( \hat{r}_2 > \hat{r}_1 \). As the value of \( \hat{r}_1 \) increase, we see that the difference between solutions decreases. In principal, one could (possibly) use these solutions to construct a Cauchy sequence and hence verify the convergence of the solution coefficients. This will not be done here, however. The difference between the solutions is shown in Fig. 13.

\[\text{It is possible to solve the ODE as an IVP in the region. However, we will not be exploring this.}\]
Figure 11: The solution $u_{15,m}$ for $r \in (0, 5000) \setminus \{\frac{1}{2}\}$, subject to the boundary conditions $(u_{15,m}^{(0)}, u_{15,m}^{(f)}) = (0, 5)$. In the region of hyperbolicity, the solution oscillates, as would be expected. Moreover in the elliptic region, we see that the solution tends to a constant.

Figure 12: Here we see $\partial_r u_{15,m}$ for $r \in (0, 5000) \setminus \{\frac{1}{2}\}$, subject to the boundary conditions $(u_{15,m}^{(0)}, u_{15,m}^{(f)}) = (0, 5)$. In the range $(0, 0.5)$ we see the derivative tends to zero. For $r \in (0.5, 5000)$ we see that $\partial_r u_{15,m}$ is approximately zero close to $r_*$. The rapid growth after $r_*$ is due to the friction term being near the singularity.
Figure 13: The error norms for two choices of $\hat{r}_1$ are shown. As $\hat{r}$ increases, the error in the numerical approximation decreases. To ensure that the two solutions share the grid points, we have that $\hat{r}_2 = 2\hat{r}_1 - 1$.

We now aim to show that the coefficients have the decay rates that would be expected. The boundary data we choose to specify here is,

$$u^{(f)} = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{|l|} e^{-\frac{\pi}{\hat{r}_1}l} Y_{l,m}(\theta, \phi).$$  \hspace{1cm} (8.71)

If the solution is to be well-defined, then one would expect to see the solution coefficients to decay exponentially. To demonstrate that this is the case we consider the averaged coefficients

$$\langle |u_{l,m}| \rangle_r = \frac{1}{N} \sum_{r=r_0}^{\hat{r}} |u_{l,m}|,$$  \hspace{1cm} (8.72)

where $N$ is the total number of grid points in the $r$-discretisation. If any of the coefficients does not experience the desired decay, then we would not expect the average too either. A plot of this is shown in Fig 14, for $l \in [0, 1000]$, where we see that the averaged coefficients decay exactly as expected.

Finally, we calculate $p_{l,m}, \tilde{p}_{l,m}$ and $q_{l,m}$. During our analysis of the long-term behaviour it was found that $q_{l,m}$ should decay as $r^{-3}$ and the quantities $p_{l,m}, \tilde{p}_{l,m}$ should decay like $r^{-2}$. For this we will make use of the mode relations that were given in the previous section. These produce Fig 15. By considering logarithmic plots we find that $q_{l,m} \propto r^{-1.97}$ and $p_{l,m}, \tilde{p}_{l,m} \propto r^{-2.97}$. The accuracy of the decay could be improved by considering a larger cut-off point. Alternatively, a finer $r$-discretisation may also increase the accuracy.
Figure 14: The solution coefficients for $l = 0, \ldots, 1000$ have been calculated. The left graph shows the decay of the averaged coefficients and the graph on the right is a semi-log plot of the same coefficients. We see that the rightmost graph produces a straight line and hence we do indeed see the desired decay behaviour.

Figure 15: The $l = 15$ mode for the curvature quantities $q_{l,m}$ and $\bar{p}_{l,m}$. These were calculated via the previously given mode relations. Moreover, due to the assumption of axial-symmetry, we have $\bar{p}_{l,m} = p_{l,m}$. Both quantities take a similar form as what was previously found.
Figure 16: Logarithmic plots of the curvature quantities. The decay rates can be calculated by looking at the slope of the line of best fit. For $p_{15,m}$ we see that the line of best fit has a slope of $-1.97$, whereas the line of best fit for $q_{15,m}$ has a slope of $-2.97$.

8.2 The parabolic-hyperbolic formulation

We now wish to consider evolutions of asymptotically hyperboloidal initial data within the framework of the parabolic-hyperbolic formulation of the constraint equations. In the strongly hyperbolic framework data with the desired properties of being both asymptotically hyperboloidal and embedded into the Schwarzschild solution was constructed via perturbations of the Minkowski space-time. In principal this approach could also be used here. However, we instead choose to directly construct hyperboloids within the Schwarzschild space-time. The $(3 + 1)$-and $(2 + 1)$-decompositions will be carried out by following the process outlined in section 7.

Regardless of our choice of $H(r)$, the parabolicity condition (Eq. (6.56)) may be checked

$$k = -\frac{2}{r},$$

it follows that the parabolicity condition is satisfied.

We will be using the standard Schwarzschild coordinates $(r, \theta, \phi)$, a drawback of which is that we will be unable to consider the region within the Schwarzschild radius. This will not be a restriction as our primary interest is the asymptotic behaviour of the perturbations.

Due to the presence of a mass term in the metric, the hyperboloids (defined by $H(r) = \sqrt{r^2 + c^2}$) that were used in the foliation of the Minkowski space-time will
not produce asymptotically hyperboloidal initial data in the Schwarzschild solution. However, it should be expected that if $M = 0$, then the standard hyperboloids are returned. Indeed this is the case, and we find that an appropriate adaptation is $H(r) = \sqrt{r^2 + c^2} + 2M \ln(r)$. A plot of these surfaces is shown in Fig. 17. The metric is

$$\gamma_{ab} = \left( \frac{r^2 - (r - 2M)^2 \left( \frac{r}{\sqrt{r^2 + c^2}} \right)^2}{r(r - 2M)} \right) dr_a dr_b + r^2 \sigma_{ab}. \quad (8.74)$$

A series expansion of $\gamma_{rr}$ around $r = \infty$ gives,

$$\gamma_{rr} = \frac{c^2 + 8M^2}{r^2} + O(r^{-3}). \quad (8.75)$$

Under the transformation $c^2 + 8M^2 \mapsto c^2$ we see that to first order this is the hyperbolic metric (as seen previously) and as such this metric will also be asymptotically hyperboloidal.

The extrinsic curvature can be written as

$$K^{ab} = \frac{1}{S r^4} \left( \frac{\hat{\kappa}}{r^2 - \hat{a}^2} \partial_r a^a \partial_r b^b + \hat{a} \partial_b a^a \partial_b b^b + \hat{a} \csc^2(\theta) \partial_\theta a^a \partial_\theta b^b \right), \quad (8.76)$$

where,

$$\hat{\kappa} = \frac{r^3(2M - r) \left[ 3Mr\hat{a} - M\hat{a}^3 - (2M - r)^2 \left( \frac{c^2}{(c^2 + r^2)^{3/2}} - \frac{2M}{r^2} \right) \right]}{\hat{a}^2 - r^2} \quad (8.77)$$

$$S = \sqrt{\frac{\hat{a}^2 - r^2}{r(2M - r)}}, \quad \hat{a} = (2M - r) \left( \frac{2M}{r} + \frac{r}{\sqrt{c^2 + r^2}} \right). \quad (8.78)$$

Calculation of the trace yields

$$K = \frac{1}{S r^2} \left( \frac{\hat{\kappa}}{r(r - 2M)} + 2\hat{a} \right). \quad (8.79)$$

The limit as $r \to \infty$

$$\lim_{r \to \infty} K = -\frac{3}{\sqrt{c^2 + 8M^2}}, \quad (8.80)$$

which is bounded away from zero. We also have,

$$K^{ab} - \frac{1}{3} K \gamma^{ab} = \begin{pmatrix} O(\hat{a}^3) & 0 & 0 \\ 0 & O(\hat{a}^5) & 0 \\ 0 & 0 & O(\hat{a}^5) \end{pmatrix}, \quad (8.81)$$

74
Figure 17: A Penrose diagram of $H(r) = \sqrt{r^2 + c^2} + 2M \ln(r)$ with $M = 1$, for varying values of $c$. Three of the resulting surfaces are shown, represented by the blue curves. As in the previous Penrose diagram (Fig. 3), the silver lines are surfaces of constant proper time. The event horizons are labelled by $\mathcal{H}^\pm$. Space-like and null infinity are represented in the diagram by the labels $\mathcal{J}^\pm$ and $i^0$, respectively.
where it has been used that one may approximate \( \varrho \) as
\[
\varrho = \varrho_0 + O(r^{-2})
\]
then, all conditions that are required for the extrinsic curvature to be asymptotically hyperboloidal are satisfied.

### 8.2.1 Linear perturbations

We now consider linear perturbations to the above data set. The linearised equations are

\[
0 = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \delta q - \frac{\hat{S}}{r^2} \left( \delta \varrho \delta p_\theta + \csc^2(\theta) \frac{\partial}{\partial \phi} \delta p_\phi + \cot(\theta) \delta p_\theta \right),
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta p_A - \frac{\hat{S}}{2} \frac{\partial}{\partial x^A} \delta q,
\]

\[
0 = \frac{\partial}{\partial r} \delta A - \frac{\hat{S}}{2} \left( \frac{\partial^2}{\partial \theta^2} \delta A + \csc^2(\theta) \frac{\partial^2}{\partial \phi^2} \delta A + \cot(\theta) \frac{\partial}{\partial r} \delta A \right)
+ \frac{1}{2r} \left( \frac{\hat{a}(\hat{a} + 2\hat{\kappa})}{r^2} - 1 \right) \delta A - \frac{r}{2} \frac{\hat{S}^2(\hat{a} + \hat{\kappa})}{r^2} \delta q.
\]

The first thing of note here is that the parabolic equation is now uncoupled from the hyperbolic equations. Furthermore, they take the same form as the equations that were dealt with previously. Following the same steps that were followed in the previous sections allows us to then express these equations in terms of the \( \mathfrak{E} \) operators:

\[
0 = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \delta q - \frac{\hat{S}}{\sqrt{2}r^2} \left( \delta \varrho + \delta p \right),
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta p - \frac{\hat{S}}{2\sqrt{2}} \delta(q),
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta \bar{p} - \frac{\hat{S}}{2\sqrt{2}} \delta(q),
\]

\[
0 = \frac{\partial}{\partial r} \delta A - \frac{\hat{S}}{r} \delta \varrho \delta(A) + \frac{1}{2r} \left( \frac{\hat{a}(\hat{a} + 2\hat{\kappa})}{r^2} - 1 \right) \delta A - \frac{r}{2} \frac{\hat{S}^2(\hat{a} + \hat{\kappa})}{r^2} \delta q.
\]

For \( q, p \) and \( \bar{p} \), we will once again make use of the expansions given by Eqs. (8.17), (8.18), and (8.19) coupled with the expansion

\[
A = \sum_{l=0}^{l=\infty} \sum_{m=-|l|}^{|l|} A_{l,m} Y_{l,m}(\theta, \phi).
\]
These allows us to express these equations in terms of each individual mode.

\[
0 = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) q_{l,m} + \frac{\hat{S}\sqrt{(l+1)}}{\sqrt{2}r^2} (\bar{p}_{l,m} - p_{l,m}), \tag{8.90}
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_{l,m} + \frac{\hat{S}\sqrt{(l+1)}}{2\sqrt{2}} q_{l,m}, \tag{8.91}
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p}_{l,m} - \frac{\hat{S}\sqrt{(l+1)}}{2\sqrt{2}} q_{l,m}, \tag{8.92}
\]

\[
0 = \frac{\partial}{\partial r} A_{l,m} - \frac{\hat{S}l(l+1)}{r} A_{l,m} + \frac{1}{2r} \left( \frac{\hat{a}(\hat{a} + 2\kappa)}{r^2} - 1 \right) A_{l,m} - \frac{r}{2}\hat{S}^2(\hat{a} + \kappa)q_{l,m}. \tag{8.93}
\]

We now wish to study the asymptotic behaviour of the coefficients. To do this, we will first solve the hyperbolic equations and then the parabolic equation. We begin with performing a Taylor expansion of \( \hat{S} \) with respect to \( r \) near infinity:

\[
\hat{S} = \frac{\sqrt{c^2 + 8M^2}}{r} + \mathcal{O}(r^{-2}), \tag{8.94}
\]

which gives

\[
0 = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) q_{l,m} + \frac{\sqrt{l(l+1)(c^2 + 8M^2)}}{\sqrt{2}r^3} (\bar{p}_{l,m} - p_{l,m}), \tag{8.95}
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_{l,m} + \frac{\sqrt{l(l+1)(c^2 + 8M^2)}}{2\sqrt{2}r} q_{l,m}, \tag{8.96}
\]

\[
0 = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \bar{p}_{l,m} - \frac{\sqrt{l(l+1)(c^2 + 8M^2)}}{2\sqrt{2}r} q_{l,m}. \tag{8.97}
\]

Multiplying by \( r \), we express the equations in matrix form:

\[
\frac{\partial}{\partial u} \begin{pmatrix} q_{l,m} \\ p_{l,m} \\ \bar{p}_{l,m} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ \frac{\sqrt{l(l+1)(c^2 + 8M^2)}}{2\sqrt{2}} & 2 & 0 \\ -\frac{\sqrt{l(l+1)(c^2 + 8M^2)}}{2\sqrt{2}} & 0 & 2 \end{pmatrix} \begin{pmatrix} q_{l,m} \\ p_{l,m} \\ \bar{p}_{l,m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8.98}
\]

where we have once again introduced the coordinate transformation \( u = -\ln \left( \frac{1}{r} \right) \).

Calculation of the eigenvalues gives \( \lambda \in \{1, 2, 2\} \), with eigenvectors

\[
\bar{\lambda} \in \left\{ \begin{pmatrix} 2\sqrt{2} \\ \sqrt{l(l+1)(c^2 + 8M^2)} \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \tag{8.99}
\]
Thus, the solution is,

\[
\begin{pmatrix}
q_{l,m} \\
p_{l,m} \\
\bar{p}_{l,m}
\end{pmatrix}
= A_q e^{-u} \begin{pmatrix}
\frac{2\sqrt{2}}{\sqrt{l(l+1)(c^2+8M^2)}} \\
-1 \\
1
\end{pmatrix} + e^{-2u} \begin{pmatrix}
0 \\
A_p \\
A_{\bar{p}}
\end{pmatrix}
= \frac{A_q}{r} \begin{pmatrix}
\frac{2\sqrt{2}}{\sqrt{l(l+1)(c^2+8M^2)}} \\
-1 \\
1
\end{pmatrix} + \frac{1}{r^2} \begin{pmatrix}
0 \\
A_p \\
A_{\bar{p}}
\end{pmatrix},
\] (8.100)

where \(A_q, A_p, A_{\bar{p}} \in \mathbb{R}\) are constants. Then, to leading order we have \((q, p, \bar{p}) = \frac{1}{r} (\hat{q}, \hat{p}, \hat{\bar{p}})\) for unknown functions \((\hat{q}, \hat{p}, \hat{\bar{p}}) = (\hat{q}(\theta, \phi), \hat{p}(\theta, \phi), \hat{\bar{p}}(\theta, \phi))\).

We now consider the leading order terms of the parabolic equation

\[
0 = r \frac{\partial}{\partial r} A_{l,m} + 4 A_{l,m} + 4 \frac{\sqrt{c^2 + 8M^2}}{r^2} q_{l,m} + O(r^{-3}),
\] (8.101)

which has general solution

\[
A_{l,m} = -\frac{4\sqrt{c^2 + 8M^2}}{3r^2} \hat{q}_{l,m}.
\] (8.102)

Thus, to leading order, we have \(A = \frac{1}{r} \hat{A}\), for some function \(\hat{A} = \hat{A}(\theta, \phi)\). Following the same steps outlined previously, we are able to reconstruct the 3-metric and extrinsic 3-curvature, to find that the solutions are asymptotically hyperboloidal.

To verify these predictions, we consider the numerical evolutions with initial data

\[
\delta A^{(0)} = \delta q^{(0)} = \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} e^{-\frac{1}{r} l} Y_{l,m}(\theta, \phi),
\] (8.103)

\[
p^{(0)} = \bar{p}^{(0)} = 0.
\] (8.104)

The averaged solutions are shown in Fig 18, and the corresponding logarithmic plot is shown in Fig. 19. The fall-off rates of the averaged solutions are

\[
<|\delta q|>_r \sim \frac{1}{r}, <|A|>_r \sim \frac{1}{r}, <|p|>_r \sim \frac{1}{r}, <|\bar{p}|>_r \sim \frac{1}{r}.
\] (8.105)

This is in agreement with the analytically predicted values. Finally, we test the \(\theta\) convergence, in the same way as before. This is shown in Fig. 20, where we see that the solutions converge as \(N_{(1)}\) increases.
Figure 18: The averaged solutions to the linearised equations.
Figure 19: The logarithmic plots of the averaged solutions. Here we see that all of the solutions fall off at $r^{-1}$. This is in agreement with what was found analytically.
Figure 20: A logarithmic plot showing convergence in the $\theta$-discretisation. We see that as the cardinality $N_{(1)}$ increases, the associated error decreases.
8.2.2 Linear perturbations as a second order system

In the strongly hyperbolic formulation, it was seen that the linearised equations were equivalent to a second order PDE, through a set of transformation rules. This is also true for the linearised equations in the parabolic-hyperbolic formulation. We will not be studying this system as it will not aid in our understanding of the linearised equations.

**Proposition 8.2.** Suppose there exists a smooth solution $u = u(x^a)$ to the equation

$$
\frac{\partial^2 u}{\partial r^2} - \left( \frac{1}{r} + \frac{1}{\hat{S}} \frac{\partial \hat{S}}{\partial r} \right) \frac{\partial u}{\partial r} - \frac{\hat{S}^2}{2 r^2} \nabla^2 u = 0.
$$

(8.106)

Then the functions

$$
\delta q = \frac{2}{\hat{S} r^2} \frac{\partial u}{\partial r}, \quad \delta p_B = \frac{1}{r^2} \frac{\partial u}{\partial x^B},
$$

(8.107)

are solutions to Eqs. (8.82) and (8.83).

Moreover, suppose $\delta p_A$ and $\delta q$ are smooth solutions to Eqs. (8.82) and (8.83) such that

$$
\omega := \frac{\partial \delta p_\theta}{\partial \phi} - \frac{\partial \delta p_\phi}{\partial \theta} = 0
$$

(8.108)

initially. Then there exists a uniquely determined family of solutions $U := \{u(x^i) + \lambda | \lambda = \text{constant}\}$ such that all elements $u \in U$ satisfy Eqs. (8.106) and (8.107).

**Proof.** This proof follows the same steps presented in Prop. 8.1:

Suppose that $u = u(x)$ is a smooth solution of Eq. (8.106) and define $\delta q$ and $\delta p_A$ as in Eq. (8.107). Then,

$$
\frac{\partial \delta A}{\partial r} = \frac{4}{\hat{S} r^2} \left( \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{2 \hat{S}} \frac{\partial \hat{S}}{\partial r} \frac{\partial u}{\partial r} \right) = -\frac{2}{\hat{S} r^3} \frac{\partial u}{\partial r} + \frac{\hat{S}}{r^4} \nabla^2 u
$$

$$
= -\frac{1}{r} \delta q + \frac{\hat{S}}{r^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \csc^2(\theta) \frac{\partial^2 u}{\partial \phi^2} + \cot(\theta) \frac{\partial u}{\partial \theta} \right)
$$

$$
\Leftrightarrow 0 = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \delta q + \frac{\hat{S}}{r^2} \left( \frac{\partial \delta p_\theta}{\partial \phi} + \csc^2(\theta) \frac{\partial \delta p_\phi}{\partial \phi} + \cot(\theta) \delta p_\theta \right).
$$

(8.109)

Similarly,

$$
\frac{\partial \delta p_I}{\partial r} = -\frac{2}{r} \delta p_I + \frac{\hat{S}}{2 \partial x^I} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \delta q - \frac{\hat{S}}{2 \partial x^I} = 0.
$$

(8.110)
These are the linearised equations.

We now define

\[ \Omega_a := \frac{\hat{S}r^2}{2} \delta q_{dr}^a + r^2 (\delta p_{\theta} d\theta_a + \delta p_{\phi} d\phi_a), \quad (8.111) \]

where \( \delta q \) and \( \delta p_I \) are solutions of the linearised equations such that \( \omega = 0 \) initially.

Calculation of the exterior derivatives gives,

\[ d\Omega = r^2 \left( \frac{\hat{S} \partial \delta q}{2 \partial x^I} - \frac{\partial \delta p_I}{\partial r} - \frac{2}{r} \delta p_I \right) dr \wedge dx^I + r^2 \omega d\theta \wedge d\phi. \quad (8.112) \]

The bracketed term is zero by assumption. To show that \( \omega \) is zero we consider its \( r \)-derivative.

\[ \frac{\partial \omega}{\partial r} := \frac{\partial}{\partial \phi} \frac{\partial \delta p_{\phi}}{\partial r} - \frac{\partial \delta p_{\theta}}{\partial r} \frac{\partial \delta p_{\phi}}{\partial r} = -\frac{2}{r} \omega \implies \omega = \frac{\omega_0}{r^2}, \quad (8.113) \]

where \( \omega_0 \in \mathbb{R} \) is a constant. Since \( r > 0 \) we have that if \( \omega = 0 \) initially then \( \omega_0 = 0 \) and hence \( \omega \) is identically zero.

\[ \text{\ } \]

8.2.3 Non-linear perturbations

Having seen that the equations are stable to linear perturbations, it is natural to ask if they are stable to non-linear perturbations. We will do this by comparing the non-linear perturbations to linear perturbations.

The full equations are

\[ \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) p_B - \frac{A}{2} \frac{\partial q}{\partial x^I} = 0, \quad (8.114) \]

\[ \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) q - \frac{A}{r^2} \sigma^{BC}(\sigma) \nabla_B p_C - 2(2M - r)\hat{\kappa} = 0, \quad (8.115) \]

\[ \frac{\partial A}{\partial r} - \frac{r}{2} A^2 (\sigma) \nabla^2 A + A^3 \left( \frac{1}{2r} + r(2M - r)\hat{\kappa} + \frac{r}{8} q^2 - \frac{r}{2} p_B p_B^B \right) - \frac{1}{2r} A = 0. \quad (8.116) \]

We wish to express these in terms of the \( \eth \) operators. For Eqs. (8.114) and (8.115) we proceed in the same way that has been outlined for the linearised equations. For the parabolic-hyperbolic equation, we note that

\[ p_B p_B^B = \frac{1}{r^2} (m_B m_C + m_C m_B) p_B p_C = \frac{2}{r^2} p \dot{p}. \quad (8.117) \]
which allows us to write
\[
\left( \frac{\partial}{\partial r} - \frac{1}{2r} \right) A - \frac{r}{2} A^2 \delta (\delta (A)) + A^3 \left( \frac{1}{2r} + r(2M - r)q\kappa + \frac{r}{8} q^2 - \frac{1}{r} p\bar{p} \right) - \frac{1}{2r} A = 0,
\]
where \( p = m^B p_B \) and \( \bar{p} = \bar{m}^B p_B \).

Let \( \delta A, \delta q \) and \( \delta p_A \) be solutions to the linearised equations (Eqs. (8.82)-(8.84)) with initial data
\[
\delta A^{(0)} = \lambda \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} e^{-\frac{1}{10} l} Y_{l,m}(\theta, \phi),
\]
\[
\delta q^{(0)} = \lambda \sum_{l=0}^{\infty} \sum_{m=-|l|}^{l} e^{-\frac{1}{10} l} Y_{l,m}(\theta, \phi),
\]
\[
\delta p_A^{(0)} = 0,
\]
where \( \lambda \in \mathbb{R}^+ \) is a constant used to control the magnitude of the perturbations. The corresponding linearly perturbed solutions are
\[
(\mathcal{L}) A := \dot{A} + \delta A, \quad (\mathcal{L}) q := \dot{q} + \delta q, \quad (\mathcal{L}) p_A := \dot{p}_A + \delta p_A,
\]
where \( \dot{A}, \dot{q} \) and \( \dot{p}_A \) are the exact Schwarzschild solutions.

We want to establish how well the linear perturbations approximate the behaviour of the fully non-linear perturbations, which we represent with a super-prescript \((F)\). The following must hold:
\[
(\mathcal{F}) A = (\mathcal{L}) A + (A)\varepsilon, \quad (\mathcal{F}) q = (\mathcal{L}) q + (q)\varepsilon, \quad (\mathcal{F}) p_A = (\mathcal{L}) p_A + (p)\varepsilon_A,
\]
where \((A)\varepsilon, (q)\varepsilon_A\) and \((p)\varepsilon_A\) are the appropriate correction terms, ignored in the linearisation. For the evolution, we will pick initial data such that the two sets of solutions agree initially:
\[
(\mathcal{F}) A^{(0)} = (\mathcal{L}) A^{(0)}, \quad (\mathcal{F}) q^{(0)} = (\mathcal{L}) q^{(0)}, \quad (\mathcal{F}) p_A^{(0)} = (\mathcal{L}) p_A^{(0)},
\]
To study the behaviour of the correction terms, we define the error quantity
\[
\| \varepsilon_s \| := \max_{\theta \in (0, \pi]} | (\mathcal{F}) s f - (\mathcal{L}) s f | = \max_{\theta \in (0, \pi]} | s\varepsilon |,
\]
for \( s f \in \{ A, q, p, \bar{p} \} \) and \( s \varepsilon \in \{ (A)\varepsilon, (q)\varepsilon, (p)\varepsilon_B m^B, (p)\varepsilon_B \bar{m}^B \} \). This will allow us to study the long term behaviour of each \( s \varepsilon \). Moreover, since Eq. (8.125) subtracts away the
Figure 21: A logarithmic plot of $\| \varepsilon \|$ for $\lambda = 1$ and $\lambda = \frac{1}{2}$. Each error quantity has been multiplied by decay factor so that it tends to a constant. We also measure the ratio of the solutions for each $\lambda$, where we see that $\| (A) \varepsilon \|$ and $\| (q) \varepsilon \|$ decrease by a factor of $\sim 3.8$. The quantities $\| (p) \varepsilon_B m^B \|$ and $\| (p) \varepsilon_{\bar{B}} \bar{m}^B \|$ decrease by a factor of $\sim 2.8$. 
linearities of the perturbations, we should expect to see that if the magnitude of $\lambda$ is halved, then the error should decrease by a factor of more than (or exactly) one fourth. Fig. 21 examines this behaviour for $\lambda = 1, \frac{1}{2}$, where we see that $(A)\varepsilon$ decays at $r^{-2}$, whereas $(q)\varepsilon$ and $(p)\varepsilon_A$ decay at $r^{-1}$. We have already seen that these fall-off rates produce an asymptotically hyperbolic geometry and hence asymptotic hyperbolicity is stable for perturbed solutions of both the linearised and the full equations.

When $\lambda$ is halved, we see that $(A)\varepsilon$ and $(q)\varepsilon$ decrease by a factor of 3.8. Within error, this is a reasonable approximation of the expected behaviour. However, $\|(p)\varepsilon_B m^B\|$ and $\|(p)\varepsilon_B \bar{m}^B\|$ decrease by a factor of $\sim 2.8$, and hence do not show the expected behaviour. It is unclear why this happens, it is possible that the perturbations considered are too large. Alternatively, the maximum radius we considered may have been too small. Due to temporal issues we will not be exploring this any further.
9 Asymptotically flat black hole(s) in the parabolic-hyperbolic formulation

Some quantities, such as the ADM mass, can only be defined for initial data sets that are asymptotically flat. In the above work, we found that non-linear perturbations to asymptotically hyperboloidal initial data were stable. This is promising and motivates us to consider asymptotically flat data.

It was mentioned previously that [29] deals with such perturbations within the framework of the strongly hyperbolic formulation of the constraints. These were found to be unstable due to the perturbed data possessing the ‘wrong mass’.

9.1 Preparations

9.1.1 Kerr Schild metrics

Motivated by the ideas in [7], we consider metrics of the Kerr-Schild form. These may be written as

\[ g_{\alpha\beta} = \eta_{\alpha\beta} - V l_\alpha l_\beta, \]  
(9.1)

where \( V \) is a smooth space-time function and \( l_\alpha \) is null with respect to both \( \eta_{\alpha\beta} \) and \( g_{\alpha\beta} \). Here, we assume that the space-time has coordinates \( (t, x^1, x^2, x^3) \).

Suppose that this space-time can be foliated by a set of smooth Cauchy surfaces \( \Sigma := \{ \Sigma_t | t = \text{constant} \} \). Each \( \Sigma_t \in \Sigma \) will have a metric of the form

\[ \gamma_{ab} = \delta_{ab} - V l_a l_b, \]  
(9.2)

which we have written in intrinsic coordinates \( (x^1, x^2, x^3) \). In this coordinate system \( l_a \) is a unit vector with respect to the Euclidean metric (i.e. \( \delta^{ab} l_a l_b = 1 \)). We then introduce the vector \( \tilde{l}_b = \delta^{ab} l_a \) which we use to define the inverse metric

\[ \gamma^{ab} = \delta^{ab} + \frac{V}{1 - V} \tilde{l}_a \tilde{l}_b. \]  
(9.3)

The lapse function and shift vector corresponding to the foliation are

\[ \alpha = \frac{1}{\sqrt{1 - V}}, \quad \beta_a = V l_a. \]  
(9.4)

The extrinsic curvature follows from the Eq. (4.21):

\[ K_{ab} = \frac{\sqrt{1 - V}}{2} \left( \nabla_a (V l_b) + \nabla_b (V l_a) - \partial_t \gamma_{ab} \right), \]  
(9.5)
where $\nabla_b$ is the covariant derivative associated with $\gamma_{ab}$. For Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$ the exact Kerr solution with mass $M$ is [37]

$$V = -\frac{4M}{2r^2 + a^2 z^2}, \quad l_a = \frac{rx + ay}{r^2 + a^2} dx_a + \frac{ry - ax}{r^2 + a^2} dy_a + \frac{z}{r} dz_a,$$

(9.6)

where $r$ is the positive and real-valued, unique solution to

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

(9.7)

### 9.1.2 Non-linear spherically symmetric perturbations of the Schwarzschild Kerr-Schild data

With the aim of understanding asymptotically flat black hole data within the parabolic-hyperbolic framework, we begin by studying non-linear perturbations of the Schwarzschild solution (in Kerr-Schild coordinates $(t, r, \theta, \phi)$) with mass $M$. To do this we will calculate the free data that arises by considering an asymptotically flat slice of the Kerr-Schild Schwarzschild solution. We will then use this as our free data in the parabolic-hyperbolic constraints and proceed to calculate the general solutions for the unknowns.

Once again we choose to exploit spherical symmetry and express our data in terms of standard polar coordinates. Moreover, the $(2+1)$-decomposition will be performed using exact 2-spheres $S := \{ S_r | r = \text{constant} \}$. For such a foliation, we are able to check that the parabolicity (Eq. (6.56)) condition is satisfied

$$\star k = -\frac{2}{r}.$$

(9.8)

The remaining free data is

$$\kappa = \frac{2M(M + r)}{(r(2M + r))^{3/2}}, \quad (2) R = \frac{2}{r^2}, \quad Q_{ab} = 0, \quad B_a = v_a = 0.$$

(9.9)

The simplest restriction we may make here is to assume that the resulting solutions will remain spherically symmetric. This means that we will not be perturbing $p_a$ from its exact Schwarzschild value of $p_A = 0$. A further consequence is that the hyperbolic and parabolic equations decouple:

$$\frac{\partial}{\partial r} q(r) = -\frac{2}{r} \left( \frac{1}{2} q(r) - \frac{2M(M + r)}{(r(2M + r))^{3/2}} \right),$$

(9.10)

$$\frac{\partial}{\partial r} A(r) = -\frac{r}{4} \left( A(r)^2 \left( \frac{2}{r^2} + \left( \frac{4M(M + r)}{(r(2M + r))^{3/2}} + \frac{q(r)}{2} \right) \right) - \frac{4}{r^2} \right),$$

(9.11)
of which the general solution is

\[
q(r) = \frac{C}{r} - \frac{4M}{r\sqrt{r(2M+r)}}, \quad (9.12)
\]

\[
A(r) = \pm \sqrt{\frac{4(2Mr + r^2)}{(2M + r)(4(2M + A) + C^2r) + 4r^2 - 8M\sqrt{r(2M+r)}C}}. \quad (9.13)
\]

The lapse function should be positive and as such we will only consider this case. Note that picking \(C = 0\) and \(A = -2M\) returns the exact Schwarzschild solution (in Kerr-Schild coordinates). Then, for each \(M\), we have a two parameter family of spherically symmetric solutions. The symmetry implies that we must be able to embed this initial data set into a Kerr-Schild Schwarzschild solution with mass parameter \(m\). In general, this mass does not have to be the same as \(M\) [29].

We do our \((3 + 1)\) foliation with the set of Cauchy surfaces \(\Sigma := \{\Sigma_T \mid T = t - H(r)\}\). We pick one of these surfaces and proceed to compare the 3-metrics to get:

\[
H_{\pm}(r) = \int \frac{1}{r - 2m} \left(2m \pm r\sqrt{\frac{16M(M + m) + 8mr + F(r)}{4(2M(2M + r) + r^2) + F(r)}}\right) dr, \quad (9.14)
\]

\[
F(r) := (2M + r)(rC^2 + 4A^2) - 8M\sqrt{r(2M+r)}C. \quad (9.15)
\]

It should be emphasized that the choice of \(\pm\) here is unrelated to the choice of \(\pm\) in Eq. (9.13). We calculate the corresponding \(q\) and perform a Taylor expansion near infinity

\[
q = \frac{C}{r} \pm \frac{2(A \pm 2(m \pm M|C|))}{|C|r^2} + O(r^{-3}). \quad (9.16)
\]

For Eq. (9.12) this is

\[
q = \frac{C}{r} - \frac{4M}{r^2} + O(r^{-3}). \quad (9.17)
\]

By comparing the above expressions, we see that we must have

\[
A = -2(m - M|C|). \quad (9.18)
\]

To gain further understanding we proceed in the same way for \(\kappa\). The Taylor expansion near infinity for the exact Schwarzschild \(\kappa\) gives

\[
\kappa = \frac{2M}{r^2} - \frac{4M^2}{r^3} + O(r^{-4}). \quad (9.19)
\]
For the $\kappa_\pm$ produced by $H_\pm(r)$,

\[
\kappa_+ = \begin{cases} 
\frac{2M}{r^2} - \frac{4M^2}{r^3} + \frac{9M^3}{C_r^4}, & \mathcal{C} < 0 \\
\frac{2M}{r^2} + \frac{4M^2}{r^3} - \frac{3M^3(16+3\mathcal{C})}{C_r^4}, & \text{else}
\end{cases}
\]

\[
\kappa_- = \begin{cases} 
\frac{2M}{r^2} - \frac{4M^2}{r^3} + \frac{9M^3}{C_r^4}, & \mathcal{C} > 0 \\
\frac{2M}{r^2} + \frac{4M^2}{r^3} - \frac{3M^3(16-3\mathcal{C})}{C_r^4}, & \text{else}
\end{cases}
\]

(9.20)

It is clear here that if we are to match these expansions to a Schwarzschild solution, then we must have $\mathcal{C} < 0$ (for $\kappa_+$) and $\mathcal{C} \geq 0$ (for $\kappa_-$):

\[
\mathcal{A} = -2(m - M(\pm |\mathcal{C}|)) = -2(m - M(\mp |\mathcal{C}|)) = -2m.
\]

(9.21)

The embedded surfaces for varying values of $\mathcal{C}$ and $M$ are shown in Fig 22, where we see that both constants $\mathcal{C}$ and $M$ affect the slope that the surfaces approach space-like infinity with.

To gain understanding of $\mathcal{C}$ we consider the Taylor expansion of $A(r)$:

\[
A(r) = \frac{2}{\sqrt{4 + \mathcal{C}^2}} + \frac{8(m + MC)}{(4 + \mathcal{C}^2)^{3/2}r} + \mathcal{O}(r^{-2}),
\]

(9.22)

We now proceed to check if the metric is asymptotically flat in the sense of Def. 5.2. To do this we note that it must be conformally related to the Euclidean metric, i.e.

\[
A(r)^2 dr^2 + r^2 d\sigma^2 = \psi^2 (dR^2 + R^2 d\sigma^2)
\]

where $\psi$ is the conformal factor. It follows that we must have

\[
A(r) = \psi \frac{dR}{dr}, \quad \psi = \frac{r}{R} \implies \ln(R) = \ln R_0 + \int A(r) \frac{dr}{r},
\]

(9.24)

where $R_0 \in \mathbb{R}$ is a constant, which we will assume to be 1 as we can always rescale $R$. Using Eq. (9.22) gives

\[
R = r \frac{2}{\sqrt{4 + \mathcal{C}^2}} \exp \left(-\frac{4(m + MC)}{(4 + \mathcal{C}^2)^{3/2}r}\right)
\]

\[
= r \frac{2}{\sqrt{4 + \mathcal{C}^2}} \left(1 - \frac{4(m + MC)}{(4 + \mathcal{C}^2)^{3/2}r} + \mathcal{O}(r^{-2})\right),
\]

(9.25)

\[
\implies r \approx \begin{cases} 
\frac{2R + m + \sqrt{4R^2 + 4RM - m^2}}{4}, & \text{if } \mathcal{C} = 0, \\
\frac{R}{\sqrt{4 + \mathcal{C}^2}}, & \text{else}
\end{cases}
\]

(9.26)

Then, the conformal factor is

\[
\psi^2 = \begin{cases} 
1 + \frac{m^2}{R^2} + \mathcal{O}(R^{-3}), & \text{if } \mathcal{C} = 0, \\
R^{4 + \mathcal{C}^2 - 2} + \mathcal{O}(R^{\sqrt{4 + \mathcal{C}^2} - 3}), & \text{else}
\end{cases}
\]

(9.27)
Figure 22: Two isometric embeddings with initial data given by Eqs. (9.12) and (9.13) for $m = 1$. The upper diagram shows surfaces for varying values of $\mathcal{C}$ with $M = 1$. The other depicts slices with fixed $\mathcal{C} = 0$, and varies $M$. In both diagrams the green dashed curve is the exact Schwarzschild Kerr-Schild data. In the top picture, the red line is $M = 2$ and in the bottom picture the red line is $\mathcal{C} = 2$. 
It is clear from this equation that the metric can only be asymptotically flat (in the sense of Def. 5.2) if $C = 0$.

For the extrinsic curvature we note that $K_{ab} = \mathcal{O}(R^{-2})$ in the limit $R \to \infty$ is equivalent to $K_{ab} = \mathcal{O}(r^{-4/\sqrt{4+r^2}})$ in the limit $r \to \infty$. Then, we have

$$K_{ab} = \begin{cases} \mathcal{O}(r^{-2}), & \text{if } C = 0, \\ \mathcal{O}(r^{-1}), & \text{else}. \end{cases} \quad (9.28)$$

Further, Eqs. (9.12) and (9.19) give

$$K = \begin{cases} \mathcal{O}(r^{-2}), & \text{if } C = 0, \\ \mathcal{O}(r^{-1}), & \text{else}. \end{cases} \quad (9.29)$$

which allows us to conclude that $C = 0$ is sufficient for asymptotic flatness.

We end by summarising the role of each constant:

$M$: This is the ‘background mass’. In the above family of solutions, it acts as a constant, and does not affect the asymptotic behaviour.

$C$: Since our data set can only be asymptotically flat if $C = 0$, we may think of this constant as having an effect on how the data set bends or curves within the space-time. For this reason we will refer to this constant as the ‘curvature parameter’.

$m$: This is the ‘embedding mass’. More specifically, it is the mass the solutions would have if we were to embed them into the Kerr-Schild Schwarzschild space-time. Note that, in general, the background and embedding masses do not have to be equal.

### 9.2 Construction of binary black hole initial data sets

It has been mentioned previously that in order to solve the constraints, we must first pick our free data. Doing so can be challenging as it is unclear how to pick that data such that one ends up with a physically reasonable solution. In what follows, we will suggest a way to pick the data such that the resulting solutions describe the behaviour of multiple black holes. In [38–40], Rácz discusses a method for doing this within his parabolic-hyperbolic formulation. In this work the $(2 + 1)$-foliation is carried out using planes. We will not use this method as we wish to foliate with topological 2-spheres.
9.2.1 Model outline

In what follows we will present our multiple black hole model. We will first outline the key steps of our method, before discussing the explicit details.

1. Constructing the free data:

(a) We will begin with a brief summary of the ideas presented in [7], where Bishop introduces a method for constructing Kerr-Schild type solutions to the constraint equations. We will focus on how this method can be adapted to construct the free data in the parabolic-hyperbolic form of the constraints.

(b) The method will rely on the introduction of a natural \((2 + 1)\) foliation in terms of topological 2-spheres. An important property of this foliation is that the surfaces will asymptotically approach the round 2-sphere.

(c) This foliation will allow us to calculate an auxiliary first and second fundamental form, which we use to construct our free data.

(d) We will then end this section by writing down some of the relevant formula produced by our method.

2. Choosing adapted coordinates:

(a) We will introduce coordinates that are adapted to the foliation and express the auxiliary data set with respect to these. We will not be able to find explicit formulae for the coordinate transformation. However, we will discuss how the coordinate transformation can be calculated numerically.

(b) In our model the space-time function \(V\) is a freedom that must be specified. In this section will discuss how we have chosen to specify it.

(c) Once all auxiliary quantities have been constructed, we will make a choice of our initial data.

3. Expected results:

Here we will discuss the symmetries of our model and how we may expect them to effect the behaviour of the solutions. Since our foliation approaches round 2-spheres asymptotically we will be able to match the solutions to the spherically symmetric data we calculated in Section 9.1.2. We will discuss how this may be done numerically, and what errors we may expect to arise.
9.2.2 Constructing the free data

In [7], a method for generating initial data of Kerr-Schild type was suggested for multiple Schwarzschild black holes. In [41, 42], the method was expanded to generate initial data for multiple Kerr black holes. Here, we will be discussing the former of the two papers. In his work, Bishop notes that \( l_a \) is orthogonal to an adapted foliation of \( 2 \)-surfaces and as such could be written in terms of a potential, i.e.

\[
(9.30) \quad l_a = (6) A \partial_a u, \quad (6) A = \left( \delta^{ab} \partial_a u \partial_b u \right)^{-\frac{1}{2}},
\]

for some known potential function \( u \). The exact Schwarzschild potential is \( u = -\frac{2M}{r} \). By making an ansatz for the form of \( u \), one may explicitly calculate \( l_a \) and hence generate a metric of Kerr-Schild form. The function \( V \), however, is left as an unknown space-time function. In this set-up Bishop solves the Einstein equations for the function \( V \). The authors also addressed how one may choose the potential \( u \), making the suggestion that one may consider the superposition of Schwarzschild potentials

\[
(9.31) \quad u = \sum_{i=1}^{N} \frac{M_i}{r_i},
\]

where \( M_i \) is the mass of the \( i \)-th black hole, \( r_i \) is the coordinate separation from the centre of the \( i \)-th black hole, and \( N \) is the total number of black holes.

We will adapt this model for use with the parabolic-hyperbolic equations. In contrast to Bishop’s model, we will treat \( V \) as a freedom and make an ansatz for its form. This will allow us to create an auxiliary metric \( [A] \gamma_{ab} \) on our three-dimensional space. The extrinsic 3-curvature associated with the auxiliary metric is

\[
(9.32) \quad [A] K_{ab} = \frac{\sqrt{1-V}}{2} \left( \partial_a (Vl_b) + \partial_b (Vl_a) - [A] \gamma_{ab} \right).
\]

So far, we have introduced three freedoms, namely, \( u \), \( V \), and \( [A] \gamma_{ab} \). Once a choice for each of these has been made, one may proceed to calculate the corresponding free data \( (Q_{ab}, v_a, B_a, \kappa, k) \). In general, the fields \( (A) \gamma_{ab}, [A] K_{ab} \) will not be solutions to the constraint equations. However, we will assume that the solutions \( (\gamma_{ab}, K_{ab}) \) produce the same free data. We will further assert that the auxiliary fields agree with the exact solutions on the initial hypersurface.

For further discussion, we must make a choice of the potential function \( u \). We will assert that Eq. (9.31) is the appropriate choice, and hence describe the foliation by the set of 2-surfaces \( S := \{ S_u \mid u = \text{constant} \} \). There are two advantages to this: Firstly, the foliation produced is adapted to the black holes, secondly, \( u = \text{constant} \).
describes a set of topological 2-spheres for sufficiently small \( u \). Moreover, as the radius goes to infinity, the foliation approaches round 2-spheres, a fact that may be exploited via the use of SWSH. We end this section by writing down some abstract formula produced from this model. Since \( l_a \) is orthogonal to the (2 + 1)-foliation, we may write

\[
N_a = (l) A l_a, \quad N^a = (l) A^{-1} l^a
\]  
(9.33)

where \((l) A \) is a proportionality function which we explicitly calculate as

\[
(l) A = \sqrt{1 - V},
\]  
(9.34)

which allows us to find the induced metric on each slice \( S_a \in S \) of the foliation:

\[
h_{ab} = \gamma_{ab} - N_a N_b = \delta_{ab} - l_a l_b.
\]  
(9.35)

We are also able to calculate the auxiliary (2 + 1) lapse as

\[
[A] A = (l) A (\delta) A.
\]  
(9.36)

The eth operators are defined with respect to the covariant derivative on the 2-sphere, \((^o) \nabla_A \). As such, we express the covariant derivative on each leaf in terms of \((^o) \nabla_A \) through use of the (1,2)-tensor, introduced in a special case by Eq. (5.7), which we calculate via the general formula:

\[
C_{AB}^E = \frac{1}{2} h^{ED} \left((^o) \nabla_A h_{BD} + (^o) \nabla_B h_{AD} - (^o) \nabla_D h_{AB}\right).
\]  
(9.37)

Furthermore, we write \( h_{AB} \) and \( h^{AB} \) in terms of the frame vectors \( (m_A, \bar{m}_A, m^A, \bar{m}^A) \):

\[
h_{AB} = h_{(-,-)} m_A m_B + h_{(-,+)} m_A \bar{m}_B + h_{(+,-)} m_B \bar{m}_A + h_{(+,+)} \bar{m}_B \bar{m}_A,
\]  
(9.38)

\[
h^{AB} = h^{(-,-)} m^A m^B + h^{(-,+)} m^A \bar{m}^B + h^{(+,-)} m^B \bar{m}^A + h^{(+,+)} \bar{m}^B \bar{m}^A,
\]  
(9.39)

where \( h^{(\pm,\pm)} \) is the set of spin-weighted functions constructed by projecting \( h_{AB} \) onto \( m^A \) and \( \bar{m}^A \). Similarly, \( h^{(\pm,\pm)} \) are spin-weighted functions, found via the contractions of \( h^{AB} \) with the frame vectors \( m_A \) and \( \bar{m}_A \). If follows that for any two co-vectors \( w_A, y_A \in T_p S \), we have

\[
h^{AB} w_A y_B = h^{(-,-)} w y + h^{(+,-)} \bar{w} y + h^{(-,+)} w \bar{y} + h^{(+,+)} \bar{w} \bar{y},
\]  
(9.40)

where \( w = m^A w_A, y = m^A y_A, \bar{w} = \bar{m}^A w_A \) and \( \bar{y} = \bar{m}^A y_A \). We also have

\[
h^{AB} D_A w_B = h^{AB (\sigma) \nabla_A w_B} - h^{AB C_{AB} w_E}
\]  
\[
= \frac{1}{\sqrt{2}} \left( h^{(-,-)} \bar{\delta}(w) + h^{(+,-)} \bar{\delta}(\bar{w}) + h^{(-,+)} \bar{\delta}(w) + h^{(+,+)} \bar{\delta}(\bar{w}) \right) - h^{AB} C_{AB} w_E.
\]  
(9.41)
Similarly, if \( w \) is a scalar field such that \( w_A = \partial_A w \), then we have
\[
D^2 w = h^{AB} (\partial_A \partial_B w - h^{AE} A_{AB} \partial_E w)
= \frac{1}{\sqrt{2}} (h^{(-,-)} \partial(\tilde{\partial}(w)) + (h^{(+,-)} + h^{(-,+)} \partial(\tilde{\partial}(w)) + h^{(+,+)} \partial(\tilde{\partial}(w))) - h^{AB} A_{AB} \partial_E w.
\]

(9.42)

In general one need not make the assumption that \( |A| \gamma_{ab} = 0 \). Nevertheless, for now we will suppose that this is the case and define the following:
\[
(A_0) K_{ab} = (A_0) \kappa N_a N_b + (A_0) p_a N_b + (A_0) p_b N_a + (A_0) q_{ab},
\]
where
\[
(A_0) \kappa = \tilde{a} \partial_a (V (l) A^{-1}),
\]
\[
(A_0) p_a = \frac{1}{2} \left( V v_a + (l) A \left( \partial_a \left( V (l) A^{-1} \right) - (A_0) \kappa_l a \right) \right),
\]
\[
(A_0) q_{ab} = \frac{(l) AV}{2(1 - V)} h^c_a h^d_b \left( (\delta) \nabla_c l_d + (\delta) \nabla d_l c \right).
\]

(9.43)

(9.44)

Here, the prefix \( (A_0) \) represents the auxiliary quantities when \( |A| \gamma_{ab} = 0 \). i.e.
\[
|A| K_{ab} = (A_0) K_{ab} - \sqrt{\frac{1 - V}{2}} |A| \gamma_{ab}
\]

(9.47)

The details of the calculations that lead to the above formulae offer little insight into the equations themselves and as such have been suppressed here. The interested reader may find the derivations in Appendix B.1. Generalising the above to \( |A| \gamma_{ab} \neq 0 \) is straight forward:
\[
\kappa = (A_0) \kappa - \frac{(l) A}{2} N^a N^b |A| \gamma_{ab},
\]
\[
|A| p_a = (A_0) p_a - \frac{(l) A}{2} N^c h^b_a |A| \gamma_{cb},
\]
\[
|A| q_{ab} = (A_0) q_{ab} - \frac{(l) A}{2} h^c_a h^d_b |A| \gamma_{cd}.
\]

(9.48)

(9.49)

It follows that
\[
|A| q = \frac{(l) AV}{2(1 - V)} \delta^{ab} (\delta) \nabla a l_b,
\]
\[
Q_{ab} = \frac{(l) AV}{2(1 - V)} h^c_a h^d_b \left( (\delta) \nabla c l_d + (\delta) \nabla d l c - \frac{1}{2} h_{cd} \delta^{ij} (\delta) \nabla i j \right).
\]

(9.50)

(9.51)

Finally, \( \hat{k} \) can be explicitly calculated as
\[
\hat{k} = - (\delta) A \delta^{ab} (\delta) \nabla a l_b.
\]

(9.52)
Figure 23: Two examples of the foliation produced by $u = \text{constant}$ with $M_+ > M_-$, embedded into $\mathbb{R}^3$. The figure on the left shows two separate distorted spheres. As $r_\pm$ increase the two 2-spheres combine, creating a surface with a bifurcation point. The surfaces appear similar to a 3D-Cassini oval, but do not share the same parametric polar equation. It is the opinion of the author that they resemble peanuts and hence all objects associated with them will be referred to as ‘peanut quantities’.

9.2.3 Choosing adapted coordinates

We will use the above model to describe a binary black holes. The coordinate system is oriented such that the two masses are joined by the $z$-axis. The first mass $M_+$ is centred at $z = Z$ and the second $M_-$ at $z = -Z$, where $Z \in \mathbb{R}^+$ is a constant. The corresponding potential is

$$u = \frac{M_+}{r_+} + \frac{M_-}{r_-}, \quad (9.53)$$

where $r_\pm$ is the coordinate distance from $i$-th black hole. Two plots of $u = \text{constant}$ are shown in Fig. 23, where we see that for $r_\pm$ sufficiently small, the corresponding $u = \text{constant}$ surface describes the union of two distorted and disjoint spheres. As the radial coordinate increases, the surfaces join to create a topological sphere. This means that there must exist a surface with a bifurcation point. A unique tangent vector cannot be associated with such a point and as such we must ensure that the evolution begins at a sufficiently large radius.

Before making a choice of $V$, it is necessary to introduce a coordinate system. The use of SWSH motivates us to use the standard polar coordinates $(r, \theta, \phi)$. This choice is further supported by the fact that it allows us to exploit axial symmetry. We will not do this. Instead we introduce adapted spherical coordinates $(\rho, \vartheta, \varphi)$ such that the constants of $\rho$ describe the foliation, and $\rho \to r$ as $r \to \infty$. These two
requirements prompt the choice

\[ \rho = \frac{M_+ + M_-}{u} \iff \frac{1}{\rho} = \frac{1}{M_+ + M_-} \left( \frac{M_+}{r_+} + \frac{M_-}{r_-} \right), \]  

(9.54)

with

\[ r_\pm = \sqrt{r^2 + Z^2 \pm 2rZ \cos(\theta)}. \]  

(9.55)

For the remaining two coordinates, we will make the associations \( \theta = \vartheta \) and \( \phi = \varphi \). The corresponding tangent vectors are

\[ \partial_\vartheta^a = \partial r \left( \frac{\rho, \vartheta}{\partial \vartheta} \right) - \partial r \partial_\vartheta \partial_\vartheta^a, \]

\[ \partial_\varphi^a = \partial_\vartheta^a. \]  

(9.56, 9.57)

An issue that arises from this coordinate system is that it is difficult to find an explicit solution for \( r(\rho, \vartheta) \). However, we are able to find such an equation for \( \theta = n\pi, n \in \mathbb{Z} \), a fact that will allow us to solve for \( r(\rho, \vartheta) \) implicitly. For each \( \rho = \text{constant} \), we numerically solve the ODE

\[ \dot{r}(\vartheta) = \frac{Z(M_+ r_-(\vartheta)^3 - M_- r_+(\vartheta)^3) r(\vartheta) \sin(\vartheta)}{M_- r_+(\vartheta)^3 (r(\vartheta) - Z \cos(\vartheta)) + M_+ r_-(\vartheta)^3 (r(\vartheta) + Z \cos(\vartheta))}, \]

subject to the initial condition

\[ r(\rho, \vartheta = 0) = \frac{1}{2} \left( \rho(M_+ + M_-) + \sqrt{\rho^2(M_+ + M_-)^2 - 4(\rho(M_+ - M_-) - Z^2)} \right). \]

(9.58, 9.59)

It should be understood that

\[ r_\pm(\vartheta) = \sqrt{r(\vartheta)^2 + Z^2 \pm 2r(\vartheta)Z \cos(\vartheta)}. \]

(9.60)

We are now in a position to choose \( V \), which we do by imposing the following two consistency conditions:

If \( M_\pm \to 0 \), then \( V \to -\frac{2M_\mp}{r_\mp} \),

(9.61)

If \( Z \to 0 \), then \( V \to -\frac{2(M_+ + M_-)}{r} \).

(9.62)

These are equivalent to saying that if either of the masses or the separation distance goes to zero, a Schwarzschild solution should be returned. One choice that satisfies these requirements is

\[ V = -\frac{2}{\rho}. \]

(9.63)
Notice that $V$ has no dependence on the mass of the system. This is due to the way that $\rho$ has been chosen. The total mass factor $M_+ + M_-$ scales $\rho$ so that we only consider the relative masses. One may view this as normalizing the masses such that $M_+ + M_- = 1$. We will always specify the magnitude of each mass before this normalization is performed.

The auxiliary metric is,

$$[A]_{ab} = \delta_{ab} + \frac{2}{\rho} l_a l_b,$$

with

$$l_a = \left( (\partial_r \rho)^2 + \left( \frac{\partial_\theta \rho}{r} \right)^2 \right)^{-\frac{1}{2}} (\partial_r \rho dr_a + \partial_\theta \rho d\theta_a).$$

In the $(\rho, \vartheta, \phi)$ coordinate system, the metric on each peanut is

$$h_{AB} = h_{\vartheta \vartheta} d\vartheta A d\vartheta B + r(\vartheta)^2 \sin^2(\vartheta) d\phi A d\phi B$$

$$= \left( \left( \frac{\partial_\rho \rho}{\partial r \rho} \right)_{r=r(\vartheta)}^2 + r(\vartheta)^2 \right) d\vartheta A d\vartheta B + r(\vartheta)^2 \sin^2(\vartheta) d\phi A d\phi B.$$ 

Due to the axial symmetry of the foliation, quantities intrinsic to the peanuts will not have cross terms:

$$Q_{AB} = Q \left( - \frac{h_{\vartheta \vartheta}}{r(\vartheta)^2} d\vartheta A d\vartheta B + \sin^2(\vartheta) d\phi A d\phi B \right),$$

$$\dot{k}_{AB} = \dot{k}_{\vartheta \vartheta} d\vartheta A d\vartheta B + \dot{k}_{\phi \phi} \sin^2(\vartheta) d\phi A d\phi B,$$

$$v_A = B_A = p_\phi = 0.$$ 

It follows that if the initial data is real (in coefficient space), then the solutions $(A, q, p, \bar{p})$ will also be real-valued. Moreover,

$$m^A p_A = \frac{1}{\sqrt{2}} p_\vartheta = \bar{m}^A p_A.$$ 

The explicit formulas produced by this foliation are listed in Appendix B.2.

### 9.2.4 Expected results

Before considering the numerical implementation of the above model, it is instructive to discuss the expected asymptotic behaviour. Recall that when $\rho$ is sufficiently small,
the resulting 2-surfaces are a set of topological spheres that loosely resemble peanuts. As \( \rho \) goes to infinity, the asymmetries of the foliation decay and the surfaces approach round 2-spheres, a consequence of which is that we would expect the solutions to approach the spherically symmetric data introduced in Section 9.1.2. We lend support to this by examining the leading order terms in a Taylor expansion of \( \kappa, \kappa^\star \) and \((2) R\). We expand near \( r \to \infty \):

\[
\kappa = \frac{2}{r^2} - \frac{4}{r^3} \left( 1 - \frac{M_- - M_+}{M_+ + M_-} Z \cos (\vartheta) \right) + \mathcal{O}(r^{-4}),
\]

\[
\kappa^\star = - \frac{2}{r} - \frac{2}{r^2} \left( 1 - \frac{M_- - M_+}{M_+ + M_-} Z \cos (\vartheta) \right) + \mathcal{O}(r^{-3}),
\]

\[
(2) R = \frac{2}{r^2} + \frac{2}{r^3} \left( 1 - \frac{M_- - M_+}{M_+ + M_-} Z \cos (\vartheta) \right) + \mathcal{O}(r^{-4})
\]

By comparing the above expansion of \( \kappa \) to Eq. (9.19), we see that this prediction can only hold true if \( M = 1 \). This makes sense and is a consequence of dividing by the total mass \( M_+ + M_- \), as was discussed previously. Even though the leading order terms in the above expansion match the leading order terms for the spherically symmetric solutions, this does not prove the claim. Nevertheless, we will suppose that it is correct. A consequence of this is that at \( \rho = \infty \) the binary black hole solutions can be matched to the spherically symmetric solutions (Eqs. (9.12) and (9.13)), allowing us to calculate the constants \( C \) and \( m \). Moreover, we note that if \( M_+ = M_- \) then the peanuts are symmetric. More specifically all of the free data on the surfaces are symmetric. It follows that we would expect the spin-zero quantities to also be symmetric. By examining the parabolic-hyperbolic evolution equation for \( p \) and \( \bar{p} \) we note that if \( q \) and \( A \) are symmetric then \( p \) and \( \bar{p} \) will be antisymmetric.

We now discuss how we will measure the constants \( C, m \), beginning with the curvature parameter. Explicitly, \( C \) can be calculated by the limit

\[
C = \lim_{\rho \to \infty} \rho \left( q + \frac{4}{\rho \sqrt{\rho (\rho + 2)}} \right),
\]

which follows from Eq. (9.12). Numerically however, we are unable to go all the way to \( \rho = \infty \). Instead, we will assume that we have calculated the numerical solution \( q \) up to a finite point \( \rho_{\text{max}} \). Since we have not yet reached infinity, the solution may still posses a dependence on \( \vartheta \), which we will remove by calculating the average

\[
< q(\rho_{\text{max}}) > = \frac{1}{N(\vartheta)} \sum_\vartheta q(\rho_{\text{max}}, \vartheta),
\]
where \( N(\vartheta) \) is the number of points in the \( \vartheta \) discretisation. This is not a geometric average. However, if we ensure that \( \rho_{\text{max}} \) is sufficiently large, such that \( q \) is approximately constant in \( \vartheta \), this will not be an issue.

We now discuss how one may calculate the embedding mass, \( m \). To do this, we first suppose that \( C \) has been calculated. Then, as with the curvature parameter, we can calculate \( m \) with the limit

\[
m = \lim_{\rho \to \infty} \frac{A(4 + C^2)^{3/2} \rho - 2\rho(4 + C^2)}{8} - C.
\]

One cannot assign particular meaning to this quantity unless the data set has a particular geometry. For example, if \( C = 0 \), then \( m \) is the ADM mass. Eq. (9.62) tells us that if \( Z = 0 \) then \( C = 0 \) and hence \( m \) is the ADM mass. As \( Z \) increases, we ‘push’ the masses apart, causing a change in the ADM mass, it follows that we may be able to calculate an explicit formula for \( m \), which we approximate by the expansion

\[
m = \hat{m} + \sum_{n} A_n Z^n = \hat{m} + \sum_{n \text{ even}} A_n Z^n + \sum_{n \text{ odd}} A_n Z^n.
\]

Under the transformation \( Z \mapsto -Z \), we should expect no change in \( m \), as re-orienting the coordinate system should not change the mass. In particular, this means that \( A_n = 0 \) if \( n \) is odd. Thus, we will not be able to use the linearised parabolic-hyperbolic equations to find a formula for small \( Z \). We also note that \( m \), will be normalised by the total mass factor \( M_+ + M_- \), in our coordinate transformation. This will mean two things: Firstly, the actual ADM mass associated with the system would be

\[
m_{\text{ADM}} = (M_+ + M_-) m.
\]

Secondly, since we scale our mass by \( M_+ + M_- \), it follows that if \( Z \) is fixed, then as the total mass change, the scale will also be adjusted and hence we will calculate the same (normalised) value of \( m \).

We end this section by discussing what numerical error may arise, and how it may be measured. First, we define \( F_E \) as the exact value (as calculated through the above limits) of the quantity we wish to measure (i.e. \( F \in \{C, m\} \)), and \( F_N \) as the numerical approximation which we calculate at a maximum radius \( \rho_{\text{max}} \). The two values can be related as

\[
F_N(\rho_{\text{max}}) = F_E + \frac{\varepsilon}{\rho_{\text{max}}} + O(\rho_{\text{max}}^{-2}),
\]

101
Figure 24: A depiction of a 2-surface produced by $\rho = \text{constant}$, embedded into $\mathbb{R}^3$. As was expected, this is spherically symmetric with respect to the $(\rho, \vartheta, \phi)$ coordinates.

where $\varepsilon \in \mathbb{R}$ is the error in our numerical measurement. Then, we define the absolute and relative errors as

$$E_A(\mathcal{F}) := 2(\mathcal{F}_N(\rho_{\text{max}}) - \mathcal{F}_N(2\rho_{\text{max}})) = \frac{\varepsilon}{\rho_{\text{max}}},$$

$$E_R(\mathcal{F}) := 100 \times \frac{E_A(\mathcal{F})}{\mathcal{F}_N(\rho_{\text{max}})} \%.$$  \hspace{1cm} (9.80) \hspace{1cm} (9.81)

9.3 Asymptotics of binary black holes

9.3.1 Convergence tests: shifted Kerr-Schild Schwarzschild initial data

We now consider the numerical implementation of the above binary black hole model. The code used is presented in Appendix C. To test the reliability of our code, we will pick our free data to coincide exactly with the known Schwarzschild space-time. We could choose $M_- = Z = [A] \gamma_{ab} = 0$, which would produce the Schwarzschild solution in Kerr-Schild coordinates. However, this is an insufficient test case for our code, as many of the auxiliary quantities are identically zero. We instead choose a coordinate system that shifts the black hole along the $z$-axis:

$$\rho = \sqrt{r^2 + Z^2 + 2rz\cos(\theta)},$$ \hspace{1cm} (9.82)

where $Z = \text{constant}$ is the distance the black hole has been shifted from the origin. This is a non-trivial test of our coordinate transformation. The 2-surfaces produced by this foliation is shown in Fig 24.

For the $(\rho, \vartheta, \phi)$ coordinate system, this is a trivial geometry with exact solutions:

$$q = -\frac{4M_+}{\rho \sqrt{\rho(2M_+ + \rho)}}, \quad A = \sqrt{1 + \frac{2M_+}{\rho}}, \quad p_C = 0,$$ \hspace{1cm} (9.83)
where $M_+$ is the mass of the black hole. In the $(r, \theta, \phi)$ coordinate system, the solutions are more complicated. The code begins its calculation in this coordinate system, and as such interprets the geometry as non-trivial. It follows that this is a better test case. However, we cannot test for convergence in the $\theta$ discretisation. This is because the solutions should be independent of $\theta$ and as such any sampling in $\theta$ is over-sampling.

The evolution is calculated through the use of the *scipy* ODE solver `odeint`. This is an adaptive numerical integrator with an absolute error tolerance $\tilde{E}$. We define the magnitude of $\tilde{E}$ as $\varepsilon$ such that the relation

$$\log_{10}(\tilde{E}) = \varepsilon$$

holds. For $s f \in \{A, q, p, \bar{p}\}$, we define the error quantity

$$\| s f \|_\theta = \max_{\theta \in (0, \pi]} | (N)_s f - (A)_s f |,$$

where $(N)_s f$ is the numerically calculated solution and $(A)_s f$ is the corresponding analytic solution. If our numerical values are correct then we would expect that the measured error should decrease as $\varepsilon$ becomes smaller. To observe this behaviour we calculate $\| s f \|$ at each value of $\rho$ for $\varepsilon = -8, -10, -12$. The result is shown in Fig. 25. Whilst the functions $q$ and $A$ demonstrate the expected convergence, the same cannot be said for $p$ and $\bar{p}$. This is because the error is below round-off error which occurs at $10^{-15}$.

### 9.3.2 Convergence tests: binary black holes

We now consider a non-trivial example of binary black holes with equal mass, $M_+ = M_- \text{ and } [A] \dot{\gamma} = 0$. We wish to demonstrate convergence in both the $\theta$ and $\rho$ discretisation. We cannot show the latter in the same way we did in the previous section as an analytic solution is not known. Nevertheless, we are still able to do convergence tests. This is done as follows: The solutions are calculated for with an error magnitude $\varepsilon_{\text{min}}$, which is used in place of $(A)_s f$ in Eq. (9.85). We further solve the equations with an error tolerance $\varepsilon > \varepsilon_{\text{min}}$. If the equations that we are solving are well-posed, then as $\varepsilon$ approaches $\varepsilon_{\text{min}}$ the associated error should decrease. This behaviour is shown in Fig. 26 with $Z = 1$. This consistency check allows us to ensure that the equations we are solving make sense but offers no insight into the convergence rate of the solutions. The ODE solver implemented here is adaptive and hence a convergence rate cannot be established. Due to the stiffness of the equations we will not swap to a non-adaptive integrator.

Once again making use of Eq. (8.43), we examine the convergence in the $\theta$-discretisation. As before, we expect to see the error decreases as the number of grid points is increased. We observe this behaviour in Fig. 27.
Figure 25: The error norms of three separate solutions with $M_- = 0$ and $Z = 1$ is shown on a log-log plot. For the top two graphs we see that the error decreases as $\varepsilon$ does. Similar behaviour is not seen in the bottom two plots.
Figure 26: The relative error of the evolution shown in a log-log plot. It can be seen for all of the solutions that $\varepsilon$ decreases with the associated error, and hence the code is self-consistent.
Figure 27: A logarithmic plot of the error associated with the $\vartheta$-discretisation. For all the unknowns, the error decreases as $N_{(1)}$ becomes finer.
A colour map of the solutions is displayed in Fig. 28, where we have used the Cartesian coordinate system

\[ x = \rho \sin(\theta) \sin(\phi), \quad y = \rho \sin(\theta) \cos(\phi), \quad z = \rho \cos(\theta). \quad (9.86) \]

Since the solutions are axisymmetric, it is enough to only view the z-y plane (i.e. \( \phi = 0 \)). Here, it can be seen that the spin-zero quantities are symmetric about the z-axis and the solutions with a non-zero spin are antisymmetric, as was predicted. The maximum error for each solution is

\[
\max_{\rho} \left\{ \max_{\vartheta \in [0, \pi/2]} |A(\vartheta) - A(\pi - \vartheta)| \right\} = 3.89 \times 10^{-1}, \quad (9.87)
\]

\[
\max_{\rho} \left\{ \max_{\vartheta \in [0, \pi/2]} |q(\vartheta) - q(\pi - \vartheta)| \right\} = 4.02 \times 10^{-1}, \quad (9.88)
\]

\[
\max_{\rho} \left\{ \max_{\vartheta \in [0, \pi/2]} |p(\vartheta) + p(\pi - \vartheta)| \right\} = 1.42 \times 10^{-2}, \quad (9.89)
\]

\[
\max_{\rho} \left\{ \max_{\vartheta \in [0, \pi/2]} |\bar{p}(\vartheta) + \bar{p}(\pi - \vartheta)| \right\} = 1.42 \times 10^{-2}. \quad (9.90)
\]

We end by calculating the corresponding curvature parameter and its errors, with \( \rho_{\text{max}} = 6000 \):

\[ C = 0.00774, \quad \mathcal{E}_A(C) = 9.7 \times 10^{-6}, \quad \mathcal{E}_R(C) = 1.126\%. \quad (9.91) \]

It is clear from these values that this data set is not asymptotically flat.

### 9.3.3 Binary black holes with a vanishing curvature parameter

From Eqs. (9.61) and (9.62) we see that if the separation distance goes to zero then the curvature parameter will also go to zero. The same statement holds true for the masses. It follows that we could model \( C \) as a function of the free parameters \((M_+, M_-, Z)\). To begin investigating this, we will first examine the dependence of the curvature parameter on \( M_- \) with both \( Z \) and \( M_+ \) fixed. For the remainder of this discussion, we will fix \( M_+ = 1^{24} \) and \( |A|^{1/2} = 0 \). Moreover, the initial and final values of the \( \rho \)-discretisation will be \((\rho_{\text{min}}, \rho_{\text{max}}) = (3, 6000)\). For \( Z = 1.5 \), we calculate \( C \) with various values of the mass \( M_- \in (0, 2] \), these are shown in Fig. 29. For the initial value of \( M_- \) we find a negative value of \( C \). As the mass increases, the curvature parameter becomes positive. This means that there exists a non-zero value

\[^{24}\text{We remind the reader that the values of } M_+ \text{ and } M_- \text{ are specified before the total mass is normalised.}\]
Figure 28: Colour scale plots of the solutions for $M_+ = M_-$ and $Z = 1$. Here we see that $q$ and $A$ are symmetric functions whilst $p$ and $\bar{p}$ are antisymmetric, as was predicted.
Figure 29: Graphs showing $C$ as a function of $M_-$ for $Z = 1.5$. From the graph on the left we can see that there exists an $M_-$ such that $C = 0$. Newton’s method gave an estimate of $M = 0.2909$. This is shown by the black dot in the right graph. We were able to obtain a more accurate estimation by considering mass values in the neighbourhood of the original guess.
Figure 30: The graph on the left depicts $M_-$ as a function of $Z$ such that $C \simeq 0$, with $M_+ = 1$. A polynomial was fitted to the measured values and used to interpolate the appropriate masses for unmeasured separation distances. The outcome of which is summarised in the table of the right.

of $M_-$ such that $C = 0$. To estimate where the zero occurs we first fit a polynomial to the data:

$$C = \sum_{n=0}^{9} C_n M_n^n, \quad \frac{dC}{dM_-} = -\sum_{n=0}^{9} n \frac{C_n}{M_-(n-1)^{1}}, \quad (9.92)$$

We use this polynomial in the application of Newton’s method to obtain an estimate of $M_-$ such that $C = 0$. We then calculate the curvature parameter for a set of mass values in the neighbourhood of the estimate. Once again we fit a polynomial and apply Newtons method to find a more accurate value of the root. The values of $C$ that were found are shown in Fig. 29. The final root was found to be $M_- = 0.3007$. The corresponding value of $C$ and its errors are

$$C = 6.44 \times 10^{-6}, \quad \mathcal{E}_A(C) = 2.75 \times 10^{-5}. \quad (9.93)$$

Clearly, the absolute error is larger than the curvature parameter. It is difficult to say exactly where this error comes from. Increasing the maximum value of $\rho$ does decrease this error, suggesting that this may be the issue. However, it may also be due to the fact that resolving around zero can be challenging and we require a cut-off point where we choose to approximate $C$ as zero.

By following the process outlined above, we were able to find a mass value for each $Z$ such that the curvature parameter was approximately zero. Plotting these values against each other then allowed us to approximate $M_-$ as a function of $Z$. The accuracy of this model is then checked by using this function to estimate $M_-$ for previously uncalculated separation distances. The outcome of this is shown in Fig.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$M_-$</th>
<th>$C$</th>
<th>$\mathcal{E}_A(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.395</td>
<td>$-7.04 \times 10^{-9}$</td>
<td>$3.13 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.35</td>
<td>0.143</td>
<td>$-5.12 \times 10^{-8}$</td>
<td>$1.69 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.55</td>
<td>0.127</td>
<td>$-3.29 \times 10^{-7}$</td>
<td>$3.64 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.75</td>
<td>0.106</td>
<td>$4.35 \times 10^{-7}$</td>
<td>$5.05 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.95</td>
<td>0.092</td>
<td>$-4.16 \times 10^{-6}$</td>
<td>$5.20 \times 10^{-6}$</td>
</tr>
<tr>
<td>1.15</td>
<td>0.152</td>
<td>$-7.18 \times 10^{-5}$</td>
<td>$-2.25 \times 10^{-5}$</td>
</tr>
<tr>
<td>1.35</td>
<td>0.253</td>
<td>$-4.63 \times 10^{-4}$</td>
<td>$5.50 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Figure 31: An illustration of the iterative process. We specify $\gamma_n$ at the initial radius, $\rho_0$, and perform the evolution. Calculation of the corresponding curvature parameter $C_n$ then allows us to construct $\gamma_{n+1}$. This process is continued until $C_{n+j} = 0$.

30. We see here that as the approximation $C \simeq 0$ gets better, the associated absolute error gets worse. This is the same sort of behaviour mentioned above.

9.3.4 Iteratively constructing asymptotically flat binary black hole data

In the above section, we saw that by choosing our freedoms appropriately, we could find cases where the curvature parameter vanishes. The mass and separation distance are not the only freedoms available to us. So far we have always assumed that $[^A] \dot{\gamma} = 0$ but we need not do this. By making an appropriate choice, it may be possible to pick our initial data such that $C = 0$. Quantities such as $p, \bar{p}$ and $\kappa$ already possess the desired fall-off rates. Since we do not wish to change these fall-off rates we make the choice

$$[^A] \dot{\gamma}_{ab} = \frac{Z}{M_- + M_+} \frac{\gamma}{\rho} h_{ab},$$

(9.94)

where $\gamma \in \mathbb{R}$ is a constant of our choice. Making the time derivative of the auxiliary metric proportional to $h_{ab}$ has the effect of only changing the initial data for $q$. The ratio of the separation distance to the total mass is a catalytic choice made for convenience (i.e. it accelerates the convergence $C \rightarrow 0$). Since $\gamma$ acts as a correction term for our initial data we call it the ’correction scalar’. Suppose that we have made an appropriate choice of $\gamma$. Then we solve for the corresponding solutions $\{^{(\gamma)} A, ^{(\gamma)} q, ^{(\gamma)} p, ^{(\gamma)} \bar{p} \}$ and calculate the curvature parameter $C_n$. In an ideal setting, it would be enough to subtract this term away. In general, this will not work. However, we claim that the absolute value of $C_{n+1}$ will be smaller than the absolute value of $C_n$, i.e.

$$|C_n| > |C_{n+1}|.$$  

(9.95)

111
This prompts us to construct $\gamma$ iteratively,

$$\gamma_n = \begin{cases} 0, & n = 0 \\ \sum_{i=0}^{n-1} C_i, & \text{else} \end{cases} \quad (9.96)$$

with $\gamma$ being the $\gamma_n$ such that $C_n = 0$. This iterative process is depicted in Fig. 31.

We will now apply this model to three cases which we summarise, before discussing the results.

**Case 1, $M_+ = 1, M_- = 1, Z = 1$:**

For this case, only two iterations were needed until $C$ had a sufficiently small magnitude:

$$C = -2.22 \times 10^{-8}, \quad E_A(C) = 3.27 \times 10^{-8}. \quad (9.97)$$

The corresponding correction scalar was:

$$\gamma = 7.78 \times 10^{-3}, \quad E_A(\gamma) = 2.40 \times 10^{-5}, \quad (9.98)$$

which had mass,

$$m = 9.50 \times 10^{-1}, \quad E_A(m) = 3.24 \times 10^{-4}, \quad m_{ADM} = 1.9. \quad (9.99)$$

**Case 2, $M_+ = 1, M_- = 1/2, Z = 1$:**

A total of eleven iterations were needed here to give a sufficiently small curvature parameter:

$$C = -6.43 \times 10^{-8}, \quad E_A(C) = 5.00 \times 10^{-8}, \quad (9.100)$$

which had correction scalar

$$\gamma = 4.88 \times 10^{-3}, \quad E_A(\gamma) = 3.20 \times 10^{-5}, \quad (9.101)$$

and mass

$$m = 9.49 \times 10^{-1}, \quad E_A(m) = 1.55 \times 10^{-4}, \quad m_{ADM} = 1.42. \quad (9.102)$$

**Case 3, $M_+ = 1, M_- = 1, Z = 3/2$:**

Nineteen iterations were used to get the values:

$$C = -8.20 \times 10^{-8}, \quad E_A(C) = 1.23 \times 10^{-8}. \quad (9.103)$$

The corresponding correction scalar was:

$$\gamma = 2.34 \times 10^{-2}, \quad E_A(\gamma) = 4.89 \times 10^{-5}. \quad (9.104)$$

The evolution gave rise to the total mass

$$m = 7.69 \times 10^{-1}, \quad E_A(m) = 2.35 \times 10^{-3}, \quad m_{ADM} = 1.54. \quad (9.105)$$
Figure 32: Colour maps for the three iterative cases considered above. The solution corresponding to each case is given in the columns.
The solutions for each of the considered cases are shown in Fig. 32. Note that they are not intended to be indicative of all possible situations, however these cases will help us investigate the dependence of \( m \) on \( Z \).

For all of the above cases the curvature parameter had the same magnitude as the corresponding absolute error. This is similar to what has been seen in the previous section.

In the Section 9.2.4, we asserted that we would expect \( m \) to be constant once the separation distance had been fixed. By comparing cases 1 and 2 we get,

\[
m = 9.49 \times 10^{-1}, \quad \mathcal{E}_A(m) = 5.15 \times 10^{-3},
\]

where the above error is measured by doubling \( M_- \) not \( \rho_{\text{max}} \). The error here is sufficiently small and hence lends support to the prediction. To study how \( m \) changes with \( Z \), we fix \( M_+ = M_- \). The equations are then solved using the iterative scheme for various separation distances. Both \( \gamma \) and \( m \) are plotted as functions of \( Z \) in Fig. 33. As the separation distance increase, the embedding mass decreases. This means that the ADM mass will always be smaller than the total mass. It is difficult to say why this is. It is of course possible that we are not dealing with binary black holes. To check this one would need to calculate the apparent horizons. Such a calculation is difficult in this model as the restriction \( r > Z \) must be made. Finally we note that the correction scalar can be modelled as a quadratic polynomial. We test this by considering \( Z = 1.35 \) and calculating the numerical value \( \langle N \rangle_{\gamma} \) and comparing it

Figure 33: The initial data and \( m \) as a function of the separation distance for fixed mass parameters \( M_+ = M_- \).
to our predicted value \((P)\gamma\):

\[
(P)\gamma = 1.76 \times 10^{-2}, \quad (N)\gamma = 1.76 \times 10^{-2}, \quad |(P)\gamma - (N)\gamma| = 6.29 \times 10^{-8}.
\]  (9.107)

The error in our prediction is significantly lower than its magnitude and hence it is reasonable to model the correction scalar in this way.
10 Summary and conclusions

We first studied the *strongly hyperbolic* formulation of the constraints (Eqs. (6.25) and (6.26)). By considering non-linear perturbations of *asymptotically hyperboloidal* initial data within the Minkowski space-time, we were able to construct a family of spherically symmetric solutions that were also asymptotically hyperboloidal. As a consequence of Birkhoff’s theorem we were able to show that this family of solutions could be embedded into a Schwarzschild solution with mass $M$. Even though these data sets were asymptotically hyperboloidal, they only satisfied the hyperbolicity condition (Eq. (6.49)) for a finite region. We studied this transition from hyperbolic to non-hyperbolic by considering linear perturbations of the data sets. The resulting data sets, found from the linear perturbations, were found to be asymptotically hyperboloidal as well. Moreover, the IVP was well-posed within the class of solutions with finitely many modes. This result was surprising for two reasons: Firstly, the same statement could not be made about perturbations to asymptotically flat data sets (due to the decay rates) [29], and secondly elliptic equations do not typically have a well posed IVP. We were able to explain this phenomenon by considering an *equivalent* second order system. In this system, the transition from hyperbolic to elliptic was emphasized, and manifested itself as a change of sign in the speed term. After the transition point, the speed term decayed to zero sufficiently fast such that the expected dynamics of exponential growth was controlled. The second order system was also used to formulate the linearised strongly hyperbolic formulation as BVP, where we showed that it was possible to solve the equations for boundary data with infinitely many modes. This is where we ended our studies of this formulation.

We then proceeded to examine linear and non-linear perturbations to spherically symmetric and asymptotically hyperboloidal data sets, within the framework of the *parabolic-hyperbolic* formulation of the constraints (Eqs. (6.40)-(6.42)). Since the parabolicity condition (Eq. (6.56)) did not depend on the solutions themselves, we were able to choose the free data such that the condition was satisfied for the entire evolution. This meant that the equations would always give rise to a well-posed IVP. However, this did not guarantee that the geometry of the data set would be stable to perturbations. We were able to show that our chosen data set was stable to both linear and non-linear perturbations. More specifically, the solutions produced by perturbing the initial data, were also asymptotically hyperboloidal.

This outcome motivated us to study *asymptotically flat* data sets within this formulation. Inspired by [7] we restricted ourselves to the class of metrics that took a Kerr-Schild form. Here, we found that non-linear perturbations to the Schwarzschild solution (in Kerr-Schild coordinates) were only asymptotically flat if the value of $q$ was unchanged (i.e. $C = 0$). In spite of this, we adapted the method presented by Bishop in [7] to generate data sets that describe binary black holes. Due to the symmetries
of our chosen foliation, we are able to show that the solutions found through the use of our model would asymptotically approach the spherically symmetric family of solutions that we found by perturbing the Schwarzschild solution (in Kerr-Schild coordinates). Even though we could not guarantee asymptotic flatness, we were able to find non-trivial binary black hole data sets that were asymptotically flat. Moreover, by adapting our initial data iteratively, we were able to construct asymptotically flat data sets.

Both formulations presented pros and cons. In the strongly hyperbolic formulation, the 3-metric is part of the free data and as such we are able to choose it such that the metric, but not necessarily the second fundamental form, will have the desired asymptotic geometry. However, since the hyperbolicity condition depends on one of the solutions of the equations it is not generally possible to guarantee that the hyperbolicity condition will remain satisfied throughout the evolution. Conversely, in the parabolic-hyperbolic formulation we are only able to pick the 2-metric, and as such we have little control over the geometry. However, the parabolicity condition depends on our choice of free data, and hence we can at least guarantee that the equations will always have a well-posed IVP. We also note that both formulations were somehow ‘more stable’ for asymptotically hyperboloidal geometries than they were for asymptotically flat ones. It remains to be seen if this is still true when the data sets are not axially symmetric.

It is the opinion of the author that, of the two formations, the parabolic-hyperbolic formulation was the most useful for numerical purposes. This is because we are able to guarantee that the equations will have a well-posed IVP, and hence we could ensure that our binary black hole model would always have well-behaved solutions. However, since we were unable guarantee the final geometry of the black holes, their physical significance is unclear. Further, the data sets we constructed that were asymptotically flat will not be stable to perturbation, as small changes to the initial data will result in a geometry that is no longer asymptotically flat.

In future work, we could impose a condition on the space-time function $V$ such that the resulting data is asymptotically hyperboloidal, as these geometries were stable under perturbation. Our code could also be generalised to describe rotating black holes, as was considered in [41, 42]. For this, an additional freedom would need to be introduced to describe the interaction of the spins of the black holes. It is also unclear if the data we found really did describe binary black holes. To confirm that our model really did generate binary black hole data we would need to calculate the apparent horizons. We may be unable to do this due to the presence of a surface with a bifurcation point. We speculate that this need not be a problem. One possible way of dealing with this is to split the space into two submanifolds $\Sigma_- := \{(x, y, z) \in \mathbb{R}^3 \mid z < 0\}$ and $\Sigma_+ := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. The constraints could then be solved in each submanifold independent of one another. A boundary
condition would then need to be imposed on the $z = 0$ plane to ensure that the solutions could be joined smoothly.
A Lie derivatives

Owing to the fact that partial derivatives are not tensorial derivatives it is necessary to define a derivative operator that does transform like a tensor. The standard way to do this is to introduce a connection term $\Gamma^a_{bc}$ such that the resulting operator has the desired transformation properties. This is not the only way to do it. Another way is to use a congruence of curves to Lie drag a tensor field from a point $P$ to a point $Q$ and then compare the dragged field to the one already there. We now endeavour to outline how this is done, and will base our discussion on [43].

Suppose $V^a(x)$ is a vector field defined over the entire manifold, $M$, and let $x$ be the coordinates. A congruence of curves can be defined by solving the ODE system,

$\frac{\partial x^a}{\partial u} = V^a(x(u)), \quad (A.1)$

where $u$ is an associated parameter. Standard results of ODE theory tell us that a unique solution to the above equation exits at least locally. We want to use $V^a(x)$ to differentiate the $(2,0)$-tensor field $T^{ab}(x)$. We have restricted ourselves to this type of tensor to ease calculations, this need not be done and it is straight forward to generalise the results. The congruence of curves produced from Eq. (A.1) to drag the tensor $T^{ab}(x)$ from a point $P$ to a neighbouring point $Q$. This is done by picking a curve from the congruence that connects the two points and using it to define the 'point transformation'

$P \rightarrow Q, \ y^a = x^a + h V^a(x), \quad (A.2)$

where $h \in \mathbb{R}^+$ is a small constant. The key to this is that the coordinate systems $y^a$ and $x^a$ are the same and hence we have,

$\frac{\partial y^a}{\partial x^b} = \delta^a_b + h \frac{\partial V^a(x)}{\partial x^b}.$ \quad (A.3)

However, we could view the dragging as a coordinate transformation, in which case we have,

$\tilde{T}^{ab}(y) = \frac{\partial y^a}{\partial x^c} \frac{\partial y^b}{\partial x^d} T^{cd}(x) = \left( \delta^a_c + h \frac{\partial V^a(x)}{\partial x^c} \right) \left( \delta^b_d + h \frac{\partial V^b(x)}{\partial x^d} \right)$

$= T^{ab}(x) + \left( T^{ad}(x) \partial_d V^b(x) + T^{cb}(x) \partial_c V^a(x) \right) h + O(h^2). \quad (A.4)$

The exact tensor field at $Q$ can be approximated by a Taylor expansion.

$T^{ab}(y^c) = T^{ab}(x^c + h V^c(x)) = T^{ab}(x^c) + h V^c \partial_c T^{ab}(x) + O(h^2). \quad (A.5)$
This finally puts us in a position to define the ‘Lie derivative’.

\[
\mathcal{L}_V T^{ab} := \lim_{h \to 0} \frac{T^{ab}(y) - \tilde{T}^{ab}(y)}{h},
\]

which has explicit formula,

\[
\mathcal{L}_V T^{ab} = V^c \partial_c T^{ab} - T^{cb} \partial_c V^a - T^{ac} \partial_c V^b.
\]

As mentioned, this formula can be generalised for a general \((p, q)\)-tensor field.

\[
\mathcal{L}_V T^{a_1 \ldots a_p b_1 \ldots b_q} = V^c \partial_c T^{a_1 \ldots a_p b_1 \ldots b_q} - T^{c \ldots a_p b_1 \ldots b_q} \partial_c X^{a_1} - \cdots + T^{a_1 \ldots a_p c \ldots b_q} \partial_{b_1} X^c + \cdots.
\]

It is always possible to introduce a coordinate system such that any curve passing through a point \(P\) is defined by only varying \(x^0\), keeping \(x^1, x^2, x^3\) constant. In this coordinate system we have,

\[
V^a \equiv (1, 0, 0, 0) \Rightarrow \mathcal{L}_V T^{ab} = \partial_0 T^{ab}.
\]

We end by summarising some important properties of the Lie derivative.

1. Linearity:

\[
\mathcal{L}_V (\alpha W^a + \beta U^a) = \alpha \mathcal{L}_V W^a + \beta \mathcal{L}_V U^a.
\]

2. Product Rule:

\[
\mathcal{L}_V (V^a U_{bc}) = W^a \mathcal{L}_V U_{bc} + U_{bc} \mathcal{L}_V W^a.
\]

3. Commutes with contraction:

\[
\delta^a_b \mathcal{L}_V T^b_a = \mathcal{L}_V T^a_a.
\]

4. Standard directional derivative for a scalar field \(u\):

\[
\mathcal{L}_V u = V^a \partial_a u.
\]
Black hole model: calculations

In this appendix we state all of the formulas that were used for modelling binary black holes (see section 9.2.3).

B.1 Coordinate free

B.1.1 Extrinsic 3-curvature

As was seen in section 9.2, the extrinsic 3-curvature can be written as

\[ K_{ab} = \frac{1}{2} A (\nabla_a (Vl_b) + \nabla_b (Vl_a) - \partial_t (\gamma_{ab})) , \]

which can be decomposed as,

\[ K_{ab} = \kappa N_a N_b + p_a N_b + p_b N_a + q_{ab}. \]

We aim to find an explicit formula for \( \kappa \), \( p_a \), and \( q_{ab} \), for \( (\partial_t \gamma_{ab}) = 0 \). We begin with \( \kappa := N^a N^b K_{ab} \).

\[ \kappa = \frac{1}{2} A N^a N^b (\nabla_a (Vl_b) + \nabla_b (Vl_a)) \]

\[ = AN^a N^b \nabla_a (Vl_b) \quad K_{ab} \text{ is symmetric} \]

\[ = AN^a N^b (VA^{-1} N_b) \quad N_a = Al_a \]

\[ = AN^a N^b (VA^{-1} \nabla_a N_b + N_b \nabla_a (VA^{-1})) \quad \text{Product Rule} \]

\[ = AN^a N^b (N_b \nabla_a (VA^{-1})) \quad N^a N^b \nabla_a N_b = 0 \]

\[ = AN^a \nabla_a (VA^{-1}) \quad N^a N_a = 1 \]

\[ = l^a \partial_a (VA^{-1}) \quad N^a = A^{-1} l^a. \]

We use this value to help define \( p_b := h^c_b N^a K_{ac} \).

\[ N^a K_{ab} = \kappa N_a + p_b = \frac{1}{2} AN^a (\nabla_a (Vl_b) + \nabla_a (Vl_a)) \quad N^a q_{ab} = N^a p_a = 0 \]

\[ = \frac{1}{2} AN^a (\nabla_a (VA^{-1} N_b) + \nabla_a (VA^{-1} N_a)) \quad N_a = Al_b \]

\[ = AN^a (VA^{-1} \nabla_a N_b + N_b \nabla_a (VA^{-1})) \quad \text{Product Rule} \]

\[ = \frac{1}{2} (Vv_b + N_b \kappa + A \nabla_b (VA^{-1})) \quad v_b := N^a \nabla_a N_b \]

\[ \Rightarrow p_b = \frac{1}{2} (Vv_b + A (\nabla_b (VA^{-1}) - \kappa l_b)) \quad N_a = Al_b \]
Finally we consider the fully spatial projection,

\[ q_{ab} := h^e_a h^d_b K_{cd} = \frac{1}{2} Ah^e_a h^d_b (\nabla_c (V l_d) + \nabla_d (V l_c)) \]

\[ = \frac{1}{2} Ah^e_a h^d_b (V \nabla_c (l_d) + l_d \nabla_c (V) + V \nabla_d (l_c) + l_c \nabla_d (V)) \quad \text{Product Rule} \]

\[ = \frac{1}{2} AV h^e_a h^d_b (\nabla_d (V l_c) + \nabla_c (l_d)) = 0 \]

\[ = \frac{1}{2} VH^e_a h^d_b \left( (\sigma) \nabla_c (l_d) + (\sigma) \nabla_d (l_c) - C^e_{de} l_e - C^e_{cd} l_e \right) \quad \nabla_a l_b = (\sigma) \nabla_a (l_b) - C^e_{ab} l_e \]

\[ = \frac{1}{2} VH^e_a h^d_b \left( (\sigma) \nabla_c (l_d) + (\sigma) \nabla_d (l_c) - 2 C^e_{cd} l_e \right) \quad C^e_{ab} = C^e_{ba} . \]  

(B.5)

We calculate \( I \) separately.

\[ 2C^e_{cd} l_e = 2 \left( \frac{1}{2} \gamma^{en} \left( (\sigma) \nabla_c \gamma_{nd} + (\sigma) \nabla_d \gamma_{cn} - (\sigma) \nabla_n \gamma_{cd} \right) \right) l_e \]

\[ = \gamma^{en} \left( (\sigma) \nabla_c \gamma_{nd} + (\sigma) \nabla_d \gamma_{cn} - (\sigma) \nabla_n \gamma_{cd} \right) l_e \]

\[ = \frac{1}{1 - V} \gamma^{en} \left( (\sigma) \nabla_c (V l_d) + (\sigma) \nabla_d (V l_c) - (\sigma) \nabla_n (V l_d) \right) \]

\[ = - \frac{1}{1 - V} \gamma^{en} \left( 2V \left( l_n (\sigma) \nabla_c l_d + l_l (\sigma) \nabla_c l_n - (\sigma) \nabla_n (l_c) l_d \right) \right) \]

\[ + 2l_n (\sigma) \nabla_d V \right) \quad \text{Product Rule} \]

\[ = - \frac{1}{1 - V} \left( 2l_d (\sigma) \nabla_c V + 2 \left( V (\sigma) \nabla_c l_d l_n (\sigma) \nabla_n (l_c) l_d \right) \right) \]

\[ + l_n (\sigma) \nabla_d l_n = 0 . \]  

(B.6)

We now consider the projections.

\[ 2h^e_a h^d_b C^e_{ab} l_e = - \frac{V}{1 - V} h^e_a h^d_b \left( \nabla_d l_d + \nabla_d l_e \right) \quad h^e_d l_e = 0 \]  

(B.7)

which gives,

\[ q_{ab} = \frac{(l)}{2(1 - V)} h^e_a h^d_b \left( \nabla_d l_d + \nabla_d l_e \right) . \]  

(B.8)
B.1.2 Extrinsic 2-curvature

The formula for the extrinsic 2-curvature is,

\[ k_{ab} = -h^c_a \nabla_c N_b = -\langle \sigma \rangle \nabla_a N_b + C_{ab}^e N_e. \tag{B.9} \]

The trace of which is,

\[ k = -h^{ab} \langle \sigma \rangle \nabla_a N_b + h^{ab} C_{ab}^e N_e. \tag{B.10} \]

We calculate these separately, beginning with I.

\[ h^{ab} C_{ab}^e N_e = \frac{1}{2} h^{ab} \gamma_{cd} \left( \langle \sigma \rangle \nabla_a \gamma_{bd} + \langle \sigma \rangle \nabla_b \gamma_{ad} - \langle \sigma \rangle \nabla_d \gamma_{ab} \right) N_e \]

\[ = \frac{1}{2} h^{ab} \left( \langle \sigma \rangle \nabla_a \gamma_{bd} + \langle \sigma \rangle \nabla_b \gamma_{ad} - \langle \sigma \rangle \nabla_d \gamma_{ab} \right) N_d \gamma^{ab} N_a = N^b \]

\[ = -h^{ab} N^d \langle \sigma \rangle \nabla_a (V_l d b) = -V^{(l)} A^{-1} h^{ab} l^d \langle \sigma \rangle \nabla_a (V_l d b) \]

\[ = -V^{(l)} A^{-1} h^{ab} l^d \nabla_{a l} = 0 \]

\[ = -V^{(l)} A^{-1} \delta^{ab} \langle \sigma \rangle \nabla_{a l} = 0. \tag{B.11} \]

Now we consider II.

\[ -h^{ab} \langle \sigma \rangle \nabla_a N_b = -h^{ab} \left( l_b \langle \sigma \rangle \nabla_a + A^{(l)} \langle \sigma \rangle \nabla_{a l} \right) = -h^{ab} l_b \langle \sigma \rangle \nabla_{a l} = 0 \]

\[ = -\delta^{ab} \langle \sigma \rangle \nabla_{a l}. \tag{B.12} \]

This finally gives,

\[ k = -V^{(l)} A^{-1} \delta^{ab} \langle \sigma \rangle \nabla_{a l} = -l^{(l)} A^{(l)} \delta^{ab} \langle \sigma \rangle \nabla_{a l} = -A^{-1} \delta^{ab} \langle \sigma \rangle \nabla_{a l}. \]

\[ \langle \sigma \rangle A = \sqrt{1 - V} \tag{B.13} \]

We also have,

\[ \hat{k}^{(\delta)} = -A \delta^{ab} \langle \sigma \rangle \nabla_{a l}. \tag{B.14} \]

Thus,

\[ V \]


B.2 Coordinates: \((r, \theta, \phi)\)

In this coordinate system we treat \(\rho = \rho(r, \theta, \phi)\) as a function.

B.2.1 Derivatives of \(\rho\)

The \((2 + 1)\)-foliation is described by the time-function

\[
\frac{1}{\rho} = \frac{M_+}{r_+} + \frac{M_-}{r_-}, \quad r_\pm = \sqrt{r^2 + Z^2 \pm 2rZ \cos(\theta)}.
\]  
(B.15)

The first order derivatives of \(r_\pm\) are,

\[
\partial_r r_\pm = \frac{r \pm Z \cos(\theta)}{r_\pm}, \quad \partial_\theta r_\pm = \mp \frac{rZ \sin(\theta)}{r_\pm}.
\]  
(B.16)

The second order derivatives are,

\[
\partial^2 r_\pm = 1 - \left(\partial_r r_\pm\right)^2 r_\pm, \quad \partial_r \partial_\theta r_\pm = \mp \frac{Z \sin(\theta) - \partial_\theta r_\pm \partial_r r_\pm}{r_\pm} r_\pm, \quad \partial_\theta^2 r_\pm = \mp \frac{Zr \cos(\theta) - \left(\partial_\theta r_\pm\right)^2}{r_\pm}
\]  
(B.17)

We now consider the derivatives of \(\rho\).

\[
\partial_i \rho = \rho^2 \sum_{\kappa \in \{+, -\}} \frac{M_\kappa}{r_\kappa^2} \partial_i r_\kappa = \rho^2 w_i.
\]  
(B.18)

The relevant derivatives of \(w_i\) are,

\[
\partial_j w_i = \sum_{\kappa \in \{+, -\}} \left( \frac{M_\kappa r_\kappa^2}{r_\kappa^4} \partial_j \partial_i r_\kappa - \frac{2r_\kappa \partial_j \partial_i r_\kappa r_\kappa + 4 \partial_j \partial_i (\partial_\kappa r_\kappa)}{r_\kappa^2} + \frac{6M_\kappa r_\kappa^2 \partial_j \partial_i r_\kappa}{r_\kappa^4} \right),
\]  
(B.19)

and

\[
\partial_k \partial_j w_i = \sum_{\kappa \in \{+, -\}} \frac{M_\kappa}{r_\kappa^2} \left( \partial_k \partial_j \partial_i r_\kappa - \frac{2r_\kappa \partial_j \partial_k r_\kappa r_\kappa + 4 \partial_j \partial_k (\partial_\kappa r_\kappa)}{r_\kappa^2} + \frac{6M_\kappa \partial_j \partial_k r_\kappa}{r_\kappa^4} \right).
\]  
(B.20)

The second and third order derivatives of \(\rho\) are,

\[
\partial_j \partial_i \rho = 2\rho w_i \partial_j \rho + \rho^2 \partial_j w_i
\]  
(B.21)

\[
\partial_k \partial_j \partial_i \rho = 2\rho \partial_j \rho \partial_k w_i + \rho^2 \partial_k \partial_j w_i + 2(\partial_k \rho \partial_j \rho + \rho \partial_k \partial_j \rho) w_i + 2\rho \partial_j \rho \partial_k w_i.
\]  
(B.22)
B.2.2 Derivatives of $l_a$

The general form of $l_a$ is,

$$l_a = (R, T, 0) = \varpi(\partial_r \rho, \partial_\theta \rho, 0),$$

where,

$$\varpi := \left((\partial_r \rho)^2 + \left(\frac{\partial_\theta \rho}{r}\right)^2\right)^{-\frac{1}{2}} = S^{-\frac{1}{2}}.$$ (B.24)

The first order derivatives of $S$ are,

$$\partial_i S = 2 \partial_i \rho \frac{\partial^2 \rho}{r} - \frac{2}{r^3} \partial_i (\partial_\theta \rho)^2 + \frac{2}{r^2} \partial_\theta \rho \partial_i \partial_\theta \rho,$$

$$\partial_i \theta S = 2 \partial_i \rho \frac{\partial_\theta \rho}{r} \partial_i \theta + \frac{2}{r^2} \partial_\theta \rho \partial_i^2 \rho.$$ (B.25)

The second order derivatives are,

$$\partial_i^2 S = 2 \left((\partial_r \rho)^2 + \partial_i \rho \partial^2 \rho + \frac{3}{r^4} (\partial_\theta \rho)^2 - \frac{2}{r^3} \partial_\theta \rho \partial_i \theta \partial_\theta \rho + \frac{1}{r^2} \left(\partial_i (\partial_\theta \rho)^2 + \partial_\theta \rho \partial_i^2 \theta \right)\right),$$

$$\partial_i \partial_j S = 2 \left(\partial_i \rho \partial_j \rho \partial_i \theta + \partial_i \rho \partial_j \theta \partial_i \rho + \frac{2}{r^3} \partial_j \rho \partial_i^2 \rho + \frac{1}{r^2} \left(\partial_i \partial_j \theta \partial_i^2 \rho + \partial_j \theta \partial_i \partial_i^2 \rho\right)\right).$$ (B.27)

Then,

$$\partial_i \varpi = -\frac{1}{2} S^3 \partial_i S,$$

and

$$\partial_j \partial_i S = -\frac{1}{2} S^2 \left(3 \partial_j S \partial_i S + S \partial_j \partial_i S\right).$$ (B.30)

The first order derivatives of $l_a$,

$$\partial_i R = \partial_i \varpi \partial_j \rho + \varpi \partial_i \partial_j \rho, \quad \text{(B.31)}$$

$$\partial_i T = \partial_i \varpi \partial_i \theta + \varpi \partial_i \partial_i \theta.$$ (B.32)

The second order derivatives of $l_a$,

$$\partial_i \partial_j R = \partial_i \varpi \partial_j \rho + \varpi \partial_i \partial_j^2 \rho + \partial_i \varpi \partial_i \partial_j \theta + \varpi \partial_i \partial_j \theta,$$

$$\partial_i \partial_i T = \partial_i \varpi \partial_i \partial_i \rho + \varpi \partial_i \partial_i \theta + \partial_i \varpi \partial_i \partial_i \theta + \varpi \partial_i \partial_i \theta.$$ (B.33)
B.3 Coordinates: \((\rho, \vartheta, \varphi)\)

The coordinate transformation is,

\[
\partial^a_\vartheta = - \left( \frac{\partial \rho}{\partial r} \right) \partial^a_r + \partial^a_\vartheta.
\]  \hspace{1cm} (B.35)

Under these transformations we have that \(r \mapsto r(\vartheta)\).

The induced metric is,

\[
h_{AB} = \left( \left( \frac{\partial \rho}{\partial r} \right)^2 + r(\vartheta)^2 \right)^{\frac{1}{2}} d\theta_A d\theta_B + r(\vartheta)^2 \sin^2(\vartheta) d\phi_A d\phi_B.
\]  \hspace{1cm} (B.36)

The derivative of \(h_{\vartheta \vartheta}\) is,

\[
\frac{dh_{\vartheta \vartheta}}{d\vartheta} = 2 \left( \frac{\partial \rho}{\partial r} \right) \left( \frac{\partial_\vartheta \partial_\theta \varrho \partial_r \varrho - \partial_\theta \partial_r \varrho \partial_\rho \varrho}{(\partial_r \varrho)^2} \right),
\]  \hspace{1cm} (B.37)

where

\[
\partial_\vartheta \partial_\theta \varrho = \partial_\theta^2 \varrho + \partial_\varrho r(\vartheta) \partial_r \varrho,
\]  \hspace{1cm} (B.38)

\[
\partial_\varrho \partial_r \varrho = \partial_\varrho \partial_r \varrho + \partial_\varrho r(\vartheta) \partial_r^2 \varrho.
\]  \hspace{1cm} (B.39)

The first order derivative of \(r(\vartheta)\) was given in Eq. (9.58). The second derivative is,

\[
\ddot{r}(\vartheta) = \sum_{\kappa \in \{+,-\}} \frac{M_N u_{\kappa}}{r_N^2} \left( \sum_{\kappa \in \{+,-\}} \frac{M_N}{r_N^2} \left( \frac{2 \dot{r}_N^2}{r_N} - f_{\kappa} \right) \right),
\]  \hspace{1cm} (B.40)

with

\[
f_{\pm} = \frac{\dot{r}(\vartheta) \left( \dot{r}(\vartheta)^2 + 2Z \sin(\vartheta) \right) - r(\vartheta) Z \cos(\vartheta) - \left( \partial_\vartheta r_{\pm}(\vartheta) \right)^2}{r_{\pm}},
\]  \hspace{1cm} (B.41)

\[
u_{\pm} = \frac{r(\vartheta) + Z \cos(\vartheta)}{r_{\pm}}.
\]  \hspace{1cm} (B.42)

Finally, the 2D Ricci scalar is,

\[
(2)^R = \frac{\csc(\vartheta) \partial_\vartheta h_{\vartheta \vartheta} \left( r(\vartheta) \cos(\vartheta) + \sin(\vartheta) \dot{r}(\vartheta) \right) + 2 h_{\vartheta \vartheta} \left( r(\vartheta) - 2 \cot(\vartheta) \dot{r}(\vartheta) - \dot{\vartheta} \right)}{r'(\vartheta) h_{\vartheta \vartheta}^2}.
\]  \hspace{1cm} (B.43)
This code uses the Axial symmetric spin-weighted function module (written by Leon Escobar who made it available at http://gravity.otago.ac.nz/wiki/index.php/People/LeonEscobar).

We divide this code into three main sections, 'Classes', 'Functions', and 'Main Code'.

Classes:
(1) PotentialFunctions defines all needed derivatives of rho (which we call t).
(2) FreeFunctions defines all quantities needed for the PDE system.
(3) r_of_t_and_theta creates r as a function of t (rho) and theta (o).

Functions:
(1) Class Functions creates functions that uses Classes to evaluate the free data at all points.
(2) Solution Functions creates functions needed to handle the solutions.
(3) Initial data functions creates the initial data for the PDE.

Main:
(1) Defines and solves the PDE for given model data.

---

```python
#---------------------------- Set up ------------------------------#
#------------------------# Imports needed modules #------------------------#
import gc
import os
import sys
gc.enable()
import numpy as np
import sympy as sy
from heat import *
from scipy.integrate import odeint
sys.path.append('/maybehome/jritchie') # To include the path where the is module Spin_Weight_Functions
from Axial_Spin_Weight_Functions_UltraFast import python_module

#---------------------------- Classes ----------------------------#

class PotentialFunctions:
    
    This class is for t and its derivatives in the (r,theta,phi) coordinate system. Moreover, r(theta) and its derivatives are in the class for the (t,theta,phi) coordinate system
    
def __init__( self, t, r, o, M = 1., m = 0., Z = 0.):
        
        Creates the class and calculates r+ and r- as well as its derivatives. Input:
        (1,2,3): t, r, o - t = constant and the coordinates (r,o).
        (4,5,6): M, m, Z - The two masses and the distance from r = 0 to m.
        
    def __call__( self, t, r, o):
        
        Returns the r+ and r- as well as their time derivative.

    ## Basic Data ##
    self.M, self.m, self.Z = float(M/m + M), float(m/m + M), float(Z)
    self.r, self.o, self.t = float(r), float(o), float(t)

    ## rplus and rmins and their squares ##
    self.rplus = np.sqrt(r**2. + Z**2. + 2.*Z*r*np.cos(o))
    self.rplus2 = r**2. + Z**2. + 2.*Z*r*np.cos(o)
```
self.rminus = np.sqrt( r**2. + Z**2. - 2. *Z*r*np.cos(o) )
self.rminus2 = r**2. + Z**2. - 2. *Z*r*np.cos(o)

## First order derivatives of rplus and rminus ##
self.drplusr2dr = (r + (Z)*np.cos(o))/self.rplus
self.drplusr2dr = (r - (Z)*np.cos(o))/self.rplus
self.drplusrdr_rplusdo_sino = -r*(Z)/self.rplus;
self.drplusrdr = self.drpulsdo_ar_sino
self.drplusrdr_sino = r/self.rminus; self.drplusrdo = self.drmrplusdo_sino*np.sin(o)

## Second order derivative of rplus and rminus ##
self.d2rplusr2dr2 = r**2. + Z**2. - 2. *Z*r*np.cos(o)

#---------------------------------#
#- Second order derivatives of t -#
#---------------------------------#

def d2t2dr2(self):
    ""
    Second derivative of t w.r.t r.
    Input:
    t, M, m = self.t, self.M, self.m
    ""
    self.rminus = np.sqrt( r**2. + Z**2. - 2. *Z*r*np.cos(o) )
    self.rminus2 = r**2. + Z**2. - 2. *Z*r*np.cos(o)
    self.drplusrdr = (r + (Z)*np.cos(o))/self.rplus
    self.drplusrdr = (r - (Z)*np.cos(o))/self.rplus
    self.drplusrdr_sino = -r*(Z)/self.rplus;
    self.drplusrdr = self.drpulsdo_ar_sino
    self.drplusrdr_sino = r/self.rminus; self.drplusrdo = self.drmrplusdo_sino*np.sin(o)

    #--------------------------#
    #- First order derivatives of t -#
    #--------------------------#

def dt2dr ( self ):
    ""
    First derivative of t w.r.t r.
    Input:
    t, M, m = self.t, self.M, self.m
    ""
    self.rminus = np.sqrt( r**2. + Z**2. - 2. *Z*r*np.cos(o) )
    self.rminus2 = r**2. + Z**2. - 2. *Z*r*np.cos(o)
    self.drplusrdr = (r + (Z)*np.cos(o))/self.rplus
    self.drplusrdr = (r - (Z)*np.cos(o))/self.rplus
    self.drplusrdr_sino = -r*(Z)/self.rplus;
    self.drplusrdr = self.drpulsdo_ar_sino
    self.drplusrdr_sino = r/self.rminus; self.drplusrdo = self.drmrplusdo_sino*np.sin(o)

    return ( t**2. )*( M*drplusrdr/rplus2 + m*drplusrdr/rminus2 )

def dt2do_sino ( self ):
    ""
    First derivative of t w.r.t o, without the sin(o) factor.
    Input:
    t, M, m = self.t, self.M, self.m
    ""
    self.rminus = np.sqrt( r**2. + Z**2. - 2. *Z*r*np.cos(o) )
    self.rminus2 = r**2. + Z**2. - 2. *Z*r*np.cos(o)
    self.drplusrdr = (r + (Z)*np.cos(o))/self.rplus
    self.drplusrdr = (r - (Z)*np.cos(o))/self.rplus
    self.drplusrdr_sino = -r*(Z)/self.rplus;
    self.drplusrdr = self.drpulsdo_ar_sino
    self.drplusrdr_sino = r/self.rminus; self.drplusrdo = self.drmrplusdo_sino*np.sin(o)

    return ( t**2. )*( M*drplusrdo_sino/rplus2 + m*drplusrdo_sino/rminus2 )

#--------------------------#
## Product rule ##

### Pre-computations of r[+/-] ###

\[
\text{rplus2, rminus2} = \text{self.rplus2, self.rminus2} \\
\text{drplusdr, dminusdr} = \text{self.drplusdr, self.dminusdr} \\
\text{d2rplusdr2, d2minusdr2} = \text{self.d2rplusdr2, self.d2minusdr2} \\
\]

### Second derivatives of t w.r.t o & r. ###

\[
\text{d2tdrdrdo_sino} = \frac{\text{self.dt2do2} - \text{2.}*\text{self.d2minusdr2}}{\text{self.d2rplusdr2} - \text{2.}*\text{self.d2minusdr2}}/\text{self.d2rplusdr2} \\
\text{return ProductOne + ProductM + ProductM} \\
\]

```python
def d2tdrdo_sino( self, DtDo_sino, DtDr ):
    ***
    Second derivative of t w.r.t o & r.
    Input:
    (1) DtDo_sino - First derivative of t w.r.t o, without the sin(o) factor.
    (2) DtDr - First derivative of t w.r.t r.
    ***
    # Basic Data #
    r, o, t = self.r, self.o, self.t
    # Basic Data #
    r, o, t = self.r, self.o, self.t
    # pow( r[+/-], 2. ) #
    rp, rm = self.rplus2, self.rminus2
    # Pre-calculated derivatives of r[+/-] #
    drplusdr, dminusdr = self.drplusdr, self.dminusdr \\
    d2rplusdr2, d2minusdr2 = self.d2rplusdr2, self.d2minusdr2 \\
    # wo := dtdo/t #
    wo_sino = self.dt2do2/rp + m*d2minusdr2/rm
    # First order derivative of wo w.r.t r #
    dwodwdr_sino = m*( self.d2rplusdr2/rm - 2.*self.d2minusdr2)/(self.d2rplusdr2)
    dwodwdr_sino = m*( self.d2minusdr2/rm - 2.*self.d2minusdr2)/(self.d2plusdr2)
    dwodwdr_sino = dwodwdr_sino + dwodwdr_sino
    return 2.*wo_sino*DtDr + pow( t, 2. )*dwodwdr_sino \\

def d2dwdr2( self, DtDo ):
    ***
    Second derivative of t w.r.t o & o.
    Input:
    (1) DtDo - First derivative of t w.r.t o.
    ***
    # Basic Data #
    r, o, t = self.r, self.o, self.t
    # Second derivatives of r[+/-] #
    d2rplusdr2, d2minusdr2 = self.d2rplusdr2, self.d2minusdr2 \\
    # Product rule #
    ProductOne = 2.*t^2*DtDo*( m*self.drplusdr/self.rplus2 + m*self.dminusdr/self.rminus2 ) \\
    ProductM = ( t^2*2. )*( m*self.drplusdr2 - 2.)*m/(self.rplus2) \\
    ProductM = m*( t^2*2. )*(d2rplusdr2 - 2.*m*(self.d2minusdr2)/self.d2plusdr2) \\
    (self.rplus2) \\
    return ProductOne + ProductM + ProductM
```

---

## Basic Data ##

\[
\]

### Basic Data ##

\[
\]

### Pre-calculated derivatives of r[+/-] ##

\[
\]

### Product One ##

\[
\]

### Second derivatives of t w.r.t o & o. ###

\[
\]

### Product rule ##

\[
\]
def d3tdr3( self, DtDr, D2tDr2 ):
    ***
    Third derivative of t w.r.t r, r & r.
    Input:
    (1) DtDr - First derivative of t w.r.t r.
    (2) D2tDr2 - Second derivative of t w.r.t r & r.
    ***
    # Basic Data#
    M, m, Z = self.M, self.m, self.Z
    r, o, t = self.r, self.o, self.t
    # pow( r[+/-], 2. )#
    rp, rm = self.rplus2, self.rminus2
    # Pre-calculated derivatives of r[+/-]#
    drplus, drminus = self.drplusdr, self.drminusdr
    d2rplus, d2rminus = self.d2rplusdr2, self.d2rminusdr2
    # Third order derivatives of r[+/-]#
    d3rplusdr3 = -(1.*self.rplus*d2rplus-(self.rplus**2.))*d2rplus/r
    d3rminusdr3 = -(1.*self.rminus*d2rminus-(self.rminus**2.))*d2rminus/r
    # wr := dtdr/t ##
    wr = M*self.drplusdr/rp + m*self.drminusdr/rm
    # First order derivative of wr w.r.t r ##
    dwrdrOne = M*self.d2rplusdr2/rp + m*self.d2rminusdr2/rm
    dwrdrTwo = 2.*M*(self.d2rplusdr)**2./self.rplus**3. + m*(self.
      drminusdr)**2./self.rminus**3.)
    dwrdr = dwrdrOne - dwrdrTwo
    # Second order derivative of wr w.r.t r & r ##
    d2wdr2M = M*( 6.*d2rplus**3.)/rp**2. - 6.*d2rplus/(self.rplus**
      3.) + d2rplusdr3/rp )
    d2wdr2m = M*( 6.*d2rminus**3.)/rm**2. - 6.*d2rminus/(self.
      rminus**3.) + d2rminusdr3/rm )
    d2wdr2 = d2wdr2M + d2wdr2m
    # Product Rule##
    PartOne = 2.*DtDr**2.*{ wr } + 2.*{DtDr2}*{ wr } +2.*w*dwdr*dDtDr
    PartTwo = 2.*w*dwdr*dDtDr + {**2.}*d2wdr2
    return PartOne + PartTwo

def d3tdrdo2( self, DtDr, D2tDr2, D3tDr3 ):
    ***
    Third derivative of t w.r.t r, o & o.
    Input:
    (1) DtDr - First derivative of t w.r.t r.
    (2) D2tDr2 - First derivative of t w.r.t o.
    (3) D3tDr3 - Second derivative of t w.r.t r & o.
    ***
    # Basic Data#
    M, m, Z = self.M, self.m, self.Z
    r, o, t = self.r, self.o, self.t
    # pow( r[+/-], 2. )#
    rp, rm = self.rplus2, self.rminus2
    # Pre-calculated derivatives of r[+/-]#
drplusdr, drminusdr = self.drplusdr, self.drminusdr
d2rplusdr2, d2drminusdr2 = self.d2rplusdr2, self.d2drminusdr2
d3rplusdr2, d3drminusdr2 = self.d3rplusdr2, self.d3drminusdr2

def d3rplusdr2do (self, drplusdo, drminusdo):
    return d3rplusdr2do

def d3rminusdr2do (self, drminusdo):
    return d3rminusdr2do

## First order derivative of wo w.r.t r & o ##

wo = (t/t + o/o) #

## Second order derivative of wo w.r.t r & o ##

dwo = (t/t + o/o) #

def d2wo(r, o, t):
    return d2wo(r, o, t)

## Third order derivative of wo w.r.t r, o & t ##

def d3wo(r, o, t):
    return d3wo(r, o, t)

## Pre-calculated derivatives of r[/ ] ##

drplusdr, drminusdr = self.drplusdr, self.drminusdr
drplusdo, drminusdo = self.drplusdo, self.drminusdo
drplusdo0, drminusdo0 = self.drplusdo0, self.drminusdo0

def d3rplusdr2do = -(2.*drplusdo*drplusdo + drplusdo*drplusdo)/self.

## Basic Data ##

M, m, Z = self.M, self.m, self.Z
r, o, t = self.r, self.o, self.t

## Pre-calculated derivatives of r[/ ] ##

drplusdr, drminusdr = self.drplusdr, self.drminusdr
drplusdo, drminusdo = self.drplusdo, self.drminusdo
drplusdo0, drminusdo0 = self.drplusdo0, self.drminusdo0

## Third order derivatives of r[/ ] ##

d3rplusdr2do = -(2.*drplusdo*drplusdo + drplusdo*drplusdo)/self.

PartOne = 2.*d2rplusdr2do(r, o) + 2.*d2rminusdr2do(r, o)
PartTwo = 2.*d2rplusdr2do(r, o) + 2.*d2rminusdr2do(r, o)

return PartOne+PartTwo

def d3tdoro(self, dt, dtdo):
    return d3tdoro(self, dt, dtdo)

## Third derivative of t w.r.t r, o & t ##

Input: (1) DDr - First derivative of t w.r.t r
      (2) D2DDr - Second derivative of t w.r.t r & r

## Basic Data ##

M, m, Z = self.M, self.m, self.Z
r, o, t = self.r, self.o, self.t

## Pre-calculated derivatives of r[/ ] ##

drplusdr, drminusdr = self.drplusdr, self.drminusdr
drplusdo, drminusdo = self.drplusdo, self.drminusdo
drplusdo0, drminusdo0 = self.drplusdo0, self.drminusdo0
```python
#--------------------------------#
# First order derivative of wo w.r.t t ##
wo = M*drpdo/rp + m*drdminusdo/rm
#
# Second derivative of wo w.r.t t ##
#--------------------------------#

def d2rdo2( self, dr):
    # Second derivative of r(theta) w.r.t theta.
    # Input:
    #   (1) dr - First derivative of r(theta) w.r.t theta.
    #--------------------------------#
    # Basic data ##
    r, o = self.r, self.o; t = self.t
    M, m, Z = self.M, self.m, self.Z
    rplus, minus = self.rplus, self.minus
    top = r**2*(M/(rplus**3.)) - m/(minus**3.)
    bottom = (M/(rplus**3.))*(r + Z*np.cos(o))
    bottom = bottom_minus + bottom_plus
    return top/bottom

def drpdo( self, rp):
    # First order derivative of rplus and minus ##
    drpdo = (dr*(r+Z*np.cos(o)) - r**2*np.sin(o))/rp
    #--------------------------------#
    #--------------------------------#
```

XIV
class FreeFunctions():
    ## Basic variables ##
    self.t, self.r, self.o = t, r, o
    ## Derivatives of r ##
    self.d2r_dr2, self.drdo_sinodo = d2r_dr2, drdo_sinodo; self.drdo = drdo_sinodo*np.sin(o)
    ## Peanut metric functions ##
    self.v, self.y = -2./t, y; self.h22 = (drdo_sinodo*np.sin(o)/dt_dr)**2. + r**2.
    ## Derivatives of v ##
    self.dvdr, self.dvdo = (2./pow( self.t, 2.))*dt_dr, (2./pow( self.t, 2. ))*dt_sinodo*np.sin(o)
    ## O derivative of derivatives of t ##
    self.dddtdo = d2t_dr2 + self.dt_drdo*drdo_sinodo*np.sin(o)
    self.dt_drdo = d2t_dodrdo*np.sin(o) + self.drdo*_ddt_drdo
    ## O derivative of h22 ##
    self.dh22_dr2 = d2r_dr2*(self.drdo_sinodo**2.) + self.drdo_sinodo*(self.d2r_dr2 + self.drdo)*np.sin(o)
    ## First order derivatives of t ##
    self.dtt_dr2, self.dtt_drdo, self.dtt_dodrdo = d2t_dr2, d2t_dodrdo, np.sin(o),
    self.dtt_dodrdo = d2t_dodrdo
    ## Second order derivatives of t ##
    self.ddt_drdo = 2.0*(self.dtt_drdo*self.dtt_dodrdo)/((self.d2t_dr2)/r**2.)
    ## First order derivatives of S ##
    self.dSdr = 2.0*(self.dtt_dr2 + self.dtt_drdo * self.dtt_dodrdo)/r**2. + self.dtt_drdo}
## Second order derivatives of $S$

```python
self.d2Sdr2 = 2.0 * (d2dt_dr2 ** 2.0 + dt_dr * d3t_dr3) + 2.0 * self.dtdo * (d2Sdrdo + self.d2tdodo) / (r ** 2.0) - 4.0 * (self.dtdo * self.dtdrdt + pow(self.d2tdrdo, 2.0)) / (r ** 2.0)
```

## Minkowski Normalisation factor

```python
self.MinkNorm = pow(np.sqrt(dt_dr ** 2.0 + dt_sinodo * np.sin(o) ** 2.0), -1.0)
```

## First order derivatives of $S$

```python
self.dMinkAdr = -(self.MinkNorm ** 3.0) * self.dSdr / 2.0
```

## Second order derivatives of $S$

```python
self.d2MinkAdr2 = -(3.0 * (self.S ** 2.0)) * (self.dSdr ** 2.0) + (self.S ** 3.0) * self.d2dSdrdo / 2.0
```

## Components of $ld$

```python
self.T_sino = dt_sinodo * self.MinkNorm; self.T = np.sin(o) * self.T_sino
self.R = dt_dr * self.MinkNorm
```

## First order derivatives of $ld$

```python
self.dRdr = self.MinkNorm * dt_dr + dt_dr * self.dMinkAdr
self.dTdr = self.dtdrsino + self.sino * self.dMinkAdr
self.dTdo = self.MinkNorm * dtdo + self.dtdo * self.dMinkAdo
```

## Second order derivatives of $ld$

```python
self.d2Rdr2 = self.d2MinkAdr2 * dt_dr ** 2.0 * self.dMinkAdr * d2t_dr2 + self.MinkNorm * d3t_dr3
self.d2Tdrdo = self.d2MinkAdo * d2t_dr2 * self.dMinkAdo + self.dtdo * self.dMinkAdo * d2tdo
```

---

## Extrinsic 3-quantities (from Kdd)

```python
def Kappa( self):
    """
    Calculates the curvature quantity kappa = NU[a]NU[b]Kdd[a,b].
    Input:
    No input required.
    """
    # Basic data
    r, t = self.r, self.t
    # t derivatives
    dt, dtdo = self.dtdt, self.dtdo
    # Pre-calculated metric pieces
    factor = (t ** 2.0) / (v ** 2.0)
    return factor * (R * dt + T * dtdo / pow(r, 2.0))
```

```python
def qddUU( self):
    """
    Calculates the trace of qdd from hUU[a,b]qdd[a,b].
    Input:
    No input required.
    """
```
## Bottom part ##

Bottom = \( \text{pow}(r, 6.0) \times \text{pow}(R, 2.0) \times v + \text{pow}(r, 4.0) \times v - \text{pow}(r, 2.0) \times v \) - \( \text{pow}(r, 4.0) \times \text{sin}\{ \text{v} \text{sqrt}(1.0 - v^2) / 2.0 \} \times \text{Top} - 2.0 \times \text{self.y} / (\text{self.t}) \)

\( \text{def} \ Q1(\text{self, Q22_sin2o }) : \)

** Calculates Q11. **

Input:

- (1) Q22_sin2o - Q22 without the sine-squared factor.

** From trace-free condition **

Output = \(-Q22_sin2o\{ \text{self.h22/pow( self.r, 2.0) } \} \)

return Output

\( \text{def} \ Q22_sin2o(\text{self, q_in } ) : \)

** Calculates Q22 without the sin(o)**2 factor. **

Input:

- (1) q_in - The trace of qdd.

** Defines q[2,2]/pow( sin(o), 2 ) **

\( v, r = \text{self.v, self.r; factor = v*\text{np.sqrt}(1.0-v) / (1.0-v) \) \)

q22_sin2o = factor*\{r*\text{self.R + np.cos( self.o )}*\text{self.T_sino} - (\text{self.y}*[r**2.0] ) / (\text{self.t}) \}

** Q[2,2] = q[2,2] - q*h[2,2]/2 **

Output = q22_sin2o - q_in*[**2.0] / 2.

return Output

\( \text{def} \ h0dKddN(\text{self, kappa }) : \)

** Calculates pd[a] := h0d[c,a]NU[b]Kdd[c,b]. **

Input:

- (1) kappa - the curvature Quantity kappa = NU[a]NU[b]Kdd[a,b].

** Basic data **

\( t = \text{self.t} \)
## Derivatives of t ##

dtdo, dtdr = self.dtdo, self.dtdr

## Metric components ##

R, T = self.R, self.T

## Derivatives of v ##

dvdr, dvdo = self.dvdr, self.dvdo

## Derivatives of V/A ##

A = np.sqrt(1. - v)
factor = (2. - v)/(2.*(1. - v))
dv_Adr, dv_Ado = factor*dvdr/A, factor*dvdo/A

## pd[a] = pr*dr[a] + po*do[a] ##

pr = dv_Adr - kappa*R
po = dv_Ado - kappa*T

## pd[A] = (po - dtdo*pr/dtdr)*dO[A] ##

return (po - dtdo*pr/dtdr)/2.

#-------------------------------#
#- Peanut curvature quantities -#
#-------------------------------#

def kstar11(self, k_star, kstar22):
    """
    Calculates kstar[1,1].
    Input:
    (1) k_star - Trace of the tensor.
    (2) kstar22 - x[3],x[3] component of kstar2d[A,B].
    """
    ## Basic data ##
    t, r, o = self.t, self.r, self.o
    
    ## t derivatives ##
    dt_dr, dt_do = self.dtdr, self.dtdo
    
    ## ld components ##
    R, T = self.R, self.T
    
    return self.h22*(k_star - kstar22/pow(self.r, 2.))

def kstar22_sino2(self):
    """
    Calculates kstar[2,2] without the pow(sin(o),2) factor.
    Input:
    No input required.
    """
    ## Basic data ##
    r, o = self.r, self.o
    
    ## Metric pieces ##
    v, R, T_sino = self.v, self.R, self.T_sino
    T = np.sin(o)*T_sino
    
    return (r**2.)*(1.-v)*(T_sino*np.cos(o)+r*R)/(v*(((r*R)**2.)+(T**2.))-r**2.)

def kstar(self):
    """
    Calculates kstar.
    Input:
    No input required.
    """
## Precomputed ld components ##
R, T_sino = self.R, self.T_sino
dRdr, dTdo = self.dRdr, self.dTdo
d2Rdr2, d2Tdo = self.d2Rdr2, self.d2Tdo

## Product Rule ##
kstar0 = -(2.*R + np.cos(o)*T_sino + dTdo + pow(r, 2.)*drdO_sinO)
PartOne = -(2.*R + 2.*np.cos(o)*T_sino + 2.*dTdo - 2.*pow(r, 2.)*drdO_sinO)
PartTwo = (np.cos(o)*dTdr_sino + d2Tdrdo + pow(r, 2.)*d2Rdr2)/pow(r, 2.)

return -self.MinkNorm*(PartOne*PartTwo) + kstar0*self.dMinkAdr

def RicciScalar(self):
    ""
    Ricci Scalar on each t = constant leaf.
    Input:
    No input required.
    ""
    r, o = self.r, self.o
    h22 = self.h22
drdo_sino = self.drdO_sinO; drdO = drdO_sinO*np.sin(o)
dh22_do2 = self.dh22_do2

    Top = dh22_do2*(np.sin(o)*r + np.sin(o)*drdO) + 2.*h22*(r - 2.*np.cos(o)*drdO_sinO - d2r_do2)

    return Top/(pow(h22, 2.)*r)

#--------------------------#
#- Needed spin & artifical functions #-
#--------------------------#

def SymContrct(self, A1, A2_sin2, B1, B2_sin2):


    Input:
    (1) A1 - The x[2],x[2] component the first tensor.
    (2) A2_sin2 - The x[3],x[3] component the first tensor without pow(sin(o),2) factor.
    (3) B1 - The x[2],x[2] component the second tensor.
    (4) B2_sin2 - The x[3],x[3] component the second tensor without pow(sin(o),2) factor.

    #--------------------------#

    h22, r = self.h22, self.r
return A1*B1*(h22**2.) + A2_sin2*B2_sin2/(r**4.)

def hUU_SpinFuncs( self ):

    """ Functions from the projections of the peanut metric onto m and m-bar vectors. 
    Input: 
    No input required. 
    """
    ## Metric data ##
    h22, r = self.h22, self.r
    ZeroSpin = ( 1./self.h22 + 1./(r**2.) )/2.
    NonZeroSpin = ( 1./h22 - 1./(r**2.) )/2.
    ## Spin is not accounted for here ##
    return ZeroSpin, NonZeroSpin

def hUUCdd_theta(self):

    """ Contraction of the CUdd[A,B,C] tensor with hUU[A,B]. 
    Input: 
    No input required. 
    """
    ## Basic Data ##
    r, o = self.r, self.o
    ## Top part of fraction ##
    TopOne = ( 2.*np.cos(o)**(2.*2.)*(np.sin(o)**3.) ) + (r**2.)*( self.dh22dO- 2.*self.drdo )
    TopTwo = 2.*g**2.*np.sin(o)*{ r*np.cos(o) - np.sin(o)*self.drdo }
    ## Bottom part of fraction ##
    Bottom = 2.*((r**3.)*g*r*(np.sin(o)**2.) )**2.
    ## hUU[A,B]CUdd[A,B,1] ##
    return (TopOne+TopTwo)/Bottom

def hUUbCUddeacQddeb_theta(self, Q1, Q2_sino2):

    Input: 
    Q1 - The x[2],x[2] component of Qdd. 
    Q2_sino2 - The x[3],x[3] component of Qdd without the 
    pow(sin(o),3) factor. 
    """
    ## Metric data ##
    h22, r = self.h22, self.r
    ## Derivatives w.r.t O ##
    dh22dO, drdo = self.dh22dO, self.drdo
    return Q1*dh22dO/{ 2.*(h22**2.) } + Q2_sino2*drdo/pow( r, 3. )

class r_of_t_and_theta():

    """ This class numerically creates r(t,o). 
    """
    def __init__( self, M, m, Z ):

        """ Stores needed quantities. 
        """
Input: 
(1,2,3) M, m, Z - The two masses and the distance from r = 0 to m.

## Normalised masses ##
self.M, self.m = float(M/(M + m)), float(m/(M + m))

## Z Distance ##
self.Z = float(Z)

def r_plus(self, rs, thetas):
    """The distance from the mass at Z.
    Input:
    (1) rs - The r value.
    (2) thetas - The theta value.
    """
    return np.sqrt(pow(rs, 2.) + pow(self.Z, 2.) + 2.*self.Z*rs*np.cos(thetas))

def r_minus(self, rs, thetas):
    """The distance from the mass at -Z.
    Input:
    (1) rs - The r value.
    (2) thetas - The theta value.
    """
    return np.sqrt(pow(rs, 2.) + pow(self.Z, 2.) - 2.*self.Z*rs*np.cos(thetas))

def drdo(self, r, o):
    """The r-derivative on each t = constant slice.
    Input:
    (1) rs - The r value.
    (2) thetas - The theta value.
    """
    ## Basic data ##
    M, m, Z = self.M, self.m, self.Z
    rplus, rminus = self.r_plus(r, o), self.r_minus(r, o)

    ## Top part of the fraction ##
    top = r*Z*(M/(rplus**3.)) - m/(rminus**3.)

    ## Bottom part of the fraction ##
    bottom_plus = (M/(rplus**3.))*(r + Z*np.cos(o))
    bottom_minus = (m/(rminus**3.))*(r - Z*np.cos(o))
    bottom = bottom_plus + bottom_minus

    return top*np.sin(o)/bottom

def Initial_r(self, t):
    """The value of r(t) at o = 0.
    Input:
    (1) t - The t value.
    """
    ## Basic data ##
    M, m, Z = self.M, self.m, self.Z

    r = (-b +/- sqrt(pow(b, 2.) - 4*ac))/2

    ac, b = t**2*(M - m) - pow(Z, 2.), t*(M + m)

    return np.array([b + np.sqrt(pow(b, 2.) - 4.*ac))/2.])

def r_of_t(self, t, thetas):
    """Solves for r(t,theta) on a t = constant leaf.
    Input:
    (1) t - The value of t on the slice.
    """
(2) thetas - All theta values.

```python
return odeint(self.drdo, self.Initial_r(t), thetas, rtol = 1e-12, atol = 1e-12).flatten()
```

---

```python
def t_to_free(t, r, o, M, m, Z, y):
    """Creates a Freedom class at a particular point.
    Input:
    (1) t, r, o - Point information.
    (2) M, m - The two masses.
    (3) Z - Distance from the origin to smallest mass.
    (4) y - ydd time-derivative coefficient.
    """
    t_stuff = PotentialFunctions(t, r, o, M, m, Z)
    Pointdtdr, Pointdtdo_sino = t_stuff.dtdr(), t_stuff.dtdo_sino()
    Pointtdto = Pointdtdo_sino*np.sin(o)
    Pointd2tdr2 = t_stuff.d2tdr2(Pointdtdr)
    Pointd2tdodr_sino = t_stuff.d2tdodr_sino(Pointdtdo_sino, Pointdtdr)
    Pointd2tdo2 = t_stuff.dt2do2(Pointtdto)
    Pointd3tdr3 = t_stuff.d3tdr3(Pointdtdr, Pointd2tdr2)
    Pointd3tdodr2 = t_stuff.d3tdodr2(Pointdtdr, Pointd2tdr2)
    Pointd3tdo2 = t_stuff.dt2do2(Pointdtdo, Pointd2tdodr)
    Pointd3tdrdo2 = t_stuff.d3tdrdo2(Pointdtdr, Pointd3tdadr2)
    Fields = FreeFunctions(t, r, o, Pointdtdr, Pointdtdo, Pointd2tdr, Pointd2tdodr, y)
    return Fields

def OnePoint(t, r, o, M, m, Z, y):
    """Evaluates all needed free data.
    Input:
    (1,2,3) t, r, o - Point information.
    (4,5) M, m - The two masses.
    (6) Z - Distance from the origin to smallest mass.
    (7) y - ydd time-derivative coefficient.
    """
    Fields = t_to_free(t, r, o, M, m, Z, y)
    # Generates Extrinsic curvature quantites (Kdd) #
```

---

XXII
def Evaluate(t, rValues, ThetaValues, Function, M, m, Z, N, y, size=11):
    """
    Evaluates a function at all points.
    Input:
    (1) t
    (2) rValues - The set of all r points.
    (3) ThetaValues - The set of all theta points.
    (4) Function - This is the function to be evaluated.
    (5,6) M, m - The masses.
    (7) N - Number of theta points.
    (8) size = 11 - Number of outputs of the Function.
    """
    # This loop runs through all values
    Output = np.array([[]])
    i = 0
    while i < N:
        Output = np.append(Output, Function(t, rValues[i], ThetaValues[i], M, m, Z, y))
        i += 1
    return (Output.reshape(N,size)).transpose()

#----------------------#
#- Solution functions -#
#----------------------#

def OneArrayCut(array):
    """
    Cuts a (n,) array into four equal pieces.
    Input:
    (1) array - The array to be split.
    """
    # Calculates splitting points
    Arlen = len(array)
    One, Two, Three = int(Arlen/4), int(Arlen/2), int(3*Arlen/4)
    # Splits the array
    XXIII
first, second = array[:One], array[:Two][One:]
third, fourth = array[:Three][Two:], array[Three:];
return (first, second, third, fourth)
def SubCut(array):
    """Cuts a (m,n) array into four equal pieces.
    Input:
    ""
    """
    One, Two, Three, four = [], [], [], [];
    for i in range(0, len(array), 1):
        alpha, beta, gamma, delta = OneArrayCut(array[i])
        One.append(alpha)
        Two.append(beta)
        Three.append(gamma)
        four.append(delta)
    return (np.array(One), np.array(Two), np.array(Three), np.array(four))

#--------------------------#
#- Initial data functions -#
#--------------------------#

def Initial_q_Point(t, r, o, M, m, Z, y):
    """Finds the value of q at one point.
    Input:
    ""
    Fields = t_to_free(t, r, o, M, m, Z, y)
    return Fields.qddhUU()

def Initial_p_Point(t, r, o, M, m, Z, y):
    """Finds the value of p at one point.
    Input:
    ""
    Fields = t_to_free(t, r, o, M, m, Z, y)
    return Fields.hUdKddNU(Fields.Kappa()) / np.sqrt(2.)

def Initial_A_Point(t, r, o, M, m, Z, y):
    """Finds the value of A at one point.
    Input:
    ""
    Fields = t_to_free(t, r, o, M, m, Z, y)
    return Fields.MinkNorm * np.sqrt(1. - Fields.v)

def InitialValues(t, N, M, m, Z, ThetaValues, rClass, y):
    """Creates all initial values.
    Input:
    ""
    (1) t - Initial t value.
    (2) N - Number of theta points.
    (3) M, m - The two masses.
    (4) Z - Distance from the origin to the smallest mass.
    (5) ThetaValues - Set of all theta points.
    (6) rClass - The pre-made numerical class r_of_t_and_theta.
    ""
    # Solves for initial r-value ##
    Rr = rClass.r_of_t_and_theta(t, ThetaValues)
    # Initial values ##
    q_start = Evaluate(t, Rr, ThetaValues, Initial_q_Point, M, m, Z, N, y, size
```python
= 1 )

p_start = Evaluate( t, Rr, ThetaValues, Initial_p_Point, M, m, Z, N, y, size = 1 )
A_Start = Evaluate( t, Rr, ThetaValues, Initial_A_Point, M, m, Z, N, y, size = 1 )

## Combines all initial values ##
p_Out = np.append( p_start, p_start ).flatten()
pq_Out = np.append( p_Out, q_start ).flatten()
pqA_Out = np.append( pq_Out, A_Start ).flatten()

return pqA_Out

####################################################################
#-------------------------- Main Code -----------------------------#
####################################################################

def main( M, m, Z, Ntheta, ts, Tolerance, Dir, Suffix, Cn ):
    ""
    Defines, solves, and saves the PDE system.
    Input:
    (1) (M,m,Z) - The two masses and the distance from the origin.
    (2) Ntheta - Number of points in the theta discretization.
    (3) ts - All points in the t-discretization.
    (4) Tolerance - Absolute error tolerance in the evolution.
    (5) Dir - Save directory.
    (6) Suffix - Save suffix.
    ""

    #------------------------#
    #- Sets up needed stuff -#
    #------------------------#

    ## initiallising the axial_symmetric_transform (for precomputation the
    wigner matrices) ##
sf = python_module.axial_symmetric_transform( Ntheta, 4 )

    ## This is the numerical theta-discretization ##
    Theta = sf.create_mesh( Ntheta )

    ## Creates r as a numerical function of t and theta ##
rClass = r_of_t_and_theta( M, m, Z )

    #-----------------#
    # Defines the ODE #
    #-----------------#

def dydr( ys, s ):
    ""
    This is the derivative function.
    Input:
    (1) ys - y values.
    (2) s - t value.
    ""

    ## Unpacks the initial data ##
p1_array, p2_array, q_array, A_array = OneArrayCut( ys )
p1, p2 = sf.function_in_s2( p1_array, -1 ), sf.function_in_s2( p2_array, 1)
q, A = sf.function_in_s2( q_array, 0 ), sf.function_in_s2( A_array, 0 )

    ## Solves for r(t,theta) ##
    Rr = rClass.r_of_t_and_theta( s, Theta )

    # Gets the free data ##
    AllData = Evaluate( s, Rr, Theta, OnePoint, M, m, Z, Ntheta, Cn )

    ## F, ESource, kstar, & kappa ##
    F, ESource, kstr, kpp = AllData[0], AllData[1], AllData[2], AllData[3]
```
Kpp_S2 = sf.function_in_s2(kpp, 0)  # Puts kappa onto 2-Sphere

## Puts hUUCUdd projections onto the 2-sphere ##
Hm_plus = sf.function_in_s2(AllData[5]/np.sqrt(2.), 1)
Hm_minus = sf.function_in_s2(AllData[5]/np.sqrt(2.), -1)

## Peanut metric functions ##
f_pp, f_mm = sf.function_in_s2(AllData[7], 2), sf.function_in_s2(AllData[7], -2)
f_0 = sf.function_in_s2(AllData[6], 0);

## Gets Q1 and Q2 ##
Q1, Q2 = AllData[8], AllData[9]

## Finds hUU[a,b]CQd[a,c] projections ##
cmQ_plus = sf.function_in_s2( Q1*AllData[5]/np.sqrt(2.), 1)
cmQ_minus = sf.function_in_s2( Q1*AllData[5]/np.sqrt(2.), -1)

cQ_plus = sf.function_in_s2( AllData[10]/np.sqrt(2.), 1)
cQ_minus = sf.function_in_s2( AllData[10]/np.sqrt(2.), -1)

## projections of Qdd onto m and m-bar ##
Q_Zero = sf.function_in_s2( Q1*Q2)/2., 0)
Q_Plus2 = sf.function_in_s2( Q1-Q2)/2., 2)
Q_Minus2 = sf.function_in_s2( Q1-Q2)/2., -2)

## Evolution for q ##
dq_p_PartOne = ( Hm_minus*p2 + Hm_plus*p1 ).map
dq_p_PartTwo = ( f_mm*sf.eth0(p2)+f_pp*sf.eth0(p1)+f_0*(sf.eth0(p1)+sf.eth0(p0) ) ).map/numpy.sqrt(2).
dq_p_part = dq_p_PartOne + dq_p_PartTwo
dq = (1./2.)*(q.map)*kstr*(A.map)*dq_p_part -(kpp*kstr) + AllData[4]

## Evolution for p1 ##
dpl_Q_PartTwo = A*[f_mm*sf.eth0( Q_zero )]+f_0*sf.eth0( Q_Minus2 )+f_0*sf.eth0( Q_Zero )]+f_pp*sf.eth0( Q_Minus2 )
dpl_Q_part = (dpl_Q_PartTwo).map/numpy.sqrt(2.) - dpl_Q_PartOne
dpl_QPart = (1./[numpy.sqrt(2.)])*( A*sf.eth0(q)).map/2.
dpl_kappa_part = (1./numpy.sqrt(2.))*{( A*sf.eth0( sf.function_in_s2( kpp, 0 ) ) )}.map
dpl_1 = dpl_kappa_part + (kstr*p1).map + dpl_q_part - dpl_Q_part

## Evolution for p2 ##
dpl2_Q_PartOne = ( A*cmQ_plus ).map + ( A*Q_plus ).map
dpl2_Q_PartTwo = A*[f_mm*sf.eth0( Q_Plus2 )]+f_0*sf.eth0( Q_Zero )]+f_0*sf.eth0( Q_Zero )]+f_pp*sf.eth0( Q_Minus2 )
dpl2_Q_part = (dpl2_Q_PartTwo).map/numpy.sqrt(2.) - dpl2_Q_PartOne
dpl2_QPart = (1./[numpy.sqrt(2.)])*( A*sf.eth0(q)).map/2.
dpl2_kappa_part = (1./numpy.sqrt(2.))*{( A*sf.eth0( sf.function_in_s2( kpp, 0 ) ) )}.map
dpl2_2 = dpl2_kappa_part + (kstr*p2).map + dpl2_q_part - dpl2_Q_part

## Evolution for A ##
d2A_PartOne = (Hm_minus*sf.eth0(A)+Hm_plus*sf.eth0(A)).map/numpy.sqrt(2.)
d2A_PartTwo = (f_mm*sf.eth0(sf.eth0(A))+f_pp*sf.eth0(sf.eth0(A))).map/2.+(f_0*(sf.eth0(sf.eth0(A))).map)
dA_2ndOrderTerm = 2.*(d2A_PartTwo - d2A_PartOne)
dA_Eq = (1./2.)*(A**2.).map + 2.*(Kpp_S2**2).map

dA_Eq = ( f_mm*p2*p1 + f_pp*p1*p1 + f_0*( p1*p2+p2*p1 ) )

ETotal = dA_Eq - 2.*dA_Eq.map + ESource

dA = numpy.divide(A.map+F*(A**2.).map)*ETotal - (A**2.).map*dA_2ndOrderTerm
,(C*kstr)

## Final output ##
output = np.array{ [dpl_1, dpl2_2, da ]}.flatten()
Output = output.real

XXVI
return Output

#----------------------#
# Solves and saves PDE #
#----------------------#

## Solves the PDE system ##
Start = InitialValues{ ts[0], Ntheta, M, m, Z, Theta, rClass, Cn }
Solutions = odeint( dydr, Start, ts, rtol = Tolerance, atol = Tolerance )
p1Sol, p2Sol, qSol, ASol = SubCut{Solutions}

## This saves the final output ##
os.chdir('%s' %Dir)
np.savetxt("A_%s.npy" %Suffix,ASol); np.savetxt("q_%s.npy" %Suffix,qSol);
np.savetxt("p1_%s.npy" %Suffix,p1Sol); np.savetxt("p2_%s.npy" %Suffix,p2Sol)
os.chdir("..")

return p1Sol, p2Sol, qSol, ASol
**References**


XXVIII


XXIX


XXX