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REFLECTIONS ON ICE:
Scattering of Flexural Gravity
Waves by Irregularities in Arctic
and Antarctic Ice Sheets

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Abstract

This thesis studies the scattering properties of different types of imperfections in large Arctic and Antarctic ice sheets. Such irregularities include cracks, pressure ridges and both open and refrozen leads. The scattering by a transition region between sea ice and a very thick ice shelf, for example as is found in the Ross Sea in Antarctica, is also treated.

Methods of solution are based on applications of Green's theorem to the appropriate situation, which leads to either a single integral equation or a pair of coupled integral equations to be solved at the boundary between the ice and the sea water. Those equations over a finite interval are solved using numerical quadrature, while those over semi-infinite ranges are solved using the Wiener-Hopf method. Results calculated using different techniques are able to be checked against each other, giving us great confidence in their accuracy. In particular, the scattering by three ice sheets of different thicknesses is confirmed analytically by mode-matching coupled with the residue calculus technique.

The scattering by the single irregularities is investigated partly for its own sake, and partly with the aim of using it to treat the scattering when large numbers of features are included in a single ice sheet. The principal objective of doing this is to observe the change in the general amounts of reflection and transmission as the background ice thickness is changed. There is enough variation in our results for us to conclude that there is definite potential for using the change in an incident wave spectrum after passing through a given ice field to estimate the background ice thickness.
Chapter 1

Introduction

This thesis is concerned primarily with the determination of the scattering of ice-coupled waves by imperfections in large expanses of sea ice, with a possible future application being the development of a technique that is able to sense ice thickness remotely. Such ice sheets are found mainly in the Arctic Ocean but they can also be found close to the Antarctic continent, although the sheets there are of a more transient nature as they are regularly broken up by incoming waves from the surrounding Southern Ocean. They may also be dispersed by strong katabatic winds from the continent.

The sea ice of both the Arctic and the Antarctic plays a very important role in the earth’s climate. Section 1.1 below presents a general overview of this. As well as describing the specific regions we intend to model, that section also describes how work involving wave interactions with sea ice can be generalized to provide results for certain situations that occur in marine engineering.

The following sections describe and justify our choice of model for polar sea ice (Section 1.2.1), review and discuss mathematical techniques used previously and currently in use in the wave scattering literature (Section 1.2.2), and provide the general outline that this thesis will follow (Section 1.3).

1.1 Arctic and Antarctic Sea Ice

This section begins by giving an overview of the important role that polar sea ice plays in the earth’s environment (Section 1.1.1). This includes the reflection of sunlight and
the production of bottom water which helps to drive global currents. Section 1.1.2 then proceeds to describe the specific regions we intend to model, while Section 1.1.3 discusses how work involving wave interactions with sea ice can be generalized to provide results for certain situations that occur in marine engineering.

Given the importance of sea ice in the environment, a convenient, remote ice-thickness-sensing technique for monitoring the condition of the polar ice sheets would be a valuable asset in monitoring the overall health of the planet. Techniques currently in use are submarine sonar, most recently coupled with LIDAR—the former is able to detect the under-sea profile of the ice, while the latter uses a laser operated from an aeroplane to detect the ice’s upper profile—and EMS, electromagnetic sensing from a specially-fitted helicopter. If ocean wave scattering could be used to deduce average ice thickness, it might be possible to measure it using suitably placed strain gauges or tilt meters, which would eliminate the need for aeroplanes or submarines to be in operation for extended periods of time.

Recall also that by breaking up sea ice, waves affect the way in which the ice interacts with the rest of the climate system; this process should be included in climate models. Consequently, any understanding that wave-ice research can contribute to such models is extremely valuable.

1.1.1 Environmental Role of Sea Ice

The ice of both the Arctic and Antarctic regions is crucial in the maintenance of the earth’s climate. Arctic ice particularly also plays a role in the lives of many human and animal populations—for example, the Sami, the indigenous people of Lapland, use the sea ice for reindeer herding, while polar bears use it as a hunting ground. Animals such as seals and penguins also use Antarctic sea ice as a base from which to search for fish.

Sea ice, the type of ice that this thesis is most concerned with, covers about 7% of the area of the world’s ocean, providing a solid barrier between the ocean and atmosphere that hinders the free exchange of heat and moisture between the two. It is also involved in helping to drive currents, most notably the massive thermohaline conveyor which links the Atlantic and Indian with the Pacific and Southern Oceans, and, aided
also by the freshwater terrestrial ice shelves of Antarctica and the Greenland ice cap, reflects a large amount of incoming solar radiation.

By reflecting so much solar energy, the ice provides some resistance to increasing global temperatures, and by melting it can absorb still more. However, the melting of the ice reduces the amount of energy that can be reflected, and so the rate of warming increases. In particular, this will allow the amount of heat entering the ocean to increase, contributing to sea level rise due to thermal expansion, a major source of sea level rise. The melting of terrestrial ice also increases sea levels, and decreases the salinity of polar waters.

Sea ice also has a role in driving the world’s currents. Its ice formation in the Antarctic causes the production of dense salty water, which then sinks to become Antarctic bottom water (AABW). Similarly North Atlantic deep water (NADW) is partly produced by sea ice formation in the Greenland Sea. Deep water produced then spreads out across the ocean floor (i.e. away from the poles), rising to the surface because of mixing due to roughness in the sea bed and wind, and returning to the poles in wind-driven currents such as the Gulf Stream. Along with the thermal forcing of warm water from the tropics travelling to cooler waters, cooling and sinking, this is the origin of the massive thermohaline conveyor current, which travels between both poles.

Before the existence of plumes was discovered, it was thought that the above process was due entirely to convection, which is impossible in fresh water. (For convection to occur, one needs density to increase as temperature decreases—this is not the case with fresh water, which has a density maximum at 4°C.) Therefore it was also thought that if the salinity of the polar oceans decreased enough, convection would shut off, with a disastrous effect on currents like the Gulf Stream which bring warm water to North Atlantic countries such as the United Kingdom and the United States of America.

However, deep water can also be formed by smaller scale events such as the refreezing of a newly formed lead. This process produces a plume—a “packet” of dense salty water that sinks to the sea floor. With the extra bottom water produced by these entities, global currents would not be as affected as was first thought by a decrease in polar salinity. Nonetheless, a significant decrease in the formation of sea ice, and thus
bottom water, would inevitably affect global currents to some extent.

1.1.2 Regions to be Modelled

We are interested in modelling the scattering of waves by imperfections in large sea ice sheets, particularly keeping in mind the possibility of remotely estimating ice thickness. In the northern hemisphere, which we are most interested in, such sheets exist mainly in the central Arctic Ocean, due to its sheltered location, although there are also some shore fast ice sheets skirting the Arctic coastline.

The sea ice which forms in places like the Bering Sea or the Greenland Sea does not normally grow into very large sheets due to the presence of high amplitude waves originating in the Pacific and Atlantic Oceans respectively. The ice in both seas is broken up into pack ice, although much of the ice in the Greenland sea is older, thicker ice that has come from the Arctic Ocean. This ice serves to attenuate waves from the Atlantic, and filter out those with shorter periods. As a result, only waves with periods greater than about 6 s penetrate into the Arctic Ocean from the Atlantic (Hunkins, 1962). The size of the Atlantic Ocean also limits the possible lengths of waves originating inside it, with a result that only periods less than about 20 s are possible.

The ice in the Bering Sea also plays a similar role. However, the main barriers to waves from the Pacific Ocean reaching the Arctic Ocean are the Aleutian Islands and the Bering Strait. These only allow very long waves (those with periods greater than about 22 s) to reach the Arctic from the Pacific (the greater fetches of the Pacific Ocean permits much longer waves to develop within it).

As well as this relative lack of short wave energy, another factor which aids in the growth of large ice sheets in the Arctic is the Beaufort Gyre current. This current circulates around the North Pole, trapping the ice in the centre of the ocean. The result is that the same ice is sometimes able to survive for several years, enabling it to grow to thicknesses of up to 10 m, although most of the undeformed sheets are around 1 m-3 m (Wadhams, 1995).

The Arctic ice sheets are criss-crossed with sometimes quite large pressure ridges, cracks and open and refrozen leads. Cracks and leads form as adjacent ice sheets respond differently to forces such as those exerted by winds and currents, causing them
to drift apart. As a lead opens up, the water exposed to the air freezes; this new, thinner ice then may contribute to the sail and keel of a pressure ridge as winds and currents change and the two ice sheets are pushed back together again. Pressure ridges may also be produced by rafting as ice sheets of similar thicknesses collide and buckle, or in shear as adjacent floes drift past one another.

Antarctic sea ice is not afforded as much protection from incoming waves and is thus generally younger and thinner than Arctic sea ice. (Except in very sheltered bays and inlets, and to the east of the Antarctic peninsula, Antarctic sea ice usually ranges from 0.5 m to 1 m in thickness.) In winter it is usually formed from roughly circular plates of pancake ice that freeze together to give larger and larger floes the further they are from the ice edge. Like the ice described above that occurs in the Bering Sea, the region near the margins of the ice cover (called the Marginal Ice Zone) is made up of pack ice which reflects and attenuates incoming waves, with the result that the sea ice closer to the continent usually only experiences low amplitude waves of periods in the range 10–20 s. The floes are thus able to reach extremely large sizes, even to the point of being able to be described as a continuous ice sheet (although vestiges of their pancake ice origins are left in the form of abundant slightly raised circular edges, giving the ice a “stony field” appearance (Lange et al., 1989). Cracks and leads can be found between neighbouring ice sheets.

The Arctic process of cracks/leads forming then contributing to ridges also occur in the Antarctic. However, much fewer ridges are observed in the Antarctic because the ice is both thinner and less constrained. Substantial ridges are only observed in the ice of the Weddell Sea, where they are formed by ice buckling against the Antarctic Peninsula.

1.1.3 Engineering Applications of Ice Research

Although this thesis is primarily concerned with providing solutions for situations involving sea ice, some of its results, as well as similar results obtained by other researchers, can also be applied in marine engineering.

A simple application could be to scattering due to a series of floating breakwaters—this could be solved by the method given by Meylan (1993), which describes the effect of a series of ice floes modelled by floating elastic plates.
Another more topical example is provided by the 1000 m x 121 m x 3 m floating “Mega-Float” platform that was constructed in 2000 in Tokyo Bay, Japan, with the intention of undertaking experiments to determine the suitability of such a structure as an airport (Kashiwagi, 2000; Watanabe et al., 2004). (Such a structure is also referred to as a Very Large Floating Structure, or VLFS.) With property in Tokyo at such a premium, such an airport would avoid using valuable space on land.

Verification experiments performed included actual take-offs and landings, with pilots reporting experiencing no mechanical differences from landing at land airports (Kanzawa et al., 2002). Landing distances and other factors in safe landing were also found to be satisfactory, as were other considerations such as under-sea aircraft noise, which was found to be at a level that would not be detrimental to the Mega-Float’s marine environment.

Given the apparent success of this first trial, the design of a 5 km long Mega-Float has been completed, including trial landings with pilots using flight simulators. In order to do this, the hydroelastic response of the airport to waves must first be calculated. By approximating the Mega-Float by a thin elastic plate, work by Meylan and Squire (1994), which models a single ice floe, or alternatively by Tkacheva (2002) could be used to estimate this response. Such models would approximate the Mega-Float as having the side perpendicular to the incident wave of infinite length. Such a solution would be most accurate closer to its centre. Three-dimensional solutions also exist (Namba and Ohkusu, 1999, who give a shallow water solution; Wang and Meylan, 2003; Peter and Meylan, 2004).

Work in this thesis could be used as well—there could easily be the possibility of a hinge joint or a similar type of joint in such a large structure, and strains near such a joint could be estimated by varying the working done in Chapter 5 (also solved in different situations by Squire and Dixon, 2000; Evans and Porter, 2003b).

Of course, when dealing with specific structures, one would need to refine our approaches somewhat. Inoue et al. (2002) use a thin plate approximation to first obtain the water pressure and instantaneous accelerations at the surface, which are then incorporated into a finite element structural analysis. Clearly techniques that
provide the hydrodynamic results quickly would be of great use in such a situation.

1.2 Mathematical Model and Existing Solution Techniques

In Section 1.2.1, this section begins by presenting and justifying the mathematical model that is generally used for ice sheets, namely the thin elastic plate model. There is now considerable evidence that this model is an accurate one.

In Section 1.2.2, we then describe the various methods that have been and are being used in the solution of scattering problems involving thin plates floating on fluids. The geometries of the plates have become successively harder, often with several different approaches being attempted for each geometry before the most efficient method is settled upon.

Before proceeding to do this, however, it is worth defining some terms. The first is dispersion. Its name follows from the fact that waves of different periods travel at different speeds. The dispersion relation is the relationship that quantifies this, and forces the wavenumber, \( a_0 \), to satisfy an equation that depends on the wave’s radial frequency, \( \omega \), and the properties of the ice (such as its Young’s modulus, thickness and density). In general, \( a_0 \) must be found numerically—an explicit formula in terms of the other quantities rarely exists. As a consequence of the dispersion relation, the phase velocity of the wave \( (\omega/a_0) \) and also the wavelength \( (2\pi/a_0) \) change with period \( (\tau = 2\pi/\omega) \) and ice properties. In particular, for constant ice properties the wavelength increases with period, and for constant period it increases with ice thickness.

The last term for the moment, which appears in the title of this thesis, is flexural gravity wave. Gravity waves are waves on a fluid surface that are driven by gravity and a restoring buoyancy force. Flexural gravity waves are waves where this interaction is modified by the inertia and the rigidity of an elastic material floating on the surface.
1.2.1 An Ice Sheet as a Thin Elastic Plate

An extensive amount of literature exists to support the modelling of an ice-sheet as a linear thin elastic plate, for which the governing equations are presented and discussed in Appendix A (also cf. Mindlin, 1951; Fung, 1965). Greenhill (1887) was the first to propose modelling a floating ice sheet by a thin elastic beam on a fluid foundation, suggesting a dispersion relation based on the Euler-Bernoulli beam theory. He also wrote a further paper, humourously entitled “Skating on thin ice” (Greenhill, 1916), in which he elaborated further on the dispersion of waves in ice, as well as touching on other related problems. (Another noteworthy comment that was made in that paper is that ice was the first material for which an experimental value of Young’s modulus was obtained.)

This elastic model for ice has since been corroborated experimentally by numerous researchers, with slight improvements to Greenhill’s theory being made with the passage of time. For example, during a series of seismic experiments carried out by Ewing et al. (1934), the generation of flexural waves in thin lake/canal ice was observed in addition to the longitudinal and transverse waves being investigated at the time. Extending the theoretical component of Greenhill’s work to include the compressibility of the water beneath the ice sheet and to model the ice as a thin plate rather than as a beam (i.e. as a three-dimensional structure rather than as a two-dimensional one), the measured group velocities of the flexural waves corresponded quite well to those predicted by their theory (Ewing, M. and Crary, A.P., 1934).

A more extensive set of seismic experiments on lake ice to create ice-coupled waves artificially was done by Press et al. (1951), who also observed air-coupled flexural waves travelling at the speed of sound (discussed in further detail by Press and Ewing, 1951b). Again, the measured flexural wave dispersion agreed quite well with the thin plate dispersion theory.

Press and Ewing (1951a) developed the elastic model of ice further by allowing for the horizontal and vertical displacements to vary arbitrarily in the vertical direction, although they neglect the effect of gravity. In the thin plate model, the horizontal displacements are neglected, while the vertical displacements are taken to be linear in the vertical coordinate. The latter approximations are valid when the thickness of the ice is small in comparison to the wavelength. Indeed, in the large wavelength limit, the
dispersion relation of Press and Ewing (1951a) explicitly converges to the thin plate relation (allowing for the fact that the gravity term also becomes negligible for large wavelengths).

In the short wave limit, where gravity does have more of an effect, their dispersion relation reproduces the relation for an infinitely thick ice sheet with unattenuated Rayleigh waves travelling along the ice-air surface and with attenuated Rayleigh waves travelling along the ice-water interface. The disadvantage of the theory of Press and Ewing (1951a) is that they were only able to solve their dispersion relation in the above long and short wave limits.

Oliver et al. (1954) did a similar seismic investigation to Press et al. (1951) in the Beaufort Sea and the Arctic Ocean amongst Arctic sea ice, and applied the theory of Press and Ewing (1951a) in their data analysis. In particular, studies of flexural wave velocities in shore-fast ice off Barter Island showed that for smaller wavelengths dispersion results were in very good agreement with theoretical predictions. Experimental results for longer waves were still quite good, although not as good as for shorter waves. The deviation was attributed to sea-bottom effects due to the relative shallowness of the water (3 m). Oliver et al. (1954) also compared measured ice thicknesses in different areas with those deduced from the group velocities of flexural waves or from the frequency at which the air-coupled waves occurred. Probably due to ice inhomogeneity, the flexural wave dispersion was not as reliable a guide to thickness as the frequency of the air-coupled wave. In general, however, the air-coupled frequency still significantly underpredicted the ice thickness (by around 11% to 15%).

Clearly, in seismological studies, the effect of water compression will become more important, especially if the charge is exploded in the liquid. However, in less destructive studies of ice-ocean interaction, such as the one carried out in Notre Dame Bay, Newfoundland by Squire and Allan (1980) involving strain gauges, neglecting that effect is less consequential. Moreover, in the analysis of Press et al. (1951), carried out before the work of Press and Ewing (1951a), the success of the thin plate model in describing the observed dispersion seems to suggest that the thin plate assumptions about the horizontal and vertical motion are not too significant. In any event, Squire and Allan (1980), were able to verify that the thin plate dispersion relation predicted wavelengths that agreed with measured ones to within experimental error. In addi-
tion, by considering the problem of a semi-infinite floating ice sheet, they were able to predict the variation of strain in the ice with distance from the ice edge, given the spectrum of the incident waves. Although no such data were available, they were able to estimate the incident wave spectrum by calculating the ratio of the measured strains to the theoretical strains given a uniform spectrum of unit amplitude. If the calculated incident wave spectrum was independent of the position of the strain-meter producing the experimental results, it would serve as a good check of the theory, although it would not confirm it absolutely. Results for three different meters were qualitatively quite similar for three different locations, and differences could possibly have been able to be attributed to the theoretical strains being based on an incomplete solution (Wadhams, 1973).

Mindlin (1951) also develops a thick plate model which allows for rotational and shear effects inside the plate. For a semi-infinite ice sheet, theoretical predictions (e.g. Fox and Squire, 1990; Balmforth and Craster, 1999) show that those effects make little difference, although they might be more important near a structure of smaller width such as a pressure ridge.

### 1.2.2 Existing Solution Techniques

In summarising the development of mathematical solution techniques, we will concentrate on two particular problems that have been the focus of a lot of previous sea ice research, before showing how these methods could be applied and extended to the treatment of irregularities in sea ice sheets.

**Shore-fast Sea Ice**

One particular problem that illustrates the progression of solution techniques for floating plate problems is that of the scattering of incoming ocean waves by shore-fast sea ice, modelled by a semi-infinite sheet of ice. The first attempt at a solution was by Weitz and Keller (1950) who solved a simplified problem by using the Wiener-Hopf technique to obtain an extremely simple expression for the modulus of the reflection coefficient, \( R \). This formula is given by

\[
|R| = \frac{|\alpha_0 - k_0|}{\alpha_0 + k_0},
\]

where \( \alpha_0 \) and \( k_0 \) are the wavenumbers of the propagating waves in the water and in the ice respectively (both are real), and was pointed out by both Shapiro and Simpson.
(1953) and Keller and Weitz (1953). (Note that although Wadhams, 1986, showed that there is an error in the paper by Shapiro and Simpson, their formula for $|R|$ is still valid as that mistake only pertained to an energy conservation theorem and an expression for the modulus of the transmission coefficient.) Their solution was obtained by ignoring the elastic properties of the ice, and simply modelling the ice as a collection of independent point masses. This model is referred to as the “mass loading model”, which, although somewhat inaccurate for ice sheets (for example it predicts that wavelengths should shorten on moving from water into ice, Squire et al., 1995), has proved reasonably successful for wave propagation through frazil or pancake ice (Wadhams and Holt, 1991). It effectively reduces the order of the thin plate boundary condition from five to one, eliminating the need to apply additional conditions at the ice-edge.

Evans and Davies (1968) subsequently used the Wiener-Hopf method (Noble, 1958; Roos, 1969) to solve the full problem. At the time the infinite products that formed a large part of the solution proved to be too difficult to compute and so they were not able to present any results, except for certain limiting cases (such as the shallow water limit). Incomplete mode-matching approaches were then taken (e.g. Hendrickson and Webb, 1963; Wadhams, 1973; Squire, 1978; Squire, 1984; Wadhams, 1986), before Fox and Squire (1990) computed the solution for the full set of eigenfunctions using a conjugate gradient technique. Fox and Squire (1994) used the same method to complete further studies on this problem, investigating the strain in the ice, and also the effect of shore fast ice on an incoming directional wave spectrum of specified structure.

Gol’dshein, R.V. and Marchenko, A.V. (1989) presented a later Wiener-Hopf solution for infinite depth, but again only certain limiting situations were discussed. However, in the late 1990’s and early 21st century improvements in computing power enabled other authors to compute results using the Wiener-Hopf solution of Evans and Davies (1968). Balmforth and Craster (1999) turned the required infinite products into integrals which were evaluated by quadrature, while Chung and Fox (2002a) showed that the products themselves could be evaluated directly with relatively little effort. Ironically Tkacheva (2001a) finally showed that if the inertia term in the thin plate equation was neglected for normally incident waves (as it can for most wavelengths), then $|R|$ could be calculated by simply using the correct value for $k_0$, the wavenumber in the ice, in the formula (1.1) given by Keller and Weitz (1953).
Other authors have returned to the mode-matching approach of Fox and Squire (1990). Sahoo et al. (2001) defined an inner product enabling the solution to be found by using $N$ eigenfunctions and inverting an $N \times N$ matrix, while Linton and Chung (2003) effectively showed that the equations Sahoo et al. (2001) had set up could be solved analytically using residue calculus. In the process they also demonstrated its equivalence to the Wiener-Hopf solution, and confirmed Tkacheva’s (2001) formula. Earlier Chakrabarti (2000) had used an infinite depth mode-matching scheme to set up a singular integral equation, the equivalent problem to a residue calculus problem when the eigenvalue spectrum is continuous. This singular integral equation was solved by transforming it into a Hilbert problem (Roos, 1969), a generalization of the Wiener-Hopf problem (although in practice both are solved in exactly the same way).

An Ice Strip of Finite Width in Open Water

A second problem that has generated a lot of the mathematics used herein is that of the scattering of water waves by an ice strip of finite width surrounded by open water. By restricting the incoming waves to normal incidence, it was initially intended as a two-dimensional model for a single ice floe such as might occur in the MIZ (Meylan, 1993; Meylan and Squire, 1994). Although modelling of ice floes has since moved away from this approach towards a more three-dimensional one (e.g. Meylan et al., 1997; Peter and Meylan, 2004), the results and techniques are still quite applicable to situations where the length of the floating object is large in comparison to other characteristic lengths of the problem. Examples include the Mega-Float discussed in Section 1.1.3 (Kanzawa et al., 2002).

The solution in the shallow water limit is relatively straightforward (Stoker, 1957) and simply involves the inversion of an $8 \times 8$ matrix. In his Ph.D. thesis, Meylan (1993) solved the problem for both finite and infinite depth by using Green’s theorem and the two-dimensional open water Green’s function to set up an integral equation over the floe (the infinite depth solution was also published in Meylan and Squire, 1994). It is also possible to extend the mode-matching technique of Sahoo et al. (2001) to treat this problem, and the author is aware of several researchers who have done this, although such a solution is still unpublished. This technique could be improved by using the residue calculus method in the same way that it is used in Appendix F of this thesis, which treats the problem of the scattering by three adjacent ice sheets of different thicknesses. With minor adjustments, either or both of the outer ice sheets may be
replaced with open water (cf. Section 8.1.3; published papers on the residue calculus method include those by Linton and Chung, 2003 and Chung and Linton, 2005).

Tkacheva (2002) solved a pair of coupled Wiener-Hopf equations to produce another approach to this problem, using a similar method to the one that is used in Chapter 8 of this thesis, and Meylan (2002) solved the time dependent problem by using a spectral approach.

Irregularities in a Continuous Ice Sheet

The first type of irregularity in a continuous ice sheet considered was an abrupt change in ice properties, i.e. the scattering by a joint/edge separating two semi-infinite ice sheets, each with different properties. Barrett and Squire (1996) solved this problem numerically by extending the method of Fox and Squire (1990) used in dealing with the shore fast ice problem, while Chung and Fox (2002b) solved it by extending their own Wiener-Hopf solution (Chung and Fox, 2002a).

A special case of the above problem is that of the scattering by a crack, i.e. by a free edge separating two identical semi-infinite ice sheets. Squire and Dixon (2000) extended the work of Meylan (1993) to derive the infinite depth Green's function for an ice sheet; this was then used to obtain an analytical expression for the reflection and transmission coefficients for a normally incident wave upon a crack (in ice floating on water of infinite depth). Williams and Squire (2002) later used the same method to derive similar expressions for the same coefficients when the incident wave arrived at an oblique angle. The finite depth problem was solved by Evans and Porter (2003b) who used two different methods to calculate the finite depth solution—the first involved using a mode-matching technique and the other used the finite depth Green's function for an ice sheet.

Chung and Linton (2005) extended this mode-matching approach to treat a crack of finite width (also known as an open lead), using the residue calculus technique to improve their solution. Williams and Squire (2004b) also presented results using a mode-matching technique for an open lead (although without detailing their method). Williams and Squire could also have used the variable thickness method outlined in the same paper, and first presented in their 2004a paper, by simply setting the thickness of the variable region to be constantly zero.
Marchenko (1997) also solved the problem of cracks in ice for infinite depth, by taking a Fourier transform directly (as opposed to finding the transform of the Green's function), allowing the ice to have arbitrarily many (parallel) cracks. The scattering by many cracks was also treated by Dixon and Squire (2001b), who considered the effect of randomising the separation of cracks, while Evans and Porter (2003a) applied Floquet's theorem to investigate the scattering by an infinite series of identically spaced parallel cracks.

An additional problem treated by Marchenko (1997) was the problem of scattering by a pressure ridge, in which he modelled the ridge as an elastic beam joined to the surrounding ice cover by an elastic hinge joint. Approaching the problem in this manner enables the solution to be found in much the same way as the crack solution was found. Williams and Squire (2004a) treated the ridge as a thin elastic plate of finite width, but with a varying thickness, and used Green's theorem to reduce the problem to an integral equation over the ridge's width (also presented in Chapter 6 of this thesis). Dixon and Squire (2001b) had previously solved this problem in the special case where the thickness of the ridge was constant by replacing the open water Green's function in the method of Meylan (1993) (also reported in Meylan and Squire, 1994) with the thin plate Green's function. Both Williams and Squire (2004a) and Dixon and Squire (2001b) neglected the submergence of the ridge/berg, although this effect could be allowed for in the latter paper in the same way that Meylan (1993) solved the problem of a submerged floe.

A different approach to the ridge problem was taken by Porter and Porter (2004), who used a variational approach and a mild slope approximation to reduce it to a sixth order ordinary differential equation. Their method allowed a pressure ridge of arbitrary thickness and keel depth to be considered by assuming that a negligible amount of evanescent wave action was produced as the incident wave was scattered. In addition, an arbitrary sea floor topography could be treated simultaneously. (Evanescent waves are waves with non-real wavenumbers whose amplitudes decay exponentially as the distance from their point of generation increases. Their wavenumbers are generally pure imaginary, although two complex wavenumbers are possible in ice; the waves corresponding to those wavenumbers are sometimes distinguished from the others by being called damped traveling waves, or simply complex modes. Traveling or propa-
gating waves have purely real wavenumbers.)

A surprising implication of the work of Porter and Porter (2004) is that for a large enough water depth, removing the submergence would have negligible impact on the scattering by a ridge as long as its overall rigidity is preserved. This means that, assuming that the gradient of the ridge's rigidity is not too large, the integral equation method of Williams and Squire (2004a) will produce accurate results for such depths, as long as the total ice thickness is increased to allow for the keel.

Williams and Squire (2004a) presented results for two ridges as well, comparing the exact solutions to a wide-spacing approximation which becomes more accurate as the separation between the ridges is increased. This approximation assumes that the evanescent waves generated at each ridge have decayed to a negligible amplitude by the time they have reached the next. For this reason it is called the No Evanescent Waves (NEW) approximation, although it is actually equivalent to the wide spacing approximation used, for example, by Newman (1965) or Evans (1990). Equivalent results were presented by Meylan (1993) for two floes.

The NEW approximation was extended by Williams and Squire (2004a) to allow for arbitrarily many ridges. In addition, when the effect of the ridge separations was averaged out, the result was extremely well approximated by the so-called serial approximation. The serial approximation is obtained by simply multiplying the transmission coefficients for each ridge to obtain an overall transmission coefficient. It was introduced to allow the general scattering properties of a given ice field to be studied without the complicated interference produced in the presence of many ridges.

Williams and Squire (2004b) extended the work of Williams and Squire (2004a) to allow the ice field to contain leads and cracks, and the serial approximation also proved to describe the average scattering behaviour of fields containing those features accurately.

This result was taken by Williams (2004) to suggest that the potential for using flexural waves in remote ice-thickness-sensing could initially be considered by simply using the serial approximation. In that paper, reflection curves for a given set of ridges were compared for different ice thicknesses and were found to be fairly different over a
relatively wide range of periods (from about 2 s to 11 s). Thus there seemed to be some hope that scattering results for different thicknesses would be sufficiently different to use those results in determining the ice thickness.

1.3 Outline of Thesis

In Chapter 2 we begin by presenting the governing equations that we wish our velocity potential to satisfy. These equations are derived and discussed in Appendix A. Then, using the Green's functions that are discussed in Chapter 3 for both finite and infinite depth, we proceed to set up an integral equation in Chapter 4 in terms of the displacement at the ice-water interface (or at the air-water interface—such as in the problem of an open lead). This integral equation may either be over a finite or an infinite interval.

Of particular interest in Chapter 3 is the presentation of calculation techniques for the infinite depth Green's function for oblique incidence. These techniques include Fast Fourier Transform (FFT), Taylor series expansions, and numerical integration. Each of these methods have particular advantages and disadvantages, depending on the number of points that the Green's function needs to be evaluated at and the distance of the points from the singularity. They are described in more detail in Appendix C. Another appendix that is related to this chapter is Appendix B, which discusses the behaviour of the roots of the shallow water, finite depth and infinite depth dispersion relations. The small period behaviour of the latter two sets of roots has never been presented before and, although it generally only occurs at physically impractical periods, is nevertheless extremely interesting mathematically.

In Chapter 4 the thickness profile of the ice is allowed to vary arbitrarily over a finite strip with the ice sheets of different but constant thicknesses. The method used to include a variable thickness profile was first reported by Williams and Squire (2004a), but the generalization of this approach to include different thicknesses on each side of the strip is new and is yet to be published. This chapter also presents a proof that the potential may always be represented by an eigenfunction expansion.

Chapters 5, 6, 7 and 8 solve the integral equation derived in Chapter 4 in a variety of different situations, and are arranged in increasing order of mathematical difficulty.
Chapters 5 and 6 can both be considered as treating special cases of the problem presented in Chapter 7, both in terms of the actual complexity of the physical problems that they solve, and in the methods of solution that they employ. In physical terms, the problem solved in Chapter 8 is also a special case of the one in Chapter 7, but mathematically the use of the Wiener-Hopf technique in particular is more involved in Chapter 8 than in Chapter 7, in which the Wiener-Hopf technique is first introduced. Consequently, Chapter 8 provides a more analytical solution method than Chapter 7, but to a smaller range of physical problems; although Chapter 7 reports a more numerical method, it can as a result deal with a wider range of problems.

The simplest problem is that of a single crack in a single uniform ice sheet. As mentioned in the previous section, this problem was solved for infinitely deep water and normally incident waves by Squire and Dixon (2000), obliquely incident waves by Williams and Squire (2002), and finitely deep water by Evans and Porter (2003b). However, the solution follows in only one or two steps from the final result in Chapter 4, which, as discussed above, solves a considerably more general problem. Moreover, it was also thought that the simplicity of the results obtained provide a good opportunity to investigate their behaviour with some of the more fundamental wave/ice parameters such as ice thickness, wave period, angle of incidence and water depth. For example, it is shown that the effects of the former two can be approximately combined into a single parameter, the nondimensional period, while in the discussion of the effect of water depth, criteria with regard to the accuracy of the shallow and infinite depth approximations are established.

Chapter 6 describes scattering by a pressure ridge. It is modelled as a thin plate of variable thickness with zero submergence (Williams and Squire, 2004a; Williams and Squire, 2004b; Williams, 2004), and, as in those papers, the problem is solved by setting up an integral equation over the interval in which the ice thickness is varying. This equation is solved numerically.

The results presented in this chapter are similar to the single-ridge results presented in the aforementioned papers. However, those results are adjusted to allow for the increased thickness due to the ridge keel. In light of the recent work of Porter and Porter (2004), changing the thickness in this way should improve the accuracy of the Williams and Squire (2004a) results for large water depths, but without significant
changes to that method. Appendix E also treats the topic of scattering by a pressure ridge, proposing a method by which submergence could be included in full, although no results are able to be presented yet.

Chapter 7 extends the variable thin plate formulation developed in Chapter 6 by allowing the ice thickness to the right of the variable region (which in this chapter we will refer to as a ramp) to differ from the ice thickness to its left. To do this we must solve two coupled integral equations—the first is much the same as the one solved in Chapter 6, but the second is over a semi-infinite range—over the ice-water interface to the right of the ramp. Proceeding in exactly the same way as in Chapter 6, we proceed to solve the first integral equation numerically for the displacement in the ramp, expressing it in terms of the displacement of the ice in the semi-infinite region. The second integral equation is one of the Wiener-Hopf type, and so we are able to write an analytical expression for the displacement in the semi-infinite region in terms of the displacement within the ramp. We can now eliminate the displacement in the right hand region from the two sets of equations, solve for the displacement in the ramp, and then regenerate the displacement to the right.

The method in Chapter 7 is rather novel, as it combines a numerical integral quadrature scheme with the analytical Wiener-Hopf technique. Previously, authors have tended to favour either one approach or the other, and so this result also represents a philosophical breakthrough in that the combined weight of both methods can be brought to bear on a single problem most effectively. The numerical approach is able to deal with the arbitrariness of the ice thickness within the ramp, while the Wiener-Hopf technique is ideal for the semi-infinite interval over which the second integral equation must be solved—such intervals are extremely hard to deal with numerically.

Unfortunately, the method does have the same drawback that our ridge model had—that submergence must be ignored. However, the method of Porter and Porter (2004) also incorporated a change in ice thickness from one side of the variable region to the other, and so we could again use their result that for large water depths the effect of submergence is negligible as long as the correct total thickness is used (and as long as the mild slope assumption is valid).

The physical problem solved in Chapter 7, i.e. a ramp connecting a thinner ice sheet
to a thicker one, has practical applications. Such a thickness profile could be taken to
describe a sea ice/ice shelf transition such as the one observed in the Ross Sea, where
the thickness of the sea ice beyond the ice shelf increases steadily to meet the thicker
ice of the ice shelf itself. Alternatively, by setting the right hand ice thickness equal
to zero, and by letting the incident wave arrive from the right instead of from the left,
the problem could represent a breakwater shielding a VLFS from waves arriving from
the open ocean.

Chapter 8 treats a special case of the problems solved in Chapters 6 and 7. When
the ice thickness over the variable region is constant, a more analytical approach may
be taken. The problem may be written as two coupled Wiener-Hopf integral equations,
producing a system of $M$ linear equations in as many variables, where $M$ is the number
of evanescent waves generated at each end of the middle region that have not decayed
sufficiently by the time they reach the other end to be able to be ignored. Since the
evanescent waves decay exponentially with distance, the order of coupling $M$ between
the two semi-infinite regions decreases as the width of the middle region increases. If
that region is wide enough, $M$ becomes 1, and only the propagating waves generated
need to be considered. Thus only a scalar matrix needs to be inverted, and the problem
reduces to the well-known wide spacing approximation (Evans, 1990), abbreviated by
Williams and Squire (2004a) to the NEW approximation (No Evanescent Waves).

In the above special case, the approach of Chapter 8 complements the numerical
approach of the two earlier chapters well. When the central region is relatively wide,
the method of Chapter 8 becomes extremely efficient, while if $a$, the width of that
central region, is decreased and less evanescent modes can be neglected, less quadra­
ture points are needed to obtain the numerical solution and that method would be
preferable. Given the wave period, angle of incidence and ice thicknesses, the coupled
Wiener-Hopf method can generate results for many different values of $a$ quite rapidly,
while if $a$ is fixed and results are required for when the thickness of the central region is
allowed to take several different values, the quadrature schemes would be more efficient.

Practically speaking again, the situation treated in Chapter 8 is a good model for
a lead separating two sheets of ice. Such features are extremely common in both the
Arctic and Antarctic regions. Different stages of refreezing may be represented, ranging
from a freshly formed lead which is still only open water, to an older lead which has
frozen over, and whose edges may have even refrozen to the two larger ice sheets.

Chapter 9 deals with the scattering by multiple features in a uniform ice sheet. After presenting some preliminary theory, it follows the structure of Williams and Squire (2004a), and starts by presenting results for the scattering at normal incidence by two pressure ridges or leads, mainly investigating the effect of the feature separation on the scattering. The results are similar to results for two cracks presented in Chapter 8, but can be taken a little further in that we can also look at the effect of giving the two features different properties from each other. (All cracks are identical, so this couldn't be done in Chapter 8.)

These results for two features are also compared with the NEW approximation, which is able to produce results for different feature separations much more quickly than can be obtained if an exact solution is sought. (This is also done in Chapter 8.) Being a wide spacing approximation, it becomes more accurate as the separation is increased, the exact and approximate curves generally coalescing when the separation is about three quarters of one wavelength.

Chapter 9 then seeks to average out the effect of separation by using a serial approximation which can produce results even more quickly than the NEW approximation can. Ridge separations are sampled from an observed type of distribution (log-normal in that case) and the median value of $|R|$ (calculated using the NEW method for two identical ridges) is calculated and compared with the serial result. The two curves are almost identical, although the variance of the results about the median is quite large.

This experiment is repeated for a larger number of identical ridges with similar success; then when the ridge sail heights are also sampled from a random distribution; and it works just as well when large numbers of cracks or leads are used, and separations and lead widths are randomly sampled.

Having established that the serial approximation summarizes the general scattering properties of a series of features for normal incidence, we also attempt to use it to predict the effect of randomising the feature orientations, and also combine the results for individual features in an attempt to model an ice field populated with cracks, pressure ridges and leads, in the manner of Williams and Squire (2004b). The scattering seems
to be dominated by the effect of the leads.

This chapter goes a little further than either Williams and Squire (2004a) or Williams and Squire (2004b), however, in that it also presents results for pressure ridges when the flexural effects of a keel are accounted for, or when semi-refrozen leads are also present.

Finally, it investigates the effect of changing the background ice thickness on scattering results when either one or 100 ridges or leads are present. (The serial approximation is used in the latter case.) A similar result for ridges was given by Williams (2004), but since the leads seem to dominate the scattering by an ice field, results for them are also added to complete the chapter. These last results are presented with the intention of showing that the scattering changes enough with the background ice thickness that there is potential for it to be used to determine that thickness.
Chapter 2

Mathematical Formulation of the Problem

In constructing a model for wave propagation in a sea-ice sheet, we are mindful of the considerable literature that suggests that the behaviour of the sheet may be represented as a thin elastic plate at the strains and strain rates involved, as originally proposed by Greenhill (1887, 1916), and discussed in Chapter 1. While some damping of the waves undoubtedly occurs—indeed viscoelastic plates have occasionally been employed—the additional complication of including viscosity is unwarranted here. We are helped considerably by the observation that the wavelength of ice-coupled, flexural-gravity oscillations is always much greater than the ice thickness or the submergence even when the wave period is quite short. Accordingly, we are generally at ease that sea ice can be described by a Euler-Bernoulli thin plate floating with zero draft on the water surface at $z = 0$. Refer to the beginnings of Chapters 6 and 8 for further discussions on the implications of this assumption.

Figure 2.1 shows the physical situation to be modelled, where the axes have been displaced to the right to avoid clutter. A monochromatic sinusoidal wave travelling beneath a uniform ice sheet of thickness $h_0$, approaches a region with variable properties at an angle $\theta$ to normal incidence. There it is partially reflected and partially transmitted into the region beneath a second uniform ice sheet, of thickness $h_2$. The purpose of this thesis is to determine how the relative amounts of reflection and transmission that occur vary with parameters such as $\theta$, $h_0$, $h_2$ and wave period, and also with the properties of the variable region.
Figure 2.1: The general physical situation to be modelled in this thesis: the scattering that occurs as an obliquely incident plane wave travelling beneath one uniform ice sheet arrives at a region of ice with a variable thickness, and is partially reflected and partially transmitted into a second uniform ice sheet. The ice on the left has a constant thickness of $h_0$, the ice on the right has a constant thickness of $h_2$, and the thickness of the central feature is represented by the function $h_1(x)$. The sea water has a finite, constant depth of $H$. The coordinate axes are oriented as shown, but should be displaced to the left so that the $x = 0$ line actually corresponds to the left hand limit of the ramp. $|I|$, which will generally be taken to be 1, is the amplitude of the velocity potential corresponding to the incident wave, while $|R|$ and $|T|$, the moduli of reflection and transmission coefficients, are the amplitudes of the velocity potentials corresponding to the reflected and transmitted waves (respectively).

2.1 Equations and boundary conditions

The water beneath the ice is taken to have constant density and to be of depth $H$. Assuming also that the fluid flow is irrotational, it may be represented by a velocity potential $\Phi$ so that the fluid velocity $\mathbf{v} = \nabla \Phi$. The potential $\Phi$ then satisfies a continuity equation (Laplace's equation), Bernoulli's equation (linearized), requirements that horizontal flow be continuous, a kinematic condition (also linearized), and a no-flow
condition at the sea-floor, as follows:

\[
\nabla_{xyz}^2 \Phi(x, y, z, t) = 0, \quad (2.1a)
\]
\[
\rho_w \Phi_t(x, y, 0, t) + P(x, y, t) - \rho_w g \eta(x, y, t) = 0, \quad (2.1b)
\]
\[
\Phi_x(x^+, y, z, t) - \Phi_x(x^-, y, z, t) = \Phi(x^+, y, z, t) - \Phi(x^-, y, z, t) = 0, \quad (2.1c)
\]
\[
\Phi_x(x, y^+, z, t) - \Phi_x(x, y^-, z, t) = \Phi(x, y^+, z, t) - \Phi(x, y^-, z, t) = 0, \quad (2.1d)
\]
\[
\Phi_z(x, y, 0, t) - \eta_t(x, y, t) = 0, \quad (2.1e)
\]
\[
\Phi_z(x, y, H, t) = 0. \quad (2.1f)
\]

In the above \(\rho_w\) is the density of water, \(P\) is the water pressure at the surface, \(g\) is the acceleration due to gravity, \(H\) is the water depth, and \(\eta\) is the vertical displacement of the free surface. In (2.1c), \(\Phi(x^\pm, y, z, t)\) is shorthand for \(\lim_{x^\pm \to x} \Phi(x, y, z, t)\), and similar notation is used for \(\Phi_x(x^\pm, y, z, t)\); similarly for \(y^\pm\) in (2.1d). These conditions generally only need to be applied at discontinuities in the ice or in its properties.

The pressure is related to the displacement by the Euler-Bernoulli thin plate equation (derived for a spatially varying flexural rigidity in Appendix A) by

\[
m(x) \eta_{tt}(x, y, t) + \mathcal{L}_{\text{plate}}(x, \partial_x, \partial_y) \eta(x, y, t) + P(x, y, t) - P_a = 0, \quad (2.2)
\]

where \(P_a\) is the atmospheric pressure, \(m(x) = [\rho h](x)\) is the mass per unit surface area of the ice, and the plate operator \(\mathcal{L}_{\text{plate}}\) is given by

\[
\mathcal{L}_{\text{plate}}(x, \partial_x, \partial_y) = \nabla_{xy}^2 (D(x) \nabla_{xy}^2) - (1 - \nu) D''(x) \partial_y^2.
\]

\(D(x) = [E h^3/12(1 - \nu^2)](x)\) is the flexural rigidity of the ice; \(E, h, \nu\) and \(\rho\) are respectively the effective (low strain rate) elastic modulus, the thickness, Poisson’s ratio, and the density of the ice. Any of these quantities may be varied but, as the most common source of variation in sea-ice is its thickness, the others will be treated as constants. The values used for the different physical constants are given in Table 2.1 below.

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Young’s modulus of ice</td>
<td>$6 \times 10^9$ Pa</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson’s ratio of ice</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of ice</td>
<td>922.5 kg m$^{-3}$</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>Density of water</td>
<td>1025 kg m$^{-3}$</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration due to gravity</td>
<td>9.81 m s$^{-2}$</td>
</tr>
</tbody>
</table>

Table 2.1: Values for physical constants used in this thesis.

The thickness will be given in a piecewise manner by

$$h(x) = \begin{cases} 
    h_0 & \text{for } x < 0 \\
    h_1(x) & \text{for } 0 < x < a \\
    h_2 & \text{for } a < x, 
\end{cases}$$ (2.3)

where $h_0$ and $h_2$ are constant, and the rigidity $D(x)$ and the mass per unit area $m(x)$ are given analogously. Inside $(0, a)$ the only restriction on possible thickness profiles is that the function $D_1(x)$ must have an integrable second derivative, be piecewise continuous itself, and must have a piecewise continuous first derivative.

Without the imposition of additional conditions, the system of equations (2.1) and (2.2) does not have a unique solution. To reduce the dimension of the problem to two instead of four we assume that our potential and displacement are harmonic in time and the $y$ coordinate. We assume we can describe the $y$ dependence of the problem in this way because it is invariant under translations in that direction. This is made more explicit in equation (2.7) below (cf. p29).

We must also apply some radiation conditions as $x \to \pm \infty$ and some conditions at any edges where $D(x)$ and/or $D'(x)$ are discontinuous. These edge conditions are described below, but we will until Section 4.1 to make the radiation conditions explicit.

### 2.1.1 Edge Conditions

Let $X_1$ be the set of points at which there is a free edge, $X_0$ be the set of points at which there is no free edge but either a discontinuity in $D(x)$ or $D'(x)$, or both, and let $X_c = X_0 \cup X_1$. The members of $X_c$ will loosely be referred to as critical points.
In most situations we will consider \( X_c = \{0, a\} \), although in Chapter 6 we will also present results for when \( X_c \), has up to three other members inside \((0, a)\). In addition, its size doubles in Chapter 9 when the scattering by two features is considered. Both sets of edge conditions used are energy conserving, i.e. no work is done at the edges, so that no energy is lost (or gained) there.

From Appendix A, at points \( x_c \in X_0 \) the displacement \( \eta \) should satisfy the following conditions, which will be referred to as the frozen edge conditions:

\[
\begin{align*}
\eta(x_c^+, y, t) &= \eta(x_c^-, y, t), \\
\eta_x(x_c^+, y, t) &= \eta_x(x_c^-, y, t), \\
\mathcal{M}_x(x_c^+, \partial_x, \partial_y) \eta(x_c^+, y, t) &= \mathcal{M}_x(x_c^-, \partial_x, \partial_y) \eta(x_c^-, y, t), \\
\mathcal{S}_{xy}(x_c^+, \partial_x, \partial_y) \eta(x_c^+, y, t) &= \mathcal{S}_{xy}(x_c^-, \partial_x, \partial_y) \eta(x_c^-, y, t),
\end{align*}
\]

where the operators \( \mathcal{M}_x \) and \( \mathcal{S}_{xy} \) are respectively the bending moment and vertical edge force operators, defined

\[
\begin{align*}
\mathcal{M}_x(x, \partial_x, \partial_y) &= -D(x) \mathcal{L}^-(\partial_x, \partial_y), \\
\mathcal{S}_{xy}(x, \partial_x, \partial_y) &= D(x) \mathcal{L}_x^+(\partial_x, \partial_y) + D'(x) \mathcal{L}_y^-(\partial_x, \partial_y), \\
\mathcal{L}_x^\pm(\partial_x, \partial_y) &= \nabla^2_{xy} \pm (1 - \nu) \partial_y^2.
\end{align*}
\]

The subscripts in the above operators are used only for consistency with Appendix A (although they do have some small significance there).

Referring to Figure A.1, the moments \( \mathcal{M}_x(x_c^\pm, \partial_x, \partial_y) \eta(x_c^\pm, y, t) \) respectively produce clockwise and anticlockwise rotations at the crack edges. Thus (2.4c) implies that these moments cancel each other out and consequently are unable to do any work. Similarly, the vertical edge forces \( \mathcal{S}_{xy}(x_c^\pm, \partial_x, \partial_y) \eta(x_c^\pm, y, t) \) respectively produce downwards and upwards displacements in the edges, so (2.4d) implies that no translational work is done there either. Conditions (2.4a) and (2.4b) require that the displacement and its slope are continuous across the extremities.

At free edges, that is at all points \( x_c \in X_1 \), the bending moments and vertical edge forces should actually vanish independently, as opposed to just cancelling. Consequently

\[
\begin{align*}
\mathcal{M}_x(x_c^+, \partial_x, \partial_y) \eta(x_c^+, y, t) &= \mathcal{M}_x(x_c^-, \partial_x, \partial_y) \eta(x_c^-, y, t) = 0, \\
\mathcal{S}_{xy}(x_c^+, \partial_x, \partial_y) \eta(x_c^+, y, t) &= \mathcal{S}_{xy}(x_c^-, \partial_x, \partial_y) \eta(x_c^-, y, t) = 0.
\end{align*}
\]

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Both sets of equations are derived in Appendix A.

2.2 Nondimensionalization

For the \( j \)th region of ice we define the characteristic length \( L_j = \sqrt{D_j/\rho_w g} \) and the characteristic time \( \tau_j = \sqrt{L_j/g} \), after Fox (2000). These quantities will be important in the parameterization of the results obtained.

Now, let \( h_M = \max\{h_j\mid j = 0, 1, 2 \& h_j(x) = 0\} \). Also denote by \( D_M, m_M, L_M \) and \( \tau_M \) the flexural rigidity, mass per unit surface area, characteristic length and characteristic time of ice with thickness \( h_M \). We shall scale times with respect to \( \tau_M \).

Then let us take the incident wave to be monochromatic, with radial frequency \( \omega \) and wave period \( \tau \). The natural length that we shall nondimensionalize lengths with respect to is defined as \( L = L_M \bar{\omega}^{-2/5} \), where \( \bar{\omega} = \omega \tau_M \). \( L \) has the property that \( L^5 = D_M/\rho_w \omega^2 \). The reason for choosing this natural length is that it allows the dispersion relation for a uniform sheet of ice to be characterized by a single parameter (other than the water depth) that combines the effects of wave period and ice thickness. This is discussed further in Sections 2.2.1 and 5.2.1.

We now define the following nondimensionalized quantities

\[
(x, y, z) = (x, y, z)/L, \quad t = t/\tau_M, \quad \Phi = \tau_M \Phi/L^2, \quad \eta = \eta/L,
\]

\[
\bar{D}_j = D_j/D_M, \quad \bar{m}_j = m_j/m_M, \quad \bar{a} = a/L, \quad \bar{H} = H/L, \quad \bar{X}_c = X_c/L
\]

(the notation \( X_c/L \) indicates that each element of the set \( X_c \) should be scaled by \( L \)), and

\[
\lambda = \bar{\omega}^{-3/5}, \quad \sigma(h_M) = m_M/\rho_w L_M, \quad \mu(h_M, \lambda) = \sigma \bar{\omega}^{3/5} = \sigma/\lambda^{1/4},
\]

We have nondimensionalized with respect to \( h_M \) because the natural length and other quantities such as the wavelength increase with ice thickness. Consequently, a distance that is quite large with respect to thinner ice, may not be as significant in comparison with the characteristic lengths of thicker ice. For example, in Section 5.2.3, we establish how large the nondimensional water depth \( \bar{H} \) must be for the infinite depth approximation to be valid, so by choosing to nondimensionalize with respect to the maximum average thickness we are ensuring that the water is deep with respect to
each different region of ice.

Since the geometry of the problem is invariant in the $y$-direction the component of the wave propagating in that direction is unaffected by the feature, so we may concentrate on velocity potentials of the form

$$
\Phi(x, y, z, t) = \Re \left[ \phi(x, z) e^{i(l_y - \omega t)} \right],
$$

(2.7)

where $\Re[.]$ denotes the real part and we have also taken into account the timewise periodicity of the forcing from the incident wave. $l$ is the wavenumber of the incident wave in the $y$ direction. Note that assuming (2.7) forces $\Phi$ to automatically satisfy (2.1d).

Dropping the overbars for clarity, the function $\phi(x, z)$ must satisfy the following system of equations

$$
\begin{align*}
(\nabla_{xz}^2 - l^2)\phi(x, z) &= 0, \quad (2.8a) \\
\mathcal{L}(x, \partial_x)\phi_x(x, 0) + \phi(x, 0) &= 0, \quad (2.8b) \\
\phi_x(x^+, z) - \phi_x(x^-, z) &= \phi(x^+, z) - \phi(x^-, z) = 0, \quad (2.8c) \\
\phi_x(x, H) &= 0. \quad (2.8d)
\end{align*}
$$

The differential operator $\mathcal{L}$ is given by

$$
\mathcal{L}(x, \partial_x) = (\partial_x^2 - l^2) (D(x)(\partial_x^2 - l^2)) + (1 - \nu) l^2 D''(x) + \lambda - m(x)\mu - i\varepsilon,
$$

(2.9)

where $\varepsilon = 0$. Although it seems pointless to define $\varepsilon$ in this way, it was found to be easier to find a $\phi$ which satisfies (2.8b) for an arbitrarily small positive value of $\varepsilon$ first, and then to take the limit as $\varepsilon \to 0$. (In particular, the application of Green's theorem in Section 4.1 is simplified greatly by doing this.) This can be done analytically, and we verify the legitimacy of this procedure in Section 4.1.

When $x \notin (0, a)$ (i.e. $j = 0, 2$), and the ice properties are constant, $\mathcal{L}$ simplifies to $\mathcal{L}_j(\partial_x) = D_j(\partial_x^2 - l^2)^2 + \lambda - m_j\mu - i\varepsilon$ and is referred to as $\mathcal{L}_1(x, \partial_x)$ when $x \in (0, a)$. It can be broken up into differential operators that are independent of $x$ by writing

$$
\mathcal{L}_1(x, \partial_x) = \mathcal{L}_0(\partial_x) + \sum_{j=1}^4 d_j(x) \mathcal{L}_{1j}(\partial_x),
$$

(2.10)

where $\mathcal{L}_{11}(\partial_x) = (\partial_x^2 - l^2)^2$, $\mathcal{L}_{12}(\partial_x) = (\partial_x^2 - l^2) \partial_x$, $\mathcal{L}_{13}(\partial_x) = \mathcal{L}_-(\partial_x)$ and $\mathcal{L}_{14}(\partial_x) = -\mu$. Their multipliers are $d_1(x) = D_0(d_0(x) - 1)$, $d_2(x) = 2D_1'(x)$, $d_3(x) = D_1''(x)$ and
\[ d_4(x) = m_1(x) - m_0, \] 
where \( d_0(x) = D_1(x)/D_0 \). Also, the operator \( \mathcal{L}^- \) in \( \mathcal{L}_{13} \) can be obtained from

\[ \mathcal{L}^\pm(\partial_x) = \mathcal{L}_{x}^\pm(\partial_x, \ii) = (\partial_x^2 - l^2) \mp (1 - \nu) l^2 \]  
(2.11)

(cf. equation 2.5c), while equation (2.8b) results from differentiating (2.1b) with respect to time, and then using (2.1e) and (2.2) to eliminate \( \eta \) and \( P \).

The edge conditions to be satisfied can also be put in terms of \( \phi_z(x, 0) \). The frozen edge conditions become

\[ \phi_z(x_c^+, 0) = \phi_z(x_c^-, 0), \]  
(2.12a)
\[ \phi_{xz}(x_c^+, 0) = \phi_{xz}(x_c^-, 0), \]  
(2.12b)
\[ \mathcal{M}(x_c^+, \partial_z) \phi_z(x_c^+, 0) = \mathcal{M}(x_c^-, \partial_z) \phi_z(x_c^-, 0), \]  
(2.12c)
\[ \mathcal{S}(x_c^+, \partial_z) \phi_z(x_c^+, 0) = \mathcal{S}(x_c^-, \partial_z) \phi_z(x_c^-, 0), \]  
(2.12d)

while the free edge conditions become

\[ \mathcal{M}(x_c^+, \partial_z) \phi_z(x_c^+, 0) = \mathcal{M}(x_c^-, \partial_z) \phi_z(x_c^-, 0) = 0, \]  
(2.13a)
\[ \mathcal{S}(x_c^+, \partial_z) \phi_z(x_c^+, 0) = \mathcal{S}(x_c^-, \partial_z) \phi_z(x_c^-, 0) = 0. \]  
(2.13b)

The non-dimensional operators \( \mathcal{M} \) and \( \mathcal{S} \) are given by

\[ \mathcal{M}(x, \partial_z) = \frac{1}{D_M} \mathcal{M}_x(x, \partial_z, \ii) = -D(x) \mathcal{L}^-(\partial_z), \]  
(2.14a)
\[ \mathcal{S}(x, \partial_z) = \frac{1}{D_M} \mathcal{S}_{xy}(x, \partial_z, \ii) = D(x) \mathcal{L}^+(\partial_z) \partial_x + D'(x) \mathcal{L}^-(\partial_z), \]  
(2.14b)

referring to (2.5) and (2.11).

### 2.2.1 Important Parameters

Of the parameters in the differential operators \( \mathcal{L}_j \), it will turn out that the ratios \( m_j = h_j/h_M \) (which also determine the nondimensional rigidities \( D_j = m_j^2 \)) will be the most important in summarizing the scattering properties of the different features modelled. Note that when talking about \( m_1(x) \) and \( D_1(x) \), we will discuss them in terms of properties such as their average value, their own and their derivatives' continuity, and the width \( \alpha \) of the region that they vary over. This will be discussed further in Chapter 6.
The properties of the wave itself and of the background ice sheet will turn out to be best described by the quantities \( t \) and \( \lambda \). Using the former as a parameter is equivalent to using the angle of incidence \( \theta \), while the latter incorporates the effects of both the wave period and the background thickness.

The remaining parameter \( \mu \) is less important because it will be shown to be almost entirely a function of \( \lambda \). From its definition it depends on the actual value of the background thickness \( h_M \) as well (through the function \( \sigma(h_M) \)); however \( \sigma \) is so weakly dependent on its argument that it may essentially be taken as being constant. \( (\sigma \propto h_M^{1/4}, \text{and we will generally only use values for } h_M \text{ between } 0.5 \text{ m and } 10 \text{ m; this corresponds to the very small range of } \sigma \text{ values of between about } 0.05 \text{ and } 0.11, \text{ compared to the much larger range for } \lambda \text{ of between about } 10^{-4} \text{ and } 4.5.) \) Accordingly, by choosing a representative value for \( h_M \), which we shall denote by \( h'_M \), \( \sigma \) may be fixed by setting it to \( \sigma' = \sigma(h'_M) \), allowing us to summarize approximately the combined effects of \( h_0 \) and wave period with the single parameter \( \lambda \). We shall generally choose \( h'_M \) so that \( h_0 \), which we shall refer to as the background ice thickness, is equal to 1 m, although certain situations will be discussed where a small or even zero value for \( h_0 \) is required; in that case a more appropriate thickness shall be forced to be 1 m. (For example, when discussing the scattering by a finite strip of ice in Section 8.1.3, in which \( h_0 = h_2 = 0 \), we shall set \( h_1 = 1 \text{ m.} \)

Using \( \lambda \) in this way is equivalent to using the nondimensional period \( \tau \) or the nondimensional frequency \( \omega \). The main consequence of being able to do this is that scattering results for a maximum thickness \( h_M \) (with characteristic time \( \tau_M \)) may be approximated extremely easily if the results for our reference thickness \( h'_M \) (with characteristic time \( \tau'_M \)) are already known. For example, say we have a plot of \(|R|\) against wave period for \( h'_M \). Since the nondimensional period is only the dimensional period divided by the characteristic time of the ice, results for the second value may be estimated by simply scaling the period axis by \( \tau_M/\tau'_M = (h'_M/h_M)^{3/8} \).

However, in situations where the three thicknesses \( h_0, h_1, \text{ and } h_2 \) are identical, such as in the single-crack problem discussed in Chapter 5, then \( \lambda \) and \( \mu \) can be combined into the single parameter \( \varpi = \lambda - \mu \), and there is no need to approximate \( \mu \). This is also useful in characterizing the Green's function derived in Chapter 3, and the roots of the dispersion relation, which are discussed in Appendix B. The solid line in Figure 2.2a plots \( \varpi \) as a function of wave period for a background ice thickness of 1 m, show-
Figure 2.2: The behaviour of some important nondimensional parameters with wave period when the maximum ice thickness $h_M$ is 1 m: (a) $\varpi = \lambda - \mu$ (solid) and $\lambda$ itself (dashed), (b) the natural length $L$, (c) the nondimensional water depth $H$ when the actual depth is 50 m.

...
Tkacheva, 2001b).

The last parameter that will prove to be important in our results is the nondimensional water depth $H$. Section 5.2.3 discusses the effect of $H$ on the scattering results for a single crack, although mainly with the intention of establishing an infinite depth criterion, as the principal goal of this thesis is to model central Arctic and Antarctic waters away from the coast, where the depth is usually at least 300 m. Appendix B also shows that the roots of the dispersion relation exhibit some quite interesting behaviour as $H$ changes.

Since the nondimensionalization used here will be relatively unfamiliar to most readers, Figure 2.2b also plots the natural length $L$ as period varies when $h_M = 1$ m while Figure 2.2c shows how the nondimensional water depth $H$ corresponding to a dimensional depth of 50 m changes with period. These figures are mostly intended for reference when reading Chapter 5.
Chapter 3

The Green's Function

In all the problems solved in this thesis, the potential $\phi$ will be found by writing it in terms of a Green's function $G$, that satisfies a simpler set of equations—namely, (3.1) below. These equations are simpler in the sense that the operator $L_0$ in (3.1b) is independent of $x$, while the corresponding operator in (2.8b) may vary with position. In addition, $G$ does not need to satisfy any edge conditions.

The Green's function that we will usually use is the finite depth Green's function, which is presented in Section 3.1. This was derived in the form given there by Evans and Porter (2003b), but it is analysed in more detail here—in particular, we discuss singularities at the origin that need to be taken into account in following chapters.

In some problems, notably the single-crack problem solved in Chapter 5, it proves more convenient to use the so-called infinite depth Green's function. Consequently, the infinite depth Green's function is presented in Section 3.2 and is intended as an approximation to the finite depth Green's function for large $H$.

An analytic expression for the full infinite depth Green's function for the case when the incident waves are normally incident was described by Dixon and Squire (2001b); the analogous oblique incidence Green's function was presented by Williams and Squire (2002), giving analytic expressions for certain derivatives at the origin. Further techniques for calculating those derivatives away from the origin (but still at the ice-water interface) are also given in Section 3.2. Although no calculations of the infinite depth Green's function away from the origin are actually undertaken in this thesis, they could be used to great advantage in problems like determining the infinite
depth scattering by a region of cracks (i.e. an oblique-incidence version of the problems considered by Dixon and Squire, 2001a; Evans and Porter, 2003a).

### 3.1 Finite Depth Green’s Function

In most of the problems solved in this thesis, the potential $\phi$ will be put in terms of a Green’s function $G$ that satisfies

$$(\nabla^2_{\xi \zeta} - l^2)G(x - \xi, z, \zeta) = \delta(x - \xi, z - \zeta), \quad (3.1a)$$

$$\mathcal{L}_0(\partial_\zeta)G_\zeta(x - \xi, z, 0) + G(x - \xi, z, 0) = 0, \quad (3.1b)$$

$$G_\zeta(x - \xi, z, H) = 0. \quad (3.1c)$$

It will be seen that the infinitesimal quantity $\varepsilon > 0$ that is contained in the operator $\mathcal{L}_0(\partial_\zeta) = D_0(\partial^2_\zeta - l^2)^2 + \lambda - m_0\mu - i\varepsilon$ forces the Green’s function to decay exponentially as $|x - \xi| \to \infty$, instead of behaving like an outward going plane wave as it will be seen to do in the limit as $\varepsilon \to 0$; this simplifies the application of Green’s theorem in Chapter 4 markedly. However, since no computations ever actually have to be made involving $G$ when $\varepsilon \neq 0$, all results in this chapter are of the limiting Green’s function.

Now, by taking the Fourier transform of the above equations with respect to $x - \xi$ we obtain the ordinary differential equation (ODE)

$$(\partial^2_\xi - \kappa^2)\hat{G}(k, z, \zeta) = \delta(z - \zeta), \quad (3.2)$$

where $\kappa = \sqrt{k^2 + l^2}$, and

$$\hat{G}(k, z, \zeta) = \int_{-\infty}^{\infty} G(x - \xi, z, \zeta) e^{ik(x - \xi)} d(x - \xi).$$

$\hat{G}$ must also satisfy initial conditions

$$\Lambda_0(\kappa)\hat{G}_\zeta(k, z, 0) + \hat{G}(k, z, 0) = 0, \quad (3.3a)$$

$$\hat{G}_\zeta(x - \xi, z, H) = 0, \quad (3.3b)$$

where $\Lambda_0(\kappa) = \mathcal{L}_0(ik) = D_0\kappa^4 + \lambda - m_0\mu - i\varepsilon$.

Evans and Porter (2003b) presented $\hat{G}$ in the form (adjusted to our notational and nondimensionalization scheme)

$$\hat{G}(k, z, \zeta) = \chi(z_-, \kappa) \frac{\varphi(z_+, \kappa)}{f_0(\kappa)}, \quad (3.4)$$
where \( z_+ = \max\{z, \zeta\} \), \( z_- = \min\{z, \zeta\} \), \( \chi(z, \kappa) = (\Lambda_0 \kappa \cosh \kappa z - \sinh \kappa z)/\kappa^2 \tanh \kappa H \), 
\( \varphi(z, \kappa) = \cosh \kappa(z - H)/\cosh \kappa H \) and \( f_0(\kappa) \) is the dispersion function for the left hand region, which will be discussed in further detail below. It is easily checked that \( \hat{\mathcal{G}} \) satisfies (3.3), and also that \( G_\zeta(k, z, z^+) - G_\zeta(k, z, z^-) = 1 \) as implied by (3.2).

Following our convention of giving a subscript of 0 to quantities specific to the left-hand region, 1 for the central region, and 2 for the right-hand region, let us now define the dispersion functions for those areas where the ice has constant properties as

\[
h(K) = \frac{1}{K \tanh KH} + \frac{h}{\kappa^2},
\]

where \( h_j(K) = D_j K^4 + \gamma_j \). Although we do not use \( h \) and \( h \) in this chapter, we define them simultaneously for overall brevity. \( h \) is required in both Chapter 7 and Chapter 8, while \( h \) is required in Chapter 8.) Let us also define the dispersion relations for those regions formally as \( h(K) = 0 \). (The ideas behind the dispersion relation were discussed in Section 1.2.)

Note that when we nondimensionalize with respect to the ice on the left hand side, \( D_0 = m_0 = 1 \), the dispersion function for that region, \( f_0 \), depends only on \( H \), and the parameters \( \varepsilon \) and \( \varpi = \lambda - \mu \). Consequently, from (3.4), the Green's function in the limit as \( \varepsilon \to 0 \), and for a given angle of incidence is also completely parameterizable in terms of those quantities (\( \varpi \) and \( H \)). Since we can obtain the Green's function in this form simply by rescaling our variables, when presenting results in this chapter (cf. Figure 3.1 and Section 3.2) we will assume that \( D_0 = m_0 = 1 \) (assuming that \( h_0 \neq 0 \)) and parameterize those results in terms of \( \varpi \) alone.

To invert the transform \( \hat{\mathcal{G}} \) and so find \( \mathcal{G} \) itself, we first note that it has (almost always simple) poles whenever \( k^2 = \alpha_n^2 = \gamma_n^2 - l^2 \), where the \( \gamma_n \) are the zeros of \( f_0(\kappa) \).

The square roots \( \alpha_n \) are taken to either be in the upper half-plane or on the positive real axis, and the distribution of the \( \gamma_n \) are described below, and illustrated by Figure 3.1; the roots are also discussed in greater detail in Appendix B.

Figure 3.1a is a similar plot to one given by Fox and Squire (1990), who demonstrated numerically typical locations for the roots \( \gamma_n \). Our figure plots contours ranging from 0.1 to 2.5 for the function \( |\tanh \kappa H f_0(\kappa)/\Lambda_0(\kappa)| \) when \( H = 1 \), \( \varpi = 0 \) and \( \varepsilon = 0 \) (this function has the same zeros as \( f_0(\kappa) \); the effect of \( \varepsilon \) on the roots' location will be discussed when we come to invert \( \hat{\mathcal{G}} \)), and shows a similar distribution to the one observed by Fox and Squire (1990), despite their figure corresponding to different values
Figure 3.1: The roots of the finite depth dispersion relation for ice-coupled waves. Figure a shows a contour plot of $|\tanh \kappa H f_0(\kappa)/(\kappa_0)|$ for contour values of 0.1, 0.5, 1.0, 2.0 and 2.5. Figure b shows the zeros of $f_0(\kappa)$ found using Newton’s method. $\varpi = 0$ and $H = 1$ for both plots; note the correspondence between clusters of contour lines in (a) and the zeros shown in (b). The positive real root is labelled $\gamma_0$, the complex roots in the first and fourth quadrants are respectively labelled $\gamma_{-1}$ and $\gamma_{-2}$, and the roots on the positive imaginary axis are labelled $\gamma_n$, for $n = 1, 2, \ldots$ (ordered in increasing modulus).

for the parameters $H$ and $\varpi$. It also validates Figure 3.1b, which plots values for the roots themselves, found by Newton’s method.

As is borne out by the symmetry observed in both of the aforementioned figures, the evenness of the dispersion function $f_0$ leads to the negatives of zeros also being zeros, and so we will only describe and label those zeros that are either on the positive imaginary axis or in the right hand half-plane. In general, for any given values of $H$ and $\varpi$, there is one zero, $\gamma_0$, on the positive real axis, a complex conjugate pair, $\gamma_{-1}$, in the first quadrant, and $\gamma_{-2} = \gamma^*_{-1}$ in the fourth, and infinitely many zeros on the positive imaginary axis, $\gamma_n$, $n = 1, 2, \ldots$. These are all evident in Figure 3.1.
Some care has to be taken, however, as period decreases, particularly as $\omega$ becomes negative (which it does for periods less than about 1.9s for 1-m-thick ice)—as that happens the four complex roots $\gamma_{-1}, \gamma_{-2}$ and their negatives may also move onto the imaginary axis (forming a pair of double or even triple roots). This phenomenon is discussed and illustrated further in Appendix B.

Having located the zeros of the dispersion relation, and assuming that the incident wave is approaching the irregularity at a non-perpendicular angle $\theta$ from normal incidence, i.e. that $l = \gamma_0 \sin \theta < \gamma_0$, it now becomes apparent that when $\varepsilon = 0$, the transform $\hat{G}$ has two real poles when $\kappa = \gamma_0$, or equivalently when the actual transform variable $k$ takes the value $k = \pm \alpha_0$. However, by letting $\varepsilon$ become positive, we can produce an infinitesimal counter-clockwise rotation in the roots, thus moving $\alpha_0$ into the upper half-plane.

$\hat{G}$ is now absolutely integrable, and since the inverse Fourier transform of an even function is even we can write

$$G(x - \xi, z, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k, z, \zeta) e^{ik|x-\xi|} \, dk. \quad (3.5)$$

By closing the inversion contour in the upper half-plane, we find that $G$ is equal to the sum of the residues at all the poles of $\hat{G}$ in the upper half-plane (i.e. at $k = \alpha_n$, $n = -2, -1, \ldots$). Hence, $G$ may be written as the following eigenfunction expansion (as by Evans and Porter, 2003b):

$$G(x - \xi, z, \zeta) = i \sum_{n=-2}^{\infty} A_n^\prime e^{i\alpha_n|x-\xi|} \varphi_n(z) \varphi_n(\zeta), \quad (3.6)$$

where $\varphi_n(z) = \varphi(z, \gamma_n)$ and if $A_n = \text{Res}(1/f_0, \alpha_n) = \gamma_n/\alpha_n f_0(\gamma_n) = -\gamma_n^2/\alpha_n C_0(\gamma_n)$ and $C_f(\kappa) = H(\Lambda_0^2(\kappa)\kappa^2 - 1) + 5D_j \kappa^4 + \lambda - m_j \mu$, then $A_n'' = A_n \Lambda_0^2(\gamma_n)$. Also, $\text{Res}(f(k), \alpha)$ refers to the residue of $f$ at the pole $k = \alpha$. However, in this thesis, the only derivative of $G$ that we ever actually need to calculate is $G_{z\xi}(x-\xi, 0, 0)$, and some of its derivatives with respect to $x$ or $\xi$. The inverted transform of that function is

$$g(x - \xi) = G_{z\xi}(x - \xi, 0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik|x-\xi|}}{f_0(\kappa)} \, dk = i \sum_{n=-2}^{\infty} A_n e^{i\alpha_n|x-\xi|}. \quad (3.7)$$

This eigenfunction expansion could also have been obtained by differentiating (3.6).
If $\varepsilon = 0$ and $\alpha_0$ is real we can see that as $|x - \xi| \to \infty$, $G$ will be asymptotically equal to

$$G(x - \xi, z, \zeta) \sim iA_0^0 e^{i\alpha_0|z - \xi|} \varphi_0(z) \varphi_0(\zeta),$$

(3.8)

while it will decay exponentially if $\alpha_0$ has a small positive imaginary part.

In addition, before beginning the next section, note the following three points:

1. Since $A_n \sim O(|\alpha_n|^{-a})$ as $|\alpha_n| \to \infty$, $g(x - \xi)$ will have seven bounded derivatives with respect to $x$ or $\xi$, all of which may be required, depending on the edge conditions applied at critical points $x_c \in X_c$ (cf. Section 2.1.1). It does turn out to have jump discontinuities in some derivatives, however, and these particular derivatives consequently have delta function singularities in their derivatives which must be allowed for when they are integrated over. Characteristics of all of $g$'s derivatives are discussed further in the following section.

2. In obtaining (3.7), we have assumed that the poles of $\hat{G}$ are all simple, and so the expansion must be adjusted if ever the pair $(\omega, H)$ is such that there exists a double or a triple root to the dispersion relation. However, in Appendix B it is shown that such pairs lie on one of a series of curves in $\mathbb{R}^2$ (cf. Figure B.4 b), and so the chances of any given pair falling on one of these curves is essentially zero. Thus we can say that roots of the dispersion relation are "almost always" simple.

3. The real poles in $\hat{G}$ could also have been dealt with by simply deforming the inversion contour slightly. While this is certainly true, introducing the exponential decay into $G$ as done above greatly simplifies the application of Green's theorem in Chapter 4.

3.1.1 Singularities in $g$

The kernel of the integral equations that will be derived in Chapter 4, which are solved in Chapters 6 and 7, contains four different derivatives of the function $g$. It will therefore be important to be aware of any singularities that they contain.
Firstly, \( g \) may be written as \( g = g_0 + g_1 \), where the transforms of \( g_0 \) and \( g_1 \) are respectively

\[
\tilde{g}_0(k) = \frac{\Lambda_0 \kappa^2 \tanh^2 \kappa H}{1 - \Lambda_0^2 \kappa^2 \tanh^2 \kappa H},
\]

\[
\tilde{g}_1(k) = \frac{\kappa \tanh \kappa H}{1 - \Lambda_0^2 \kappa^2 \tanh^2 \kappa H}.
\]

Immediately, we can say that since \( \tilde{g}_1 \sim k^{-8}/|k| \) as \( |k| \to \infty \), \( g_1 \) will have seven continuous derivatives, but that \( g_1^{(8)}(x) \sim \log |x|/\pi \) as \( x \to 0 \). This can be seen by noting that \(-\pi/|k|\) is the Fourier transform of \( \log |x| \) (Lighthill, 1958, p43); adding \( 1/|k| \) to \( k^8 \tilde{g}_1 \) and moving the real poles off the real axis slightly leaves an absolutely integrable function, so \( g_1^{(8)}(x) - \log |x|/\pi \) will be continuous. We also note that

\[
\Lambda_0(\kappa) \tilde{g}_0(k) = -1 + \frac{1}{1 - \Lambda_0^2 \kappa^2 \tanh^2 \kappa H},
\]

so

\[
\mathcal{L}_0(\partial_x)g_0(x) = -\delta(x) + \frac{1}{2\pi} \int_0^\infty \frac{e^{-ikx}dk}{1 - \Lambda_0^2 \kappa^2 \tanh^2 \kappa H}, \tag{3.11}
\]

which implies that \( g_0^{(4)} \) has a delta function type singularity. Consequently, since \( g_1^{(n)} \) is well-behaved for \( n \leq 7 \), so does \( g^{(4)} \). In fact, the analogous equation to (3.11) for \( g \) is

\[
\mathcal{L}_0(\partial_x)g(x) = -\delta(x) - G_\zeta(x, 0, 0). \tag{3.12}
\]

That \( G_\zeta(x, 0, 0) \) should appear in the above equation should not be so surprising as equations (3.6) and (3.7) imply that \( \mathcal{L}_0 g(x) = -G_\zeta(x, 0, 0) \) have the same eigenfunction expansions. (The former has coefficients \( A_n \Lambda_0(\gamma_n) \), which can be simplified by using the dispersion relation \( f_0(\gamma_n) = 0 \) to give \( A_n^a \varphi'_n(0) \times \Lambda_0(\gamma_n) \varphi'_n(0) = -A_n^a \varphi'_n(0) \), the coefficients of the latter.)

Continuing, however, (3.11) and (3.12) imply that

\[
g_0^{(3)}(0^+) - g_0^{(3)}(0^-) = g^{(3)}(0^+) - g^{(3)}(0^-) = -1/D_0,
\]

so both \( g_0^{(3)} \) and \( g^{(3)} \) have jump discontinuities at the origin. Equation (3.11) implies that this will also be true for \( g_0^{(7)} \) and \( g^{(7)} \), although the size of those jumps will be equal to \( (\lambda - m_0 \mu)/D_0^2 \). Consequently, we can immediately predict what values odd derivatives of \( g_0 \) and \( g \) will take as they approach the origin. (Since \( g \) is even its odd derivatives are odd functions.) In terms of \( g \), they are

\[
g^{(1)}(0^+) = g^{(5)}(0^+) = 0, \quad g^{(3)}(0^+) = -\frac{1}{2D_0}, \quad g^{(7)}(0^+) = \frac{\lambda - m_0 \mu}{2D_0^2}, \tag{3.13}
\]
and similarly for $g_0$. (Since $g_1$ has seven continuous derivatives, $g_1^{(n)}(0^\pm) = 0$ for $n = 1, 3, 5$ and 7.)

The properties of $g_0$ and $g_1$ described in this section are demonstrated in Figures 3.2 and 3.3, respectively, for the infinite depth Green's function, which is discussed in the following section.

3.2 Infinite Depth Green's Function

Letting $H \to \infty$ in (3.4), the Fourier transform of the infinite depth Green's function can be written in the form derived by Williams and Squire (2002)

$$\hat{G}(k, z, \zeta) = -\frac{1}{2\kappa} \left( e^{-\kappa|z-\zeta|} + e^{-\kappa(z+\zeta)} - \frac{2e^{-\kappa(z+\zeta)}}{\kappa f_0(\kappa)} \right),$$  \hspace{1cm} (3.14)

where the approximate dispersion relation for the water beneath the left hand ice sheet is $f_0(\kappa) = 1/\kappa + i\varepsilon - \Lambda_0 = -p_\infty(\kappa)/\kappa$, and the polynomial $p_\infty$ is given by $p_\infty(\kappa) = \bar{\Lambda}_0 \kappa - 1$. Note that this can simply be obtained by making the approximation $\tanh \kappa H \approx 1$ in the finite depth dispersion relation (Kinsman, 1984).

As mentioned in Section 3.1, assuming that $D_0 = m_0 = 1$ simply involves rescaling our variables. If we do this then $p_\infty$ is simply $p_\infty(\kappa) = \kappa^5 + (\varepsilon - i\varepsilon)\kappa - 1$, and so we shall continue this section under this assumption.

Proceeding to invert (3.14), we can write $G$ itself as

$$G(x - \xi, z, \zeta) = -\frac{1}{2\pi} \left( K_0({lr}_-) + K_0({lr}_+) - \int_{-\infty}^{\infty} \frac{e^{-\kappa|z-\zeta|}}{\kappa^2 f_0(\kappa)} e^{ik|z-\xi|} dk \right),$$  \hspace{1cm} (3.15)

where $r_\pm = \sqrt{(x-\xi)^2 + (z \pm \zeta)^2}$, and $K_0$ is the modified zeroth order Bessel function of the second kind (Abramowitz and Stegun, 1965; Gray and Matthews, 1966; Stakgold, 1968), which is the fundamental solution for the modified Helmholtz equation (2.8a).

The sea-floor condition that $G$ now satisfies is

$$\lim_{\zeta \to \infty} G(z)(x - \xi, z, \zeta) = 0,$$  \hspace{1cm} (3.16)

which replaces (3.1c).

As in the case of finite depth, $g(x - \xi) = G(x - \xi, 0, 0)$ can be split into a sum $g_0 + g_1$ as well. In this case this is done by multiplying $\hat{g}(k) = \kappa/(1 - \Lambda_0(\kappa))$ by
\( (1 + \Lambda_0(\kappa))/(1 + \Lambda_0(\kappa)) \). \( g_0 \) and \( g_1 \) can also be obtained directly from their finite depth counterparts; by letting \( H \to \infty \) in (3.9a), \( g_0 \) can be written

\[
g_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Lambda_0 \kappa^2}{1 - \Lambda_0^2 \kappa^2} e^{-ik|x|} \, dk = \frac{1}{2} \sum_{n=-2}^{2} A_n e^{i\alpha_n|x|}, \tag{3.17}
\]

while doing the same in (3.9b) gives

\[
g_1(k) = \frac{\kappa}{1 - \Lambda_0^2 \kappa^2}. \tag{3.18}
\]

The \( A_n \) in (3.17) are defined as \( A_n = \text{Res}(1/f_0, \alpha_n) = -\gamma_n^3/\alpha_n(4\gamma_n^5 + 1) \). Note that these formulae could have been deduced directly from the expressions for the finite depth residues—letting \( H \) become large in the \( C_0(\gamma_n) \) terms for the non-imaginary roots, \( C_0(\gamma_n) \approx 5\gamma_n^3 + \omega + O(H \exp(-2H |\text{Re}[\gamma_n]|)) \). Since \( \Lambda_0(\gamma_n)\gamma_n - 1 = 0 \) for the infinite depth roots, \( 5\gamma_n^5 + \omega = (4\gamma_n^5 + 1)/\gamma_n \), and the above formulae for the infinite depth residues follow.

For \( \varepsilon = 0 \), the roots \( \gamma_0, \gamma_{-1} \) and \( \gamma_{-2} \) are in much the same positions as before: \( \gamma_0 > 0 \) is real, \( \gamma_{-1} \) is in the first quadrant, and \( \gamma_{-2} = \gamma_{-1}^* \). However, \( \gamma_1 \) is now defined to be in either the second quadrant or on the negative real line. It is only negative for values of \( \omega \leq \omega_\infty = -5/4^{4/5} \approx -1.65 \) (cf. Appendix B.3). For \( \omega > \omega_\infty \) the root \( \gamma_2 \) is the complex conjugate of \( \gamma_1 \); otherwise it is another negative root. For \( \varepsilon > 0 \), the negative real line becomes an asymptote for these latter two roots as \( \omega \to -\infty \), while \( \gamma_0 \) again moves into the upper half-plane. Likewise, \( \gamma_{-1} \) and \( \gamma_{-2} \) experience a small counter-clockwise rotation.

Although there is an obvious similarity in the positions of the infinite depth roots in the right hand half plane and their finite depth counterparts, the position and behaviour of the infinite depth roots in the left hand half-plane is, at first sight, completely unexpected. However, their behaviour can be explained by considering the roots of the equation \( \Lambda_0(\kappa)\kappa \tanh \kappa H + 1 = 0 \) instead of the equation that the finite depth roots satisfy, namely \( \Lambda_0(\kappa)\kappa \tanh \kappa H - 1 = 0 \). Appendix B.3 discusses this further, explaining in full the transition from the finite depth situation to the infinite depth situation.

Having located the infinite depth roots, we can now proceed to calculate \( g_0 \). The final expansion in (3.17) assumed that all the \( \alpha_n \) were in the upper half plane, which they are when \( \varepsilon \) is positive. In that case \( g_0(x) \) will decay exponentially as \( |x| \to \infty \).
Figure 3.2: Properties of \( g_0 \) and its derivatives. The solid lines show the real parts of the \( g_0^{(n)} \) plotted against \( \gamma_0 x / 2 \pi \) (\( x \) divided by the wavelength), for indicated values of \( n \). \( \omega = 0.1, \theta = 0 \), and the water depth is infinite. Note that the jump discontinuities in \( g_0^{(3)} \) and \( g_0^{(7)} \) are \(-1\) and \( \omega \) respectively, as predicted in Section 3.1.1. The dashed lines show the real parts of the propagating components of the \( g_0^{(n)} \)—note that in each graph the two curves have all become indistinguishable by one wavelength.

In practice, however, we only have to calculate the limit of \( g_0 \) as \( \varepsilon \to 0 \), and it will have a finite amplitude for large \( |x| \). For \( \omega > \omega_\infty \), \( \alpha_0 \) and \( \gamma_0 \) become positive real so that as \( |x| \to \infty \), \( g_0 \) has a propagating component \((i/2)A_0 e^{i\omega_0 |x|}\). When \( \omega \leq \omega_\infty \), \( \alpha_2 \) becomes positive real in the limit, and \( \alpha_0 \) and \( \gamma_0 \), and \( \alpha_1 \), \( \gamma_1 \) and \( \gamma_2 \) become negative real, implying that as \( |\xi| \to \infty \), \( g_0 \) is instead asymptotically equal to

\[
g_0(x) \sim \begin{cases} 
\frac{i}{2} A_0 e^{i\omega_0 |x|} - A_1 \sin \alpha_1 |x| & \text{for } \omega = \omega_\infty, \\
\frac{i}{2} \sum_{n=0}^{2} A_n e^{i\alpha_n |x|} & \text{for } \omega < \omega_\infty.
\end{cases}
\]

The \( n = 1 \) and \( n = 2 \) terms rarely have to be accounted for as values of \( \omega \) that are less than or equal to \( \omega_\infty \) correspond to such small periods, and when the periods are small enough, they cancel out similar terms in \( g_1 \) when \( g = g_0 + g_1 \) itself is calculated. (This should be clear from the Fourier transform \( \hat{g}(k) = 1/f_0(k); \quad \kappa = \sqrt{k^2 + \ell^2} \geq 0 \).
can never equal $\gamma_1$ or $\gamma_2$ when $k$ is real.) $g(x)$ itself always has propagating component $iA_0e^{j\omega|x|}$, which follows from taking the limit as $H \to \infty$ in the finite depth formula for $g$, equation (3.7).

The real part of $g_0$, and the real parts of some of its derivatives are plotted as solid lines in Figure 3.2, for $\omega = 0.1 > \omega_\infty$ and $\theta = 0$, which corresponds to a period of about 2.7 s for 1-m-thick ice. (Their imaginary parts are simply sine/cosine waves, and are thus not very interesting.) Particularly note the jump discontinuities in $g_0^{(3)}(x)$ and in $g_0^{(7)}(x)$, and that the size of the jumps are as predicted in the previous section.

The dashed lines plot the propagating components of those derivatives—note that the solid lines have become indistinguishable from them by the time $x$ has moved about one wavelength from the origin, showing that using only this component is a good approximation to $g_0$ even for only moderately large values of $x$.

The inversion of (3.18) is a little more complicated, as $\hat{g}_1$ is no longer meromorphic due to the $\kappa = \sqrt{k^2 + l^2}$ term in its numerator, which is a multi-valued function of $k$. We will define $\kappa$ as taking values in the right hand half plane, which corresponds to it having branch cuts on the lines $k = \pm i \times [l, \infty)$.

However, an analytical solution is still possible for normally incident waves—using $\theta = 0$, Squire and Dixon (2000) were able to write $G(x-\xi, z, \zeta)$ (for all values of $z$ and $\zeta$) in terms of the auxiliary sine and cosine functions, while it is shown in Appendix C.2.3 that $g(x-\xi) = G_\zeta(x-\xi, 0, 0)$ may be put in terms of the related exponential integral function. Figure 3.3 below illustrates the behaviour of its seven bounded derivatives (plotted as solid lines). Note that $g_1^{(7)}(x)$ has a vertical slope at $x = 0$ implying that $g_1^{(8)}(x)$ is singular there, as was also shown in the previous section.

As with Figure 3.2, note that the exact solutions have converged to their propagating components (dashed lines, $iA_0e^{j\omega|x|/2}$ in this case as $\omega > \omega_\infty$) by about one to one-and-a-half wavelengths from the origin, with the higher derivatives decaying faster. This implies that for moderately large $x$, like $g_0$, $g$ itself (and its derivatives) may be approximated well by its propagating part, and the parts of the inverse transform that are more difficult to calculate quickly become negligible.

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Figure 3.3: Properties of $g_1$ and its derivatives. The solid lines show the real parts of the $g_1^{(n)}$ plotted against $\gamma_0 x / 2\pi$ ($x$ divided by the wavelength), for indicated values of $n$. $\omega = 0.1$, $\theta = 0$, and the water depth is infinite. Note that $g_1^{(7)}$ is vertical at the origin due to its derivative having a logarithmic singularity. (cf. Section 3.1.1.) The dashed lines show the real parts of the propagating components of the $g_1^{(n)}$—as in Figure 3.2, note that in each graph the two curves have all become indistinguishable by about one wavelength. The solid curves are calculated analytically using the method of Appendix C.2.3, which is equivalent to using the formulae of Dixon and Squire (2001b).

However, Figure 3.3 does show that the non-propagating components make significant contributions near the origin, and so they cannot be safely ignored without introducing large errors. Although these components may only be calculated exactly for normally incident waves, relatively straightforward numerical techniques are available which make their approximation both accurate and efficient for obliquely incident waves also.

A first approach to inverting (3.18) for obliquely incident waves is to use the Fast Fourier Transform (FFT) technique (Press, 1992). Running the FFT routine on $\hat{g}_0(k)$ as written in that equation would be quite inefficient due to the presence of poles at
Figure 3.4: Convergence of the series expansion for $g$ to its inverse Fourier transform calculated directly using a Fast Fourier Transform (FFT) algorithm. The solid lines show the fully converged FFT results for the real parts of $g$ and $g'$, and are taken to be exact. They are plotted against $\gamma_0 x/2\pi$, or $x$ divided by the wavelength of the incident wave, for the angles of incidences indicated. The dotted, chained and dashed lines show series approximations using 5, 10, and 15 terms in each respectively. $\varpi = 0.1$ and the water depth is infinite.

$k = \pm \alpha_0$ (and also at $k = \pm \alpha_1, \pm \alpha_2$ for $\varpi \leq \varpi_\infty$) on the real line. However, the convergence can be improved by subtracting sufficient meromorphic terms to cancel those poles, leaving a relatively smooth transform to be inverted. For example, when $\varpi > \varpi_\infty$ we can write (3.18) as

$$\hat{g}_1(k) = \frac{\kappa - \gamma_0}{1 - \Lambda_0^2\kappa^2} + \frac{\gamma_0}{1 - \Lambda_0^2\kappa^2}. \quad (3.19)$$

The correction term here is meromorphic and may be determined in the same way that $g_0$ was. Other terms may be subtracted if necessary (i.e. if $\varpi \leq \varpi_\infty$).

The FFT approach is most advantageous when a large number of points are required to be generated at once, while an alternative approach would be more efficient
if only a few points are required. A further disadvantage of the FFT is that the higher
derivatives of \( g_1 \) take longer to converge around \( x = 0 \), due to their transforms decaying
more slowly as \( k \to \infty \). \( (g_1^{(n)}(k)) \sim O(|k|^{n-9}) \) as \( k \to \infty \).

One alternative method, which takes the singularities around the origin into account, is to derive a series expansion for \( g_1 \). This is described in detail in Appendix C.1, and its convergence is illustrated in Figure 3.4, which compares the FFT results (solid curves) with those obtained by using an expansion using 5, 10, and 15 terms (dotted, chained and dashed curves respectively). This method is exact at the origin.

The FFT results agree at normal incidence with both the analytic solution using
the exponential integrals (cf. Appendix C.2.3), and also Gaussian quadrature results calculated from (C.19a), or alternatively (3.21) below, with weight function \( k^{n+1}e^{-k|x|} \) for its \( n^{th} \) derivative. (This type of quadrature scheme is called generalized Gauss-Laguerre quadrature, cf. Press, 1992) Hence, they (the FFT results) will also be taken to be accurate for oblique incidence, and our alternative methods will be shown to converge to them (which also confirms their validity). In the graphs below, \( 2^8 \) points were used, which was probably a little conservative but was still quite fast, and the inverse transform was assumed to have vanished to zero for values of \( |x| \) greater than eight wavelengths. Convergence was checked at the origin as it would be slowest there, and so was the assumption about the vanishing of the transform (to check that there was no aliasing).

The solid curves in Figure 3.4 show the real parts of \( g \) and \( g' \) computed using FFT
for angles of incidence \( \pi/12, \pi/6, \pi/2 \) and \( \pi/3 \), and when \( \omega = 0.1 \). (Again, their imaginary parts are just cosine or sine waves, and are thus not shown.) They are plotted against \( \gamma_0 x/2\pi \), which is \( x \) divided by the wavelength of the incident wave.

The real part of \( g \) is very similar to a sine wave of wavelength \( 2\pi/\alpha_0 \), which is its asymptotic form as \( x \to \infty \). Since \( \alpha_0 \to 0 \) as \( \theta \to \pi/2 \), the wavelength of the sine wave increases with angle of incidence, and has doubled by the time \( \theta \) has reached \( \pi/3 \) (\( \cos \pi/3 = 0.5 \)). Similarly, its amplitude, \( |A_0| \), has also doubled by that angle of incidence.

Similar observations may be made about the real part of \( g' \): its asymptotic be-
haviour is that of a cosine wave, the wavelength of which also increases with $\theta$. However, its amplitude is $|A_0\alpha_0|$, which is independent of the angle of incidence. Since it is an odd, continuous function, it is zero at the origin.

In terms of the convergence of the series, the figure shows more accuracy near the origin (exact at the origin) and for smaller angles of incidence. Both of these effects are a result of using a series in powers of $lx$—a larger value of $x$ makes $lx$ larger for a given $l$ and will thus require more terms to make it converge; similarly, if $x$ is fixed, larger values of $l$ also make $lx$ larger.

A second alternative method for calculating $g$ can be found directly from (3.7), which was derived by contour integration. As $H \to \infty$ in that equation, the imaginary $\gamma_n$ become closer and closer together (the difference between successive zeros $\gamma_{n+1} - \gamma_n$ tends towards $\pi i/H$), so that the sum from $n = 1$ to $\infty$ becomes an integral from 0 to $\infty$ with respect to the roots. The $n = -2, -1$ and 0 terms in the sum remain unchanged, giving

$$g_2(x) = i \sum_{n=-2}^{0} A_n e^{i\gamma_n |x|},$$

while the resulting integral along the positive imaginary line is

$$g_3(x) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{\kappa^2/ke^{i\kappa |x|}d\kappa}{\Lambda_0^2(\kappa) - 1} = -\frac{1}{\pi} \int_{0}^{\infty} \frac{ke^{-\kappa |x|}d\kappa}{\Lambda_0^2(k)k^2 + 1},$$

(3.21)

where $k$ in both cases is $\sqrt{\kappa^2 - l^2}$ and is always located on either the positive real or the positive imaginary axis. This gives us a different partition for $g$, namely

$$g(x) = g_2(x) + g_3(x),$$

(3.22)

which is also confirmed in Appendix C.2 by doing the contour integral for infinite depth explicitly. Note that $g_3$ incorporates both the more difficult inverse transform $g_1$ and a similar sum to (3.17).

When $x = 0$, $g_3$, or indeed any of its derivatives may be evaluated analytically. Odd derivatives are trivial to calculate, although unnecessary in light of (3.13), but the more useful even derivatives are slightly harder to find. Nevertheless, surprisingly simple formulae still eventuate for those derivatives:

$$g_3^{(2n)}(0) = -\frac{(-1)^n}{2\pi} \sum_{m=-2}^{2} A_m \alpha_m^{2n} \log \Theta_m,$$

(3.23)
where \( \Theta_n = (\gamma_n + \alpha_n)/(\gamma_n - \alpha_n) \). These formulae are derived in Appendix C.2.2.

For nonzero values of \( x \) equation (3.21) transforms to (C.19b) in Appendix C.2 by integrating with respect to \( t = (\kappa - l)|x| \) from 0 to \( \infty \). This enables it to also be evaluated by generalized Gauss-Laguerre quadrature, although in this case the weighting function used is \( t^{1/2} e^{-t} \).

Figure 3.5: Convergence of the integral form of \( g \), calculated using Gauss-Laguerre quadrature, to its inverse Fourier transform, calculated directly using an FFT algorithm. As in Figure 3.4, the solid lines show the fully converged FFT results for the real parts of \( g \) and \( g' \), plotted against \( \gamma_0 x/2\pi \), or \( x \) divided by the wavelength of the incident wave, for the angles of incidences indicated. The dotted, chained and dashed lines show generalized Gauss-Laguerre quadrature results using 5, 10, and 15 points respectively. \( \omega = 0.1 \) and the water depth is infinite.

Figure 3.5 compares results obtained by the quadrature method, using five, ten and fifteen points (dotted, chained and dashed lines respectively), with the same FFT results (solid lines) that were presented in Figure 3.4. The integrand in (C.19b) has a square-root singularity at \( t = -2l|x| \), so we expect that more points will be needed.
for smaller values of $x$ and smaller angles of incidence. Indeed this is what is observed in Figure 3.5. If we define $x_N$ so that the quadrature results using $N$ points can be said to have converged sufficiently for $x > x_N$, then, referring again to Figure 3.5, $x_5 > x_{10} > x_{15}$ for any angle of incidence, and for a given $N$, $x_N$ decreases as the angle increases.

The two alternatives to the FFT method are thus complementary to each other—the series expansion method converges better for smaller values of $lx$, while the contour integral method converges better for larger values of $lx$. A combination of the two is thus quite efficient in calculating $g$ for oblique angles of incidence—for example, when $\varpi = 0.1$ and the angle of incidence is $\pi/6$, sufficiently accurate results could be obtained by simply using five points in a series expansion if $0 \leq \gamma_0x/2\pi \leq 0.75$, or by just using five points in a Gaussian quadrature scheme if $\gamma_0x/2\pi > 0.75$. Such results can be calculated extremely quickly.

As can be seen in Appendix C, negative values of $\gamma_1$ and $\gamma_2$ do not cause any difficulties in either method. Consequently, both techniques are correct for all values of $\varpi$. It is also comforting that they both converge to the FFT results—giving us three alternative ways of calculating $g$ which independently confirm the accuracy of the others.
Chapter 4

Formulation of General Integral Equation

Using the Green's function derived in the previous chapter, Green's theorem is now used to set up a pair of coupled integrals equation over the two intervals \((0, a)\) and \((a, \infty)\). In the process we verify the legitimacy of assuming that we can take the limit as \(\varepsilon \to 0\) in (2.8b), and demonstrate the existence of an eigenfunction expansion for the solution. The existence of such an expansion is usually assumed a priori in the floating elastic plate literature (e.g. Fox and Squire, 1990; Linton and Chung, 2003), perhaps being justified by comparing to another solution method for the same problem, so its proof will be of interest to researchers in the area.

4.1 Application of Green's Theorem

Before we apply Green's theorem we first reformulate it from its classical form into the form in which it is usually used in fluid dynamics.

We also specify the radiation conditions that \(\phi\) should satisfy in order to represent the physical situation being modelled (i.e., a wave from the left being partially reflected and partially transmitted by the central region).

We then divide the fluid region into three different regions and apply Green's theorem to each of them individually. Doing this allows us to check that finding \(\phi\) for \(\varepsilon > 0\) and then taking the limit produces the \(\phi\) that would be found by solving for it directly when \(\varepsilon = 0\). It also allows us to derive eigenfunction expansions for the solutions in
4.1.1 Statement of Theorem

Green's theorem in the form that it is usually stated outside fluid dynamics (Anton, 1992, p.1170) states that if \( \Omega \) is a simply connected plane region whose boundary \( \partial \Omega \) is a simple closed, piecewise smooth, positively oriented curve, then if \( u(x, z) \) and \( v(x, z) \) have continuous first partial derivatives on some open set containing \( \Omega \), then

\[
\iint_{\Omega} (\partial_x v - \partial_z u) \, dx \, dz = \oint_{\partial \Omega} (u(x, z) \, dx + v(x, z) \, dz).
\] (4.1)

The proof of the above theorem simply involves integrating by parts. Consequently, it still holds if the conditions on \( u \) and \( v \) are relaxed to requiring them to have integrable first derivatives, instead of needing them to be continuous.

If we then let \( u = u_* \partial_y v_* - v_* \partial_y u_* \) and \( v = v_* \partial_x u_* - u_* \partial_x v_* \), where \( u_* \) and \( v_* \) have integrable second derivatives, (4.1) implies that

\[
\iint_{\Omega} (v_* \nabla^2 u_* - u_* \nabla^2 v_*) \, dx \, dz = \oint_{\partial \Omega} (v_* \partial_n u_* - u_* \partial_n v_*) \, ds
\] 
\[ - \oint_{\partial \Omega} (v_* \partial_s u_* - u_* \partial_s v_*) \, dn, \tag{4.2}
\]

where \( s \) is the arc length travelling around \( \partial \Omega \) clockwise and \( n \) is the coordinate with its axis perpendicular to the direction of travel, and pointing out from \( \Omega \). \( \partial_s \) and \( \partial_n \) denote partial differentiation with respect to these coordinates. Clearly, \( dn = 0 \) on \( \partial \Omega \), so (4.2) can now be written in the form that Green's theorem usually takes in fluid dynamics:

\[
\iint_{\Omega} (v_* \nabla^2 u_* - u_* \nabla^2 v_*) \, dx \, dz = \oint_{\partial \Omega} (v_* \partial_n u_* - u_* \partial_n v_*) \, ds. \tag{4.3}
\]

4.1.2 Radiation Conditions

Before applying Green's theorem, let us first consider the expected asymptotic behaviour of \( \phi(x, z) \). As \( |x| \to \infty \), we require that

\[
\phi(x, z) \sim \begin{cases} 
 e^{i\alpha_0 z} \varphi_0(z) + Re^{-i\alpha_0 z} \varphi_0(z) & \text{as } x \to -\infty, \\
 Te^{i\beta_0 z} \tilde{\varphi}_0(z) & \text{as } x \to \infty,
\end{cases} \tag{4.4}
\]

where \( \varphi_0(z) = \varphi(z, \hat{\gamma}_0) \). For future reference, \( \varphi_n(z) = \varphi(z, \hat{\gamma}_n) \) for \( n = -2, -1, 0, 1, \ldots \) (The \( \varphi_n(z) \) were defined in Section 3.1.) The \( \hat{\gamma}_n \) are the roots of the dispersion relation.
for the right hand side \( f_2(\kappa) = 0 \), and the square roots \( \hat{\alpha}_n = \sqrt{\pi^2 - \kappa^2} \) are again taken from the upper half-plane or the positive real axis. Equations (4.4) are the radiation conditions referred to in Section 2.1 that are the last conditions needed to specify the solution.

When \( \varepsilon = 0 \), and \( \alpha_0 \) and \( \hat{\alpha}_0 \) are real, (4.4) represents a plane wave of unit amplitude arriving from the left, and being partially reflected (producing one plane wave of amplitude \( R \)) and partially transmitted (producing another plane wave of amplitude \( T \)). \( R \) is called the reflection coefficient and \( T \) is called the transmission coefficient; their values form part of the solution.

However, for \( \varepsilon > 0 \), when the two roots have positive imaginary parts, a \( \phi \) which is asymptotically like (4.4) will become infinite as \( x \to -\infty \). Although it is still possible to find such a \( \phi \), things are simpler if we search for a bounded function instead. For this reason we now define the scattering potential \( \psi(x, z) = \phi(x, z) - \phi_0(x, z) \), where \( \phi_0(x, z) = e^{i\alpha_0 x} \varphi_0(z) \), which is the result of subtracting the incident wave potential from the velocity potential. \( \psi \) is asymptotically equal to

\[
\psi(x, z) \sim \begin{cases} 
R e^{-i\alpha_0 x} \varphi_0(z) & \text{as } x \to -\infty, \\
T e^{i\hat{\alpha}_0 x} \hat{\varphi}_0(z) - e^{i\alpha_0 x} \varphi_0(z) & \text{as } x \to \infty,
\end{cases}
\]

and so, when \( \varepsilon > 0 \), it will decay exponentially as \( |x| \to \infty \).

From (2.8b), and using the fact that \( \phi_0(x, 0) = -\mathcal{L}_0(\partial_x) \phi_0, x(x, 0) \), the differential equation that it satisfies at \( z = 0 \) is

\[
-\psi(x, 0) = \mathcal{L}(x, \partial_x) \psi_0(x, 0) - \mathcal{L}_0(\partial_x) \phi_0, x(x, 0) \\
= \mathcal{L}(x, \partial_x) \psi_0(x, 0) + (\mathcal{L}(x, \partial_x) - \mathcal{L}_0(\partial_x)) \phi_0, x(x, 0).
\]

Equation (4.6a) is the most useful form to apply when \( x > 0 \), while (4.6b) is the most useful form when \( x < 0 \). (Recall that when \( x < 0 \), \( \mathcal{L} = \mathcal{L}_0 \).)

### 4.1.3 Application to the Current Problem

To find an expression for \( \psi(x, z) \) we divide the fluid region into three parts, as done in Figure 4.1. We then consider \( \psi \) in each of the three regions separately, using Green’s theorem to write it as a contour integral around the edges of each region.
Figure 4.1: The regions and contours of integration used in the application of Green’s theorem. Note that we travel clockwise around each region instead of anti-clockwise, as the $\zeta$ axis points downwards instead of upwards. Also note that the right hand limit of $\Omega_0$ should be the $\zeta$ axis, but it has been displaced to the left for clarity; both horizontal limits of $\Omega_1$ and the right hand limit of $\Omega_2$ have also been displaced slightly for the same reason.

Doing this allows us to check that finding $\phi$ for $\varepsilon > 0$ and then taking the limit produces the $\phi$ that would be found by solving for it directly when $\varepsilon = 0$. It also allows us to derive eigenfunction expansions for the solutions in the three regions.

We then combine the three regions by applying the conditions (2.8c) to eliminate some of the integrals. This final expression will be used to set up an integral equation in $\phi_z(x,0)$ for $x > 0$ (cf. Section 4.2).

Central Region

We will consider the central region first. If $(x, z)$ is inside this region, which we shall denote $\Omega_1 = \{ (\xi, \zeta) \mid 0 < \xi < a \& 0 < \zeta < H \}$ (cf. Figure 4.1), then from (3.1a) and (2.8a), $\psi(x, z)$ can be written as the following integral over that region:

$$
\psi(x, z) = \int_{\Omega_1} \psi(\xi, \zeta)(\partial_\xi^2 + \partial_\zeta^2 - l^2)G(x - \xi, z, \zeta)d\xi d\zeta
- \int_{\Omega_1} G(x - \xi, z, \zeta)(\partial_\xi^2 + \partial_\zeta^2 - l^2)\psi(\xi, \zeta)d\xi d\zeta
= \int_{\Omega_1} (\psi \nabla_\xi^2 G - G \nabla_\xi^2 \psi)d\xi d\zeta.
$$

(4.7a)
While the second integrand in (4.7a) is zero, and is consequently integrable, the first is highly singular at \((\xi, \zeta) = (x, z)\) and does not meet the criteria for Green’s theorem. However, adding the \(\psi(x, z)\) term on the left hand side of the equation accounts for the singularity, and indeed if the integral is evaluated by removing a circle of radius \(\varepsilon'\) centred at \((x, z)\), applying Green’s theorem (the delta function is zero outside this circle), and taking the limit as \(\varepsilon' \to 0\), one gets the same result as if we assume \(\delta(x - \xi, z - \zeta)\) is integrable and satisfies the requirements of Green’s theorem. (Note that although although the region produced by removing the circle is no longer simply connected, we can split this region into two simply connected regions, apply Green’s theorem to each one individually, and then add the results, giving one line integral around \(\partial \Omega_1\) and another around the edge of the circle. This latter line integral results in a \(\psi(x, z)\) term when the limit as \(\varepsilon' \to 0\) is taken.) The final result is that \(\psi(x, z)\) can now be written as

\[
\psi(x, z) = \int_{\partial \Omega} (\psi \partial_n G - G \partial_n \psi) \, ds, \tag{4.8}
\]

which equates to

\[
\psi(x, z) = \int_0^a (G(x - \xi, z, 0)\psi_z(\xi, 0) - G_\zeta(x - \xi, z, 0)\psi(\xi, 0)) \, d\xi \\
- \int_0^H (G_\xi(x, z, 0)\psi(0^+, \zeta) - G(x, z, 0)\psi_x(0^+, \zeta)) \, d\zeta \\
+ \int_0^H (G_\xi(x - a, z, 0)\psi(a^-, \zeta) - G(x - a, z, 0)\psi_x(a^-, \zeta)) \, d\zeta \tag{4.9}
\]

when we use the sea floor conditions (3.1c) and (2.8d).

The first integral may also be simplified by substituting for \(-G(x - \xi, z, 0)\) and \(-\psi(\xi, 0)\) using (3.1b) and (4.6a), giving

\[
\int_0^a (G\psi_z - G_\zeta \psi) \, d\xi = \int_0^a (G_\zeta (\mathbf{L} \phi_z - \mathbf{L}_0 \phi_{0,z}) - \psi_z \mathbf{L}_0 G_\zeta) \, d\xi. \tag{4.10}
\]

If there are no discontinuities in \(D\) and/or \(D'\), integrating the above expression by
parts and substituting into (4.9) lets us write

$$\psi(x, z) = \int_0^\alpha \left( L(\xi, \partial_\xi) - L_0(\partial_\xi) \right) G_\xi(x - \xi, z, 0) \phi_x(\xi, 0) d\xi$$

$$+ \left[ E(\xi, \partial_\xi; \phi) G_\xi(x - \xi, z, 0) \right]_{\xi = 0^+}^{\xi = 0^-} - \left[ E_0(\xi, \partial_\xi; \phi_0) G_\xi(x - \xi, z, 0) \right]_{\xi = 0^+}^{\xi = 0^-}$$

$$- \int_0^H \left( G_\xi(x, z, 0) \psi(0^+, \xi) - G(x, z, 0) \psi_x(0^+, \xi) \right) d\xi$$

$$+ \int_0^H \left( G_\xi(x - a, z, 0) \psi(a^-, \xi) - G(x - a, z, 0) \psi_x(a^-, \xi) \right) d\xi,$$  \hspace{1cm} (4.11)

where the operator $E$ is given by

$$E(x, \partial_x; \phi) = \partial_x \left( D(x) \left( \partial_x^2 - l^2 \right) \right) \phi_x(x, 0) - \left( D(x) \left( \partial_x^2 - l^2 \right) \right) \phi_x(x, 0) \partial_x$$

$$+ \phi_x(x, 0) \left( D(x) \left( \partial_x^2 - l^2 \right) \right) - \phi_x(x, 0) \partial_x \left( D(x) \left( \partial_x^2 - l^2 \right) \right)$$  \hspace{1cm} (4.12a)

$$= S(x, \partial_x) \phi_x(x, 0) - \phi_x(x, 0) S(x, \partial_x)$$

$$+ \phi_x(x, 0) M(x, \partial_x) - M(x, \partial_x) \phi_x(x, 0) \partial_x,$$  \hspace{1cm} (4.12b)

and $E_0$ is the same, but with $D(x)$ in (4.12a) replaced by the constant $D_0$. The bracketed term involving $E(\xi, \partial_\xi; \phi) G_\xi$ is the result of moving the $L$ off the $\phi_x(\xi, 0)$ in (4.10), while the $E_0(\xi, \partial_\xi; \phi_0) G_\xi$ term is the result of moving the $L_0$ off $\phi_{0x}(\xi, 0)$. Note that if there are discontinuities in $D$ and/or $D'$ at $x = x_c \in X_c$, then we simply get additional edge terms

$$- \left[ E(\xi, \partial_\xi; \phi) G_\xi(x - \xi, z, 0) \right]_{\xi = x_c^+}^{\xi = x_c^-} = \left( E(x_c^+, \partial_x; \phi) - E(x_c^-, \partial_x; \phi) \right) G_\xi(x - x_c, z, 0)$$

appearing. Note that the right hand expressions for the edge terms result from the fact that $G_\xi(x, z, 0)$ is continuous for $z > 0$ (it is actually analytic, which can be deduced most easily from its Fourier transform $\hat{G}_\xi(k, z, 0) = \varphi(z, \kappa)/(\Lambda_0(\kappa) \kappa \tanh \kappa H - 1)$ which behaves like $e^{-\kappa k}/(D_0 \kappa^5)$ as $|k| \to \infty$; it will have a logarithmic singularity in its fourth derivative when $z = 0$, as might have been predicted from equation 3.1b), and observing that $E$ is an odd function with respect to its second argument, and that $E_0(x_c^+, \partial_x; \phi_0) - E_0(x_c^-, \partial_x; \phi_0) = 0$.

However, for the sake of simplicity, we will assume for the moment that $D$ and $D'$ are continuous, and that we can ignore such terms.

Note that the above working did not assume anything about the size of $\varepsilon$ at all. Consequently, since $G$ and $\phi_0$ are continuous in $\varepsilon$, setting $\varepsilon = 0$ in (4.11) directly
and taking the limit as $\varepsilon \to 0$ give the same result for $\psi$. More precisely, it gives
the same relationships between $\psi(x, z)$ inside the main region, and the unknown functions $\phi_z(x, 0)$, $\partial^2_x \psi(x, z)$ ($j = 0, 1, x = 0, a$) and the unknown constants contained in $E(x, \partial_z; \phi)G_z$.

Our expressions for $\psi$ in the left and right hand regions will also depend on unknown functions and constants; however, if the relationships in both those regions between the general $\psi(x, z)$ and these unknowns are also the same whether $\varepsilon$ is set to zero directly or whether a limit is taken, then the solutions for the unknowns will also be the same. This is checked in the following two sections. (The solution for the unknowns corresponding to each individual region will necessarily depend on the unknowns from the other regions; hence, to show that $\psi$ converges everywhere as $\varepsilon \to 0$, we must show that the relationships in all three regions are the same regardless of whether $\psi$ is found directly or in the limit. Theoretically, those relationships could then be used to solve for all the unknowns, although in practice not all of them are actually used; they are just used to check that our limiting process is legitimate.)

Now, the last step that will be taken in this section is to check whether $\phi$ as implied by (4.11) satisfies (2.8b). To do this, we add $\phi_0(x, z)$ to (4.11), apply the operator $L_0(\partial_x)\partial_z$ and take the limit as $z \to 0$, giving

$$L_0(\partial_z)\phi_z(x, 0) = L_0(\partial_x)\phi_{0,z}(x, 0) + \int_0^\alpha L_0(\partial_z)(L(\xi, \partial_x) - L_0(\partial_\xi))g(x - \xi)\phi_z(\xi, 0)d\xi$$

$$+ \left[ E(\xi, \partial_\xi; \phi)L_0(\partial_x)g(x - \xi) \right]_{\xi \to 0^+}^{\alpha^-} - \left[ E_0(\xi, \partial_\xi; \phi_0)L_0(\partial_x)g(x - \xi) \right]_{\xi \to 0^+}^{\alpha^-}$$

$$- \int_0^H L_0(\partial_x)(G_z(x, 0, 0)\psi(0^+, \zeta) - G_z(x, 0, 0)\psi_z(0^+, \zeta))d\zeta$$

$$+ \int_0^H L_0(\partial_z)(G_z(x - a, 0, 0)\psi(a^-, \zeta) - G_z(x - a, 0, 0)\psi_z(a^-, \zeta))d\zeta,$$

recalling that $g(x - \xi) = G_z(x - \xi, 0, 0)$. We can now use (3.12) and the fact that $L_0(\partial_z)G_z(x - \xi, \zeta, 0) = -G(x - \xi, \zeta, 0)$ (since $G$ is symmetric in $z$ and $\zeta$, as can be seen most easily from equation 3.6) to eliminate $G_z$ and $g$. Hence, after integrating out the delta function in the first integral, which arises because of (3.12) and gives rise
to \((\mathbf{L}_0 - \mathbf{L})\phi_z(x, 0)\), the above equation simplifies to

\[
\mathbf{L}(x, \partial_z)\phi_z(x, 0) = -\phi_0(x, 0) - \int_0^a (\mathbf{L}(\xi, \partial_z) - \mathbf{L}_0(\partial_z)) G_\xi(x - \xi, 0, 0) \phi_z(\xi, 0) d\xi
\]

\[
- \left[ \mathbf{E}(\xi, \partial_z; \phi) G_\xi(x - \xi, 0, 0) \right]_{\xi=0^+}^{a^-} - \left[ \mathbf{E}_0(\xi, \partial_z; \phi_0) G_\xi(x - \xi, 0, 0) \right]_{\xi=0^-}^{a^-}
\]

\[
+ \int_0^H (G_\xi(x, 0, 0)\psi(0^+, \zeta) - G(x, 0, 0)\psi_x(0^+, \zeta)) d\zeta
\]

\[
- \int_0^H (G_\xi(x - a, 0, 0)\psi(a^-, \zeta) - G(x - a, 0, 0)\psi_x(a^-, \zeta)) d\zeta
\]

\[
= -\phi(x, 0),
\]

as required. This result is also demonstrated numerically in Section 6.2.2 for a ridge with a variable thickness profile (if the thickness is constant, equation 2.8b is easily verifiable by comparison with solutions found using other techniques such as mode-matching).

**Left Hand Region**

To find the potential in the left hand region, we proceed in the same way as with the central region, writing \(\psi(x, z)\) as

\[
\psi(x, z) = \int \int_{\Omega_0} \psi(\xi, \zeta)(\partial_\xi^2 + \partial_\zeta^2 - l^2) G(x - \xi, z, \zeta) d\xi d\zeta
\]

\[
= \int \left( \partial_n G \psi - \partial_n \psi \right) ds,
\]

(4.14a)

where \(\Omega_0 = \{ (\xi, \zeta) \mid \xi_0 < \xi < 0 \& 0 < \zeta < H \} \) (cf. Figure 4.1). Applying the sea floor conditions again, we obtain a similar expression to (4.9):

\[
\psi(x, z) = \int_0^a (G(x - \xi, z, 0)\psi_x(\xi, 0) - G_\xi(x - \xi, z, 0)\psi(\xi, 0)) d\xi
\]

\[
+ \int_0^H (G_\xi(x, z, 0)\psi(0^+, \zeta) - G(x, z, 0)\psi_x(0^+, \zeta)) d\zeta
\]

\[
- \int_0^H (G_\xi(x - \xi_0, z, 0)\psi(\xi_0^+, \zeta) - G(x - a, z, 0)\psi_x(\xi_0^+, \zeta)) d\zeta.
\]

(4.15)
Again we can simplify the first integral by substituting for \(-G\) and \(-\psi\) using (3.1b) and (4.6b), and then integrating by parts. Hence,

\[
\psi(x, z) = \left[ \mathcal{E}(\xi, \partial\xi; \psi)G_\xi(x - \xi, z, 0) \right]_{\xi = \xi_0}^{\xi_0^+} + \int_0^H (G_\xi(x, z, 0)\psi(0^+, \zeta) - G(x, z, 0)\psi_x(0^+, \zeta))d\zeta
- \int_0^H (G_\xi(x - \xi_0, z, 0)\psi(\xi_0^+, \zeta) - G(x - \xi_0, z, 0)\psi_x(\xi_0^+, \zeta))d\zeta. \tag{4.16}
\]

If \(\varepsilon > 0\), then \(G\) and \(\psi\) both decay exponentially if we let \(\xi_0 \to -\infty\), and so the final integral vanishes. On the other hand, if \(\varepsilon = 0\) then from (3.8) and (4.5) the integrand will be asymptotically equal to

\[-iA_0'\Re e^{i\alpha_0 z}\varphi_0(z) \times \left( ( -i\alpha_0 e^{-i\alpha_0 \zeta}) \times e^{-i\alpha_0 \zeta_0} - e^{-i\alpha_0 \zeta_0} \times ( -i\alpha_0 e^{-i\alpha_0 \zeta}) \right) \varphi_0^2(\zeta) = 0,
\]
and the final integral still vanishes.

Similarly, the bracketed term will vanish as \(\xi_0 \to -\infty\) regardless of whether \(\varepsilon = 0\) or not, letting us finally write

\[
\psi(x, z) = -\mathcal{E}(0^-, \partial_x; \phi)G_\xi(x, z, 0) + \mathcal{E}_0(0^-, \partial_x; \phi_0)G_\xi(x, z, 0)
+ \int_0^H (G_\xi(x, z, 0)\psi(0^-, \zeta) - G(x, z, 0)\psi_x(0^-, \zeta))d\zeta. \tag{4.17}
\]

In the above, we have also used the fact that for \(x < 0\), \(\mathcal{E} = \mathcal{E}_0\) and the linearity of \(\mathcal{E}\) which implies that

\[
\mathcal{E}(0^-, \partial_x; \phi) = \mathcal{E}(0^-, \partial_x; \phi + \phi_0) = \mathcal{E}(0^-, \partial_x; \psi) + \mathcal{E}(0^-, \partial_x; \phi_0).
\]

The \(\mathcal{E}_0(0^-, \partial_x; \phi_0)\) in (4.17) will cancel out the \(-\mathcal{E}_0(0^+, \partial_x; \phi_0)\) term in (4.11) when we come to combine the results for the three region.

As with the expression for \(\phi\) in the central region, we arrived at (4.17) independently of our choice of \(\varepsilon\), and the remaining terms are continuous in \(\varepsilon\), we again we can be confident of obtaining the correct solution in the limit as \(\varepsilon \to 0\). The last step we must take in verifying our solution method is to check it for the right hand region, which is slightly more complicated than it was for the other regions.

**Right Hand Region**

Going through the same reasoning that was used in the previous sections, applying Green's theorem to the region \(\Omega_2 = \{ (\xi, \zeta) \mid a < \xi < \xi_1 \text{ & } 0 < \zeta < H \} \) (cf. Figure 4.1),
and letting $\xi_1$ become infinite implies that for $\varepsilon > 0$ we can write $\psi$ in the right hand region as

$$
\psi(x, z) = (E(a^+, \partial_z; \phi) - (E_0(a^+, \partial_z; \phi_0))G_\xi(x - a, z, 0)
+ \int_{0}^{\infty} (\mathcal{L}(\xi, \partial_\xi) - \mathcal{L}_0(\partial_\xi))G_\xi(x - \xi, z, 0)\phi_z(\xi, 0)d\xi
- \int_{0}^{H} (G_\xi(x - a, z, 0)\psi(a^+, \xi) - G(x - a, z, 0)\psi_x(a^+, \xi))d\xi.
$$

(4.18)

This is the expression that we will use in our solution method. However, it is not obvious from it that taking the limit as $\varepsilon \to 0$ will produce the same solution as the one obtained by finding $\psi$ when $\varepsilon = 0$ directly. That the limit as $\varepsilon \to 0$ (which does exist) is the $\varepsilon = 0$ can be seen most easily by first defining a new Green’s function $G_*$ which satisfies the thin plate equation corresponding to the right hand plate, and applying Green’s theorem with $G$ replaced by $G_*$.

Now, the new Green’s function $G_*$ satisfies (3.1) but with (3.1b) replaced by

$$
\mathcal{L}_2(\partial_\xi)G_*\xi(x - \xi, z, 0) + G_*(x - \xi, z, 0) = 0.
$$

(4.19)

Applying Green’s theorem directly to $\phi$ itself, as opposed to the intermediary $\psi$, we obtain another expression for $\phi(x, z)$

$$
\phi(x, z) = (E(a^+, \partial_z; \phi)G_*\xi(x - a, z, 0)
- \int_{0}^{H} (G_*\xi(x - a, z, 0)\phi(a^+, \xi) - G_*(x - a, z, 0)\phi_x(a^+, \xi))d\xi.
$$

(4.20)

Like (4.17), this expression is independent of our choice of $\varepsilon$: it it is positive, the integrals and boundary terms $E G_\xi, \phi$ decay exponentially as $\xi_1 \to \infty$ and can be ignored; and if it is zero, substituting (4.4) and the equivalent expression to (3.8) for $G_*$ into those terms again cause them to vanish, in the same way that they did in the analogous terms at $\xi_0$ which arose when Green’s theorem was applied to the left hand region. Since all the remaining terms are continuous in $\varepsilon$, we can conclude that finding the solution for $\varepsilon > 0$, and then taking the limit as $\varepsilon \to 0$, does indeed produce the $\varepsilon = 0$ solution.

**Eigenfunction Expansions**

Before we combine the results for the three individual regions, we will demonstrate that one additional consequence of treating them separately is a proof of the existence of eigenfunction expansions for the solution. These expansions are generally
assumed a priori in mode-matching solutions, and can lead to quite simple solutions when combined with certain integration rules (e.g., Linton and Chung, 2003; also cf. Appendix F).

We will assume that \( \varepsilon = 0 \) throughout this section, as no limits need to be taken in eigenfunction-matching solutions.

Let us begin with the left hand region. Substituting (3.6) into (4.17) means that if \( D_0 > 0 \), \( \psi \) can be written as

\[
\psi(x, z) = \sum_{n=-2}^{\infty} a'_n e^{-i\alpha_n x} \varphi_n(z),
\]

(4.21)

where

\[
a'_n = i A''_n \left( \int_0^H \left( i\alpha_n \varphi(0^-, \zeta) - \varphi_x(0^-, \zeta) \right) \varphi_n(\zeta) d\zeta - \mathcal{E}(0^-, -i\alpha_n; \varphi) \varphi'_n(0) \right).
\]

(4.22)

It can be shown, by applying Green's theorem directly to \( \phi \) in the left hand region, that the \( a'_n \) can be written (for the purpose of comparison with Appendix F) as

\[
a'_n = i A''_n \left( \int_0^H \left( i\alpha_n \varphi(0^-, \zeta) - \varphi_x(0^-, \zeta) \right) \varphi_n(\zeta) d\zeta - \mathcal{E}(0^-, -i\alpha_n; \phi) \varphi'_n(0) \right).
\]

(4.23)

Now let us turn to the right hand region. If we define \( \hat{A}''_n = \hat{A}_n A''_n(\gamma_n) \), where, for \( n = -2, -1, \ldots \), \( \hat{A}_n = \text{Res}[1/f_2(\kappa), k = \alpha_n] = -\gamma_n^2/\alpha_n C_2(\gamma_n) \), then we can write the Green's function used in the previous section, \( G_x \), as the eigenfunction expansion

\[
G_x(x - \xi, z, \zeta) = i \sum_{n=-2}^{\infty} \hat{A}''_n e^{i\alpha_n|x-\xi|} \phi_n(z) \varphi_n(\zeta).
\]

(4.24)

When substituted into (4.20), this gives

\[
\phi(x, z) = \sum_{n=-2}^{\infty} d'_n e^{i\alpha_n(z-x)} \hat{\varphi}_n(z),
\]

(4.25)

where

\[
d'_n = i A''_n \left( \int_0^H \left( i\alpha_n \phi(a^+, \zeta) + \phi_x(a^+, \zeta) \right) \hat{\varphi}_n(\zeta) d\zeta + \mathcal{E}(a^+, i\alpha_n; \phi) \hat{\varphi}'_n(0) \right).
\]

(4.26)

Both (4.21) and (4.25) were derived fairly simply from the results of the previous two sections. However, the situation in the central region is more complicated. Substituting (3.6) into (4.11) gives the following expression for \( \psi \):

\[
\psi(x, z) = \sum_{n=-2}^{\infty} \left( \hat{a}_n(x) + \hat{b}_n e^{i\alpha_n z} + \hat{c}_n e^{i\alpha_n(a-z)} \right) \varphi_n(z),
\]

(4.27)
where
\[
\hat{a}_n(x) = iA_n'' \varphi_n'(0) \sum_{j=1}^{4} \mathcal{L}_{ij}(-i\alpha_n) \int_0^x d_j(\xi) \phi_\delta(\xi,0) e^{i\alpha_n(x-\xi)} d\xi \\
+ iA_n''' \varphi_n'(0) \sum_{j=1}^{4} \mathcal{L}_{ij}(i\alpha_n) \int_x^a d_j(\xi) \phi_\delta(\xi,0) e^{i\alpha_n(\xi-x)} d\xi, \tag{4.28a}
\]
\[
\hat{b}_n = iA_n'' \int_0^H (i\alpha_n \psi(0^+,\zeta) + \psi_x(0^+,\zeta)) \varphi_n(\zeta) d\zeta \\
+ iA_n'''(\mathcal{E}(0^+,i\alpha_n;\phi) - \mathcal{E}_0(0^+,i\alpha_n;\phi_0)) \varphi'_n(0), \tag{4.28b}
\]
\[
\hat{c}_n = iA_n'' \int_0^H (i\alpha_n \psi(a^-,\zeta) - \psi_x(a^-,\zeta)) \varphi_n(\zeta) d\zeta \\
- iA_n'''(\mathcal{E}(a^-,i\alpha_n;\phi) - \mathcal{E}_0(a^-,-i\alpha_n;\phi_0)) \varphi'_n(0). \tag{4.28c}
\]

The above result implies that any function that satisfies (2.8a) and (2.8d) and a fourth-order (or less) boundary condition at \(z = 0\) (since \(\mathcal{L}\) could have been an arbitrary operator) can be expanded in the form
\[
\sum_{n=-2}^{\infty} \hat{a}_n(x) \varphi_n(z),
\]
where the continuity of the \(\hat{a}_n(x)\) depends on the continuity properties of the coefficient functions in \(\mathcal{L}\). For example, if \(\mathcal{L}\) retains the form given by (2.9), then they will be continuous if \(D\) and \(D'\) are continuous (the coefficients in equations 4.28 are all continuous, but extra terms produced at any discontinuities when the integration by parts between equations 4.9 and 4.11 is done).

A special case that is relevant to Chapter 8 and Appendix F arises when \(D_1(x)\) and \(m_1(x)\) are constant. In that case we can define another Green’s function, \(G_1\), which satisfies
\[
\mathcal{L}_1(\partial_\xi)G_1(x - \xi, z, 0) + G_1(x - \xi, z, 0) = 0, \tag{4.29}
\]
and which may be written
\[
G_1(x - \xi, z, \zeta) = i \sum_{n=-2}^{\infty} B_n'' e^{ik_n|x - \xi|} \psi_n(z) \varphi_n(\zeta), \tag{4.30}
\]
where \(B_n'' = B_n\alpha_n^3(\kappa_n), B_n = -\kappa_n^2/k_n C_1(\kappa_n), \psi_n(z) = \varphi(z, \kappa_n), k_n = \sqrt{\kappa_n^2 - l^2}\) (where the square root is either positive real or in the upper half-plane), and where the \(\kappa_n\) satisfy the dispersion relation \(f_1(\kappa_n) = 0\). It can be shown that
\[
\phi(x, z) = \sum_{n=-2}^{\infty} (b_n' e^{ik_n x} + c_n' e^{ik_n(a-x)}) \psi_n(z), \tag{4.31}
\]

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where

\[ b_n' = iB_n'' \left( \int_0^H (ik_n \phi(0^+, \zeta) + \phi_x(0^+, \zeta)) \psi_n(\zeta) d\zeta + \mathcal{E}(0^+, ik_n; \phi) \psi_n'(0) \right), \]

\[ c_n' = iB_n'' \left( \int_0^H (ik_n \phi(a^-, \zeta) - \phi_x(a^-, \zeta)) \psi_n(\zeta) d\zeta - \mathcal{E}(a^-, -ik_n; \phi) \psi_n'(0) \right). \]

The above results all assumed that ice was present in all three regions. When any of the regions do not possess an ice cover, the sums in the eigenfunction expansions (4.21), (4.25) and (4.31) should only be from \( n = 0 \) to \( \infty \), as the dispersion relation for open water \( 1/(\kappa \tanh \kappa H) - \lambda = 0 \) does not have the two complex roots that the ice-coupled dispersion relations have.

Combination of Results for Individual Regions

At this point, we could combine the various results for the three regions in several different ways. For example, if \( D_1 \) and \( m_1 \) are constant, we could use the eigenfunction expansions obtained using \( G_* \) and \( G_1 \), and apply (2.8c) at \( x = 0 \) and \( x = a \) in the manner of Appendix F. Alternatively, we could derive a system of integral equations in \( \psi \) and \( \psi_x \) which could be solved numerically. (This could also be done if \( D_1 \) and \( m_1 \) were variable.)

However, we choose to add equations (4.11), (4.17) and (4.18), which were all obtained using \( G \), the Green's function for the left hand side, and if we apply (2.8c) then the integrals along the joints between adjacent regions will cancel each other out. This gives

\[
\psi(x, z) = \sum_{x_0 \in X_c} (\mathcal{E}(x_c^+, \partial_x; \phi) - \mathcal{E}(x_c^-, \partial_x; \phi)) G_\zeta(x - x_c, z, 0) \\
+ \int_0^\infty (\mathcal{L}(\xi, \partial_\xi) - \mathcal{L}_0(\partial_\xi)) G_\zeta(x - \xi, z, 0) \phi_z(\xi, 0) d\xi. \tag{4.33}
\]

There are no \( \mathcal{E}_0 \) terms in the above equation because the terms from the three different regions cancel each other out as \( \mathcal{E}_0(x_c^+, \partial_x; \phi_0) - \mathcal{E}_0(x_c^-, \partial_x; \phi_0) = 0 \). Note that this formula depends on \( h_0 \) being nonzero, otherwise the four derivatives required of \( G_\zeta \) will be singular, although the singularities are cancelled out by terms in \( \mathcal{E}G_\zeta \). If \( h_0 = 0 \), the solution proceeds more smoothly if we integrate by parts to reduce the number of derivatives required.
To separate the unknown constants implicit in the $\mathcal{E}G_\xi$ terms in (4.33), and to lead into the section on the application of the edge conditions (Section 4.2.3), we now rewrite it as

$$\psi(x, z) = \sum_{x \in X_e} P_{zc}^T \mathcal{L}_{\text{edge}}(\partial_x)G_\xi(x - x_c, z, 0)$$

$$+ \int_0^\infty (\mathcal{L}(\xi, \partial_\xi) - \mathcal{L}_0(\partial_\xi))G_\xi(x - \xi, z, 0)\phi_z(\xi, 0)d\xi, \quad (4.34)$$

where

$$\mathcal{L}_{\text{edge}}^T(\partial_z) = -(\mathcal{L}^+(\partial_z)\partial_x, \mathcal{L}^-(\partial_z), \partial_z, 1), \quad (4.35)$$

and

$$P_{zc} = \begin{pmatrix} P_0(x_c) \\ P_1(x_c) \\ M(x_c) \\ S(x_c) \end{pmatrix} = \begin{pmatrix} p_0(x_c^+) - p_0(x_c^-) \\ p_1(x_c^+) - p_1(x_c^-) \\ M^+(x_c) - M^-(x_c) \\ S^+(x_c) - S^-(x_c) \end{pmatrix}, \quad (4.36)$$

The functions $p_0(x)$ and $p_1(x)$ are given by

$$p_0(x) = D(x)\phi_z(x, 0), \quad p_1(x) = D(x)\phi_{zz}(x, 0) - D'(x)\phi_z(x, 0),$$

while the $M^\pm(x)$, given by

$$M^\pm(x) = \mathcal{M}(x^\pm, \partial_z)\phi_z(x^\pm, 0),$$

define the non-dimensional bending moments (with $y$ and $t$ dependence removed) acting on plate edges with normals pointing in either the opposite ($M^+$) or the same ($M^-$) directions as the $x$ axis. Their effects are illustrated in Figure A.1. That figure also shows the $S^\pm(x)$, defined similarly as

$$S^\pm(x) = S(x^\pm, \partial_z)\phi_z(x^\pm, 0),$$

which are the vertical edge forces on left or right-facing plate edges.

From equation (4.34), and also from the definitions of the $P_{zc}$, it is clear that $\phi(x, z)$ depends entirely on the values that the function $\phi_z(x, 0)$ takes for $x > 0$. In the next section we will use (4.34) to derive an integral equation for this unknown function.
The four constants contained in each vector $\mathbf{P}_{xc}$ are, at this point, undetermined since we have not found $\phi_z(x, 0)$ yet, and so they will form part of the overall solution. These four constants can generally be reduced to two by using the edge conditions that $\phi$ should satisfy at the point $x_c$ in consideration. For example, the conditions (2.12c) and (2.12d), which require that the bending moments and edge forces on each side of a discontinuity should cancel each other out, are equivalent to demanding that $M(x_c) = S(x_c) = 0$. This leaves only the $P_j(x_c)$ ($j = 0, 1$) to be determined by applying the two remaining conditions (2.12a) and (2.12b), once an expression involving them for $\phi_z(x, 0)$ has been found. Section 4.2.3 explains how these two edge conditions are applied.

The conditions (2.13a) and (2.13b) also imply that $M(x_c) = S(x_c) = 0$. However, in this case the edge conditions and the constants vanishing are not equivalent—we must also require that either $M^+(x_c)$ or $M^-(x_c)$ vanish to satisfy (2.13a) completely; similarly we need either $S^+(x_c)$ or $S^-(x_c)$ to vanish, in addition to their difference $S(x_c)$ vanishing, to satisfy (2.13b). Hence, in the free edge situation we will still have two equations with which to determine the remaining unknown constants; Section 4.2.3 also explains how these two edge conditions are applied.

Thus, since the only edge conditions that we will consider in this thesis imply that the moments and edge forces vanish, we can simply set $M(x_c) = S(x_c) = 0$ now and proceed without concerning ourselves with them. This implies that each discontinuity in $D$ and/or $D'$ only produces two unknowns, $P_0(x_c)$ and $P_1(x_c)$.

There are other possible edge conditions, however, in which $M(x_c)$ and $S(x_c)$ cannot be eliminated so easily. For example, Marchenko (1997) models a pressure ridge as a line irregularity separating two ice sheets at which certain other conditions involving the moments and edge forces must be applied.
4.2 Integral Equations

In general we will only have two critical points \( x_c = 0, a \). Assuming this, differentiating equation (4.34) and letting \( \zeta \rightarrow 0 \) lets us write

\[
\psi_z(x, 0) = \psi^T(x) P_0 + \psi^T(x - a) P_a \\
+ \int_0^a (\mathcal{L}_1(\xi, \partial_\xi) - \mathcal{L}_0(\partial_\xi)) g(x - \xi) \phi_z(\xi, 0) d\xi \\
+ \int_a^\infty (\mathcal{L}_2(\partial_\xi) - \mathcal{L}_0(\partial_\xi)) g(x - \xi) \phi_z(\xi, 0) d\xi, \tag{4.37}
\]

where \( \psi(x) = \mathcal{L}_{\text{edge}}(\partial_z) g(x) \). The two integrals above could have been combined into a single one from zero to \( \infty \), but it is sensible to separate them as they will both be treated rather differently as we undertake to find \( \phi_z(x, 0) \). In fact, we will think of (4.37) as a pair of coupled integral equations in the two unknown functions \( \phi_z(x, 0) |_{x \in (0, a)} \) and \( \phi_z(x, 0) |_{x > a} \), rather than as a single integral equation.

Having said that, however, certain simplifications are possible in some circumstances. For example, when \( h_2 = h_0 \) as in Chapter 6 which discusses the Scatteringscattering by a pressure ridge, \( \mathcal{L}_2 = \mathcal{L}_0 \) and the second integral vanishes leaving only a single integral equation. If \( h_2 = h_1(x) = h_0 \) as well, both the integrals disappear and all that remains is to find the unknown constants in \( P_0 \) and \( P_a \). Applying the free edge conditions at both \( x_c = 0 \) and \( x_c = a \) then gives the solution when two cracks are present (cf. Chapter 9), while just applying them at one of the two points gives the solution for a single crack, which is discussed in Chapter 5. This is effectively the same as applying the frozen edge conditions at the other point, which as we will see from Section 4.2.3, makes the \( P \) vector corresponding to that point identically zero. (That chapter shows that if we apply the frozen edge conditions, then \( P_{x_c} = 0 \) if \( D \) and \( D' \) are continuous at \( x = x_c \).

We can also make (4.37) more general without complicating our working overly much. In that equation we have taken the set \( X_c \) to be \( \{0, a\} \) as normally they are the only critical points but, as pointed out in Section 2.1.1, results will be presented in Chapter 6 when there are up to three interior critical points.
4.2.1 Reflection Coefficient

The reflection coefficient is the main result that we are interested in, so it is useful at this stage to write down an expression for it to give us some idea of the particular quantities that we will need to calculate. Once $\phi_z(x, 0)$ has been found for $x > 0$, $\psi_z(x, 0)$ for $x < 0$ will be given by the eigenfunction expansion

$$\psi_z(x, 0) = \sum_{n=-2}^{\infty} a_n e^{-i\alpha_n x},$$  \hspace{1cm} (4.38)

where, from (4.21), $a_n = a_n' \varphi'_n(0)$. A more useful expression for the $a_n$ is

$$a_n = i A_n \left( p^T(\alpha_n) \left( P_0 + P e^{i\alpha_n} \right) + \hat{\phi}(\alpha_n) - f_2(\gamma_n) \Phi^+(\alpha_n) e^{i\alpha_n} \right),$$  \hspace{1cm} (4.39)

where

$$p^T(k) = \mathcal{L}_\text{edge}^T(-ik) = \left( -ik f_+(\kappa), f_-(\kappa), ik, -1 \right), \hspace{0.5cm} f_{\pm}(\kappa) = -\mathcal{L}^\pm(ik) = \kappa^2 \pm (1 - \nu)^2.$$ 

Equation (4.39) comes from substituting (3.7) into (4.37) when $x < 0 < \zeta$. The transforms $\hat{\phi}$ and $\Phi^+$ are given by

$$\hat{\phi}(k) = \sum_{j=1}^{4} \mathcal{L}_{ij}(ik) \int_0^a d_j(\xi) \phi_z(\xi, 0) e^{i\kappa \xi} d\xi, \hspace{1cm} \Phi^+(k) = \int_a^\infty \phi_z(\xi, 0) e^{i\kappa (\xi - a)} d\xi.$$  \hspace{1cm} (4.40)

Letting $x \to -\infty$ in (4.21) and comparing with (4.4) suggests the reflection coefficient will be given by

$$R = a_0' = a_0 / \varphi'_0(0),$$  \hspace{1cm} (4.41)

and we can see from (4.39) that the main quantities we will need to determine it are the values of the transforms $\hat{\phi}$ and $\Phi^+$ when $k = \alpha_n$.

4.2.2 Transmission Coefficient

It will be shown in later chapters that a similar expression to (4.38) exists for $\phi_z(x, 0)$ when $x > a$:

$$\phi_z(x, 0) = \sum_{n=-2}^{\infty} d_n e^{i\alpha_n (x-a)} \text{ for } x > a.$$  \hspace{1cm} (4.42)

From (4.25), $d_n = d_n' \varphi'_n(0)$ but again, this is not particularly useful, and other as-yet-underived formulae will be used to compute them. Once they are found, though, the transmission coefficient will be able to be written

$$T = d_0 e^{-i\alpha_0 a} = d_0 e^{-i\alpha_0 a} / \varphi'_0(0).$$  \hspace{1cm} (4.43)
An alternative means of calculating the modulus of the transmission coefficient is provided by the relationship between $|R|$ and $|T|$ given in Section 4.2.4. (This relationship could equally be used as an alternative way to calculate $|R|$ if it turned out that $T$ was the easier coefficient to obtain.) However, no information about its phase can be deduced by this method.

4.2.3 Application of the Edge Conditions

Once we have derived an expression for $\phi_z(x,0)$, we can now proceed to apply the appropriate edge conditions. We have already established that our edge conditions (2.12) or (2.13) both imply that $M(x_c) = S(x_c) = 0$, leaving only the $P_j(x_c)$ ($j = 0, 1$) to be determined. Depending on which set of edge conditions are used, those constants must be found in different ways; these are described below. Also, depending on the properties of $D$, it may either be easier or harder to apply the boundary conditions from inside the variable region; alternatively we may apply them from outside it using the eigenfunction representations (4.38) and (4.42). Sometimes, however, having both options available is necessary as only one approach is applicable, such as when open water is present (the edge conditions must be applied from within the ice/plate; cf. the problem of a breakwater with a variable thickness profile treated in Section 7.1.4), or when $D$ and/or $D'$ have discontinuities inside the variable region. If we do have a choice then having both methods working also provides a useful check of our theory.

Frozen Edge Conditions

At points where the frozen edge conditions need to be applied, we can substitute (2.12a) and (2.12b) directly into the definitions of $P_0$ and $P_1$ (4.36) to give

$$P_0(x_c) = (D(x_c^+) - D(x_c^-)) \phi_z(x_c^\pm, 0), \quad (4.44a)$$
$$P_1(x_c) = (D(x_c^+) - D(x_c^-)) \phi_{zx}(x_c^\pm, 0)$$
$$- (D'(x_c^+) - D'(x_c^-)) \phi_z(x_c^\pm, 0), \quad (4.44b)$$

for all $x \in X_0$. These equations do guarantee that (2.12) is satisfied, but when a simple expression for either of the derivatives $\phi_{zx}(x_c^\pm, 0)$ is not available, (2.12b) is sometimes more efficiently applied in a different manner.

When $x_c \in \{0, a\}$, we have no such problem. In that case $\partial_x^j \phi_z(x_c^\pm, 0)$ ($j = 0, 1$) are easily calculated using (4.38) and (4.42). However, if $D$ is sufficiently smooth, even
this may be unnecessary. For example if $D$ and $D'$ are both continuous at $x = x_c$, then (4.44) implies that both the $P_j(x_c)$ are identically zero. When $D$ is continuous but $D'$ is not, then $P_0(x_c) = 0$ still, but

$$P_1(x_c) = -\left(D'(x_c^+) - D'(x_c^-)\right)\phi_z(x_c^\pm, 0).$$

(4.45)

Since we would only be applying the above condition in a situation where $h_1(x)$ is varying over $(0,a)$, it does not represent any extra effort as we will have already determined the values of the $\phi_z(x_c, 0)$ in the process of solving (4.37) for $\phi_z(x, 0)|_{x \in (0,a)}$ using a quadrature scheme.

For critical points $x_c \notin \{0, a\}$, we can also exploit the continuity of $D$ and $D'$ in the same way that we did when $x_c$ was in that set. However, when $D$ is not continuous then we must somehow apply (2.12b) without the convenience of having the eigenfunction expansions (4.38) and (4.42).

One method that could be used is simply to approximate the derivatives numerically. For example, $\phi_{xx}(x_c^+, 0) \approx (\phi_z(x_c^+ \pm \Delta, 0) - \phi_z(x_c^+, 0))/\Delta$. Alternatively, we can differentiate (4.37) analytically to find it, as described in Section 6.1.3. Using the numerical approach, one needs to increase the number of panels that $(0,a)$ is divided up into so that $\Delta$ is small enough for the finite difference formula to converge, while the analytical approach requires more terms to be used in the expansion (3.7) for $g(x - \xi)$. Although the latter approach requires slightly more algebra, it is generally preferable to the numerical method, as increasing the number of modes in (3.7) does not affect the size of the matrix that must be inverted, while decreasing $\Delta$ does.

**Free Edge Conditions**

In this case, we require $M^\pm(x_c) = 0$ and $S^\pm(x_c) = 0$. Since we have already required their sums $M(x_c) = M^+(x_c) + M^-(x_c)$ and $S(x_c) = S^+(x_c) + S^-(x_c)$ to vanish, we now only have to make the moments and edge forces vanish on one side of $x_c$. As with applying the frozen edge conditions when $D(x)$ is discontinuous, this is also most easily done by first calculating the eigenfunction expansions (4.38) and (4.42), and making the aforementioned quantities vanish outside the variable region $(0,a)$.

If either or both of $h_0$ or $h_2$ are zero, however, then the free edge conditions must be applied from within $(0,a)$, in which $h_1(x)$ may be varying. If $h_1$ is constant, such
as in the problem of a finite ice strip surrounded by open water or a breakwater (cf. Section 8.1.3), then an eigenfunction expansion also exists for $\phi_s(x,0)|_{x<0}$, and the edge conditions may still be applied relatively easily. If $h_1(x)$ is not constant, however, we must do a little more work to apply them.

As when we were trying to apply the frozen edge conditions within the variable region, differentiating (4.37) produces delta function type singularities and the algebra can become quite complicated when two or three derivatives are required. However, it turns out that these singularities combine to produce relatively simple expressions for the moments and vertical edge forces. This is discussed in more detail in Section 7.1.4.

### 4.2.4 Energy Conservation Theorem

In Appendix D Green’s theorem, with some assumptions about the nature of the edge conditions to be applied, is used to derive an energy conservation result that can be used as a simple check that our results, and especially their numerical implementation, are not incorrect. Further checks must also be performed to verify that they are in fact correct, but nevertheless this still proves quite useful. It states that the reflection and transmission coefficient must satisfy the relation

$$|R|^2 + s|T|^2 = 1,$$

(4.46)

where $s = \frac{A_0^0}{A_0^0}$ is called the intrinsic admittance. Note that when $h_2 = h_0$ then (4.46) simplifies to the more familiar $|R|^2 + |T|^2 = 1$ (e.g. Squire and Dixon, 2000).

This equation was first derived by Evans and Davies (1968), although their formula for the intrinsic admittance was incorrect for oblique angles of incidence (the error was corrected by Fox and Squire, 1994). However, the derivation in Appendix D is a variation of the one given by Balmforth and Craster (1999).
Chapter 5

Scattering by a Single Crack

The problem to be discussed in this chapter is the scattering of flexural-gravity waves by a single crack. It is illustrated in Figure 5.1. Such cracks are extremely common throughout the ice sheets of the Arctic and Antarctic, along with other features such as pressure ridges and open and refrozen leads, which are treated in later chapters.

An alternative application of this work could be to a crack or weld in a floating airport. Although it is unlikely that an open crack would exist in such a structure, a suitably modified set of edge conditions could be applied in much the same way as the free edge conditions are in this chapter. At present, however, it is unclear how such a crack might be represented realistically.

Returning to ice modelling, the crack edges are usually taken to be free edges (cf. equations 2.13). This problem was first solved for infinite depth by Marchenko (1997), who presented results for arbitrarily many parallel cracks by writing the solution as an inverse Fourier transform. In 2000, Squire and Dixon treated the special case of the scattering of normally incident waves by a single crack (also for infinite depth); Williams and Squire (2002) later extended their work to allow for obliquely incident waves. Both of these papers used the infinite depth Green’s functions and were able to produce exact analytical formulae for the reflection and transmission coefficients.

For sea water of finite depth two equivalent methods of solution were reported by Evans and Porter (2003b). The first was an eigenfunction matching method, and the second was a Green’s function method. Both used the inherent symmetry of the problem to divide it into two parts—a symmetric and an antisymmetric problem, each of
which could be solved by finding a single unknown constant.

Figure 5.1: The situation to be modelled in this chapter: the scattering that occurs as an obliquely incident wave arrives at a single crack in a uniform sheet of ice. The sea water has a finite constant depth of \( H \).

In this chapter we present the single crack problem as a special case of the general problem introduced in Chapter 2. The integral equation (4.37) that was derived in Chapter 4 (using the Green’s function presented by Evans and Porter, 2003b) in this case also reduces to an algebraic one involving two unknown constants. These constants are able to be found independently of each other, in the same way that Squire and Dixon (2000) and Williams and Squire (2002) found them when the water was taken to be infinitely deep. As this method is relatively simple, it is presented for finite depth in the following section. Infinite depth results may be obtained simply by substituting the infinite depth Green’s function for the finite depth one into the given working, as can another approximation to the Green’s function that applies when the water is sufficiently shallow.

Since the problem itself is also relatively straightforward, we will take the opportunity in the results section of this chapter to investigate the effects of different parameters such as the wave period, ice thickness, angle of incidence and water depth more fully than in subsequent chapters, in which variations in ice thickness and the scattering by combinations of irregularities are introduced.
5.1 Solution Method

In the first case we treat the problem to be solved as a special case of the integral equation (4.37). This leads to a solution that is essentially a finite depth version of those given by Squire and Dixon (2000) and Williams and Squire (2002). In addition, we also present some approximations to our results—namely infinite and shallow water depth approximations. A Chebyshev polynomial approximation is also demonstrated which speeds up the production of reflection and transmission graphs markedly.

5.1.1 Full Solution

Here \( h_0 = h_1 = h_2 \), so \( \mathcal{L}_1(x, \partial_x) = \mathcal{L}_2(\partial_x) = \mathcal{L}_0(\partial_x) \), which make the kernels in both the integrals in (4.37) vanish. In addition we do not need a free edge at \( x = a \) at this stage, so applying (2.12), we can see from its definition (4.36) that \( \mathbf{P}_a = 0 \) identically. Hence (4.37) reduces to

\[
\phi_z(x, 0) = e^{i\alpha x} \varphi_0(0) + \psi_T^T(x, 0) \mathbf{P}_0, \tag{5.1}
\]

and so the solution is complete once the unknown vector \( \mathbf{P}_0 \) is found. This is done by first noting that the transverse edge force and bending moment vanish from both sides, making \( \mathbf{M}(0) = \mathbf{S}(0) = 0 \), and then solving for the remaining constants \( P_0(0) \) and \( P_1(0) \). The expected 2 \( \times \) 2 matrix we must invert turns out to be diagonal as

\[
\lim_{\xi \to \pm} \mathbf{S}(x, \partial_x) \mathbf{M}(\xi, \partial_\xi) g(x - \xi) = \lim_{\xi \to \pm} \mathbf{M}(x, \partial_x) \mathbf{S}(\xi, \partial_\xi) g(x - \xi) = 0.
\]

This follows from the fact that \( g \) can be written as the sum \( g_0 + g_1 \), where \( g_1 \) has seven continuous derivatives (by inspection of its Fourier transform, given in equation 3.9b), and where differentiating equation (3.11) shows that \( (\partial_x^2 - l^2)^2 g_0(0^+) = (\partial_x^2 - l^2)^2 g_0(0^-) \) (since we are not integrating over the \( \delta'(x) \) term produced by doing this, we can ignore it as it is zero when \( x \neq 0 \)). Thus, \( \mathbf{M} \mathbf{S} g(x) \) is continuous and odd, and consequently, the two constants \( P_0(0) \) and \( P_1(0) \) can effectively be found independently of each other, as follows:

\[
P_0(0) = -\frac{i\alpha_0 f_+(\gamma_0)}{\Lambda_0(\gamma_0) Q_0}, \tag{5.2a}
\]

\[
P_1(0) = -\frac{f_-(\gamma_0)}{\Lambda_0(\gamma_0) Q_1}, \tag{5.2b}
\]

where \( f_\pm(\kappa) = \kappa^2 \pm (1 - \nu) l^2 \) (cf. equation 4.39), and

\[
Q_0 = -\lim_{x \to 0^\pm} \mathcal{S}^2(x, \partial_x) g(x), \quad Q_1 = -\lim_{x \to 0^\pm} \mathcal{M}^2(x, \partial_x) g(x).
\]
The numbers $Q_0$ and $Q_1$ can be calculated from (3.7).

Noting that $f_2(\gamma_n) = f_0(\gamma_n) = 0$ (since $h_2 = h_0$), we can see that there will be no contribution from the $\Phi^+ (\alpha_n)$ terms in (4.39); the $\phi (\alpha_n)$ terms are also zero because $h_1(x) = h_0$; and of course $P_a = 0$ for the reasons given above, so the reflection coefficient follows from (4.41). If we also use (3.7) again to write $\phi_+(x,0)|_{x>0}$ as

$$\phi_+(x,0) = \sum_{n=-2}^{\infty} b_n e^{i\alpha_n x} \quad \text{for } x > 0,$$

where

$$b_n = \varphi'_0(0) \delta_{n0} + i A_n P^T (-\alpha_n) P_0,$$

we can also obtain the transmission coefficient, allowing us to write

$$R = i A_0 \left( \frac{\alpha_0^2 f_1^2 (\gamma_0)}{Q_0} + \frac{f_2^2 (\gamma_0)}{Q_1} \right), \quad (5.4a)$$

$$T = 1 - i A_0 \left( \frac{\alpha_0^2 f_1^2 (\gamma_0)}{Q_0} - \frac{f_2^2 (\gamma_0)}{Q_1} \right). \quad (5.4b)$$

These can easily be shown (cf. Williams and Squire, 2002) to satisfy the conservation of energy theorem (4.46, D.11) which in this case is simply $|R|^2 + |T|^2 = 1$.

5.1.2 Approximations

In this section we will introduce some approximations that will be used in the following chapters and sections. The first approximation is a purely numerical one but, although it is completely unrelated to either the mathematics or physics of the problems at hand, it is nevertheless able to speed up the reproduction of figures presented later considerably. When investigating parameters such as the wave period or the angle of incidence, this approximation simply involves calculating the reflection coefficient for only a few values of the parameter of interest (usually about 50–100, compared to about 150–200 for a graph of sufficient resolution), and interpolating between those points using Chebyshev polynomials. The efficacy of this technique is demonstrated in Section 5.2.1.

The second approximation we will use is the infinite depth approximation—which holds if the sea water is deep enough that disturbances at the surface have become negligible by the time they reach the sea floor. In that case (2.8d) is satisfied automatically. This approximation will be used in all later chapters, firstly to eliminate the effect of water depth so that we can isolate the effects that the different types
of irregularities have on the scattering of the incident wave, and also because it will be shown that it will generally hold in the regions we are attempting to model. It is introduced formally below, and an appropriate deep water criterion is established in Section 5.2.3 while investigating the effect of water depth on wave scattering.

The final approximation used, apart from those introduced when studying multiple irregularities in Chapter 9, is the shallow water approximation. This approximation is less applicable to the aforementioned regions, i.e. central Arctic waters away from the coast, and the waters beyond the Antarctic ice shelves, than the infinite depth approximation, and thus it will not be used in following chapters. Nevertheless, it was still thought of interest to investigate in this chapter the effect that shallow water would have on the scattering by a crack, and also to attempt to establish a suitable criterion that would determine when water beneath an ice sheet could be considered shallow. This is also introduced formally below and results are presented in Section 5.2.3. One surprising result is that for certain period ranges (larger periods), the two water depth approximations coincide.

One advantage of the shallow water approximation, though, is that when it is applicable results can be obtained extremely quickly. In addition, results can often be obtained exactly as the Green’s function for the problem, has only three terms in its eigenfunction expansion, which is of the same form as (3.7). This also contributes to the high speed of the solution.

One further point to make about the shallow water Green’s function is that it has basically the same structure as the Green’s function for an “isolated” elastic plate that is considered without air or water pressure acting on it. Consequently, techniques for the solution of shallow water problems can easily be transferred to isolated plate problems. This is also true when two-dimensional variation is permitted.

Infinite Depth Approximation

This approximation is obtained by requiring \( \phi \) to satisfy the following condition as \( z \to \infty \)

\[
\lim_{z \to \infty} \phi_z(x, z) = 0
\]

(5.5)
in place of the usual sea-floor condition (2.8d). This is done by simply replacing the finite depth Green’s function (3.4) with the infinite depth one (3.15) in the working in
Chapter 4 and above in Section 5.1.1. In practice this means that the constants $Q_0$ and $Q_1$ may be evaluated exactly using (3.20), (3.22) and (3.23), making the single crack problem one in which the infinite depth results are more convenient to calculate than the finite depth ones.

**Shallow Water Approximation**

For shallow enough water, an approximation to the system (2.8), can be obtained by integrating (2.8a) from 0 to $H$ with respect to $z$ and neglecting $O(H^2)$ terms (Stoker, 1957). This leads to the relation $\phi_z(x, 0) = H(\partial_x^2 - \ell^2)\phi(x, 0)$, which on substituting into (2.8b) gives the following ODE in $\phi(x, 0)$:

$$\left(H \mathcal{L}(x, \partial_z)(\partial_x^2 - \ell^2) + 1\right)\phi(x, 0) = 0,$$

(5.6)

with the six initial conditions required in this case given by

$$\phi_z(0^+, 0) - \phi_z(0^-, 0) = \phi(0^+, 0) - \phi(0^-, 0) = 0,$$

(5.7a)

$$\mathcal{L}^+(\partial_x^2 - \ell^2)\phi_x(0^+, 0) = 0,$$

(5.7b)

$$\mathcal{L}^-(\partial_x^2 - \ell^2)\phi_x(0^-, 0) = 0.$$

(5.7c)

Equation (5.6) leads to the simplified dispersion relation for the left hand side

$$\Lambda_0(\kappa)\kappa^2H - 1 = 0,$$

(5.8)

which can be obtained from the finite depth relation by replacing $\tanh \kappa H$ with $\kappa H$ (Kinsman, 1984). The behaviour of its roots are described in Appendix B.1—there is one positive real root $\gamma_0$, two roots in the upper complex half-plane, $\gamma_- 1$ and $\gamma_- 2$, and their negatives. Usually $\gamma_- 1$ and $\gamma_- 2$ are complex, in which case $\gamma_- 2 = \gamma_- 1^*$, but as for the finite depth case, these two roots become purely imaginary as period decreases.

The point made above about the Green’s function for the shallow water problem, $g_\delta$, which satisfies

$$\left(H \mathcal{L}_0(\partial_\xi)(\partial_x^2 - \ell^2) + 1\right)g_\delta(x - \xi) = \delta(x - \xi),$$

and is given by

$$g_\delta(x - \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik(x-\xi)}dk}{\Lambda_0(\kappa)\kappa^2H - 1} = \sum_{n=-2}^{0} \frac{\gamma_n e^{\gamma_n|x-\xi|}}{2\alpha_n(2D_0\gamma_n^6 + 1)},$$

(5.9)
is that it is very similar to the Green's function for an isolated plate, which has Fourier transform $\hat{g}_p(k) = 1/(\alpha_p k^4 - 1)$, where $\alpha_p = (D_0 L)/(m_0 h_M)$. (The dispersion function for the isolated plate, i.e. the denominator of $\hat{g}_p$, comes from setting $P = P_a = 0$ and $\mathcal{L} = \mathcal{L}_0$ in equation (2.2), and nondimensionalizing as in Chapter 2.) Like the shallow water dispersion relation, the isolated plate dispersion relation has two real roots, $\pm \alpha_p^{-1/4}$, but only two non-real roots, $\pm i \alpha_p^{-1/4}$. Thus $g_p$ will have two terms in its eigenfunction expansion, instead of three, which is only a trivial adjustment. When two-dimension problems are treated, the eigenfunctions change from exponential functions to Hankel functions (a type of Bessel function), but we would still be able to transfer our shallow water solution methods to isolated plate problems simply.

Appendix B.2 describes how the shallow water situation is produced from the finite depth one by the purely imaginary roots $\{\gamma_n | n \geq 1\}$ becoming larger and larger as the value of $H$ decreases, implying that their residues, the $A_n$ coefficients, will become less and less significant. Consequently, when calculating $g$ for small $H$, we can truncate the series (3.7) after only three terms, and simply use the shallow water roots instead of the finite depth ones. Note, however, that we still use the original formulae for the $A_n$ residues, only replacing the finite depth roots in those formulae with the zeros of (5.8). Having done this, the constants $Q_0$ and $Q_1$ can be easily approximated, and the reflection and transmission coefficients obtained from them using (5.4) as in the full finite depth solution.

In addition, an approximation to $\phi$ itself can be obtained, using our values for $Q_0$ and $Q_1$ to determine $P_0(0)$ and $P_1(0)$, and substituting them into (4.34). We also approximate the $\varphi_n(z)$ in (3.6) by $1 + \gamma_n^2 z(z - 2H)/2$. We shall call this method of solution the partial shallow water solution, as, although $\phi$ obtained in this way does satisfy (5.7b) and (5.7c) exactly, it only satisfies (5.7a) approximately. Nevertheless, it does provide quite accurate values for $R$ and $T$ at smaller water depths—even more accurate than the full shallow water solution, which would usually be found with a mode-matching technique (requiring the inversion of a $6 \times 6$ matrix—see Stoker, 1957, for the full shallow water solution to a similar problem). Section 5.2.3 compares the two approximations with the finite depth results for shallower water.
5.2 Results

In this situation where there is only a single crack in an otherwise uniform sheet of ice, the form of our results can be attributed to the effect of four parameters: $h_M$, the dimensional wave period, $\theta$, and $H$. Section 5.2.1 examines the effect of the former two, Section 5.2.2 investigates the effect of angle of incidence, while Section 5.2.3 looks at how our results change with decreasing water depth. Section 5.2.1 also demonstrates the effectiveness of the Chebyshev interpolation technique, and Section 5.2.3 seeks to establish suitable criteria for when the infinite depth and the shallow water approximations hold.

5.2.1 The Effect of Ice Thickness and Wave Period

In this section we concentrate on the effect of changing $h_M$ and period. Since the infinite depth results are so easily calculated for the current single-crack situation, we shall only use those results here and investigate the effect of decreasing the water depth in Section 5.2.3. We will also only use results for normal incidence, saving any oblique incidence results until Section 5.2.2.

Figure 5.2a illustrates, for normally incident waves on water of infinite depth, the behaviour of $|R|$ with period for four different thicknesses. Since the size (height) of the crack becomes larger as the ice thickness increases, we might expect that the amount of reflection would also increase with it, and indeed this is what the figure shows. And, as might also be expected, there is a general decrease in reflection as the waves become longer (as period increases). What is more surprising is the finer structure of the curves, especially the way in which each $|R|$ curve drops from showing near total reflection to showing a particular period at which transmission is total. Having done this it rises to a maximum before dropping asymptotically to zero again. These results are identical to those of Squire and Dixon (2000).

Figures 5.2b and c show similar patterns to Figure 5.2a, but illustrate how the effects of ice thickness and wave period can be described in terms of different single parameters—both exactly (b), and approximately (c). These parameterizations are described below.
Figure 5.2: Parameterizations combining the effect of ice thickness and wave period on the scattering by a single crack. The figures show the behaviour of $|R|$ with (a) wave period, (b) $\varpi = \lambda - \mu$, and (c) $\tau$ (nondimensional period) for four different thicknesses. The ice thicknesses used are 0.5 m (solid), 1 m (dashed), 2 m (chained) and 10 m (dotted). Note that the different curves are mapped onto each other exactly by plotting against $\varpi$, while they are mapped approximately onto each other by plotting against $\tau$. The incoming waves are normally incident and the water depth is taken to be infinite.

Parameterization of Results

Our solution in this case depends entirely on the roots $\alpha_n$ of the dispersion function $f_0$. These are themselves described entirely by the three parameters $\varpi(h_M, \lambda) = \lambda - \mu$, $\theta$ and $H$. Since we have kept $\theta = 0$ and also let $H \to \infty$, their only other source of variation is $\varpi$, and so we might expect the four curves in Figure 5.2a to coalesce if $|R|$ is plotted against that parameter instead of against period. And indeed, Figure 5.2b shows that this is exactly what happens, although the same period range gives a slightly different range in $\varpi$. Since $\varpi$ may be written as $\varpi = \lambda - \sigma/\lambda^{1/4}$, and $\lambda$ is entirely a function of $\omega$, or equivalently of $\tau \propto \text{period} \times h_M^{-3/8}$, an increase in period corresponds
to an increase in $\lambda$, and so $\varpi$ becomes more positive, as might have been guessed from the shape of the $|R| = \varpi$ curve. Conversely, increasing the thickness decreases the value of $\lambda$, forcing $\varpi$ to become more negative and generally producing more reflection. This is confirmed by the figure as only the 10 m curve is able to be seen for the lower values of $\varpi$.

It was also thought that since the zero in reflection noted earlier can now be mapped to a single value of $\varpi$ (of approximately 0.90), this might be of some use in attributing some physical significance to the period at which it occurs for each thickness. Unfortunately, this only proved to be useful as a tool for negating various proposals. For example, for each different thickness tried, the zero turned out to be tantalisingly close to the period at which the group velocity of a propagating wave $d\omega/da_0$ (Stoker, 1957) coincides with its phase velocity $\omega/a_0$. However, this explanation could be ruled out by virtue of it not keeping $\varpi$ constant as the ice thickness was changed.

That there should be a zero at all was quite puzzling. As can be seen from the results given in later chapters, such zeros occur for several periods when there are two critical points present, and thus could be attributed to “resonances" where the two waves resulting from reflection at the two points interfere destructively to produce zero reflection. However, it was not obvious why in the current situation of a single crack, with only one critical point, there should also be a zero in $R$. This was recently resolved by Vaughan et al. (unpublished manuscript, 2005), who showed that by treating the crack as an open lead with increasingly smaller width, the wave reflected by the crack could be decomposed into two waves—one from the first edge and another from the second. Thus it is still possible for destructive interference to occur for a single crack. This is also mentioned in Chapter 8, when the scattering by a single lead is discussed.

A given value of $\varpi$ and $h_M$ (which in this chapter is simply the background ice thickness $h_0$) describes a quintic equation in $\lambda^{1/4}$:

$$\lambda^{5/4} - \varpi \lambda^{1/4} - \sigma(h_M) = 0. \quad (5.10)$$

A single positive real solution to this equation always exists, letting us then deduce the corresponding period. However as was pointed out in Section 2.2.1, $\sigma$ only varies as $h_M^{1/4}$ with the actual value of the ice thickness and this fact can be exploited to generate a simpler, although approximate, parametrization. If we define $h'_M$ to be a
reference ice thickness of, say, 1 m, and then approximate $\sigma$ by $\sigma' = \sigma(h'_M)$, we can eliminate the dependence on $h_M$ of the value of $\lambda$ solving (5.10). In other words, $\omega$ can be approximated by a function of $\lambda(\tau)$ alone—implying that we may use $\lambda$ itself, or equivalently $\tau$, as our parameter of choice, thus combining the effects of both period and ice thickness in an extremely simple fashion. Figure 5.2c shows that this approach, which effectively just scales the period axis in Figure 5.2a, is also quite effective.

Given that results for different thicknesses may be obtained both accurately and easily from $h'_M$ by referring to the nondimensional period $\tau$, from now on we will only present results for $h_M = h'_M$, which will generally be chosen so that the background thickness $h_0 = 1$ m (although, as mentioned in Section 2.2.1, there will be some exceptions to this policy). This enables dimensional periods to be presented, which are more easily interpreted than nondimensional periods, and will also provide typical results for both the Arctic and the Antarctic as 1-m-thick ice is very common in both regions.

The scaling factor to be used in translating results to a different maximum ice thickness $h_M$ (with associated characteristic time $\tau_M$) is simply $\tau_M/\tau'_M = (h_M/h'_M)^{3/8}$ (where $\tau'_M$ is the characteristic time corresponding to an ice thickness of $h'_M$)—results for $h_M$ are well approximated by simply multiplying the period axis by that factor. Alternatively, if the scattering is plotted against another variable such as angle of incidence for a given wave period, those results should be taken to apply to that period multiplied by $(h_M/h'_M)^{3/8}$.

Approximation of Results by Chebyshev Polynomial Interpolation

To speed up our calculation of results, both in this and following chapters, we will use the Chebyshev approximation mentioned earlier. Although it is not specific to the physics or even the mathematics of the problems at hand, it still proves to be extremely effective at reproducing graphs such as Figure 5.2 and with a significant reduction in computing time as well.

Figures 5.3a, b show successive Chebyshev approximations to $|R|$ and $|T|$ respectively (plotted exactly as solid curves). The solid curve in (a) is the same as the solid curve in Figure 5.2a, while the solid $|T|$ curve is presented to demonstrate its behaviour with period. As period increases and $|R|$ decreases to zero, $|T|$ increases to 1, so as to satisfy the energy conservation law $|R|^2 + |T|^2 = 1$. Once it has reached 1 it approximately stays there, dipping slightly when $|R|$ has its maximum at about
Figure 5.3: The approximation of the scattering by a single crack using Chebyshev interpolation. Figure a shows the behaviour of $|R|$ with wave period. The exact curve is plotted as a solid line, while the dotted, chained and dashed lines correspond to the modulus of fits to $R$ itself, approximating it by expansions using 9, 19 and 29 Chebyshev polynomials respectively (which require 10, 20 and 30 evaluations of $R$ respectively). Figure b is the same as a, but shows successive approximations to $|T|$ instead of $|R|$. In both graphs, the ice thickness is 1 m, the water depth is infinite and the incident waves are normally incident.

When the scattering patterns are simpler, such as those observed in Chapter 6, less function evaluations are required to obtain a sufficiently accurate approximation; plots against angle of incidence generally need about 20–60 points, more being needed for larger periods.
Note however, more complicated curves such as those in later chapters need slightly more points (usually from 50–100), with extra allowance being made at short periods for multiple zeros in $|R|$ (cf. Figure 6.4).

5.2.2 The Effect of Angle of Incidence

Having now found a simple way to combine the effects of both ice thickness and wave period (in the nondimensional period $\tau$), let us now investigate the effect that varying the angle of incidence produces. We will also retain the assumption of infinite depth here and investigate the effect of changing the water depth in the next section.

Figure 5.4: The effect of angle of incidence on the scattering by a single crack. The figures show the behaviour of $|R|$ with wave period for angles of incidence (a) $0$, (b) $2\pi/9$, (c) $\pi/3$, (d) $7\pi/18$ or $70^\circ$ in dimensional units, (e) $4\pi/9$, (f) $0.458\pi$ or $82.5^\circ$, (g) $0.472\pi$ or $85^\circ$, (h) $0.483\pi$ or $87^\circ$ and (f) $0.494\pi$ or $89^\circ$. The ice thickness is 1 m, and the water depth is infinite.
Figure 5.4 shows plots of $|R|$ against period for a series of angles of incidence, chosen to best illustrate their evolution with $\theta$. Figure a, the figure showing the result for normal incidence ($\theta = 0$), is the same as the solid curve in Figure 5.2a and has already been discussed in the previous section. The same general structure is preserved as $\theta$ is increased, although two gradual changes do begin to take place—the reflection in the short wave region gradually decreases, as does the height of the maxima following the zero at about 7 s. Once $\theta$ has reached about $4\pi/9$ (cf. Figure e) however, it is apparent that the values of $|R|$ at the maxima have begun to increase again and that, in addition, the zero has begun to shift further and further to the left. This process continues as the angle of incidence moves towards $\pi/2$—by the time $\theta = 0.472\pi$ (85°) in g the zeros have actually disappeared, while the height of the maxima have nearly reached their maximum value of 1 by $\theta = 0.494\pi$ (89°; cf. Figure i). Indeed it can be shown from (5.4) that $R \to -1$ as $\theta \to \pi/2$ for a given wave period and ice thickness, and this is borne out by Figure 5.4 and further results plotted in following sections and chapters (e.g. Figure 5.6, partially described below).

Later, Figure 5.6c shows $|R|$ plotted against angle of incidence for wave periods of 2, 6.8, 8.5 and 11.3 s (solid, dashed, chained and dotted lines respectively) for infinite depth. The 2 s curve is the simplest. It begins at about $|R| = 0.95$ at normal incidence and arcs down to a zero at about $0.461\pi$ (83°), before climbing up to 1 as $\theta \to \pi/2$ (as expected). The curves corresponding to periods nearer to the zero in the $|R|$-period normal incidence curve (cf. Figure 5.4a) are much more interesting. Their structure remains the same as that shown by the 2 s curve from that period up to about 5.8 s, before the characteristics shown by the 6.8 s curve begin to develop. That curve begins at the much lower value of 0.16, and proceeds to climb to a local maximum of 0.26 when $\theta \approx 2\pi/9$, before falling to a zero near $4\pi/9$ and climbing to 1.

As period increases from 6.8 s to about 7.5 s the normal incidence value drops towards zero, and only a single zero is observed when $\theta$ is around $4\pi/9$. However, as period increases beyond where the normal incidence zero occurs, a second zero appears at the $|R|$ axis and proceeds to move to higher angles of incidence, giving the structure observed in the 8.5 s graph. The maximum apparent in the 6.8 s curve also decreases as period increases. It continues to decrease, and the original zero moves to the left until it meets the second one when the period is about 9.8 s and $\theta \approx \pi/3$, forming a local minimum which lifts off the $\theta$ axis as period increases, giving the 11.3 s structure. This
structure persists as the period increases further, although, as it does, the minimum moves towards \( \pi/2 \) and values of \(|R|\) to the left of those minima move closer and closer to zero.

5.2.3 The Effect of Water Depth

Section 1.1.2 explained that the results presented in this thesis are mainly applicable to the central Arctic Ocean in the northern hemisphere and to the sea ice beyond the Antarctic ice shelves in the south. The Antarctic waters are generally at least 300 m deep, as are the Arctic waters, although they can get quite shallow near the coasts of Canada and Siberia. Around the North Pole, however, the water can be up to 4.5 km deep. Hence the infinite depth results will be those that we are most interested in. However, in calculating infinite depth results, it is often easier to run the finite depth programs at a sufficiently large depth instead of calculating them directly, which necessitates determining the depth beyond which they are accurate. This is particularly true in later chapters in which the actual infinite depth results are much harder to obtain than for the specific problem being solved here.

It is also interesting in itself to see the effect that decreasing the depth has on the scattering at the surface, as well as being useful in predicting what effect the shallower parts of the Arctic Ocean might have on our results. Attempting to establish a criterion for when shallow results are valid is less useful, but also allows us to investigate further the effect of decreasing the water depth on the scattering results.

An Infinite Depth Criterion

Kinsman (1984, pp 128–129) shows that for a given wave period, the wavelength of a wave travelling in open water of finite depth agrees to within 0.37% accuracy with the wavelength of a wave travelling in deep open water if the water depth is greater than half the latter wavelength. Hence the largest wavelength that we will be interested in will determine the depth beyond which water may be taken to be infinitely deep. Since we are mainly interested in waves with periods less than about 20 s, Kinsman's criterion implies that open water could be taken to be deep if it were greater than about 312 m.

In terms of the amount of reflection when ice is present, however, Chung and
Fox (2002a) found that this condition could be relaxed somewhat. A depth of $2\pi L_0$ (approximately 94 m for 1-m-thick ice, or 150 m for 2-m-thick ice) was sufficiently large that their scattering results stopped changing if the depth was increased beyond this value. Accordingly, it would seem that the infinite depth results are even more applicable to the Antarctic and we can extend them closer to the coast in the Arctic Ocean.

Figure 5.5: The effect of the nondimensional water depth on the scattering by a single crack for angles of incidence $(a)$ $0$, $(b)$ $2\pi/9$, $(c)$ $\pi/3$, $(d)$ $7\pi/18$ or $70^\circ$ in dimensional units, $(e)$ $4\pi/9$, $(f)$ $0.458\pi$ or $82.5^\circ$, $(g)$ $0.472\pi$ or $85^\circ$, $(h)$ $0.483\pi$ or $87^\circ$ and $(f)$ $0.494\pi$ or $89^\circ$. The figures show the behaviour of $|R|$ with wave period for water depths of $H = 1$ (dotted), $H = 2$ (chained), $H = 3$ (dashed), and $H = 4$ (solid).

Referring to our own nondimensionalization scheme, from Figure 2.2c, for a period of 20 s a depth of $2\pi L_0$ corresponds to a nondimensional depth of $H \approx 4$. The wavelength results for open water (Kinsman, 1984) imply that shorter waves require a smaller depth before the water can be considered infinitely deep. Since $H$ decreases with period (cf. Figure 2.2b, which shows how $L$ varies with period), it was thought
that the infinite depth criterion of Chung and Fox (2002a) could be relaxed further to $H$ being greater than or equal to a sufficiently large constant $H_0$, of about 4, say. In situations where infinite depth results were most easily found by calculating finite depth results, knowing that increasing the water depth past 4 wouldn’t change the reflection and transmission would enable us to be as efficient in our calculations as possible, as the smaller the water depth is, the less evanescent modes that are needed in the finite depth solution. It would also mean that shorter period, infinite depth results would be valid even closer to the Arctic coastline.

Figure 5.5 is the finite depth equivalent to Figure 5.4, plotting $|R|$ against period for a series of angles of incidence, and for values of $H$ of 1 through to 4 (dotted, chained, dashed and solid lines respectively). All curves show the same basic structure as the infinite depth curves, although some trends are apparent as water depth is decreased. As is most obvious in the graphs corresponding to higher angles of incidence, the curves seem to move to the right as $H$ becomes smaller, generally increasing the amount of reflection for larger periods, but with a drop in reflection at smaller periods. This can be seen especially well in Figure 5 when $\theta = 0.494\pi$ (89°), which shows both the $H = 1$ and the $H = 2$ curves departing noticeably from the deeper water curves. On closer inspection, the $H = 3$ curve shows the same tendency. The $H = 1$ curve has become clearly visible by the time $\theta$ has reached $2\pi/9$, while the $H = 2$ curve does not differ appreciably from the deeper water results until the $0.494\pi$ figure, although it can be distinguished from the others at angles above $\pi/3$.

Figure 5.5 seems to confirm our proposition that $H = 4$ would be large enough to be considered infinitely deep—in fact, it seems to suggest that even a depth of $H = 3$ would be large enough. This is also confirmed by Figure 5.6. Plot (e), described in the previous section when discussing the effect of angle of incidence, corresponds to the formal infinite depth results, and is virtually identical to plots (c) and (d), which show results for $H = 3$ and $H = 4$ respectively. However, in the following chapter where pressure ridges are modelled, the thickness in the variable region is significantly larger than the background ice thickness, $h_0$, which would enable the disturbances at the ice-water surface to penetrate to larger water depths. Thus there might still be some interactions with the sea floor, even though there would not normally be any if none of the ice was thicker than $h_0$. Consequently, to be on the safe side, $H > 5$ will be used in later chapters as our infinite depth criterion.
Figure 5.6: The effect of the nondimensional water depth on the scattering by a single crack and comparison with infinite water depth results. The figures show the behaviour of $|R|$ with angle of incidence for wave periods of 2 s (solid curves), 6.8 s (dashed curves), 8.5 s (chained curves) and 11.3 s (dotted curves). Nondimensional water depths are (a) $H = 1$, (b) $H = 2$, (c) $H = 3$, (d) $H = 4$, and (e) infinite.

Figure 5.6 also gives results that are consistent with the pattern observed in Figure 5.5. Moving from the deeper water plots to the $H = 2$ graph (b), and then onto the $H = 1$ graph (a), the reflection decreases when the period is 2 s and increases when the period is 6.8 s. This might have been predicted from the behaviour of reflection with period since decreasing the depth pushes the reflection curves towards higher periods while at the same time pushing them downwards at low periods. Similar predictions could have been made for the 8.5 s and 11.3 s curves—the left-hand zero in the former curve moves toward the $|R|$ axis, almost reproducing the single-zero structure observed for the lower two periods, while the latter changes from the general long period structure to the beginnings of the two-zero structure originally observed in the 8.5 s curve.
A Shallow Water Criterion

In the previous section, it was shown that the scattering results converged fairly uniformly over the whole range of periods we are interested in to the infinite depth results as the nondimensional water depth $H$ was increased to about 4. Consequently, it was hoped that $H$ might also help in determining a similarly "objective" criterion (with respect to period) for when the shallow water approximations would hold.

In fact, it was found that simply choosing a constant value of $H$ meant that for the shallow water results to hold over the whole range of periods, one had to make the water unnecessarily shallow for larger periods.

Kinsman (1984) proposed that open water could be considered shallow if the depth was less than $5 \times 10^{-3}$ of the infinite depth wavelength. Since we are dealing with ice-covered water, the wavelength of an infinite depth ice-coupled wave, which we shall denote by $\lambda_\infty$ when nondimensionalized, was also trialled as a possible length scale with which to compare the water depth. Again, however, choosing a sufficiently small constant value of $H/\lambda_\infty$ meant that one unfortunately had to be too stringent at large periods to get the results to converge over the whole period range.

After a little experimentation, a nondimensional length scale, $H_s = 4/(5\gamma_\infty)^2$ was proposed, where $\gamma_\infty = 2\pi/\lambda_\infty$ is the infinite depth wavenumber of a propagating wave. For 1-m-thick ice, this ranges from about 0.16 (corresponding to a dimensional depth of 1.4 m) when the period is 2 s to about 3.4 (dimensional depth of 72 m) when the period is 20 s. Figure 5.7 shows the behaviour of $|R|$ with various values of $H/H_s$, comparing the full finite depth results (solid curves) to both the partial (dashed curves) and the full (chained curves) shallow water approximations, and it also demonstrates that both approximations converge fairly uniformly to the finite depth ones as $H/H_s$ is decreased, although the full shallow water results take slightly longer to converge than the partial ones.

Even for $H = 4H_s$, the partial shallow water results, which are shown in plot (d), are virtually indistinguishable from the finite depth ones. However, as $H/H_s$ is increased to 5, small but noticeable differences begin to appear between the partial shallow water curve and the finite depth curve—particularly for periods of about 10 s, in the vicinity of the local maximum in $|R|$. In general though, the dashed curve is still a good fit.
There is little difference between the two sets of results for the smaller values of $H/H_s$, $(0.5, 1$ and $2$, which correspond to plots (a), (b) and (c) respectively).

![Figure 5.7: The effect of decreasing the nondimensional water depth on the scattering by a single crack and comparison with shallow water results.](image)

The figures show the behaviour of $|R|$ with wave period for nondimensional water depths of (a) $H_s/2$, (b) $H_s$, (c) $2H_s$, (d) $4H_s$ and (e) $5H_s$, where $H_s = 4/(5\gamma_\infty)^2$ and $\gamma_\infty$ is the (also nondimensional) infinite depth wavenumber for an ice-coupled flexural-gravity wave. The full finite depth results are plotted as solid curves, the partial shallow water results are plotted as dashed curves, and the full shallow water results are plotted as chained curves, while the waves are normally incident and the ice is 1 m thick.

One comment that should be made here about the partial shallow water results are that for a given period, there is a certain value of $H$ at which $C_0(\gamma_0) = 0$, making the finite depth residue $A_0$ infinite when the shallow water root is used instead of the finite depth root. If the ratio $H/H_s$ is large enough, $H$ itself will be large enough that it will zero $C_0(\gamma_0)$ for some period in our range of interest when plotting $|R|$ against period. Results can be quite unstable around that period, sometimes departing quite markedly
from the finite depth results they are approximating, even if for the rest of the period range the approximation still holds quite well.

This problem can be partially resolved by restricting the water depth to being less than or equal to $H_\infty$—if the finite depth results can be taken to have converged to the infinite depth results by then, and the shallow water results are still giving a good approximation to the finite depth results, then it also seems reasonable to take the shallow water results calculated at $H_\infty$ to be a good approximation to the infinite depth results.

However, this solution does not prove as successful for smaller periods, since the water depth at which $C_0(\gamma_0)$ becomes zero also decreases with period. Consequently, $H/H_s$ cannot be increased much further than 5 without being affected by that phenomenon. Results for larger values could possibly be obtained by only calculating results for unaffected areas, and interpolating between them, using a spline perhaps, but as can be seen from Figure 5.7e, the possible accuracy of such an approximation would have begun to decrease by then, making the additional effort unwarranted.

The full shallow water solution does not experience the numerical difficulties of the partial solution, but unfortunately takes significantly longer to converge to the finite depth results. Increasing the depth on moving from Figure 5.7a to Figure 5.7c shows the chained curve only just becoming distinguishable from the other two when $H = H_s$, and becoming distinctly separate when $H = 2H_s$, although they are still very similar.

Consequently, depending on the degree of accuracy required, we could perhaps assume that $H \leq H_s$ and $H \leq 4H_s$ were sufficient conditions for the full and the partial shallow water approximations to hold respectively—otherwise we could replace them by $H \leq 2H_s$ and $H \leq 5H_s$ if less precision was acceptable—if we were already constrained by measurement accuracy, for example.

Now, $H = 4H_s$ corresponds to a range of 5.1 m to 271 m when considering a period range of 2 s to 20 s (for 1-m-thick ice), so it would seem that a given region of ice-covered sea (off the coast of Canada or Siberia, say) could be considered shallow (with respect to the partial approximation) if it were less than about 5 m deep. This maximum depth could be increased depending on how large a range of periods was being modelled. However, there is an underlying assumption in our working that the
submergence of the ice, including the depth penetration of pressure ridge keels, is small in comparison to the water depth, and so we are also limited in how shallow we can reasonably make the water. Hence, the partial shallow water results will only hold for depths between this minimum depth and depths corresponding to $4H_s$ (or $5H_s$). And, similarly, the full results will only hold for depths between the minimum depth and depths corresponding to $H_s$ itself (or maybe $2H_s$). For both approximations, this will effectively produce a lower limit to the range of periods over which they are valid.

For larger periods when $H_s$ corresponds to a relatively large water depth, there are some surprising applications of the partial shallow water solution particularly. As alluded to earlier when discussing the numerics of that solution, for longer periods it sometimes holds for water depths greater than those required to produce infinite depth (when $H_\infty < 4H_s$ or $5H_s$). Taking the more stringent shallow water condition, and values of $H_\infty$ of 3 or 4, such an approximation would be valid for periods greater than 12.7 s or 14.0 s respectively, while if the infinite depth condition were relaxed slightly to values of $H$ greater than $H_\infty = 2$, the approximation would apply down to about 11.0 s. Hence, we have a range of between 6 and 9 s, depending on the accuracy required, where the calculation of infinite depth results could be markedly sped up; this range could be increased further, although only by an extra second in period, by relaxing the partial shallow water condition to $H$ being less than $5H_s$.

Finally, some general statements about the behaviour with decreasing water depth can also be made from Figure 5.7. As in the previous section, decreasing the water depth still has the effect of moving the curves towards longer periods while reducing the reflection at the smaller period end.
Chapter 6

Scattering by a Single Ridge

The problem addressed in this chapter is scattering by a pressure ridge. Mathematically, it is the simplest after the crack, although numerically the solution is actually harder to implement than the solution for a lead, which is given later in Chapter 8. However, in obtaining a more analytic solution the lead solution sacrifices some generality by requiring that the thickness over $(0, a)$ is constant, while the method presented in this chapter and the next allows the thickness to vary arbitrarily over that interval.

Figure 6.1a illustrates the physical make-up of a pressure ridge (Matsuo et al., 2002), although the figure is slightly idealized in that the width of the keel is generally much larger than that of the sail (cf. Figure E.1 for a more realistic representation). This is the result of the physical process of ridge formation, described in Section 1.1.2, coupled with buoyancy effects (the mass of the ice in the keel should be $\rho/(\rho_w - \rho) = 9$ times greater than the mass of the ice in the sail). To begin with, two adjacent sheets are separated by a differential response to a change in wind or current; this produces an open lead, which quickly refreezes; another wind or current change may then force the two large sheets together again, with the result that the newer, thinner ice in between them is broken up. In this process some of the rubble is thrust up onto the top of the two ice sheets to form a sail but most is pushed down to form a keel. The rubble of both the sail and the keel soon freezes to form a pair of solid masses above and below the two main ice sheets. The large pressures involved also produce a consolidated layer between the sail and keel that may be about one to two times the thickness of the originating ice sheets.
Figure 6.1: (a) The physical make-up of a pressure ridge. The sail and the keel are made up from rubble, and there should be nine times as much ice in the keel than in the sail to make the ridge neutrally buoyant. (b) The model used in this chapter—the submergence of the keel is neglected but its rigidity can be included by preserving the total ice thickness. As in Figure 2.1 the width of the variable region is $a$ and we model the scattering of a plane wave arriving from the left at an angle $\theta$ from the normal to the ridge (which is parallel to the $x$ axis). The ice surrounding the ridge has a constant thickness of $h_0$, and the sea water has a finite depth of $H$. The coordinate axes are oriented as shown but are displaced to the left so that the $x = 0$ line corresponds to the left hand limit of the ridge.

It is unclear at present how to model such a ridge precisely. A full solution might treat the consolidated layer as a thin plate, and model how the pressure on the keel from the water is transmitted through to it. This pressure, along with the pressure
due to the sail above could then be substituted into the thin plate equation. However, this would be extremely difficult.

A simpler alternative might be simply to treat the whole ridge as a thin plate with variable thickness. Porter and Porter (2004) recently used a variational approach coupled with a mild slope approximation, to produce results that can be used to approximate a pressure ridge in this way. Since the flexural rigidity increases to the third power of the thickness, this would make the ridge extremely rigid. Qualitatively, this would model the expected behaviour of the keel in particular—being made up of an inverted pile of small blocks frozen together one would presume that it would only experience minimal flexure. Rotational and shear deformational effects could also be included by using the thick plate model of Mindlin (1951). By adding two further terms to the Lagrangian of Porter and Porter (2004), their model could be adapted to include these extra effects.

The paper of Williams and Squire (2004a) took the approach of only modelling the sail and the consolidated layer of the ridge as a single thin plate (with varying thickness) as at the time including a keel seemed to complicate the solution considerably, and also neglecting the submergence seemed reasonable given its small size in comparison to the typical length scales of the problem. (Keel depths are typically up to around 4-5 m, compared to infinite-depth wavelengths of between 50 m and 530 m for waves beneath 1-m-thick ice in our period range.)

If the assumption about submergence is valid, then the effects of the increased rigidity—that is, impedance effects (the wavelength and thus phase velocity increases as ice thickness is increased; there is more effect the more abrupt the change) could be incorporated by changing our thickness profiles accordingly, as illustrated in Figure 6.1b. This chapter will present some results obtained by doing this, and the work of Porter and Porter (2004) gives us further confidence in the accuracy of this approach for the general water depths we are interested in. In their paper they derived a sixth order ordinary differential equation that depends only upon the overall thickness of the ridge and on the distance between the bottom of the keel and the sea floor. Consequently, if the mild slope approximation for the keel is valid, moving the ice in the keel upwards so that its bottom is at the same level as the rest of the ice should make no difference as long as the sea floor is moved up by the same amount. If, in addition,
the water is deep enough that the infinite depth approximation holds, changing the sea floor in this way would only have a negligible effect.

The method of Williams and Squire (2004a) could also be adjusted to produce results for a thick plate. However, it is unable to be extended in its entirety to include submergence due to its reliance on the high differentiability of $G_{z}(x - \xi, 0, 0)$. For that method to work on a keel profile of $z = d(x)$ over $(0, a)$ the function

$$(d'(x)\partial_z - \partial_z)\left(d'(\xi)\partial_z - \partial_z\right)G(x - \xi, d(x), d(\xi))$$

would need the same level of differentiability. (It doesn’t—it has a singularity of order $1/(x - \xi)^2$ since $G$ has a logarithmic singularity when $x = \xi$, and $z = \zeta > 0$. Also note that $d'(x)\partial_z - \partial_z$ would be the derivative with respect to the outward normal for the contour that would be used when applying Green’s theorem, which is why it appears in the above expression.)

An extension of the method of Dixon and Squire (2001b), which solves the problem of a flat iceberg embedded in an otherwise uniform ice sheet, is more promising, especially since it is based on the method of Meylan (1993) that was successfully used to treat the problem of a submerged floe. They also used Green’s theorem, but instead of eliminating $\phi(x, 0)$ beneath the berg and writing it in terms of $\phi_z(x, 0)$ as Williams and Squire (2004a) did (and as we do in Chapter 4), they used a second Green’s function (the Green’s function for a thin plate uncoupled to any fluid) to write $\phi_z(x, 0)$ as an integral operator acting on $\phi(x, 0)$. This allowed them to eliminate $\phi_z(x, 0)$ and solve an alternative integral equation for $\phi(x, 0)$.

It was thought that the method of Dixon and Squire (2001b) would not easily be able to be extended to allow for a variable thickness profile, but it seems now that in that case, one would simply have to solve an integral equation for $\phi_z(x, 0)$ in terms of $\phi(x, 0)$, and the solution would proceed in exactly the same way. The second Green’s function would have all the differentiability we need, and we would only require that the first Green’s function be integrable and have an integrable normal derivative. This idea is described in full in Appendix E.

Alternative approaches to the problem include setting up an integro-differential equation for $\partial_n\phi(x, d)$, or removing the mild slope approximation from the work of
Porter and Porter (2004) by allowing for the production of evanescent waves when that assumption is no longer as valid. Marchenko (1997) also models a ridge as a point irregularity by making the reasonable assumption that if the submergence of the keel is negligible, then its width is also negligible. He treats the ridge as a thin elastic beam that is also permitted to have some rotational motion, and is joined to the surrounding ice by elastic hinge joints. If such a representation is correct, then the problem simply reduces to the algebraic one of finding three unknown constants. In infinitely deep water, these could be found exactly, as they can in the crack problem solved in the previous chapter. At the very least, some comparison with his work should be done in the future as, if a full solution and his solution gave similar results, his method would serve as a highly efficient approximation.

### 6.1 Solution Method

We now proceed to solve (4.37) when \( h_2 = h_0 \) but with \( h_1(x) \) permitted to vary. This is the method used by Williams and Squire (2004a). As in the previous section, the integral over \((a, \infty)\) in (4.37) will again vanish (since \( h_2 = h_0 \) again). However, since \( h_1 \) now varies, we do actually have to solve an integral equation over \((0, a)\). Recalling from (3.11) that \( g^{(4)} \) has a delta function singularity, we find that

\[
\phi_z(x, 0) = e^{j\alpha_0 x} \varphi'_0(0) + \psi^T(x) P_0 + \psi^T(x - a) P_a \\
+ \left(1 - \frac{D_1(x)}{D_0}\right) \phi_z(x, 0) + \int_0^a K(x, \xi) \phi_z(\xi, 0) d\xi, \tag{6.1}
\]

or

\[
d_0(x) \phi_z(x, 0) = e^{j\alpha_0 x} \varphi'_0(0) + \psi^T(x) P_0 + \psi^T(x - a) P_a \\
+ \int_0^a K(x, \xi) \phi_z(\xi, 0) d\xi, \tag{6.2}
\]

where \( d_0(x) = D_1(x)/D_0 \), the kernel function is

\[
K(x, \xi) = (\mathcal{L}_1(\xi, \partial \xi) - \mathcal{L}_0(\partial \xi)) g(x - \xi) = \sum_{j=1}^4 d_j(\xi) \mathcal{L}_{1j}(\partial \xi) g(x - \xi), \tag{6.3}
\]

and we have used the Cauchy principal value symbol

\[
\int_0^a d\xi = \lim_{\varepsilon \to 0} \left( \int_0^{x-\varepsilon} + \int_{x+\varepsilon}^a \right) d\xi
\]
as a theoretical reminder to us that we have already integrated out the delta function singularity (the integrand is not actually Cauchy singular). In practice we simply treat the difference kernels

\[ K_j(x - \xi) = L_{1j}(\partial_1)g(x - \xi), \]

and in particular \( K_1(x - \xi) \), as if they had no generalized function components (having allowed for these already), computing them from their eigenfunction expansions obtained from (3.7).

Equation (6.2) is now solved approximately using the quadrature scheme described in the following section.

### 6.1.1 Quadrature Scheme

The quadrature scheme below uses a discretization method. Collocation was considered initially, but we decided against it due to singularities in the kernels (e.g., \( K_2(x - \xi) \) has a jump discontinuity at \( x = \xi \)), which would slow down the convergence. A Galerkin method could possibly also have been used, but they are generally slower than quadrature methods (Press, 1992).

Proceeding, the integral in (6.2) can be written in terms of the integral operators

\[ [\mathcal{W}_jv](x) = \int_0^a K_j(x - \xi)v(\xi)\,d\xi, \quad (6.4) \]

for any function \( v \) that is integrable on \((0, a)\). For the purposes of solving (6.2) we seek to approximate these integral operators by vector inner products. We first divide the interval \((0, a)\) into \( M \) panels of length \( \Delta = a/M \), with end-points contained in the vector \( x \), where \([x]_m = x_m = m\Delta \) for \( m \in \mathcal{M} = \{0, 1, \ldots, M\} \). Defining \( v_m = [v]_m = v(x_m) \), we then assume that \( v \) can be approximated on the \( m^{th} \) panel by the linear function

\[ v^{(m)}(x) = v_{m-1} + (v_m - v_{m-1}) \frac{x - x_m}{x_m - x_{m-1}} \quad (\text{for } m \in \mathcal{M}), \quad (6.5) \]

allowing us to write (6.4) as

\[
[\mathcal{W}_jv](x) \approx \sum_{m=1}^{M} \int_{x_{m-1}}^{x_m} K_j(x - \xi)v^{(m)}(\xi)\,d\xi \\
= \sum_{m=1}^{M} \left[ u_{m-1}w_j0(x - x_{m-1}) + v_m w_j1(x - x_{m-1}) \right], \quad (6.6)
\]
where
\[ w_{j0}(x) = \int_0^\Delta K_j(x - \xi) \left( 1 - \frac{\xi}{\Delta} \right) d\xi, \quad w_{j1}(x) = \int_0^\Delta K_j(x - \xi) \frac{\xi}{\Delta} d\xi. \]

After truncating the series (3.7) at \( n = N \) for a sufficiently large value of \( N \), the above integrals may be evaluated exactly; since the coefficients \( A_n \) are \( O(|\alpha_n|^9) \sim O(n^{-9}) \) for large \( n \), the coefficients of \( K_1(x - \xi) \) will be of the order of \( |A_n\alpha_n^4| \sim O(n^{-5}) \), and so its eigenfunction expansion will converge reasonably rapidly; the other \( K_j \) should converge even faster as they require lower derivatives of \( g \). (In practice, we usually need \( N \) to be between about 30 for the \( K_j \) to converge; however, we often need up to about 80 as further derivatives of the \( K_j \) are usually needed to be evaluated when the edge conditions are applied—one for frozen edges and three for free edges.)

If we then define the vector-valued function \( w_j(x) \) by
\[
[w_j]_{n}(x) = \begin{cases} 
  w_{j0}(x) & \text{for } m = 0, \\
  w_{j0}(x - m\Delta) + w_{j1}(x - (m-1)\Delta) & \text{for } m = 1, \ldots, M - 1, \\
  w_{j1}(x - a + \Delta) & \text{for } m = M,
\end{cases}
\]
we have the vector inner product representations for the integral operators that we seek
\[ \mathcal{W}_j v(x) \approx w_j^T(x)v. \]

In particular, defining the matrices \( \mathbf{W}_j \) so that its \( m^{th} \) row is \( w_j^T(x_m) \) (for \( m \in \mathcal{M} \)), we have the following expression for the values that \( \mathcal{W}_j v \) takes at the endpoints of each of the \( M \) panels
\[ \mathcal{W}_j v(x_m) \approx [\mathbf{W}_j v]_m. \]
Consequently, if \([\phi_x]_m = \phi_x(x_m, 0)\), approximating \( \phi_x(x, 0) \) by a linear function over each panel in the same way that \( v \) was approximated, the integral equation (6.2) may be written as the \((M + 1) \times (M + 1)\) matrix equation
\[ (D_0 - K)\phi_x = M_0 \phi_x = f_0, \]
where for \( m, n \in \mathcal{M} \),
\[ [f_0]_m = e^{i\alpha \omega n} \varphi_0(0) + \psi^T(x_m) P_0 + \psi^T(x_m - a) P_a, \]
and

\[ K = \sum_{j=1}^{4} W_j D_j, \quad [D_j]_{mn} = d_j(x_m) \delta_{mn}. \]

Solving (6.10) has put \( \phi_z \) in terms of the as yet unknown vectors \( P_0 \) and \( P_a \); the solution is completed by applying the edge conditions as described in Section 4.2.3.

The number of panels required for convergence, \( M \), depends on the variability of the operands \( d_j(x) \phi_z(x, 0) \), and also of the forcing functions \( e^{i\alpha_x} \) and \( \psi(x) \), inside the interval \((0, a)\). The variability of the forcing functions depend on the wavenumber of the incident wave, \( \alpha_0 \), which increases as period becomes smaller (as the waves become shorter)—consequently, we need a larger \( M \) as period decreases. We also need it to be larger if the width \( a \) is increased, or if the thickness profile becomes steeper. In practice, \( M \) ranged from about 50 to 150 at the smaller periods, depending on the thickness profile (e.g., constant thickness profiles required fewer panels); although fewer panels could have been used for the larger periods, in order to avoid having to determine the convergence at each individual period, we just used the \( M \) that was needed for the small periods to converge across the whole period range. (The convergence criteria used was that \(|R|\) have converged to three decimal places. This is more than was needed for graphical purposes as on a scale of 0 to 1, differences of even about 0.05 are very difficult to resolve).

### 6.1.2 Scattering Coefficients

In the situation modelled in this chapter where \( h_1 \neq h_0 \), the reflection and transmission coefficients can no longer be calculated in closed form, as they were able to be in the previous chapter. In particular, to calculate these coefficients, some Fourier integrals must be approximated in the same way as the integral operators \( W_j \) were in Section 6.1.1.

**Reflection Coefficient**

In this situation, where \( h_2 = h_0 \), the coefficients \( a_n \) in the eigenfunction expansion (4.38) for \( \psi_z(x, 0) \) are given by

\[ a_n = iA_n \left( p^T(\alpha_n) (P_0 + P_a e^{i\alpha_n a}) + \dot{\phi}(\alpha_n) \right), \quad (6.11) \]
since \( f_2(\gamma_n) = 0 \). We are only able to calculate a finite number of the above coefficients, so we shall assume that we only need to include the effects of \( N \) evanescent modes in order for our results to converge—i.e., we shall assume that \( n \in \mathcal{N} = \{-2, -1, \ldots, N\} \).

To calculate the integrals in the transform \( \hat{\phi}(\alpha_n) \), we must once again use an approximation for \( \phi_z(x, 0) \) over \((0, a)\). First, we construct an \((N + 3) \times (M + 1)\) matrix \( F \) so that

\[
[F \mathbf{v}]_n \approx \int_0^a u(x)e^{i\alpha_n x} \, dx \quad \text{for } n \in \mathcal{N}. \tag{6.12}
\]

As in the previous section, the approximation is exact when \( u(x) \) is a piecewise linear function which is divided into \( M \) parts. Having constructed \( F \), we can now say

\[
\hat{\phi}(\alpha_n) \approx [F^+ \phi_z]_n, \tag{6.13}
\]

where, for \( n, m \in \mathcal{N} \),

\[
F^+ = \sum_{j=1}^4 L_j^+ F D_j, \quad [L_j^+]_{nm} = L_{ij}(\pm i\alpha_n)\delta_{nm}.
\]

We can now use equation (6.13) to approximate the \( \alpha_n \), and so calculate the reflection coefficient \( R \).

### Transmission Coefficient

The coefficients \( d_n \) in the expansion (4.42) for \( \phi_z(x, 0) \) when \( x > a \) are given by

\[
d_n = \delta_{n,0}\phi'_0(0)e^{i\alpha_n a} + iA_n \left( P^T(-\alpha_n) \left( P_0 e^{i\alpha_n a} + P_a \right) + \hat{\phi}(-\alpha_n)e^{i\alpha_n a} \right), \tag{6.14}
\]

and so to calculate \( d_0 \) and \( T \), we must calculate integrals of the type

\[
\int_0^a u(x)e^{i\alpha_n (a-x)} \, dx = \int_0^a u(a-x)e^{i\alpha_n x} \, dx \approx [F \mathbf{I} \mathbf{v}]_n \quad \text{(for } n \in \mathcal{N}),
\]

where \( \mathbf{I} \) is an upside down \((M + 1) \times (M + 1)\) identity matrix, i.e.

\[
[F \mathbf{I}]_{nm} = \delta_{m,M+1-n} \quad \text{(for } m, n \in \mathcal{M}).
\]

Thus for \( n \in \mathcal{N} \) we can now write

\[
\hat{\phi}(-\alpha_n)e^{i\alpha_n a} \approx [F^- \phi_z]_n, \tag{6.15}
\]

where

\[
F^- = \sum_{j=1}^4 L_j^- F \mathbf{I} D_j.
\]

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6.1.3 Internal Application of Frozen Edge Conditions

As discussed in Section 4.2.3, for \( x_c \in (0, a) \) the frozen edge conditions may be applied by substituting the eigenfunction expansions (4.38) and (4.42) into equations (4.44). However, when \( D \) is discontinuous inside \((0, a)\), we must use a different approach to applying (2.12b).

For simplicity of presentation, in the following working we will assume that \( X_0 \) contains only the elements 0 and \( a \); the procedure to follow at other critical points is essentially the same. If we let

\[
b_n = \varphi'_n(0)\delta_n + iA_n\mathbf{p}^T(-\alpha_n)\mathbf{P}_0, \quad c_n = iA_n(\mathbf{p}^T(\alpha_n)\mathbf{P}_a - f_2(\gamma_n)\Phi^+(\alpha_n)),
\]

then (6.2) may be written as

\[
\frac{D(x)}{D_0}\phi_x(x, 0) = \sum_{n=-2}^{\infty} \left( b_n e^{i\alpha_n x} + c_n e^{i\alpha_n (a-x)} \right) + \int_0^a K_1(x - \xi) d_1(\xi) \phi_x(\xi, 0) d\xi + \sum_{j=2}^{4} \int_0^a K_j(x - \xi) d_j(\xi) \phi_x(\xi, 0) d\xi.
\]

Note that by allowing for the presence of \( \Phi^+(\alpha_n) \) in the definition of \( c_n \), our working in this section is applicable to when \( h_2 \neq h_0 \), as well as \( h_2 = h_0 \).

The integrals for \( j = 2, 3, 4 \) are actually ordinary integrals because they contain no delta function type singularities. However, differentiating (6.16) produces a singularity in \( K'_2(x) \): noting that \( K_{2,\xi}(x - \xi) = -K_{2,\xi}(x - \xi) \),

\[
\partial_\xi L_{12} \partial_\xi g(x - \xi) = -\left( \partial_\xi^2 - l^2 \right) g(x - \xi) - l^2 (\partial_\xi^2 - l^2) g(x - \xi),
\]

which from (3.11) has singular part \( \delta(x - \xi)/D_0 \). Allowing for this, the first derivative of (6.16) is

\[
\frac{1}{D_0} \left( D(x)\phi_{xx}(x, 0) + D'(x)\phi_x(x, 0) \right) = 2\frac{D'(x)}{D_0}\phi_x(x, 0) + \sum_{n=-2}^{\infty} \alpha_n \left( b_n e^{i\alpha_n x} - c_n e^{i\alpha_n (a-x)} \right)
\]

\[
+ \sum_{j=1}^{4} \int_0^a K'_j(x - \xi) d_j(\xi) \phi_x(\xi, 0) d\xi,
\]

(6.17)
or
\[
\frac{p_1(x)}{D_0} = i \sum_{n=-2}^{\infty} \alpha_n (b_n e^{i\alpha n x} - c_n e^{i\alpha (a-x)}) + \sum_{j=1}^{4} \int_0^a K_j'(x - \xi) d_j(\xi) \phi_2(\xi, 0) d\xi. \tag{6.18}
\]

Letting \(x \to 0^+\) and \(x \to a^-\) in (6.18) we can now calculate \(p_1(x^\pm_c)\) and apply (2.12b) by eliminating \(\phi_{xx}(x^\pm_c, 0)\) from (4.44b) to give
\[
D(x^\pm_c) P_1(x_c) = (D(x^+_c) - D(x^-_c)) p_1(x^\pm_c) + (D(x^+_c) D'(x^-_c) - D(x^-_c) D'(x^+_c)) \phi_2(x^\pm_c, 0). \tag{6.19}
\]

### 6.2 Results

This section begins in Section 6.2.1 by defining some different types of ridges—in particular, concentrating on the continuity properties of the rigidity function \(D(x)\)—and comparing their scattering properties.

Section 6.2.2 verifies as much as is possible the theory contained in this chapter. The most definitive way of doing this is by checking that the solutions obtained satisfy the equations they are intended to—namely (2.8) and the applicable choice of edge conditions, either (2.12) or (2.13). As pointed out in Section 4.1, (2.8a), (2.8c) and (2.8d) are satisfied by construction. That section also showed theoretically that (2.8b) is satisfied, but this chapter also demonstrates it numerically.

Independent checks are available for the case when \(h_1(x)\) is constant—other researchers have produced results for that situation (e.g. Meylan, 1993; Dixon and Squire, 2001b), and the method used in Chapter 8 may also be used to directly verify those results. When the thickness of the ridge is variable, there are no other results that may directly confirm our results. However, if our thickness profile satisfies the mild slope approximation used by Porter and Porter (2004), and if our neglect of the submergence is justified, then results obtained by using the theory of that paper should be in reasonable agreement with results presented in this chapter. To this end a figure from the paper of Porter and Porter (2004) is reproduced later using our method, showing great similarity to the original.

A further check of our results is provided, which establishes that as a non-constant ridge profile approaches rectangularity the scattering behaviour of the ridge also ap-
proaches the validated results smoothly.

Section 6.2.3 attempts to approximate a triangular ridge by a rectangular one. When the ridge width is not too large (less than about 15 m) simply taking the rectangular ridge as having half the height of the triangular one seems to be adequate. The advantages of doing this when using numerical quadrature is that less panels are required for our results to converge when the thickness is constant, and that we only need to construct a single quadrature matrix, instead of the four $W_j$ matrices required for a general thickness profile. In addition, the method of Chapter 8 may also be used if that proves to be faster than quadrature.

The last section (Section 6.2.4) briefly treats the flexural effects of a keel, although submergence effects are still neglected. As might be expected, the increased rigidity produced generally leads to more reflection and less transmission. The increased width of the variable region also leads to more opportunity for resonances between the wavelength of the incoming wave and the edges of the ridge, thus producing more complicated reflection patterns than observed for the ridge when the keel is neglected. For the same reasons that are given in Section 6.2.3, a ridge with a triangular sail and keel is also approximated by another with a rectangular sail and keel. When reasonable ridge parameters are used—in particular when the ridge approximated is not too wide—there is good agreement between the results for the two shapes.

As in the latter plots of Chapter 5, all results in this chapter will be calculated for ice where $h_0 = h_2 = 1$ m. All results are for infinite depth (calculated by using a finite nondimensional depth of $H = 5$ for the reasons given in the previous chapter), except for the graph comparing our results with those of Porter and Porter (2004), where the water depth used is 20 m.

### 6.2.1 Types of Ridge Profiles

Three different types of ridge profile will be considered in this section: type 0, where $h(x)$ is $C^1$ (a $C^k$ function is one whose $k$th derivative exists and is continuous); type 1, where $h$ is $C^0$, but only piecewise $C^1$; and type 2, where $h$ is only piecewise $C^0$. Figure 6.2 illustrates examples of each type (in Figures 6.2a, d and g respectively), as well as demonstrating the scattering behaviour that those ridges produce. The Type
0 profile is based on the function

\[ u_{\sin}(x) = \sin^2(\pi x) \quad \text{for} \quad 0 \leq x \leq 1, \quad (6.20) \]

so that if \( h_{\text{sail}} \) is the height of the ridge sail (which in Figure 6.2 is 1 m), \( h(x) \) is given by the relation

\[ h(x) = h_0 + h_{\text{sail}} \cdot u(x/a) \quad \text{for} \quad 0 \leq x \leq a \quad (6.21) \]

with \( u(x) = u_{\sin}(x) \), which represents a ridge with a sinusoidal sail.

Figures b, e, and h show normal incidence results for type 0, 1 and 2 ridges respectively. All three show very similar patterns—reflection rising rapidly to a maximum as the wave period is increased from zero to about 0.5 s, before dropping monotonically to zero as the period increases further. However, the type 2 ridge shows noticeably more reflection than the other two—its maximum value is approximately 0.4, compared to about 0.3 for the type 1 ridge and about 0.35 for the type 0 ridge. \(|R|\) also has a zero at about 0.2 s, which we will see in later graphs is very common in the Type 2 plots, especially for wider ridge widths. These zeros also appear in later Type 0 and Type 1 figures but, near a period where a zero in \(|R|\) occurs for a rectangular ridge, it often only drops to a nonzero local minimum when the ridge thickness is not constant.

Figures c, f and i show how the reflection varies with angle of incidence for four different periods: 2 s (solid curves), 5 s (dashed curves), 10 s (chained curves) and 15 s (dotted curves). Again, the Type 2 ridge curves show more reflection than those of the Type 0 and 1 ridges—beginning from a higher normal incidence value of \(|R|\) at \( \theta = 0 \), and rising more quickly to 1 as \( \theta \to \pi/2 \). Between the regions of higher reflection near normal and grazing incidence, there is a region of low reflection which shows some interesting fine structure. In all four curves in Figure 6.2i this region contains a pair of zeros with a maximum located between them. The spacing between the pair of zeros increases with period, and the height of the maxima between them in general decreases with period. However as period is decreased below 2 s it also drops and the zeros move closer together before meeting to form a double zero and lifting off the horizontal axis.

This has already happened to the 2 s curve in Figure f, in which a single local minima can be observed at about \( 2\pi/9 \). However, the higher period curves in that figure still show the two-zero structure, as can all four curves in Figure c.
The general trend of the Type 2 ridge producing greater reflection than the Type 0 ridge, which itself produces slightly more reflection than the Type 1 ridge, is possibly correlated to their decrease in average masses (cross-sectional areas), as less mass means less rigidity and thus a greater ease of wave transmission. Section 6.2.3 investigates the effect of approximating a triangular ridge by a rectangular one of half the height, and shows that for small widths this approximation is quite satisfactory.
6.2.2 Verification of Results

The most definitive test of our theory is a check of whether our solution \( \phi \) satisfies the system of equations that it was intended to, namely (2.8) and the applicable set of edge conditions, (2.12) or (2.13). Figures 6.3b and c show the modulus of the displacement \(|\eta(x,y,t)| = |\phi_z(x,0)/\phi_0(0)|\) and the potential at the surface \(|\phi(x,0)|\) when a wave of period 5 s and amplitude 1 m arrives at the ridge with profile shown in Figure a at an angle of \( \pi/4 \). The frozen edge conditions are clearly satisfied at the ends of the ridge by the displacement shown in Figure 6.3b, while the two curves in Figure c corresponding to different methods of calculating \( \phi(x,0) \) show excellent agreement. The solid curve shows \( \phi(x,0) \) calculated analytically from (4.13), while the dashed curve shows it calculated directly from (2.8b) as \( \phi(x,0) = -\mathcal{L}(x,\partial_x)\phi_z(x,0) \) using numerical differentiation.

The very slight difference between the two curves at the top of the ridge is due to numerical error in the analytical calculation of the integral

\[
\int_0^a (\mathcal{L}(\xi, \partial_\xi) - \mathcal{L}_0(\partial_\xi)) G_\zeta(x - \xi, 0, 0) \phi_z(\xi, 0) d\xi,
\]

the kernel of which has a logarithmic singularity since

\[
\mathcal{L}_0(\partial_\xi) G_\zeta(x - \xi, 0, 0) = -G(x - \xi, 0, 0)
\]

(cf. equation 3.1b), and we know \( G \) is log-like when \( x - \xi = z - \zeta = 0 \) from the expansion (C.5) of the fundamental solution \( K_0(lr) \). Consequently a large number of modes is needed in the eigenfunction expansion for \( G_\zeta \) to make the integral converge. The error is greatest at the top of the ridge since the fourth derivative of \( G_\zeta \) is multiplied by \( D(\xi) - D_0 \).

Similar agreement (in fact better) was also observed for rectangular and triangular ridges, and for ridges of different widths and heights. However, for smooth ridges with smaller widths the function \( D''(x) \) becomes so large that for practical purposes \( \phi(x,0) \) becomes impossible to calculate numerically. This is because the differential operator we have to calculate is given by

\[
\mathcal{L}(x, \partial_x) = D(x)(\partial_x^2 - l^2)^2 + 2D'(x)(\partial_x^2 - l^2)\partial_x + D''(x)\mathcal{L}^{-}(\partial_x) + \lambda - m(x)\mu,
\]

and so a large \( D''(x) \) amplifies the error in \( \phi_{zz}(x,0) \) so much that convergence is slowed down significantly.
Figure 6.3: Verification that the solution satisfies equation (2.8b) for variable thickness. Figure a shows the thickness profile of the ridge in question, which is 50 m wide and has a sinusoidal sail 1 m high. Figure b shows $|\eta|$ when a wave of period 5 s and amplitude of 1 m arrives at an angle of $\pi/4$ from normal incidence. The solid curve in Figure c shows $|\phi(x,0)|$ calculated analytically from equation (4.13), while the dashed curve shows $|\phi(x,0)|$ calculated by numerically applying the operator $-\mathcal{L}$ to $\phi_z(x,0)$, the solution of (6.2). The water depth is infinite.

The second way we can check our results is by calculating the scattering by some ridges with constant thicknesses and comparing it with results calculated independently by other methods. Figure 6.4 plots the scattering by a constantly 2-m-thick ridge embedded in 1-m-thick ice which is either 15 m wide (a) or 100 m wide (b). The solid curves show the scattering when the free edge conditions are applied and may be directly compared with the solid curves in Figures 8.4a and b, calculated using the method of Chapter 8. The free edge result for the 100-m-wide ridge may also be compared with the chained curve in Figure 4 in the paper by Dixon and Squire (2001b). The different methods of calculation show excellent agreement.
Figure 6.4: The scattering by some Type 2 ridges of constant thickness, producing results that are able to be verified independently by other methods. The figure also illustrates the effect of the ridge width and the choice of edge conditions. The curves show the scattering of a 2-m-thick rectangular ridge located between two 1-m-thick semi-infinite ice sheets. The ridge is either 15 m wide (figure a) or 100 m wide (b), and both the free edge and the frozen edge conditions are applied (solid and dashed curves, respectively). The solid curves may be compared to the solid curves in Figures 8.4a and b; the solid curve in b may also be compared favourably to the chained curve in Figure 4 in the paper of Dixon and Squire (2001). The water beneath the ice is taken to be infinitely deep in both figures.

The dashed curves show the scattering by the same ridges when the frozen edge conditions are applied. These results have also been checked against the Chapter 8 results and agree well.

At this point we can also make some comments about the effect of changing the ridge width and/or the edge conditions that are applied. Both Figures a and b show that in general using the free edge conditions produces significantly more reflection.
than using the frozen edge conditions. Exceptions to this rule occur near periods where \( R = 0 \), or equivalently where \( |T| = 1 \). At these periods perfect transmission is obtained, and \( |R| \) can drop quite sharply, invariably causing the free edge curves to drop beneath the frozen edge curves.

Increasing the ridge width has the effect of moving the curves to the right, allowing several very closely spaced zeros that occurred at periods too small to be included in our period range in (a) to appear at low periods in both curves in (b). Interestingly, though, the two left hand zeros present in the free edge curve in (a) are not present in (b). By considering curves at intermediate ridge widths it can be seen that as \( a \) is increased the two zeros move closer together at the same time as the two left hand maxima drop to zero. All three points eventually meet when the ridge width is about 45 m before lifting off the period axis to form a local minimum, and eventually developing into the point of inflexion apparent in (b).

It is shown in Section 8.1.5 when discussing the NEW approximation that for larger ridge widths these zeros in \( |R| \) are due to resonances between the wavelength of the propagating component of the wave travelling beneath the central ridge and the ridge’s width. However for smaller ridge widths, the evanescent modes interfere with these resonances, modifying the period at which they occur and also even producing others that would not otherwise have occurred. For example the two zeros discussed in the previous paragraph that disappear as the ridge width increases must be attributed to the effect of evanescent waves. It can also be hypothesized that the absence of these zeros when the frozen edge conditions are applied is the result of a relative lack of evanescent wave production in comparison to when the free edge conditions are applied. This is consistent with the lower amount of reflection observed in the frozen edge curves.

Further confirmation of our variable (i.e. non-constant) thickness results can be obtained by comparing them to those of Porter and Porter (2004). Although different assumptions are made in their theory to ours—they assume a mildly sloped thickness profile, while we neglect submergence—comparing Figure 6.5 with Figure 1 of the Porter and Porter paper gives excellent agreement for seven different ridge profiles. This is extremely reassuring for us, especially given the relatively small water depth used (20 m).
Figure 6.5: Further confirmation of the variable thickness results. The plots shown are reproductions of plots presented in the paper by Porter and Porter (2004) created using a variational approach and a mild slope approximation. Figures a and c show different (sinusoidal) profiles, and the scattering corresponding to each is plotted in the curve to its right (either Figures b and d) using the same linestyle. Figures b and d show the behaviour of $|R|$ with wavelength when normally incident waves are incident upon the ridges in question. The water depth used is 20 m.

Figures 6.5a and b also confirm the results of Figure 6.4 that increasing the ridge width again leads to increasing fine structure in the scattering curves for small periods, including a small number of zeros in the $|R|$ curves corresponding to the ridges that are 60 m and 80 m wide. However, there are not as many zeros observed as for the constant thickness profiles. Certainly, since the wavelength beneath a non-constantly thick ridge is also non-constant, the interactions between it and the ridge width become more complicated. This probably makes the resonances required for perfect transmission of a wave of a certain period less likely.

Figures 6.5c and d show the predictable result that thicker ridges produce more
reflection. It also shows that the thinner ridges (the 1.25-m and 1.5-m-thick ridges) have zeros in $|R|$ which is also as expected since waves travelling beneath them have smaller wavelengths that are more similar to or smaller than the ridge width of 40 m, and thus produce resonances.

Figure 6.6: Confirmation that the scattering by a sequence of smooth Type 0 ridges (indicated in Figure a) approaches the scattering of a rectangular ridge as their profiles approach that of the rectangular Type 2 ridge also indicated in (a). The solid, dashed, chained and dotted Type 0 profiles are defined by (6.21) using $u(x) = u^2\sin(x)$, $u\sin(x)$, $u_0(x)$ and $u_{0.4}(x)$ respectively (cf. equations 6.20 and 6.22 for the definitions of $u\sin(x)$ and $u_a(x)$), and their scattering curves are plotted using the same linestyles in the two lower figures. Figure b plots the scattering of normally incident waves by each ridge when the ridge width $aL$ is 15 m and Figure c plots the scattering when $aL = 100$ m. The Type 2 scattering curves are indicated in each of the latter figures to distinguish them from the solid Type 0 curves, and the water depth is infinite.

The agreement between the results presented in this section, and between the results
of Chapter 8, Dixon and Squire (2001b) and Porter and Porter (2004) is reassuring. However, we will do one further test by investigating whether or not there is a smooth transition in terms of scattering properties as a Type 0 profile becomes more and more similar to a rectangular profile, as shown in Figure 6.6a. Physically it seems unlikely that a wave travelling beneath an ice sheet would be able to distinguish between two ridges of similar mass, but where one had sharp corners and in the other the corners were rounded off. Thus this test is a little like the energy conservation theorem (4.46) in that while a positive results doesn’t provide direct confirmation of our results it does allow us to say that they are not obviously incorrect. Indeed, like the energy rule, it has proved most useful in debugging programs.

The solid and dashed Type 0 profiles in Figure 6.6a are defined by (6.21) using $u(x) = u_{\sin}^2(x)$ and $u_{\sin}(x)$ respectively, while the chained and dotted profiles use $u(x) = u_0(x)$ and $u_{a,4}(x)$ (also respectively). The profile function $u_{a}(x)$ is defined as follows: if $u_q(x) = 1 - (1 - x)^3(3x + 1)$ for $0 \leq x \leq 1$, and if $\beta = (1 - \alpha)/2$ ($0 \leq \alpha \leq 1$),

$$u_{a}(x) = \begin{cases} u_q(\beta x) & \text{for } 0 \leq x \leq \beta, \\ 1 & \text{for } \beta \leq x \leq 1 - \beta, \\ u_q(\beta(1 - x)) & \text{for } 1 - \beta \leq x \leq 1. \end{cases}$$

$u_{a}(x)$ represents a quartic curve ramping up from 0 to 1 as it travels from $x = 0$ to $x = \beta$, retaining continuity in itself and its first two derivatives as it flattens out for a distance of $\alpha$, before ramping smoothly down to 0 again. It has zero derivative at $x = 0$ and $x = 1$. Obviously, $u_{a}$ becomes more rectangular as $\alpha \rightarrow 1$.

The two lower plots in Figure 6.6 show the changes in the $|R|$-period curves as we move from the solid Type 0 curve to the rectangular curve for two different ridge widths, 15 m (b) and 100 m (c). As in Figure 6.2, when the ridge width is 15 m and the curves are still relatively uncomplicated, the Type 2 ridge shows more reflection than any of the Type 0 ridges. And, as might have been expected, the reflection from the Type 0 ridges increases steadily towards the Type 2 reflection as their cross-sectional area increases.

When the ridge width is 100 m the same trend persists at higher wave periods (above about 12 s) but, as in earlier plots the lower period behaviour is much more complicated.
Nevertheless, for intermediate periods (between about 4 s and 12 s) general features such as the positions of zeros or local minima in $|R|$, or local maxima can still be seen to also move continuously towards their positions in the Type 2 curve as the sail area increases. At lower periods the scattering behaviour of the Type 2 ridge is very complex with a large number of closely spaced zeros and so it is difficult to ascertain from Figure 6.6c how, or even if, the Type 0 curves are progressing towards the Type 2 curve. By zooming in, however, it can be seen that the zeros in the Type 2 curve all have corresponding zeros, or local minima for the smaller sails, and that the maxima between the zeros/minima are increasing in height towards the height of the corresponding Type 2 maxima. This provides more evidence of continuity in profile and thus more confidence that our results are correct.

6.2.3 Approximation of Results

The scattering by a rectangular ridge can be calculated significantly more quickly than the scattering by a triangular ridge, the type most commonly observed in ice fields. Consequently, it is expedient to investigate whether there is a case for approximating the latter type of ridge by an appropriately chosen rectangular ridge.

Figure 6.7 shows the results obtained by approximating a triangular ridge by a flat one with the same width and average height, as illustrated in Figure a. The sail height of 0.5 m used in this graph is a more realistic height, given the background ice thickness is $h_0 = 1$ m, than the sail height of 1 m used in Figures 6.8, 6.4 and 6.5 which was chosen for the entirely aesthetic reason that it produces larger reflections.

Figures b, c and d plot $|R|$ for three different widths: 2 m, 5 m and 15 m respectively. The 2-m-wide ridge is more realistic than the other two ridges, which were included to demonstrate the limits of the approximation.

Figures b and c show that the scattering by the triangular ridge for the two smallest widths is approximated well by the rectangular one; however, as the width increases to 15 m in Figure d, the fine structure observed in the $|R|$ graphs of the previous section begins to develop at shorter periods and the two curves begin to diverge. This is because the aforementioned fine structure—and in particular, the zeros in $|R|$ and the heights of the maxima in between them—depend on resonances between the wavelength in the ridge and the ridge width and thus it will be more sensitive to differences
Figure 6.7: Approximation of a triangular ridge by a rectangular one with the same average height, as illustrated in Figure a. Figures b, c and d plot $|R|$ against wave period for ridge widths of $aL = 2\text{ m}$, 5 m and 15 m respectively (the $L$ factor redimensionalizes the nondimensional quantity $a$). The water depth is infinite.

Nevertheless, there is still reasonably good agreement for wave periods greater than about 4 s (i.e. for most physically significant periods), and we can still conclude from Figures b and c show that for realistic ridge widths the approximation is a very good one.

6.2.4 Flexural Effects of a Keel

This section attempts to model the flexural effects of a keel on the way that a pressure ridge scatters ice-coupled waves. Note that our results still neglect the effect of the keels' submergence, but in light of the work of Porter and Porter (2004) we believe
that flexural effects will dominate if their slopes are mild enough. In the results below we present results for Type 0 (smooth), 1 (triangular) and 2 (rectangular) ridges—the results for the former two are probably reasonably accurate since the transitions are only gradual, but the abrupt changes caused by the rectangular keels could lead to an underestimation of the reflection. (If the slopes of the Type 0 and 1 keels are not quite mild enough we would also expect a slight underestimation of $|R|$. ) However, our main use of the calculations from the rectangular ridges will be as an approximation for the more common triangular ridges—consequently, if the Type 1 results are taken to be accurate, then the Type 2 results are still useful if they can generate approximations to them, even if they produce less reliable results for the amount of scattering that genuine, submerged (and less common), rectangular ridges would produce. (Rectangular ridges require fewer panels for $R$ and $T$ to converge because of the comparative simplicity of their thickness profiles; cf Section 6.1.1.)

Therefore, we will continue on the assumption that only considering flexural effects and neglecting submergence will give us reasonably accurate results for more realistic ridges (Types 0 and 1). We expect that the increased rigidity due to the inclusion of a keel would permit less transmission and thus produce more reflection, and we also expect that the increased ridge width would produce more complicated scattering behaviour as it becomes comparable to the wavelengths of the incoming waves. This is exactly what is shown by Figure 6.8, which for each of the three different sail types shown in Figure 6.2 shows the effects of three differently proportioned keels on the ridge’s scattering behaviour. The no-keel results are also plotted as dotted lines for additional comparison.

The keels chosen reach to maximum depths of 2.4 m (chained curves), 3.2 m (dashed curves) and 4 m (solid curves) and their widths are such that the cross-sectional area of the keel is nine times that of the sail to produce neutral buoyancy (assuming average ice density of $\rho = 0.9\rho_w$ throughout the entire ridge; cf. Table 2.1).

As in Figure 6.2, Figure 6.8f shows much more reflection for the Type 2 ridges than for the Type 0 and 1 ridges shown in Figures b and d. The Type 0 ridges produce slightly more reflection than the Type 1 features, which also mirrors the results when no keels are present.
For all three types of ridge, the most obvious effect produced by including a keel is that the amount of reflection is increased. In addition, the extra width allows some zeros at smaller periods for the Type 2 ridges, and minima for the other two types of ridges.

Figure 6.8: The scattering by differently shaped ridges when the flexural effects of the keel are included. The profiles used are plotted in Figures a, c and e. The sails are the same as those used in Figure 6.2, while the keels have depths 4 m (solid curves), 3.2 m (dashed curves) and 2.4 m (chained curves) and widths calculated to make the ridge neutrally buoyant. The reflection produced by each keel shape for normally incident waves is plotted in the same linestyle in the figure to its right, and for comparison the no-keel results are plotted as dotted lines (cf. the second column of plots in Figure 6.2). The water depth is infinite.

Comparing the different choices of keels, these zeros/minima are more numerous and occur at higher periods for the wider keels, but the reflection maxima are greater for the deeper ones. However, as the wave period increases and the keel dimensions become smaller in comparison with the wavelength, the results for the three keels con-
These results suggest that when a keel is present, an “average height” approximate approach such as the one used in Section 6.2.3 might be more successful for ridges that are not as wide as the ones used in Figure 6.8. Indeed, as pointed out in the previous section, those ridges would be extremely difficult to find in 1-m-thick ice.

Figure 6.9: Approximation of a triangular ridge by a rectangular one when the flexural effects of the keel are included, as illustrated in Figure a. Recall that z is the nondimensionalized vertical coordinate. Figures b, c and d show |R| plotted against wave period for sail widths of 2 m, 5 m and 15 m respectively. The sail height is 0.5 m, the keel depth is 1.5 m, and the keel width is three times that of the sail in all three scattering figures. The incident waves are normally incident and the water is infinitely deep.

Figure 6.9 seeks to approximate the scattering by a ridge that has a sail height of 0.5 m, a keel depth of 1.5 m, and keel width three times larger than the sail width. As in Figure 6.7, when only the ridge sail was considered, a rectangular sail of half the
original sail height is used in the approximation; however, a rectangular keel of half the depth of the triangular keel is also added to complete the ridge, as illustrated in Figure 6.9a.

As Figure b shows, when the sail is only 2 m wide, the approximation is reasonably good. It is still adequate at most periods when the sail width is increased to 5 m in Figure c, with slight differences for shorter incident waves. However, if its width is made much wider, Figure d shows that the total ridge width becomes large enough for the individual resonance properties of the different shapes to take effect, and the curves become quite different for periods less than about 7 s.

However, as noted in Section 6.2.3, for the background ice thickness used, the 2-m-wide ridge is the most realistic and so we can conclude that the approximation is still relatively accurate when physical limitations on ridge size are taken into account. In Chapter 9 when ice fields containing multiple ridges are discussed, this will speed up our results-gathering considerably. (Constant thickness profiles take fewer panels to converge than variable-thickness profiles; cf. Section 6.1.1.)
Chapter 7

The "General Solution": Scattering by a Sea Ice/Ice Shelf Transition

The problem tackled in this chapter is illustrated in Figure 2.1. It is the most general of the problems treated in this thesis, as it allows for a variable thickness profile over the finite interval \((0, a)\) and also allows for the thickness \(h_2\) to the right of that interval to differ from the thickness \(h_0\) to its far left. For brevity, in this chapter we shall refer to the variable region as a ramp, although we emphasize from the outset that there is no requirement for the ramp to be a simple linear function.

A thickness profile such as the one shown in Figure 2.1 could be used to model a sea ice/ice shelf transition such as the one observed in the Ross Sea, where the thickness of the sea ice beyond the ice shelf increases steadily to meet the thicker ice of the shelf itself. Alternatively, by setting \(h_2\) to zero, and by letting the incident wave arrive from the right instead of from the left, the problem could equally represent a breakwater shielding a VLFS from waves arriving from the open ocean. Results are presented in this chapter for both these situations.

As in our treatment of pressure ridges in the previous chapter, the model does have the drawback that submergence is ignored. However, the method of Porter and Porter (2004) also incorporated a change in ice thickness from one side of the variable region to the other, and so we can again use their result that for large water depths the effect of submergence is negligible as long as the ramp slope is mild enough and the correct total thickness is used. (In this case their solution method could be used instead of the one presented here, and has the advantage that submergence is accounted for.)
7.1 Solution Method

To complete this problem we must solve two coupled integral equations. The first is much the same as the one solved in Chapter 6, but the second is over a semi-infinite range—for \( \phi_2(x, 0) \) to the right of the ramp.

How the two are related is most easily shown by presenting each in turn, beginning with the integral equation over \((0, a)\). Since \( h_2 \neq h_0 \), the second integral in (4.37) no longer vanishes; by substituting (3.7) into it we can see that for \( x < a \) it can be written as

\[
\sum_{n=-2}^{\infty} \beta_n^+ e^{i\alpha_n (a-x)},
\]

where \( \beta_n^+ = -iA_n f_2(\gamma_n) \Phi^+(\alpha_n) \), and the Fourier transform \( \Phi^+ \) was defined in equation (4.40).

Hence, proceeding as in Section 6.1, the integral equation we need to solve over \((0, a)\) is

\[
d_0(x) \phi_2(x, 0) = e^{i\alpha_0 x} \varphi_0'(0) + \psi^T(x) P_0 + \psi^T(x-a) P_a \\
+ \sum_{n=-2}^{\infty} \beta_n^+ e^{i\alpha_n (a-x)} + \int_0^a K(x, \xi) \phi_2(\xi, 0) d\xi,
\]

where we have used the Cauchy principal value symbol \( f \) in the same way that we used it in (6.2) to formally avoid the delta function singularity in \( K(x, \xi) \). Since the \( \beta_n^+ \) coefficients depend on the values that \( \phi_2(x, 0) \) takes for \( x > a \), they are unknowns that will be eliminated by solving the second integral equation analytically.

To derive this second integral equation, we substitute (3.7) into the first integral of (4.37); when combined with the incident wave forcing and the edge terms, it evaluates to

\[
\sum_{n=-2}^{\infty} \beta_n^- e^{i\alpha_n (x-a)},
\]

where

\[
\beta_n^- = iA_n (i\delta_0n + p^T(-\alpha_n)(P_0 e^{i\alpha_n a} + P_a) + \hat{\phi}(-\alpha_n) e^{i\alpha_n a}),
\]
\[ \dot{e} = -i\varphi'(0)e^{i\alpha a}/A_0, \] and \( \dot{\phi} \) was also defined in (4.40). It contains Fourier integrals over \((0, a)\) involving \( \phi_z(x, 0) \), and so provides the dependence of \( \phi_z(x, 0) \) for \( x > a \) upon (7.1).

Consequently, we must solve the following Wiener-Hopf type integral equation over the semi-infinite interval \((a, \infty)\):

\[ \phi_x(x, 0) = \sum_{n=-2}^{\infty} \beta_n^{-} e^{i\alpha_n(x-a)} + \int_{a}^{\infty} (\mathcal{L}_2 - \mathcal{L}_0) g(x - \xi) \phi_z(\xi, 0) d\xi. \] (7.2)

Note that when \( h_0 = h_2 \) and \( f_2(\gamma_n) = 0 \), the \( \beta_n^+ \) are also zero, so (7.1) reduces to (6.2) again, while the kernel of the integral operator in (7.2) vanishes, and so the solution proceeds in the same way as it did in Chapter 6.

Using the Wiener-Hopf technique, equation (7.2) may be solved analytically in terms of the \( \beta_n^{-} \) constants. This enables the \( \beta_n^+ \) constants to be written in terms of the \( \beta_n^{-} \) and consequently to be eliminated from (7.1). Accordingly, (7.1) may be solved by adjusting the quadrature scheme developed in Section 6.1.1 to allow for the presence of the quantities \( \dot{\phi}(-\alpha_n)e^{i\alpha a} \) in the \( \beta_n^{-} \).

We will now proceed to describe how the Wiener-Hopf solution proceeds, and then indicate how the quadrature scheme of the previous chapter can be adapted to this new situation.

7.1.1 Solution of Wiener-Hopf Type Integral Equation

In order to solve the Wiener-Hopf type integral equation (7.2), we must take its Fourier transform. Since that transform involves an integral over an infinite interval, we must somehow extend (7.2) into \((-\infty, a)\). To do this we take the functions outside the integral to be zero for \( x < a \), and then write the extension of the integral into that region as an inverse Fourier transform:

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_a^-(k) e^{-ikx} dk = H(a-x) \int_{a}^{\infty} (\mathcal{L}_2 - \mathcal{L}_0) g(x - \xi) \phi_z(\xi, 0) e^{i(k(x-a))} d\xi. \] (7.3)

Taking the Fourier transform of (7.2) with respect to \( x - a \) then gives

\[ \Psi^-(a) + \Phi^+(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^-}{k + \alpha_n} + \left( 1 - \frac{f_2(\kappa)}{f_0(\kappa)} \right) \Phi^+(k), \] (7.4)
which can be rearranged into

\[ \Psi_a^-(k) + \frac{f_2(\kappa)}{f_0(\kappa)} \Phi^+(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^-}{k + \alpha_n}. \]  

(7.5)

In the above working we have used "\(^+\)" and "\(^-\)" superscripts to refer to functions that are analytic in the upper and lower complex half-planes respectively; in the following such functions will be referred to as "plus" or "minus" functions (again respectively). \( \Phi^+ \) and \( \Psi_a^- \) satisfy the above definition if we assume a bounded displacement and the exponential decay that we introduced when letting \( \varepsilon \) be nonzero. Assuming a bounded displacement also implies that both the above transforms will be \( O(k^{-1}) \) as \( |k| \to \infty \).

Equation (7.5) is now a typical Wiener-Hopf equation (Roos, 1969; Noble, 1958; Chung and Fox, 2002a; Chung and Fox, 2002b). To solve it we must write the quotient \( \frac{f_2}{f_0} \) as the product of a plus and a minus function, i.e.

\[ \frac{f_2(\kappa)}{f_0(\kappa)} = K^+(k)K^+(-k) = K^+(k)K^-(k). \]  

(7.6)

Since the quotient is meromorphic, this can be done by using the Weierstrauss factorization theorem (Noble, 1958) to express the entire functions \( \kappa f_j(\kappa) \sinh \kappa H \) as infinite products (\( j = 0, 2 \)). For example, \( \kappa f_2(\kappa) \sinh \kappa H \) can be written (e.g. Chung and Fox, 2002a):

\[ \kappa f_2(\kappa) \sinh \kappa H = \prod_{n=-2}^{\infty} \left( 1 - \frac{\kappa^2}{\gamma_n^2} \right). \]  

(7.7)

Hence

\[ K^+(k) = \prod_{n=-2}^{\infty} \frac{k + \bar{\alpha}_n}{\gamma_n} \prod_{n=-2}^{\infty} \frac{k + \alpha_n}{\gamma_n} = \sqrt{\frac{D_2}{D_0}} \prod_{n=-2}^{\infty} \frac{k + \bar{\alpha}_n}{k + \alpha_n}. \]  

(7.8)

and (7.5) becomes

\[ K^+(k)\Phi^+(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^- / K^+(\alpha_n)}{k + \alpha_n} \]

\[ = i \sum_{n=-2}^{\infty} \frac{\beta_n^-}{k + \alpha_n} \left( \frac{1}{K^-(k)} - \frac{1}{K^+(\alpha_n)} \right) - \Psi_a^-(k) \frac{K^-(k)}{K^-(k)}. \]  

(7.9)

We have subtracted the sum

\[ i \sum_{n=-2}^{\infty} \frac{\beta_n^- / K^+(\alpha_n)}{k + \alpha_n} \]
from both the right and the left hand sides of equation (7.9) to cancel the poles in the
original sum at \( k = -\alpha_n \), which makes the right hand side into a minus function. Since
the left-hand side of the equation is clearly a plus function, the Riemann principle
implies that both sides must be equal to a single entire function. (This principle
states that two functions that are analytic in adjacent regions are the same function if
they agree on the boundary between the regions; cf. Roos, 1969.) In addition, since
\( K^+(k) \sim O(1) \) as \( |k| \to \infty \), both sides of (7.9) must decay like \( k^{-1} \). Consequently,
Liouville's theorem implies that the entire function must be equal to zero. (Liouville's
theorem states that an entire function whose modulus is bounded by a constant must
itself be constant.) Hence, we have the result we need to solve (7.1), namely

\[
\Phi^+(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^- / K^+(\alpha_n)}{K^+(k)(k + \alpha_n)},
\]

which can now be incorporated into the quadrature scheme below by setting \( k = \alpha_n \) in
(7.10) and multiplying through by \(-iA_n f_2(\gamma_n)\) to evaluate the \( \beta_n^+ \) in terms of the \( \beta_n^- \).

7.1.2 Incorporation into Quadrature Scheme

From (7.10), after truncating the sums after \( n = N \) (so that \( n \in \mathcal{N} = \{-2, -1, \ldots, N\} \)),
we can now write

\[
\beta^+ = iM_\beta (f_\beta + F^- \phi_z),
\]

where

\[
[M_\beta]_{nm} = \frac{A_n A_m f_2(\gamma_n) / K^+(\alpha_m)}{K^+(\alpha_n)(\alpha_n + \alpha_m)}, \quad [f_\beta]_n = i\delta_{nm} + p^T(-\alpha_n)(P_0 e^{i\alpha_n} + P_a),
\]

and where we have discretized the interval \((0, a)\) in the same way that we did in
Section 6.1.1 in order to use the result (6.15) to evaluate the values of \( \Phi(-\alpha_n) e^{i\alpha_n} \)
that appear in the \( \beta_n^- \) using the matrix \( F^- \). Once again \( \phi_z \) is an unknown vector with
elements \( [\phi_z]_m = \phi_z(x_m, 0) \) (for \( m \in \mathcal{M} = \{0, 1, \ldots, M\} \)). To find it, we approximate
equation (7.1) at \( x = x_m \) by

\[
M_0 \phi_z = f_0 + E \beta^+,
\]

where \( M_0 \) and \( f_0 \) were defined in the previous chapter (in Section 6.1.1), and the \( [E]_{nm} \)
are given by \( [E]_{nm} = e^{i\alpha_n(a-x_m)} \). Hence we can obtain \( \phi_z \) from the following equation:

\[
(M_0 + M_1) \phi_z = (f_0 + f_1),
\]

where

\[
M_1 = -i E M_\beta F^-, \quad f_1 = i E M_\beta f_\beta.
\]
7.1.3 Scattering Coefficients

From (4.39), the coefficients $a_n$ in the expansion (4.38) can be written

$$a_n = iA_n \left( p^+(\alpha_n) (p_0 + p_a e^{i\alpha_n}) + \hat{\phi}(\alpha_n) \right) + \beta_n^+ e^{i\alpha_n}.$$  \hspace{1cm} (7.14)

Recalling that the $\hat{\phi}(\alpha_n)$ are able to be calculated from (6.13), these can be calculated straightforwardly. If the edge conditions have already been applied to calculate $P_0$ and $P_a$, $R$ can be calculated immediately from (4.41); otherwise we must also calculate the $d_n$ in the expansion (4.42) for $\phi_2(x, 0)$ when $x > a$.

This is done as follows. We invert the transform (7.10) by closing the inversion integral in the lower half plane, extracting the residues at the poles at $k = -\hat{\alpha}_n$; the $d_n$ are equal to these residues multiplied by $-i$. Thus,

$$d_n = -\hat{A}_n f_0(\gamma_n)K^+(\hat{\alpha}_n) \sum_{m=-2}^{\infty} \frac{\beta_m^{-}/K^+(\alpha_m)}{\alpha_m - \hat{\alpha}_n},$$  \hspace{1cm} (7.15)

or

$$d = iM_d (f_\beta + F^- \phi_2),$$  \hspace{1cm} (7.16)

where

$$[M_d]_{nm} = \frac{A_m \hat{A}_n K^+(\hat{\alpha}_n) f_0(\gamma_n)}{K^+(\alpha_m)(\hat{\alpha}_n - \alpha_m)} \text{ for } n, m \in \mathcal{N}.$$  

This result can be used to complete the application of the edge conditions if they have not been applied already; otherwise we can simply calculate the transmission coefficient $T$ from (4.43).

7.1.4 Solution for a Flexible Breakwater

When $h_2 = 0$, our solution method must be adjusted slightly. Firstly, the purpose of a breakwater is generally to reflect wave energy from the open ocean, and so to model this, we ought to have our waves arriving from the right instead of from the left (i.e. instead of propagating into the open water from beneath the VLFS). Fortunately, there is an extremely simple relationship between the scattering coefficients for the two cases (Meylan, 1993), and so we will continue to complete our solution for the wave arriving from the left, and present this relationship in Section 7.1.4. The reason we cannot simply set $h_0 = 0$ and solve in the same manner that we did in the previous
section is that in that case \( g \) has a logarithmic singularity and so its required derivatives 
\( \mathcal{L}_j(\partial_x) - \lambda, j = 1, 2 \) have \( 1/x^3 \) type singularities. As a result sums like 
\[
\sum_{n=0}^{\infty} \frac{\beta_n^j}{k + \alpha_n}
\]
in (7.5) don't converge. Theoretically problematic terms should combine with others 
in the unknown transforms, but the solution proceeds more smoothly when the singularities 
are dealt with explicitly. However, this extra effort is unwarranted when results 
can be obtained by just using the method of the previous section.

The second adjustment we must make when \( h_2 = 0 \) is due to the dispersion relation 
for the right hand side no longer having two complex roots. Consequently, the infinite 
product \( K^+ \) must be adjusted, and the size of the matrix \( M_d \) must be changed from 
\((N + 3) \times (N + 3)\) to \((N + 1) \times (N + 3)\). The latter change is trivial, while the new 
form of \( K^+ \) can be written down immediately (using the infinite product expansions 
for \( f_0 \) and \( f_2 \); cf. equation 7.7 for the expansion of \( f_2 \)—however, the product will now 
only be from \( n = 0 \) to \( \infty \)):
\[
K^+(k) = \Pi_b \prod_{n=0}^{\infty} \frac{k + \hat{\alpha}_n}{k + \alpha_n} \prod_{n=-2}^{-1} k + \alpha_n,
\]
where 
\[
\Pi_b = -|\gamma_1|^2 \prod_{n=0}^{\infty} \frac{\gamma_n}{\gamma_n} = \sqrt{\frac{\lambda}{D_0}}.
\]
Although the right hand expression for \( \Pi_b \) is exact, it takes a long time for the truncated 
product to converge to that number, and so it was found that computing it 
directly gave better results.

Lastly, since there is water to the right of the breakwater, we are unable to exploit 
the expansion (4.42) in applying the right hand edge conditions and instead we must 
apply the free edge conditions (2.13) from within the breakwater using the method 
described below.

**Internal Application of Free Edge Conditions**

To apply the free edge conditions from within the breakwater we use the same method 
used in Section 6.1.3. Applying the operator \( \mathcal{L}^- (\partial_x) \) to (4.37) and allowing for the presence 
of further delta function type singularities from \( \partial^n_x (\partial_x^2 - l^2)^2 g \) type terms eventually
yields a relatively simple formula for the bending moment, while applying $\mathbf{L}^+(\partial_x)\partial_x$ gives a similar formula for the vertical edge force. Respectively, these formulae are

\[
\mathbf{M}(x, \partial_x)\phi_2(x, 0) = -\sum_{n=-2}^{\infty} f_-(\gamma_n) \left( b_n e^{i\alpha_n x} + c_n e^{i\alpha_n (a-x)} \right) + \sum_{j=1}^{4} \int_0^{a} \mathbf{L}^-(\partial_\xi)\mathbf{L}_{1j}(\partial_\xi)g(x - \xi)\phi_2(\xi, 0) d\xi, \quad (7.18a)
\]

\[
\mathbf{S}(x, \partial_x)\phi_2(x, 0) = -i \sum_{n=-2}^{\infty} \alpha_n f_+(\gamma_n) \left( b_n e^{i\alpha_n x} - c_n e^{i\alpha_n (a-x)} \right) - \sum_{j=1}^{4} \int_0^{a} \mathbf{L}^+(\partial_\xi)\partial_\xi\mathbf{L}_{1j}(\partial_\xi)g(x - \xi)\phi_2(\xi, 0) d\xi, \quad (7.18b)
\]

where we have used the relation $\partial_\xi g(x - \xi) = -\partial_\xi g(x - \xi)$ to simplify the terms under the integrals.

Let us now define the vectors $\mathbf{e}_M$ and $\mathbf{e}_S$, and the diagonal matrix $D_\alpha$ as follows:

\[
[e_M]_n = -iA_n f_-(\gamma_n), \quad [e_S]_n = A_n \alpha_n f_+(\gamma_n), \quad [D_\alpha]_{nn} = e^{i\alpha_n} \delta_{nm}, \quad \text{for } n, m \in \mathcal{N}.
\]

Truncating the sums in (7.18) after $n = N$, we can then write the required moments and edge forces as

\[
M^+(0) = \mathbf{e}_M^T (\mathbf{b} + D_\alpha \mathbf{c} + F^+ \phi_2),
\]

\[
S^+(0) = \mathbf{e}_S^T (\mathbf{b} - D_\alpha \mathbf{c} - F^+ \phi_2),
\]

\[
M^-(a) = \mathbf{e}_M^T (D_\alpha \mathbf{b} + \mathbf{c} + F^- \phi_2),
\]

\[
S^-(a) = \mathbf{e}_S^T (D_\alpha \mathbf{b} - \mathbf{c} + F^- \phi_2).
\]

Setting each of these to zero gives us four equations in the four unknowns $P_j(x_c)$ ($j = 0, 1, x_c = 0, a$).

Wave Arriving from the Right

This section presents a result which is well-known in the water wave context (p322–323 Kreisel, 1949; Kinsman, 1983); it has also been presented in the sea ice literature by Meylan (1993) and Meylan and Squire (1993). We wish to find a function $\tilde{\phi}(x, z)$ that satisfies the same equations (2.8) and edge conditions (2.13) as $\phi$ does, but has the following asymptotic behaviour as $|x| \to \infty$:

\[
\tilde{\phi}_<(x, z) \sim \begin{cases} \bar{T} e^{-i\alpha_0 x} \varphi_0(z) & \text{as } x \to -\infty, \\ e^{-i\alpha_0 x} \tilde{\phi}_0(z) + \bar{R} e^{i\alpha_0 x} \tilde{\phi}_0(z) & \text{as } x \to \infty. \end{cases} \quad (7.20)
\]
Since $\phi^*$, the complex conjugate of $\phi$, satisfies the same equations that $\phi$ itself does, so will

$$\tilde{\phi}(x, z) = \frac{1}{T^*} (\phi^*(x, z) - R^* \phi(x, z)),$$

(7.21)

which satisfies the radiation conditions (7.20). $\tilde{R}$ and $\tilde{T}$ may now be written explicitly as

$$\tilde{R} = -\frac{R^* T^*}{T^*},$$

(7.22a)

$$\tilde{T} = \frac{1 - |R|^2}{T^*} = sT,$$

(7.22b)

where the energy conservation law (4.46) has been used to simplify (7.22b). Also note that (7.22a) implies that $|\tilde{R}| = |R|$, and so if we are only interested in that quantity we effectively only need to calculate $|R|$ as if the wave were arriving from the left.

One thing we should take care of when the incident waves are arriving at an oblique angle is that the wave will be refracted as it passes from the water into the ice/VLFS. If the wave from the right arrives at an angle $\theta$ from normal incidence, then the parameter $l$, which contains the information about the incident angle should be calculated from $l = \gamma_0 \sin \theta$. The angle of incidence $\theta$ that the corresponding wave approaching from the left arrives at can be obtained from the relation

$$\sin \theta = l/\gamma_0 = \gamma_0 \sin \tilde{\theta}/\gamma_0.$$

### 7.2 Results

The beginning of our results section takes the same structure as the beginning of the previous chapter. First Section 7.2.1 demonstrates the different types of ramp profiles we will use, which, as in Chapter 6, are classified in terms of the continuity properties of $D(x)$ and $D'(x)$ at the ends of the variable region. And second, Section 7.2.2 presents results that verify the accuracy of our theory and its numerical implementation.

After those two sections we present a pair of applications that are more specific to the theory of this chapter. Section 7.2.3 attempts to model a sea ice/ice shelf transition such as occurs where the Ross sea sea ice meets the Ross Ice Shelf, while Section 7.2.4 presents the effectiveness of breakwaters with different widths and thickness profiles. The latter results could be used to protect a Very Large Floating Structure (VLFS) such as a floating airport from incoming ocean waves.
7.2.1 Types of Ramp Profiles

As in Section 6.2.1, our ramp profiles will be classed as either Type 0, Type 1 or Type 2. The Type 0 ramps have the property that \( D(x) \) and \( D'(x) \) are continuous at their edges and thus no edge conditions need to be applied there (assuming frozen edge conditions are applied, one must always apply two conditions at each free edge); for the Type 1 ramps \( D(x) \) is continuous but \( D'(x) \) is not, and so one edge condition must be applied; for the Type 2 ramps \( D(x) \) and/or \( D'(x) \) are discontinuous, meaning two edge conditions must be applied. Examples of each type are plotted in Figures 7.1a, d and g, and the scattering behaviour of each is plotted in the two graphs to their right. The middle column of plots show the scattering of normally incident waves, while the right hand column shows the scattering of obliquely incident waves for a series of periods.

Unlike Figures 6.2 and 6.8, all three profiles have the same average thickness and consequently produce similar amounts of reflection at normal incidence. Compared with the Type 2 ridges of the previous chapter, the Type 2 ramp scattering curve has a lot more fine structure at low periods, a fact which can be attributed to the shorter wavelengths of waves travelling in 1.5-m-thick than in ice with thicknesses of 2 m or greater. However, we still see a decrease in this fine structure when the ice thickness is allowed to vary over the ramp's width, although the linear Type 1 ramp does have a zero in \(|R|\) at about 0.3 s. Thus, we can say that the more complicated the thickness profile, the less fine structure in the scattering curve, and the less reflection. This also explains why the smooth ridges in the previous chapter always seemed to produce more reflection than the linear ones.

There are two main differences between the scattering of obliquely incident waves by ramps and by ridges—both due to \( h_2 \) being different to \( h_0 \). The first is that the only zeros in \(|R|\) are in the three 15 s curves (dotted curves). The curves corresponding to other periods either have minima or monotonic increase (cf. the 2 s curve in Figure i). The second is that each period has a critical angle, less than \( \pi/2 \), above which any incident waves will be completely reflected. This is due to the wavenumber in the thicker ice \( \gamma_0 \) being larger than \( \gamma_0 \). If \( l \), the component of the wavenumber in the \( y \) direction, exceeds \( \gamma_0 \), no wave is able to propagate into the right hand region in the \( x \) direction. (This corresponds to \( \alpha_0 \) either vanishing or becoming imaginary.)
Figure 7.1: The scattering of waves incident on three different types of ramp—a smooth Type 0 ramp, with profile shown in figure a; a linear Type 1 ramp (d); and a double step Type 2 ramp (g). Scattering results for each different type of ramp are shown in the two plots to the right of the corresponding thickness profile—figures in the second column (figures b, e and h) illustrate the variation in the amount of reflection of normally incident waves with wave period, while figures in the right hand column (figures c, f and i) show the behaviour of $|R|$ with angle of incidence for a series of different wave periods. The periods used are 2 s (solid curves), 5 s (dashed curves), 10 s (chained curves) and 15 s (dotted curves), and the water depth is infinite.

Figure 7.2 plots this critical angle as a function of period for a series of values of the ratio $h_2/h_0$. The two smaller values of $h_2/h_0$ show that as the ratio becomes smaller and smaller there is a “critical period”, at which the critical angle reaches $\pi/2$, that moves closer to the vertical axis. In the limit, when $h_2$ reaches $h_0$, the critical angle is constantly $\pi/2$ since the wavenumbers on both sides of the ramp are equal (this can be checked by referring to Figure 6.2). Conversely, this critical period moves to larger periods as $h_2$ becomes greater in comparison with $h_0$, and has moved out of the plotted range for the three curves corresponding to $h_2/h_0 = 3, 5$ and 10. A result of this is that the critical angle curves move downward as $h_2/h_0$ increases, so that when $h_2$ is
very thick very little transmission of obliquely incident waves will be possible at all.

Figure 7.2: The variation of the critical angle of reflection with period for different values of \( h_2/h_0 \): 1.5 (upper solid curve), 2 (dashed curve), 3 (chained curve), 5 (dotted curve) and 10 (lower solid curve). The water depth is infinite.

7.2.2 Verification of Results

This section is devoted to the verification of our ramp results, in much the same way that Section 6.2.2 was devoted to checking the ridge results. Most of the figures are similar to Figure 6.6 and establish that our variable thickness results converge to independently verifiable results as their profiles converge to ones that permit alternative solutions. The first, however, Figure 7.3, attempts to reproduce Figure 5a of the paper by Porter and Porter (2004). As mentioned in the previous section, that paper treats the variable thickness problem by using a mild slope approximation and a variational method. It does not assume that there is no submergence as we do.
Figure 7.3: Verification of linear ramp results. The plot shown is a reproduction of Figure 5a of the paper by Porter and Porter (2004). Figure a of this plot shows three different thickness profiles that ramp up linearly from 1 m to either 1.5 m, 2 m or 3 m. The scattering by each ramp is plotted in Figure b using the same line-style that was used for its profile’s line-style in a. In that figure $|R|$ is plotted against its dimensional wavenumber for the left hand region, $\alpha_0/L$. The incident waves are normally incident and the water depth is 20 m.

Figure 7.3a plots the thickness profiles of the ramps that Porter and Porter used, and Figure b plots the corresponding scattering, calculated by our method, as a function of the incident wavenumber. The two methods appear to be in reasonably good agreement, although they disagree in the long wave limit. As $\alpha_0 \to 0$, Figure 7.3b predicts that $|R|$ will vanish for all three profiles, while Porter and Porter’s figure shows it tending towards a small finite value which decreases as $h_2$ decreases. This value is predicted by Lamb (1932, Section 176), and is the result of narrowing of the right hand fluid region due to the greater submergence of the ice on that side. This example reveals one of the limitations of the no-submergence assumption at finite depths but it will not affect our infinite depth results.
Figure 7.4: Convergence of the scattering by a series of increasingly steep Type 1 linear ramps to the scattering by a step, which may be considered as an infinitely steep ramp. The curve corresponding to the step is pointed out in figure (b), as is the curve corresponding to the ramp profile plotted in figure (a) with a solid line; the scattering curves for the other ramps are also plotted using the line-style used for their profile in figure (a). The scattering curves for the other ramps are also plotted using the line-style used for their profile in figure (a). The incident waves are normally incident, and the water depth is infinite.

Figure 7.4 illustrates the convergence of the scattering by a series of increasingly steep Type 1 linear ramps to the scattering by a step. The step results can be checked by comparing them to Figure 8 in the paper by Barrett and Squire (1996), and $|R|$ displays a monotonic decrease from about 0.22 as period increases.

As the steepness of the ramps increase, the progression of their reflection curves towards the ramp’s reflection can be especially seen at large periods. It can also be seen in the period below which each ramp curve starts to significantly depart from the step curve. For the 100-m-wide ramp this happens at about 14 s, while the 25-m-wide ramp’s reflection is still quite close until the period drops to about 10.5 s. After each
curve drops away, it has a maximum followed by either a minimum or a zero, and the height of this maximum increases towards the step curve as steepness increases, while the period at which the zero occurs moves to the left. It has almost moved out of the plotted range by the time the ramp width has dropped to 25 m.

Figure 7.5: The same as Figure 7.4, but with a series of increasingly steep sinusoidal Type 0 ramps instead of a series of linear ramps.

These results are consistent with those of the previous chapter, which showed that the larger a feature's width is in relation to the incident wavelength, the more opportunity there is for resonance and the more fine structure that is observed in the corresponding scattering curves.

Figure 7.5 shows the analogous results for Type 0 ramps, and is similar to Figure 7.4. However, as was seen in Figure 7.1, the smooth ramp curves display less fine structure than the linear ones, although their reflection is similar to that of the step for larger period ranges. That is, the reflection of each smooth ramp begins to depart from the
step’s at a period about 1 s lower than the equivalent period for the linear ramp of the same width.

Figure 7.6: Convergence of the scattering by a series of smooth ramps to the scattering by a double step. Figure a shows the profiles used—the smooth ramps’ shapes are given by equations (7.23), with $\alpha = 0$ (solid curve), 0.25 (dashed curve), 0.5 (chained curve) and 0.75 (dotted curve). The lower figures show the scattering by each ramp plotted in the same line-style that was used for its profile in a when the ramp width is either (b) 15 m, or (c) 100 m. The double step reflection curve is indicated on each figure to distinguish it from the $\alpha = 0$ curves, and the water depth is infinite.

Figure 7.6 is the analogous figure to Figure 6.6, and shows the scattering by some smooth ramps as their profiles become more and more similar to a double step. The
thicknesses of the variable regions are plotted in Figure 6.6 and are given by

\[ h_1(x) = h_0 + (h_2 - h_0) \times u_\alpha(x/a), \]  

\( \mu_\beta(x) = \begin{cases} 
\frac{1}{2}u_q(\beta x) & \text{for } 0 \leq x \leq \beta, \\
\frac{1}{2} & \text{for } \beta \leq x \leq 1 - \beta, \\
1 - \frac{1}{2}u_q(\beta(1 - x)) & \text{for } 1 - \beta \leq x \leq 1, 
\end{cases} \]  

where \( u_q \) is the quartic from Section 6.2.2, \( u_q(x) = 1 - (1 - x)^3(3x + 1) \) for \( 0 \leq x \leq 1 \), and \( \beta = (1 - \alpha)/2 \) (\( 0 \leq \alpha \leq 1 \)).

Figure 7.6b shows the scattering by each ramp when the ramp width is 15 m. There is a very clear progression in the curves as their profiles become more and more rectangular and, at periods above about 15 s, all the ramps produce about the same reflection. For lower periods, the double step produces the least reflection while the \( \alpha = 0 \) ramp produces the most. This is probably due to its constant central thickness being able to cause resonances more easily than the variable thicknesses of the smooth ramps, so that its reflection drops down to a minimum sooner than the others (at a higher period).

The scattering curves are more complicated in Figure c, when the ramp width is 100 m, but we can still see a similar progression in the reflection curves as \( \alpha \) increases. This is especially true for larger periods, but even at lower periods, when there is more fine structure, features like maxima and minima can be seen to converge towards their double-step positions as the ramps become more rectangular.

The previous three graphs have established that as the shape of a ramp becomes more similar to another, the amount of reflection that it produces also becomes more similar. As discussed in Chapter 6, this makes physical sense. In addition, the single and double step results can be checked independently, so the fact that the variable thickness results converge to them as their shape converges to its shape is reassuring in that respect also.

However, these latter results do all make the assumption that submergence is negligible and, comparing Figure 7.3 with the equivalent figure of Porter and Porter (2004) showed that this made a slight difference for long waves. This should be borne in mind at smaller depths, but will not cause problems in the calculation of results for deeper water, and especially not in the infinite depth limit.
The situation modelled in the two previous sections, of one ice thickness ramping up to a different thickness over a given distance, lends itself most to a ramp connecting relatively thin sea ice to a much thicker, fresh water ice shelf. Such a situation can be found in the Ross sea, where the slope from the sea ice up to the Ross ice shelf is gentle enough that one can walk up it.

Figure 7.7 plots the scattering by such a transition, when the sea ice has thickness $h_0 = 1\,\text{m}$ and the thickness of the ice shelf, $h_2$, is (a) 5\,\text{m}, (b) 10\,\text{m}, (c) 15\,\text{m}$ or (d) 20\,\text{m}. Results are plotted when the width of the transition is 250\,\text{m}, 500\,\text{m}$ or 750\,\text{m}$.

A typical sea ice/ice shelf transition would ramp up from 1\,\text{m}$ to 15\,\text{m}$ over about 500\,\text{m}$ (cf. the dashed curve in Figure c), but the other plots are included to give us more of an idea about the effect that changing the different dimensions has.

As we have already seen in Chapter 6 and in Sections 7.2.1 and 7.2.2, increasing the width produces more complicated reflection patterns due to more resonances between it and the wavelength, and all four figures testify to this. In addition, it also lowers the slope of the ramp, making the change in the thickness less abrupt, and this reduces the amount of reflection.

However, we have not discussed the result of changing $h_2$ as much. As might have been expected, increasing it while keeping the width constant increases the slope and thus causes more reflection. The next most noticeable quality is that the scale over which the $|R|$ curves change increases. The spacings between successive maxima and minima become wider, and $|R|$ takes a lot longer to begin dropping to zero.

The increase in spacings can be attributed to the fact that as the ice becomes thicker the waves beneath the ramp are becoming much longer in comparison to the ramp's width. The slow decay in reflection is due to the overall size of the features taking longer to become insignificant in relation to the incident wavelength.
In general it seems that sea ice/ice shelf transitions of this type have the most marked effect on the longer waves (in comparison to other features). A value of $|R| = 0.19$ is the largest reflection of a 20 s wave produced by any feature modelled in previous figures, and none of the features to be discussed later will produce such a large reflection for so long a wave either.

However, the lower period waves are not unaffected by the considerable increase in thickness as they travel into the region beneath the ice shelf. We have already mentioned how the wavelength of a wave of a given period increases with thickness. This increase is actually quite dramatic. For example, when the wave period is 5 s, the wavelength increases from 85 m to 205 m as $h$ increases from 1 m to 5 m. If $h$ increases further to 20 m, the wavelength becomes 450 m.
Figure 7.8: The relative amplitudes of normally incident waves transmitted by a sea ice/ice shelf transition. If a wave has amplitude $A$ in the sea ice region, its amplitude when it reaches the ice shelf will be $AT$, where $T = \alpha T$, and $\alpha = \varphi'_0(0)/\varphi'_0(0)$ is the relative amplitude of a perfectly transmitted wave. $\alpha$ is plotted for reference as a dotted line when $h_0 = 1$ m and $h_2$ takes values of (a) 5 m, (b) 10 m, (c) 15 m and (d) 20 m. In each figure (i.e. for each value of $h_2$), $|T|$ is also plotted when the widths of the transition regions are 250 m (solid curves), 500 m (dashed curves) and 750 m (chained curves), and the water depth is infinite. Note that the dashed and chained curves are almost indistinguishable from the solid ones.

Another effect is that wave amplitude decreases under thicker ice, due to greater rigidity and, to a lesser extent, greater mass. If an incident wave of amplitude $A$ was perfectly transmitted into the region under the ice shelf, its amplitude would drop to $\alpha A$, where $\alpha = \varphi'_0(0)/\varphi'_0(0) < 1$. Figure 7.8 plots $\alpha$ as function of period for the four different values of $h_2$ used in Figure 7.7. It can be seen that the drop in amplitude is greatest at lower periods, but that as the period increases, the difference diminishes. As might have been expected, $\alpha$ takes a lot longer to climb to 1 as the ice shelf becomes
thicker.

Figure 7.8 also plots the modulus of $T = \alpha T$, the amplitude transmission coefficient, for the twelve ramps featured in Figure 7.7. If we also allow for imperfect transmission, the final amplitude of a wave that initially has amplitude $A$ is $AT$.

We can see instantly that the width of the sea ice/ice shelf transition region has only a negligible effect on the amplitude transmission coefficient. The only exception is in Figure 7.8d when $h_2 = 20 \text{ m}$, where the three curves diverge very slightly around 19s. In addition, the longer scale over which the curves for the thicker ice shelves take to reach perfect transmission (the dotted curve) is also apparent.

One further point to make on comparison of Figure 7.8 with Figure 7.7 is that the $|T|$ curves show no evidence at all of the oscillations, maxima and minima that were evident in the $|R|$ curves. This might have been expected from the conservation of energy law $s|T|^2 = 1 - |R|^2$ (equation 4.46); if two values of $|R|$ are both quite small, their squares will be even smaller, and the corresponding values of $|T|$ will both be very close to $1/\sqrt{s}$.

### 7.2.4 The Scattering by a Flexible Breakwater

Another physical application of the method of this chapter is to the situation of a breakwater shielding a VLFS from waves arriving from the open ocean. Although these structures would not normally be made from ice, for convenience we shall take the building material to have the same density, Young's modulus and Poisson's ratio.

Figure 7.9 shows the amounts of reflection produced by three different profiles, and for three different widths: 5 m ($b$), 15 m ($c$) and 30 m ($d$). The solid, dashed and chained curves correspond to the profiles in Figure $a$ that are plotted in the same line-style. The outer thicknesses are $h_0 = 1 \text{ m}$ and $h_2 = 0$, and the incident wave arrives normally from the right.

The differences in widths are not great enough for huge differences in the scattering patterns, but we can see that the maxima in $|R|$ at about 4s in Figure $b$ moves to the right and drops in height as $a$ increases. The region of very high reflection at low periods also moves to the right.
The breakwater that is sloping away from the open water (plotted as a solid curve) seems to give noticeably more reflection than the other breakwaters in the intermediate period range (about 5 s to 10 s), although the size of this interval depends on the breakwater’s width. When \( a \) is larger, it takes longer to become small in comparison to the incident wavelength, at which point the reflection by the three shapes converges.

![Diagram](image.png)

Figure 7.9: The scattering by a breakwater. Figure \( a \) shows the three different shapes used. The incident wave arrives at a normal angle from the open water region to the right, and the breakwater is located between the \( x = 0 \) and \( x = a \) planes, and is separated from the floating structure to the left that it shields by a free edge. The reflection that each different shape produces is plotted in the same line-style that was used for its profile in Figure \( a \) when the width of the breakwater is either \( b \) 5 m, \( c \) 15 m or \( d \) 30 m wide. The water depth is infinite.

However, as in the previous section, when we consider the transmission into the region under the VLFS, the differences between the three different widths and shapes become less significant. This is demonstrated by Figure 7.10, which investigates the
effect that a breakwater has on a common open water wave spectrum.

![Graph](image)

Figure 7.10: The effect of a breakwater on a Pierson-Moskowitz incident wave spectrum. The dotted lines in each figure are reference lines plotting the spectrum that would result if the Pierson-Moskowitz spectrum was transmitted perfectly from the open water region to beneath the VLFS. If $f_{sd}$ is the SDF for the incident wave spectrum, this reference SDF is given by $f_{sd} = \alpha^2 f_{sd}$. The dotted, dashed and chained curves show the SDF's resulting from the breakwater profiles plotted in Figure 7.9 in the same line-styles, and their widths are either (a) 5 m, (b) 15 m or (c) 30 m. (Note that the dashed and chained curves are very hard to distinguish from the solid curves.) Their SDF's are given by $f_{sd} = \tilde{f}_{sd} \times |T|^2$. The incident waves are all taken to arrive normally, and the water depth is infinite.

The spectral density function (SDF) we will use is for a Pierson-Moskowitz wave spectrum (Pierson and Moskowitz, 1964), and is given by

$$f_{sd}(\tau) = \beta(\tau_0 \tau)^3 \times \exp (- \gamma(\tau_0 \tau)^4), \quad (7.24)$$

where $\tau_0 \tau$ is the dimensional wave period ($\tau$ is nondimensional), $\beta = 7.4 \times 10^{-4}$ m$^2$ s$^{-4}$, and $\gamma = 3.0 \times 10^{-4}$ s$^{-4}$. The arrow over the $f$ is intended to indicate that this is the
incident wave spectrum. In open water, this density function is very similar in shape
to the dotted curves, climbing from zero in the short wave limit to a peak period at
8 s, before dropping to zero again as the period increases further.

The dotted lines in Figure 7.10 actually plot $\alpha^2 f_{\text{sd}}(\tau)$, which is intended as a ref­
erence to show what the wave spectrum beneath the VLFS would look like if the
breakwater did not produce any reflection. The solid, dashed and chained curves plot
the spectra resulting from using the different shaped breakwaters plotted in Figure 7.9a,
while the different sub-plots, Figures 7.10a, b and c, correspond to the three different
widths used in Figure 7.9b, c and d respectively (Note that the dashed and chained
curves are very hard to distinguish from the solid curves.).

Figure 7.11: Further investigations of the reflection and transmission produced by a
breakwater. Figures a and c show $|R|$ and $f_{\text{sd}}$ for three 5-m-wide breakwaters 1 m thick
(solid curve), 2 m thick (dashed curve), and 3 m thick (chained curve), and Figures b
and d show the analogous results for 15-m-wide breakwaters. The incident waves are
normally incident, and the water depth is infinite.
The spectra are calculated by $f_{sd}(\tau) = \tilde{f}_{sd}(\tau) \times |T|^2$, and it can be seen that the different breakwater shapes produce only a negligible difference in the wave spectra underneath the VLFS. The width does not seem to make a large amount of difference either, although the 5-m-wide breakwaters seem to filter out slightly more wave amplitude.

Similarly, Figure 7.11 shows that using a breakwater with a larger average thickness does not affect the transmitted wave spectrum markedly, although in general it does increase $|R|$. Figures a and c show $|R|$ and $f_{sd}$ for three 5-m-wide breakwaters 1 m thick (solid curve), 2 m thick (dashed curve), and 3 m thick (chained curve), and Figures b and d show the analogous results for 15-m-wide breakwaters.

The results of this section lead one to conclude that should one be required to construct a breakwater to protect a VLFS, the extra expense and effort needed to make it wider and/or thicker would not filter out a significant proportion of the incoming wave amplitude. Of course, if the thickness was decreased too much there would start to be a decrease in reflection, but keeping it at 1 m would seem to be sufficient.

The most economical approach to improving the amount of protection that is afforded the VLFS would probably be to have several smaller breakwaters side by side—and possibly having them separated by small regions of open water. Increasing the number of abrupt changes in surface properties that the incident waves must cross could help to produce more significant amounts of reflection.
Chapter 8

Scattering by a Single Lead

The problem addressed here is a special case of the problem solved in the previous chapter. It is again illustrated by Figure 2.1, although the central thickness profile is limited to being constant. However, the physical situation being modelled—that of a lead in an Arctic or Antarctic ice sheet—is common enough that a solution method specially tailored to its peculiarities is useful. More specifically, a method that takes advantage of the constant central thickness will often speed up the generation of results significantly.

Section 1.1.2 described how leads were produced—by two adjacent ice sheets responding differently (due to different sizes, thicknesses, etc.) to a change in wind and/or current and so drifting apart. This produces an open lead—a region of open water between the two separated ice sheets, which are submerged relative to the water surface. With the values for the densities of ice and sea water used in Table 2.1, the ratio $\rho_i/\rho_w$ will be $0.9$ exactly, and so the air-water interface will be located in the $z = -0.9h_0$ plane.

On exposure to cold air the water in the lead proceeds to freeze over. Once this process has started, new ice continues to form beneath the original layer causing the ice in the lead to grow downwards until all the water between the two ice sheets has frozen. This process is illustrated by Figure 8.1a.

As in the previous two chapters, we will model the scattering by a lead by neglecting the submergence of the two ice sheets, and take the air/ice-water interface to be at $z = 0$. Results for the lead at various stages of refreezing can then be calculated by
letting the central thickness $h_1$ take a constant value between 0 and $0.9h_0$. In effect, for the purposes of predicting wave scattering anyway, we model the ice in the lead as growing upwards until it reaches $z = -0.9h_0$, as shown in Figure 8.1b. This means that our approximation becomes more accurate the further the refreezing process progresses, although for the reasons outlined in Chapter 6, we anticipate that submergence would only have a negligible effect on our scattering results.

Figure 8.1: The process of lead refreezing. (a) shows the physical process, while (b) illustrates our model, which neglects the submergence of the two ice sheets. The coordinate axes are displaced to the right to avoid clutter—the left hand limit of the lead corresponds to the $x = 0$ plane; the $y$ axis points out of the page. The dashed line in the latter figure shows the plane that the top of the lead actually lies in (the $z = -0.9h_0$ plane); this plane is reached when the lead has fully refrozen.
8.1 Solution Method

To obtain the solution for the scattering by a lead, we first rearrange the integral equation (4.37) to derive an alternative system of integral equations to the one solved in the previous chapter. Both turn out to be of the Wiener-Hopf type, implying that solutions to each equation may be solved analytically in terms of the other. The level of coupling depends on the width of the lead—the wider the lead is, the greater the number of evanescent waves that will have decayed sufficiently by the time they have travelled the width of the lead to be neglected without significantly affecting our results.

If $M$ evanescent waves still have significant amplitude by the time they have crossed the lead, then a system of $M$ linear equations must be solved to approximate the solution. For wide enough leads, sufficiently accurate results may be obtained by simply using $M = 1$; these results are similar to the well-known wide spacing approximation (Newman, 1965; Evans, 1990), or the equivalent NEW (No Evanescent Waves) approximation (Williams and Squire, 2004a), which is derived in Section 8.1.5, but the two approximations only seem to be equivalent when $h_1 = 0$. (This is merely an observation, but it could be related to the application of the edge conditions—for example, in the full solution for the two-crack problem, the edge conditions may be applied exactly, without ignoring any of the evanescent wave modes, but in the NEW approximation the potential is written as a linear combination of two solutions, one of which satisfies the edge conditions at the first crack, while the other satisfies them at the second.)

8.1.1 Alternative System of Integral Equations

Taking the Fourier transform of (4.37) gives

$$
\Psi(k) = \int_{-\infty}^{\infty} \psi_x(x, 0)e^{ikx}dx = \frac{P(k)}{f_0(\kappa)} + \hat{\phi}(k) + \left(1 - \frac{f_2(\kappa)}{f_0(\kappa)}\right)\Phi^+(k),
$$

(8.1)

where $P(k) = p^T(k)(P_0 + P_a e^{ika})$, and $\hat{\phi}$ and $\Phi^+$ were defined in (4.40).

Now, when $h_1(x) = h_1$ is a constant function, $\hat{\phi}$ may be written as

$$
\hat{\phi}(k) = (\Lambda_1 - \Lambda_0) \int_{0}^{a} \psi_x(x, 0)e^{ikx}dx

= (f_0(\kappa) - f_1(\kappa)) \left(\frac{i\hat{\phi}(0)}{k + \alpha_0} - \Psi^-(k) - \Phi^+(k)e^{ika}\right),
$$

(8.2)
where
\[ \Psi_-(k) = \int_{-\infty}^{0} \psi_z(x,0)e^{ikx}dx. \]

Consequently, multiplying (8.1) through by \( f_0(\kappa)/f_1(\kappa) \), eliminating \( \hat{\phi}(k) \) with (8.2), and rearranging implies that
\[
\Psi(k) = \frac{P(k)}{f_1(\kappa)} + \left( 1 - \frac{f_0(\kappa)}{f_1(\kappa)} \right) \left( \Psi_-(k) - \frac{i\varphi_0(0)}{k + \alpha_0} \right) + \left( 1 - \frac{f_2(\kappa)}{f_1(\kappa)} \right) \Phi^+(k)e^{ik\alpha}. \tag{8.3}
\]

This can be inverted to give an alternative integral equation to (4.37):
\[
\psi_z(x,0) = \psi_T^T(x) P_0 + \psi_T^T(x-a) P_a
+ \int_{0}^{\infty} (\mathcal{L}_1 - \mathcal{L}_0) g_t(x-\xi)\phi_{0,\xi}(\xi,0)d\xi
+ \int_{-\infty}^{0} (\mathcal{L}_0 - \mathcal{L}_1) g_t(x-\xi)\psi_z(\xi,0)d\xi
+ \int_{0}^{\infty} (\mathcal{L}_2 - \mathcal{L}_1) g_t(x-\xi)\phi_+(\xi,0)d\xi, \tag{8.4}
\]
where \( \psi_1(x) = \mathcal{L}_{\text{edge}}(\partial_x) g_1(x) \), and
\[
g_1(x) = G_1,\xi(x,0,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{f_1(\kappa)} dk = i \sum_{n=-2}^{\infty} B_n e^{ik_n|x|}. \tag{8.5}
\]
The operator \( \mathcal{L}_{\text{edge}} \) was defined in equation (4.35), while the \( \kappa_n \), given by \( \kappa_n = \sqrt{\kappa_n^2 - l^2} \) \( (n = -2, -1, \ldots) \), are on the positive real axis or in the upper complex half-plane. The \( \kappa_n \) are the zeros of the dispersion relation for the lead \( f_1(\kappa) \), and the \( B_n \) are the equivalents of the \( A_n \) in the left hand region or the \( \tilde{A}_n \) in the right hand region—i.e. \( B_n = \text{Res}(1/f_1, k = k_n) = -\kappa_n^2/k_n C_1(\kappa_n) \).

\( G_\dagger \) is the Green’s function defined in (4.30) for the case when \( h_1 \) was constant—it satisfied
\[
\mathcal{L}_1(\partial_\xi)\partial_\xi G_\dagger(x-\xi,z,0) + G_\dagger(x-\xi,z,0) = 0 \tag{8.6}
\]
instead of (3.1b), the equation that \( G \) satisfies, and (8.4) could have been obtained by progressing through Sections 4.1.3 and 4.2 with \( G \) replaced by \( G_\dagger \).

To begin the solution of (8.4), we split it into three different equations, correspond-
ing to each region, as follows:

\[
\psi_z(x, 0) |_{x < 0} = \sum_{n=-\infty}^{\infty} \beta_n e^{-ik_n x} + \int_{-\infty}^{0} (\mathcal{L}_0 - \mathcal{L}_1) g_t(x - \xi) \psi_z(\xi, 0) d\xi, \quad (8.7a)
\]

\[
\phi_z(x, 0) |_{0 < x < a} = \sum_{n=-\infty}^{\infty} \left( b_n e^{ik_n x} + c_n e^{ik_n(a-x)} \right), \quad (8.7b)
\]

\[
\phi_z(x, 0) |_{x > a} = \sum_{n=-\infty}^{\infty} \beta_n e^{ik_n(x-a)} + \int_{a}^{\infty} (\mathcal{L}_2 - \mathcal{L}_1) g_t(x - \xi) \phi_z(\xi, 0) d\xi, \quad (8.7c)
\]

where if \( F_0(k) = p^T(k) P_0 + i f_0(\kappa)c_0(0)/(\alpha_0 + k) \), then

\[
\beta_n^+ = iB_n F_0(k_n) + c_n e^{ik_n a}, \quad (8.8a)
\]

\[
b_n = iB_n (F_0(-k_n) - f_0(\kappa_n) \Psi^{-}(-k_n)), \quad (8.8b)
\]

\[
c_n = iB_n (p^T(-k_n) P_a - f_2(\kappa_n) \Phi^{+}(k_n)), \quad (8.8c)
\]

\[
\beta_n^- = iB_n p^T(-k_n) P_a + b_n e^{ik_n a}. \quad (8.8d)
\]

Both of the integral equations (8.7a) and (8.7c) are of the Wiener-Hopf type, and so they may be solved analytically in terms of the \( \beta_n^\pm \) in exactly the same way that (7.2) was. This is done in Section 8.1.2. The \( \beta_n^\pm \) provide the coupling between the two semi-infinite regions through their dependence on the unknown \( b_n \) and \( c_n \) coefficients. (They are unknown because they depend on the unknown transforms \( \Psi^{-} \) and \( \Phi^{+} \).) However, if we suppose that for some subset \( \mathcal{M} \subset \{-2, -1, \ldots, \infty\} \), \( \text{Im}[k_n] \times a \) is large enough that \( \exp(ik_n a) \) is negligible for \( n \notin \mathcal{M} \), then those \( \beta_n^\pm \) may be approximated as

\[
\begin{align*}
\beta_n^+ & \approx iB_n F_0(k_n) \\
\beta_n^- & \approx iB_n p^T(-k_n) P_a 
\end{align*}
\]

for \( n \notin \mathcal{M} \). (8.9)

If \( \mathcal{M} \) has \( M \) members, then \( M \) \( b_n \) and \( c_n \) coefficients will still remain to be determined—it will turn out that this will require us to invert an \( M \times M \) matrix. However, if \( M = 1 \) produces adequate results, then only \( b_0 \) and \( c_0 \) need to be determined. This allows us to find a solution in closed form that is similar to using the NEW approximation discussed in Section 8.1.5. However, the two approximations only seem to be equivalent when \( h_1 = 0 \).

One thing that should be kept in mind when only considering the size of \( \exp(ik_n a) \) in determining what value of \( M \) is appropriate, however, is that we have ignored differences in the magnitudes of the \( b_n \) and \( c_n \). Although the size of \( \text{Im}[k_n] \), \( |b_n| \) and \( |c_n| \) are generally correlated (for \( n \neq 0 \)), an important exception is that \( |b_{-1}| , |b_{-2}| \),
and $|c_{-2}|$ are often an order or two of magnitude greater than $|b_1|$ and $|c_1|$, even if $\text{Im}[k_{-1}] = \text{Im}[k_{-2}] > \text{Im}[k_1]$. Consequently, if $e^{ik_1\alpha}$ is above our tolerance level, but the $n = -1$ and $n = -2$ exponentials fall below it, then our policy is to include the complex modes anyway to allow for them potentially having larger amplitudes. This is consistent with the approach of Tkacheva (2002), in her discussion of the problem of a finite strip surrounded by open water, does not attempt to improve on a four-mode approximation which uses the real mode, the two complex modes and the first imaginary mode.

8.1.2 Wiener-Hopf Solution

As in Chapter 7, we now take the Fourier transform of the equations (8.7) to give two Wiener-Hopf equations

$$\frac{f_0(\kappa)}{f_1(\kappa)}\Psi^{-}(k) + \Phi_0^{+}(k) = -i \sum_{n=-2}^{\infty} \frac{\beta_n^+}{k - k_n}, \quad (8.10a)$$

$$\Psi_{-}^{-}(k) + \frac{f_2(\kappa)}{f_1(\kappa)}\Phi^{+}(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^-}{k + k_n}. \quad (8.10b)$$

$\Psi_{-}^{-}$ is defined in the same way as in the previous chapter (except with the kernel defined in terms of $g_1$ instead of $g$), while $\Phi_0^{+}$ is the transform of the extension of the integral in (8.7a) into the $x > 0$ region:

$$\Phi_0^{+}(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} H(-x)(\mathcal{L}_0 - \mathcal{L}_1)g_1(x - \xi)e^{ik\xi}d\xi dx.$$

Again, we must first factorize the two quotients $f_j/f_1$ ($j = 0, 2$) by defining two functions

$$K_0^{+}(k) = \sqrt{\frac{D_0}{D_1}} \prod_{n=-2}^{\infty} \frac{k + \alpha_n}{k + k_n}, \quad K_2^{+}(k) = \sqrt{\frac{D_2}{D_1}} \prod_{n=-2}^{\infty} \frac{k + \hat{\alpha}_n}{k + k_n}, \quad (8.11)$$

so that $f_j(\kappa)/f_1(\kappa) = K_0^{+}(k)K_{-}^{+}(-k) = K_{-}^{+}(k)K_{-}^{+}(-k)$ ($j = 0, 2$). Again, these formulae follow from the infinite product expansions for the dispersion relations (Chung and Fox, 2002a). The solution now follows in much the same way as in the previous chapter.
We first write

\[ K_0^-(k) \Psi^-(k) + i \sum_{n=-2}^{\infty} \frac{\beta_n^+ / K_0^+(k_n)}{k - k_n} \]

\[ = \Phi_0^+(k) - i \sum_{n=-2}^{\infty} \frac{\beta_n^+}{k - k_n} \left( \frac{1}{K_0^+(k)} - \frac{1}{K_0^+(k_n)} \right), \quad (8.12a) \]

\[ K_2^+(k) \Phi^+(k) - i \sum_{n=-2}^{\infty} \frac{\beta_n^- / K_2^+(k_n)}{k + k_n} \]

\[ = i \sum_{n=-2}^{\infty} \frac{\beta_n^-}{k + k_n} \left( \frac{1}{K_2^+(k)} - \frac{1}{K_2^+(k_n)} \right) - \Psi_0^-(k), \quad (8.12b) \]

to give two equations where one side is a plus function and the other is a minus function.

By the Riemann principle and Liouville's theorem, both sides of each equation must be zero, and so

\[ \Psi^-(k) = -i \sum_{n=-2}^{\infty} \frac{\beta_n^+ / K_0^+(k_n)}{K_0^+(k)(k - k_n)}, \quad (8.13a) \]

\[ \Phi^+(k) = i \sum_{n=-2}^{\infty} \frac{\beta_n^- / K_2^+(k_n)}{K_2^+(k)(k + k_n)}. \quad (8.13b) \]

We can invert the above transforms to put the first \( N + 3 \) \( a_n \) and \( d_n \) coefficients \((n \in \mathcal{N} = \{-2, -1, \ldots, N\})\) in terms of the \( M \) unknown \( b_n \) and \( c_n \) (recall that the set \( \mathcal{M} \) defined at the end of Section 8.1.1 has \( M \) members, which correspond to the modes that still have significant amplitude after having travelled the width of the lead), as follows

\[ a = M_a (f_a + D_k c), \quad (8.14a) \]

\[ d = M_d (f_d + D_k b). \quad (8.14b) \]

Here

\[ [f_a]_n = i B_n F_0(k_n), \quad [f_d]_n = i B_n p^T(-k_n) P_a \]

come from the definitions of the \( \beta_n^\pm \), as does the diagonal matrix \( D_k \) with elements given by \([D_k]_{nm} = e^{ik_n \alpha} \delta_{nm} \) \((n \in \mathcal{N}, m \in \mathcal{M})\). The elements of the matrices \( M_a \) and \( M_d \) are given by

\[ [M_a]_{nm} = \frac{A_n K_0^+(\alpha_n) f_1(\gamma_n)}{K_0^+(k_n)(\alpha_n - k_n)}, \quad [M_d]_{nm} = \frac{\hat{A}_n K_2^+(\hat{\alpha}_n) f_1(\hat{\gamma}_n)}{K_2^+(k_n)(\hat{\alpha}_n - k_n)} \]

(for \( n, m \in \mathcal{N} \)). However, the \( M \) \( b_n \) and \( c_n \) are still unknown, and so we must find further expressions for them using (8.8) and (8.13). Setting \( k = -k_n \) in (8.13a), and
\( k = k_n \) in (8.13b), we now have

\[
\begin{align*}
\mathbf{b} &= \mathbf{b}_0 + \mathbf{M}_b (\mathbf{a} + \mathbf{D}_k \mathbf{c}), \\
\mathbf{c} &= \mathbf{c}_0 + \mathbf{M}_c (\mathbf{d} + \mathbf{D}_k \mathbf{b}),
\end{align*}
\]

where for \( n, m \in \mathcal{M} \),

\[
\begin{align*}
[f_b]_n &= iB_n f_0(-k_n), & [f_c]_n &= iB_n \mathbf{p}_t^T(k_n)\mathbf{p}_a, \\
[M_b]_{nm} &= \frac{B_n f_0(k_n)/K_0^+(k_n)}{K_0^+(k_m)(k_n + k_m)}, & [M_c]_{nm} &= \frac{B_n f_2(k_n)/K_2^+(k_n)}{K_2^+(k_m)(k_n + k_m)}.
\end{align*}
\]

Equations (8.15) give us two simultaneous equations to solve in \( b \) and \( c \). We can eliminate \( c \) from these to write \( b \) as

\[
b = (\mathbf{I} - \mathbf{M}_b \mathbf{D}_k \mathbf{M}_c \mathbf{D}_k)^{-1} (\mathbf{b}_0 + \mathbf{M}_b \mathbf{a} + \mathbf{M}_b \mathbf{D}_k (\mathbf{c}_0 + \mathbf{M}_c \mathbf{d})),
\]

and then use (8.15b) to calculate \( c \), completing our solution. As mentioned in Section 8.1.1, the size of the matrix that must be inverted in (8.16) is \( M \times M \).

### 8.1.3 Solution for a Finite Strip or a Breakwater

As in the solution for a breakwater presented in Chapter 7, allowing for the presence of open water in either of the semi-infinite regions is an almost trivial extension. If \( D_2 = 0 \), then \( K_2^+ \) in (8.11) must be adjusted in the same way that \( K^+ \) was in (7.17) to allow for the non-existence of the two roots \( \tilde{\gamma}_{-1} \) and \( \tilde{\gamma}_{-2} \) of the dispersion relation \( f_2(\kappa) = 0 \).

Similarly, when \( D_0 = 0 \), \( K_0^+ \) must be adjusted in a similar manner; the application of the Riemann principle and Liouville's theorem proceeds as in Section 7.1.4, and the remainder of the solution (solving for the \( b_n \) and \( c_n \)) follows in the same way as in the previous section, culminating once again in (8.14), (8.15) and (8.16). (Apart from the adjustments to the formulae for the \( K_j^\pm \), the only differences are that the rows corresponding to the complex modes in \( \mathbf{M}_a \) and/or \( \mathbf{M}_d \) must be left out.) We can use the representation (8.7b) to apply the edge conditions.

Although the above situations are not as relevant to scattering by irregularities in an ice sheet, they do have possible engineering applications such as to the scattering by a VLFS such as a floating airport or, as discussed in the previous chapter, to the
scattering by a breakwater that shields one. Meylan (1993) used results corresponding to \( h_0 = h_2 = 0 \) and \( \theta = 0 \) to model a two-dimensional ice floe in the MIZ; this corresponds to the solution of Tkacheva (2002) that was also found by using the Wiener-Hopf technique.

### 8.1.4 Solution for an Open Lead

The scattering by an open stretch of water between two large ice sheets is of more immediate relevance to the overall goal of modelling an ice field than the previous two applications. A solution has been presented by Chung and Linton (2005), who solved this problem when \( h_2 = h_0 \) using residue calculus, but we wish to generalize their results to allow \( h_2 \) and \( h_0 \) to differ, and also to generalize the method of Section 8.1.2, which presented results for three different nonzero thicknesses, to allow \( h_1 \) to become zero.

Unfortunately, the solution does not follow quite as straightforwardly from our working in Section 8.1.2 as those did, as when \( h_1 = 0 \), \( g_t \) contains a logarithmic singularity and so the derivatives required in (8.7) are highly singular. This is the same problem that prevented us from setting \( h_0 = 0 \) in Section 7.1.4. To deal with this we integrate (8.7) by parts to move the derivatives onto the more highly differentiable functions \( \psi_z(x, 0)|_{x<0} \) and \( \phi_z(x, 0)|_{x>a} \). This gives us the following set of equations

\[
\psi_z(x, 0)|_{x<0} = \sum_{n=-2}^{\infty} c_n e^{i k_n (a-x)} - \int_{0}^{\infty} (L_0 - \lambda)\psi_0(z, 0)g_t(x - \xi) d\xi, \tag{8.17a}
\]

\[
\phi_z(x, 0)|_{0<x<a} = \sum_{n=-2}^{\infty} (b_n e^{i k_n x} + c_n e^{i k_n (a-x)}), \tag{8.17b}
\]

\[
\phi_z(x, 0)|_{x>a} = \sum_{n=-2}^{\infty} b_n e^{i k_n x} + \int_{a}^{\infty} (L_2 - L_1)\phi_z(z, 0)g_t(x - \xi) d\xi, \tag{8.17c}
\]

where

\[
\begin{align*}
    b_n &= f_1(\gamma_0)\varphi_0'(0) \frac{B_n}{\alpha_0 - k_n} - B_n \sum_{m=-2}^{\infty} \frac{a_m f_1(\gamma_m)}{k_n + \alpha_m}, \tag{8.18a} \\
    c_n &= -B_n \sum_{m=-2}^{\infty} \frac{d_m f_1(\gamma_m)}{k_n + \alpha_m}, \tag{8.18b}
\end{align*}
\]
and the $a_n$ and $d_n$ come from (4.38) and (4.42).

Proceeding in much the same way as in Section 8.1.2 we now take the Fourier transforms of (8.17a) and (8.17c) and rearrange to give

$$\Psi^-(k) = \frac{j\varphi_0(0)}{k + \alpha_0} + \frac{P_T(k)P_0}{f_0(\kappa)} - \frac{f_1(\kappa)}{f_0(\kappa)} \left( \Phi^+_0(k) + i \sum_{n=-2}^{\infty} c_n e^{i k n} \right), \quad (8.19a)$$

$$\Phi^+(k) = \frac{P_T(k)P_a}{f_2(\kappa)} - \frac{f_1(\kappa)}{f_2(\kappa)} \left( \Psi^-_a(k) - i \sum_{n=0}^{\infty} b_n e^{i k n} \right), \quad (8.19b)$$

We can invert the above transforms to put the $a_n$ and $d_n$ in terms of the $b_n$ and $c_n$:

$$a_n = i A_n \left( p^T(\alpha_n)P_0 - f_1(\gamma_n)\beta^+_n \right), \quad (8.20a)$$

$$d_n = i \tilde{A}_n \left( p^T(-\alpha_n)P_a - f_1(\tilde{\gamma}_n)\beta^-_n \right), \quad (8.20b)$$

where

$$\beta^+_n = \Phi^+_0(\alpha_n) + i \sum_{m=-2}^{\infty} \frac{c_m e^{i k m}}{\alpha_n - k_m}, \quad (8.21a)$$

$$\beta^-_n = \Psi^-_a(-\alpha_n) + i \sum_{m=0}^{\infty} \frac{b_m e^{i k m}}{\tilde{\alpha}_n - k_m}, \quad (8.21b)$$

Thus, our solutions are entirely formulated in terms of the auxiliary functions $\Phi^+_0$ and $\Psi^-_a$. To find them we first note that to complete the factorization step we must adjust both the $K^+_1$ in a similar way that we adjusted $K^+_j$ in (7.17) when solving the breakwater problem in Section 7.1.4. In this case $K^+_0$ is given by

$$K^+_0(k) = \Pi_0 \prod_{n=-2}^{-1} k + \alpha_n \times \prod_{n=0}^{\infty} \frac{k + \alpha_n}{k + k_n}, \quad (8.22)$$

where

$$\Pi_0 = -\frac{1}{|\gamma-1|^2} \prod_{n=0}^{\infty} \kappa_n = \sqrt{\frac{D_0}{\lambda}},$$

and $K^+_2$ must be changed in a similar fashion.

We now proceed as we did in Section 7.1.1 and Section 8.1.2, and are eventually
able to write final expressions for $\Phi^+_0$ and $\Psi^-_a$:

$$
\Phi^+_0(k) = K^+_0(k) \left( K^+_0(\alpha_0) \frac{i \varphi_0'(0)}{k + \alpha_0} - \sum_{n=-\infty}^{\infty} A_n K^+_0(\alpha_n) \frac{p^T(-\alpha_n) p_0}{k + \alpha_n} \right) \\
- i \sum_{n=0}^{\infty} c_n e^{ik_n a} \left( 1 - \frac{K^+_0(k_n)}{K^+_0(k)} \right),
$$

\hspace{1cm}(8.23a)

$$
\Psi^-_a(k) = K^-_2(k) \sum_{n=-\infty}^{\infty} \hat{A}_n K^+_2(\hat{\alpha}_n) \frac{p^T(\hat{\alpha}_n) p_a}{k + \hat{\alpha}_n} + i \sum_{n=0}^{\infty} b_n e^{ik_n a} \left( 1 - \frac{K^+_2(k_n)}{K^+_2(k)} \right).
$$

\hspace{1cm}(8.23b)

Consequently,

$$
\beta^+_n = K^+_0(\alpha_n) \left( K^+_0(\alpha_0) \frac{i \varphi_0'(0)}{\alpha_n + \alpha_0} - \sum_{m=-\infty}^{\infty} A_m K^+_0(\alpha_m) \frac{p^T(-\alpha_m) p_0}{\alpha_m + \alpha_n} \right) \\
+ i \sum_{m=0}^{\infty} c_m e^{i k_m a} \frac{K^+_0(\alpha_m)}{K^+_0(k_m)},
$$

\hspace{1cm}(8.24a)

$$
\beta^-_n = -K^+_2(\hat{\alpha}_n) \sum_{m=-\infty}^{\infty} \hat{A}_m K^+_2(\hat{\alpha}_m) \frac{p^T(\hat{\alpha}_m) p_a}{\hat{\alpha}_m + \hat{\alpha}_n} + i \sum_{m=0}^{\infty} b_n e^{i k_m a} \frac{K^+_2(\hat{\alpha}_n)}{K^+_2(k_m)}.
$$

\hspace{1cm}(8.24b)

Substituting (8.24) back into (8.20) give us the two matrix-vector equations

$$
a = f_a + M_a D_k c,
$$

\hspace{1cm}(8.25a)

$$
d = f_d + M_d D_k b,
$$

\hspace{1cm}(8.25b)

where for $n \in \mathcal{N}$ and $m \in \mathcal{N}' = \{0, 1, \ldots, N\}$

$$
[f_a]_n = A_n f_1(\gamma_n) K^+_0(\alpha_0) K^+_0(\alpha_n) \frac{\varphi_0'(0)}{\alpha_n + \alpha_0} + i A_n p^T(\alpha_n) p_0 \\
+ i A_n f_1(\gamma_n) K^+_2(\alpha_n) \sum_{m=-\infty}^{\infty} A_m K^+_0(\alpha_m) \frac{p^T(-\alpha_m) p_0}{\alpha_m + \alpha_n},
$$

$$
[f_d]_n = i \hat{A}_n p^T(-\alpha_n) p_a + i \hat{A}_n f_1(\gamma_n) K^+_2(\hat{\alpha}_n) \sum_{m=-\infty}^{\infty} \hat{A}_m K^+_2(\hat{\alpha}_m) \frac{p^T(\hat{\alpha}_m) p_a}{\hat{\alpha}_m + \hat{\alpha}_n},
$$

$$
[M_a]_{nm} = \frac{A_n K^+_0(\alpha_n) f_1(\gamma_n)}{K^+_0(k_m)(\alpha_n - k_m)},
$$

$$
[M_d]_{nm} = \frac{\hat{A}_n K^+_2(\hat{\alpha}_n) f_1(\gamma_n)}{K^+_2(k_m)(\hat{\alpha}_n - k_m)}.
$$

Meanwhile (8.17b) gives us

$$
b = f_b + M_b a = f_b + M_b f_a + M_b M_a D_k c,
$$

\hspace{1cm}(8.26a)

$$
c = M_c d = M_c f_d + M_c M_d D_k b,
$$

\hspace{1cm}(8.26b)

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where

\[ [f_b]_n = f_1(\gamma)\varphi'_0(0)\frac{B_n}{\alpha_0 - k_n}, \quad [M_b]_{nm} = \frac{B_n f_1(\gamma_m)}{k_n + \alpha_m}, \quad [M_c]_{nm} = \frac{B_n f_1(\gamma_m)}{k_n + \hat{\alpha}_m}, \]

for \( n, m \in \mathcal{M} \subseteq \mathcal{N}' \). Thus, for an open lead, \( b \) can be written as

\[ b = (I - M_b M_a D_k M_c D_{D_k})^{-1}(f_b + M_b f_a + M_b M_a D_k M_c f_d). \] (8.27)

The vector \( c \) can then be found from (8.26b), and \( a \) and \( d \) can in turn be found from (8.25a). Applying the edge conditions (2.13) using the \( a_n \) and \( d_n \) completes the solution.

Note that, as in Section 8.1.2, the presence of the \( \exp(ik_an) \) terms in the \( D_k \) matrix again reduces the dependence of (8.25a) and (8.26) on the \( b_n \) and \( c_n \) to the point that either only \( M + 3 \) are needed (\( M \geq 2 \)), or only \( b_0 \) and \( c_0 \) are needed. The latter case is dealt with in the following section.

Also note that \( D_k \) contains all the information about the width of the lead—none of the other matrices or vectors depend on \( a \) at all. This enables us to generate results for a large number of lead widths extremely rapidly. We are also helped considerably by the fact that we can also make a NEW approximation once the widths become large enough.

### 8.1.5 A NEW Approximation

Like Section 7.1.4, this section presents another well-known result, which is commonly called the wide-spacing approximation (Newman, 1965; Evans, 1990; Meylan, 1993; Meylan and Squire, 1993).

Suppose \( a \) is large enough in (8.7c) that no evanescent waves are able to travel the width of the lead and that the only wave reaching the right hand edge at \( x = a \) from the left hand edge has a potential with amplitude \( b'_0 = -b_0 \Lambda_1(\kappa_0) \). Likewise, the only wave reaching the left hand edge from the right has a potential with amplitude \( c'_0 = -c_0 \Lambda_1(\kappa_0) \), as shown in Figure 8.2. One might think that this is equivalent to supposing that \( \mathcal{M} = \{0\} \) or that \( M = 1 \). However, it will be shown that in general this is not the case as the \( M = 1 \) result does not necessarily guarantee conservation of energy, while the NEW approximation does. The two approximations do appear to always agree when \( h_1 = 0 \) though, implying that it is in the application of the edge conditions in the central region that they differ. Indeed, the NEW method applies the
edge conditions at each edge independently of what is happening at the other edge, while the edge conditions at both edges are still applied together even when only one mode is carried through in the "full" solution.

Figure 8.2: Figure used to derive the NEW (No Evanescent Waves) approximation for the scattering by a lead or an ice strip of constant thickness. The left hand edge is located at \( x = 0 \) and the right hand edge is at \( x = a \). The incident wave may be approaching at an angle that is not perpendicular to the lead, which stretches into and out of the page.

Continuing with the derivation, we divide the situation described above into three separate but similar problems, as follows:

1. An incident wave travelling beneath ice of thickness \( h_0 \) arriving from the left at the edge of another ice sheet of thickness \( h_1 \). The resulting scattering coefficients will be denoted \( R_0 \) and \( T_0 \).

2. An incident wave travelling beneath ice of thickness \( h_1 \) arriving from the right at the edge of another ice sheet of thickness \( h_0 \). The resulting scattering coefficients will be denoted \( \tilde{R}_0 \) and \( \tilde{T}_0 \).

3. An incident wave travelling beneath ice of thickness \( h_1 \) arriving from the left at the edge of another ice sheet of thickness \( h_2 \). The resulting scattering coefficients will be denoted \( R_1 \) and \( T_1 \).

We will take each of the edges to be located at \( x = 0 \). We saw in Section 7.1.4 that the solution for the second problem follows easily from the solution to the first problem: the scattering coefficients \( \tilde{R}_0 \) and \( \tilde{T}_0 \) for the second problem may be written in terms of \( R_0 \) and \( T_0 \) by using (7.22). Meanwhile, the third problem is essentially the same as the first, but using different thicknesses for the left hand and right hand ice sheets. Accordingly, we will only provide the solution for Problem 1. This will then enable us to derive a wide spacing approximation for the scattering by a lead.
The Scattering by an Edge Separating Two Adjacent Semi-infinite Ice Sheets

The scattering coefficients $R_0$ and $T_0$ may be found analytically by adjusting the method of this chapter. Published solutions to the general problem are given by Barrett and Squire (1996) and Chung and Fox (2002b), while several solutions exist to the special case when $h_0 = 0$ (e.g. Chung and Fox, 2002a). If $h_1 = 0$, the $a_n$ coefficients of (4.38) would simply be given by the vector $f_a$ in Section 8.1.4, and we could immediately apply the free edge conditions to solve for the unknown vector $P_0$, and thus determine $R_0$. We can then write $\phi_s(x, 0)$ to the right of the edge in much the same form that we did in the single crack problem—in this case it can be found by replacing the $a_n$ of (5.3) with the open water roots $k_n$, and by using the $b_n$ of Section 8.1.4 instead of the original coefficients used in Section 5.1.1. This will also allow us to determine the transmission coefficient $T_0$.

For nonzero $h_1$, we should use the vector $M_a f_a$ of Section 8.1.2 to give us the $a_n$ coefficients, and $f_b + M_b a$ to provide the $b_n$. By applying the appropriate edge conditions, we can again find $R_0$ and $T_0$.

NEW (Wide Spacing) Approximation for the Scattering by a Lead

In this section, we will follow the derivation of Meylan (1993), although other researchers, e.g. Evans (1990), have also used similar approximations.

Referring to Figure 8.2 again, if we assume that none of the evanescent waves generated at each edge of the lead have significant amplitudes after travelling its width, the coefficients $b'_c$ and $c'_c$ satisfy the following relationships which can be decoupled to give

$$b'_c = T_0 + \bar{R}_0 c'_c,$$
$$c'_c = R_1 b'_c e^{2ika},$$

which can be decoupled to give

$$b'_c = \frac{T_0}{1 - \bar{R}_0 R_1 e^{2ika}},$$
$$c'_c = \frac{R_1 T_0 e^{2ika}}{1 - \bar{R}_0 R_1 e^{2ika}}.$$

We can now approximate the scattering coefficients $R$ and $T$ to give the final result of
this section:

\[ R = R_0 + c_0 T_0 e^{2ik_0a} + \frac{R_1 T_0 e^{2ik_0a}}{1 - \frac{R_0 R_1 e^{2ik_0a}}{1 + R_0 R_1}}, \quad (8.30a) \]

\[ T = T_1 T_0 e^{i(k_0 - \delta_0)a} + \frac{T_0 T_1 e^{i(k_0 - \delta_0)a}}{1 - \frac{R_0 R_1 e^{2ik_0a}}{1 + R_0 R_1}}, \quad (8.30b) \]

This approximation is also discussed in Chapter 9, where it is used to approximate the scattering by two or more ridges or leads. It is clearly periodic in \( a \), with the moduli repeating themselves every half wavelength \( (\pi/k_0) \). Similarly, if \( a \) is kept constant, the oscillations will be closer together in wavenumber space for large values than for small ones. This is consistent with the observations of Chapter 6 and Chapter 7, where a lot more fine structure was observed for large ridge or ramp widths than for smaller ones. (This was especially true for flat features.)

It can be shown that the formulae for \( R \) and \( T \) (8.30) imply that energy is conserved (cf. Equation 4.46); it can also be shown that (8.30b) implies that \(|T|\) is bounded as follows:

\[ \frac{|T_0 T_1|}{1 + |R_0 R_1|} \leq |T| \leq \frac{|T_0 T_1|}{1 - |R_0 R_1|}. \quad (8.31) \]

In addition, we can simplify the corresponding limits for \(|R|\) (using conservation of energy to help us) to

\[ \frac{|R_0| - |R_1|}{1 - |R_0 R_1|} \leq |R| \leq \frac{|R_0| + |R_1|}{1 + |R_0 R_1|}. \quad (8.32) \]

The latter equation tells us that \(|R|\) has periodic minima that are nonzero in general, unless \( h_2 = h_0 \) or the minima occur at a period when \(|R_0| = |R_1|\), which is much less likely.

If \( h_2 = h_0 \), we can simplify \( R \) and \( T \) further, since \( R_0 = \tilde{R}_1, T_0 = \tilde{T}_1, \tilde{T}_0 = T_1 \) and \( \tilde{R}_0 = R_1 \), giving

\[ R = \frac{T_1 e^{ik_0a}}{T_1^* e^{i\alpha_0a}} \times \frac{r_1 - r_1^*}{1 - r_1^2}, \quad (8.33a) \]

\[ T = \frac{T_1 e^{ik_0a}}{T_1^* e^{i\alpha_0a}} \times \frac{1 - |r_1|^2}{1 - r_1^2}, \quad (8.33b) \]

where \( r_1 = R_1 e^{ik_0a} \). It is easily seen from (8.33a) that the zeros in \(|R|\) occur whenever \( r_1 \) is real. At such points it is also apparent that \(|T| = 1\) also, as we would expect from
As well as information about the zeros in $|R|$, the formulae (8.33) also provide an additional result pertaining to the relative phases of $R$ and $T$. Since

$$R/T = 2ie^{i\omega_0} \text{Im} [r_1]/(1 - |r_1|^2),$$

we can deduce the following relationship between the phases of the two coefficients, i.e.

$$\frac{R}{|R|} = \text{sgn}(\text{Im}[r_1]) \times \frac{IT}{|T|} e^{i\omega_0}.$$ 

Note that as $r_1$ moves from the upper complex half plane to the lower, or vice versa, the above formula predicts that the argument of $R$ will either increase or decrease by $\pi$. Recalling that $R$ becomes zero as $r_1$ becomes real, this is consistent with $R$ travelling through the origin on a smooth curve (which in the limit as one approaches the origin from either side is essentially a straight line).

### 8.2 Results

From equations (4.21), (4.31) and (4.25) we may write $\phi$ as the following eigenfunction expansion

$$\phi(x, z) = \begin{cases} \phi_0(x, z) + \sum_{n=-2}^{\infty} a_n e^{-i\kappa_n x} \varphi_n(z) & \text{for } x < 0, \\ \sum_{n=-2}^{\infty} (b_n e^{ik_n x} + c_n e^{ik_n(a-x)}) \psi_n(z) & \text{for } 0 < x < a, \\ \sum_{n=-2}^{\infty} d_n e^{ik_n(x-a)} \hat{\varphi}_n(z) & \text{for } x > a, \end{cases}$$

(8.34)

where $\psi_n(z) = \varphi(z, \kappa_n)$, and the coefficients in the expansion may be calculated from the relations

$$a'_n = a_n/\varphi'_n(0), \quad b'_n = b_n/\psi'_n(0), \quad c'_n = c_n/\psi'_n(0), \quad d'_n = d_n/\hat{\varphi}'_n(0).$$

Consequently, we are able to check that our solution satisfies all of the equations (2.8), as well as the appropriate edge conditions, explicitly and straightforwardly. In addition, Appendix F uses residue calculus to show analytically that $\phi(x, z)$ as calculated in (8.34) above satisfies a weak form of (2.8c). Hence, we do not need to spend the time verifying our results that we did in the Results sections of the Chapters 6 and 7.

In addition, for the sake of brevity, only results for normally incident waves will be presented. This enables us to concentrate on how the scattering is affected by changes
in other parameters such as the thicknesses of the two semi-infinite regions and the central strip, which may now be varied independently of one another. Infinite depth results will be used, and unless otherwise stated, the free edge conditions will be applied.

First, however, we investigate the convergence of the method and the accuracy of the NEW approximation (Section 8.2.1). This essentially involves a discussion of the number of modes $M$ that need to be included for our solution to converge (the matrix that we must invert is $M \times M$), and also of the convergence of the NEW approximation as the crack is made wider.

These results may then be used implicitly in the results of the following sections, which discuss the refreezing of a lead (Section 8.2.2), and the application of this chapter's method to other problems such as the scattering by a finite strip or a breakwater (Section 8.2.3). Some of the results of the latter section are used to confirm results for the same situations obtained by the methods of the previous two chapters.

### 8.2.1 Numerical Considerations

The first aspect of the convergence of the method presented in this chapter that we will consider is the number of modes $M$ that we need to use to give sufficiently accurate results. Figures 8.3a and b show the effect that the choice of $M$ can have on the predicted scattering by a strip of 2-m-thick ice surrounded by 1-m-thick ice sheets on both sides. The strip is separated from the sheets by free edges, and is either 5 m (a) or 15 m (b) wide, while Figures 8.3c and d show the analogous results for a 0.5-m-thick strip.

In Figures 8.3a and b, the solid, dashed, chained and dotted curves correspond to values for $M$ of either 7, 6, 5 or 4 respectively. When the strip is 5 m wide (a), both the dotted and chained curves are clearly different from the other two, while only the dotted curve is distinguishable when the width is increased to 15 m in Figure b. Surprisingly, the lower $M$ approximations seem to differ mainly at shorter periods—one might have thought that they would have been more likely to differ at larger periods and longer wavelengths but this is apparently not the case. Also note that the approximations do not guarantee the conservation of energy (cf. Equation 4.46), as can be seen by $|R|$ sometimes exceeding 1.
Figure 8.3: The effect of the choice of the integer $M$ on the convergence of the results for the scattering of normally incident waves by a strip of ice that is either 2 m thick (figures $a$ and $b$) or 0.5 m thick ($c$ and $d$) located between two 1-m-thick semi-infinite ice sheets. The free edge conditions are applied at each end of the strip, which is either 5 m wide ($a$ and $c$) or 15 m wide ($b$ and $d$). The different curves show the effects of the choice of $M$: respectively, the solid, dashed, chained and dotted curves correspond to values for $M$ of either 7, 6, 5 or 4 ($a$ and $b$) or 6, 5, 4 or 1 ($b$ and $d$). The water depth is infinite.

Similar results are observed in Figures $c$ and $d$, in which the aforementioned line-styles correspond to values for $M$ of either 6, 5, 4 or 1 respectively—only the $M = 1$ curve is visible in Figure $c$, while all the curves are identical in $d$.

These results indicate that even for the relatively small choices of strip width, for a 2-m-thick strip we only need at most six modes to get the solution to converge, while even fewer are needed for the thinner strip—four when the strip width is 5 m, just one when the width is 15 m. This trend of needing less modes as the strip is made thinner continues to the point of it being made into an open lead—in that case the $M = 1$ curve is indistinguishable from the true solution even when the width is only 5 m wide.
This rapid convergence is confirmed on comparing the $M = 7$, 6 and 5 curves in Figure 8.3b with the full solution calculated numerically by the method of Chapter 6 for the same conditions and shown in Figure 6.4a. This demonstrates that the method described in this chapter may produce accurate results extremely efficiently.

The actual reflection characteristics of the fully converged results (solid curve) shown in Figure 8.3b were discussed in Section 6.2.2 while discussing Figure 6.4a. And, as was also observed in Figures 7.9 and 7.11, decreasing the strip width causes the prominent features of the curve (zeros, maxima and minima) to move to the left. The maxima have also increased in height.

The scattering by a strip with the same material properties but with the smaller thickness of 0.5 m is very similar (cf. Figure 8.3a, solid curve). Decreasing the strip thickness appears to have a similar effect to decreasing its width. Comparing Figure 8.3c with a, and d with b, the same process has occurred—the two left hand maxima have gained in height and have moved to the left, and the zeros have also moved to the left (although the right hand maxima has become slightly smaller). Note that this has also occurred on moving from d to c.

Figures 8.4 and 8.5 demonstrate the convergence of the NEW approximation, which is a genuine approximation. In these and other graphs that follow in this chapter, the process of choosing the smallest $M$ that still gives sufficiently accurate results is automated by defining a tolerance $\varepsilon'$ so that the $n^{th}$ mode may be neglected if $|e^{ik_n a}| < \varepsilon'$. A tolerance of $\varepsilon' = 0.01$ seemed to be small enough to produce results that were adequate for the purpose of graphing results; $\varepsilon' \approx 5 \times 10^{-5}$ produces results that are accurate to four significant figures.

In Figure 8.4, the exact results are plotted as solid curves while the NEW approximate results are plotted as dashed curves. In all graphs there is a central strip of ice with thickness of either 2 m (figures a and b), 1 m (c and d) or 0.5 m (e and f). The strip widths are either 15 m (figures a, c and e) or 100 m (b, d and f). Note that having a 1-m-thick ice strip simply involves having two cracks (cf. Marchenko, 1997; Dixon and Squire, 2001a; Evans and Porter, 2003a; Williams and Squire, 2004b).
Figure 8.4: Comparison of NEW approximate results (dashed curves) with exact results (solid curves) for the scattering of normally incident waves by a strip of ice that is either 2 m thick (Figures a and b), 1 m thick (Figures c and d) or 0.5 m thick (Figures e and f) located between two 1-m-thick semi-infinite ice sheets. The free edge conditions are applied at each end of the strip, which is either 15 m wide (Figures a, c and e), or 100 m wide (Figures b, d and f). The water depth is infinite.

As might be expected from its alternative name (wide spacing approximation), retaining the same properties of the ice in the central strip and increasing the strip width causes the NEW results to move closer to, if not onto, the exact results. In the case of the 1-m-thick and 0.5-m-thick strips, the approximation is exact at 100 m; there are still visible differences when the strip is 2 m thick.

The faster convergence of the NEW results to the true results as the thickness of the central strip is decreased is similar to the convergence observed in Figure 8.3, which also depends on a certain number of the evanescent waves having decayed sufficiently as they travel the width of the strip to be ignored. This might have been expected due to the decrease in the wavelength of the ice in that region as it becomes thinner (the evanescent waves have usually decayed sufficiently to be ignored by about one
wavelength; cf. Figure 8.5). This trend continues as the strip thickness is zeroed—for an open lead the NEW approximation is exact when its width is either 15 m or 100 m.

The actual scattering observed in Figures 8.4a and e was discussed in regard to Figures 8.3b and d, while the solid curve in Figure 8.4b is the same as the solid curve in Figure 6.4b and was discussed in Chapter 6. It was noted in that chapter that the first two zeros apparent in the plot for the 15-m-wide, 2-m-thick strip merge and lift off the period axis as the width increases to 45 m. Thus they are not present in Figure 8.4b, but manifest themselves in the point of inflection at about 9 s. Other features of the curve have moved to the right—for example, the zero at 9 s in Figure 8.4a occurs when the period is about 15 s in figure b—and the two left-hand zeros in the latter figure occurred at periods below 2 s.

Neither the 1-m-thick strip nor the 0.5-m-thick strip display the phenomenon described above where a point of inflection was produced, but they both show the same movement of zeros and maxima to the right as strip width is increased. They also show the arrival of more lower-period zeros, which were initially outside the plotted period range but which also moved to the right and into the plot in going from c to d or from e to f.

The 1-m-thick and 0.5-m-thick strips show significantly more zeros in general than the 2-m-thick strip. Since the NEW results agree with the exact results for the higher strip widths, we can use the formulae (8.33) to help us interpret those results: the formulae are periodic in $2k_0a$ and so any zeros will be separated in wavenumber space by a spacing of $\pi/a$. There are consequently infinitely many high wavenumber zeros (equivalent to low wavelength or period). Increasing $a$ means both that these zeros become more and more bunched up in wavenumber space (and thus period space also), and consequently that more zeros move into the plotted range of wavenumbers/periods. Similarly, since the under-ice wavelengths become smaller as the thickness is decreased, the wavenumbers are bigger, and so the plotted period range corresponds to a larger range of wavenumbers and thus contains more zeros in $|R|$. Figure 8.5 shows plots of $|R|$ against $k_0a/2\pi$, the ratio of the strip width to the wavelength inside the strip, for two different periods, 4.5 s (left hand column) and 6.5 s (right hand column). Exact results are plotted as solid curves, and NEW results are
plotted as dashed curves. The thickness of the strip is either 2 m (a and b), 1 m (c and d), 0.5 m (e and f) or 0 m (g and h; these are open lead results). The two periods were chosen to produce a reasonable amount of reflection—the first and second maxima in Figures 8.4a, c and e occur roughly at these periods.

Figure 8.5: Comparison of NEW approximate results (dashed curves) with exact results (solid curves) for the scattering of normally incident waves by strips of ice either 2 m thick (a and b), 1 m thick (c and d), 0.5 m thick (e and f), or 0 m thick (g and h) located between two 1-m-thick semi-infinite ice sheets. The latter results correspond to those for an open lead. The free edge conditions are applied at each end of the strip, the water depth is infinite, and the incident wave period is either 4.5 s (figures a, c, e and g, wavelength 80 m) or 6.5 s wide (figures b, d, f and h, wavelength 103 m). $|R|$ is plotted against $k_0a/2\pi$, the ratio of the width to the wavelength inside the strip; its limit as $a \to 0$ is also compared to the value of $|R|$ for a crack at the appropriate period.

For the 2-m-thick strip at 4.5 s, there are two zeros extremely close together, fol-
ollowed by a comparatively wide arch before the exact results settle down to the periodic behaviour (with period 0.5) predicted by the NEW approximation. As the thickness is decreased, the first two zeros move to the right and their separation increases. The height of the maximum between those two zeros also increases towards the NEW curve, and the width of the second arch decreases towards 0.5. Once the ice thickness has decreased to zero in \( g \), the exact and the approximate curves are essentially indistinguishable, even in the limit as \( a \to 0 \). This could possibly be attributed to the non-existence of any damped travelling waves in open water (modes \( n = -2, -1 \)); the greater fine structure in the 2m plot (a) could also be attributed to these modes having more influence as their imaginary part becomes smaller and smaller as the thickness is increased, and becomes larger and larger as the thickness is decreased, so that the two roots may eventually be discarded when \( h_1 \) becomes zero.

Figures 8.5 b, d and f show similar trends with decreasing thickness—the thicker ice figures show more complexity for smaller strip widths, with the curves again moving towards the periodic behaviour of the NEW approximation as the ice thins.

The “o’s” plotted at \( k_0 a/2\pi = 0 \) in each graph correspond to \(|R|\) for a single crack for the relevant period. It might be expected that since the free edge conditions are applied at each edge of the strip, it might behave more and more like a crack as its width is reduced, and it is useful confirmation of our theory that it always does. What is surprising, however, is that for an open lead the zero-width limits of the NEW approximate results—the wide spacing results—are so close to those of a crack. This was noticed by Vaughan et al. (2005), who showed that the results for a single crack were well approximated by simply setting \( a = 0 \) in (8.33), with \( R_1 \) being the reflection coefficient for a wave travelling in a semi-infinite region of open water impinging on a semi-infinite ice sheet.

Having this approximation for the crack results does not improve their calculation at all, but it does provide an explanation of how a zero in \(|R|\) such as the one seen in Figure 5.2 can be produced, even if it does not explain why it occurs at the particular period that it does. Equation (8.30a) expresses the reflected wave as the sum of one that is reflected from the left hand edge, and another that is reflected from the right hand edge. At a certain period these waves are able to cancel each other out entirely—as was pointed out by Vaughan et al. (unpublished manuscript, 2005).
The final comment to be made about Figure 8.5 is that for the graphs where ice is present in the central region, the NEW results all converge to the exact results when \( k_0 a / 2\pi \approx 0.75 \). This value decreases as the ice is made thinner than 0.5 m to where it is about 0.2 or 0.25 when the strip is an open lead.

This information can also be used to increase the speed at which our results are gathered—we will take the conservative approach of only using the NEW approximation for strip widths greater than one wavelength when ice is present, or one half wavelength when open water is present. This can save a lot of time, especially when it is necessary to calculate \( R \) and \( T \) for several different values of \( a \); the widths greater than the relevant threshold can be dealt with virtually simultaneously.

**8.2.2 The Refreezing of a Lead**

This section is the most relevant to the topic of imperfections in ice sheets as leads—both open and partially or fully refrozen—are very common features in both the Arctic and the Antarctic.

Figure 8.6 shows the different scattering patterns as refreezing progresses. In figures a, b and c (top row), the edges are frozen to the larger ice sheets, while they are free to move independently in figures d, e and f (bottom row).

Figure 8.6a shows the scattering by a one-quarter-frozen (solid curve), semi-frozen (dashed curve), three-quarters-frozen (chained curve) and fully frozen (dotted curve) 15-m-wide lead. Since the edges are frozen, increasing the lead thickness \( h_1 \) makes the situation closer and closer to that of a uniform ice sheet without any imperfection, and so we would expect that the reflection would decrease as the refreezing progresses. Indeed this is what is apparent in both figure a, and figures b and c, which contain the same results but for a 100-m-wide lead. For both widths the fully frozen lead has considerably less reflection; also, as discussed with regard to Figure 8.4, the fully frozen curves do not show as many zeros in \( |R| \) (due to the longer wavelengths/shorter wavenumbers). In addition, as in Figure 8.4, increasing the lead width to 100 m produces more zeros at lower periods.
Figure 8.6: The scattering of normally incident waves by a lead at different stages of refreezing and the effect of the edge conditions applied at its boundaries. The frozen edge conditions are applied at the edges in figures a, b and c, and the free edge conditions are applied at the edges in figures d, e and f. In all figures the water depth is infinite, and a fully frozen lead is $\rho h_0 / \rho_w = 0.9 h_0$ thick. Frozen edge conditions: figure a shows the scattering by a 15-m-wide lead one-quarter-frozen (solid curve), half-frozen (dashed curve), three-quarter-frozen (chained curve) and fully frozen (dotted curve); figure b shows the scattering by a 100-m-wide one-quarter-frozen lead, while c shows the scattering by a 100-m-wide lead half-frozen (solid curve), three-quarter-frozen (dashed curve) and fully frozen (chained curve). Free edge conditions: figure d shows the scattering of normally incident waves by an open 15-m-wide lead (solid curve) and a fully frozen 15-m-wide lead (dashed curve); the scattering by an open 100-m-wide lead is shown in figure e, while the reflection produced by a fully frozen 100-m-wide lead is shown in figure f.

Figure 8.6d compares the scattering by a 15-m-wide open lead to that produced by a fully frozen lead of the same width. The general reflection pattern of the two
curves are extremely similar, with the main difference being the large number of zeros apparent in the open lead curve. Similarly, increasing the lead width in figures e and f only increases the number of these zeros, with the open lead figure (e) showing so many it is difficult to distinguish them all.

Since in general the scattering results for a refrozen lead depend quite strongly on the edge conditions used, the question of how to tell which ones to use at which stage of refreezing arises. Obviously, we can only use the free edge conditions when the lead is open, and we would expect that the edges would remain free when there is only a very small degree of refreezing.

Now, there will almost certainly be periods of very little wave action, refrozen leads of most thicknesses would probably freeze to the main ice bodies during those times. Consequently, the question of which edge conditions to use then becomes “Will the strain produced in the edges be so great that the ice will break when waves begin arriving again?”

One thing that would need to be taken into account when addressing this question is the way that our results are affected by the material properties \((E, \rho \text{ and } \nu)\) of the ice inside the lead, which would no doubt differ from the ice of the main sheets.

We can make some initial predictions based on our experiments in Section 5.2.1. Since the \(\mu\) term in (2.9) was shown in that section to have the least effect on our results, one would not anticipate changing the density to have much effect. The rigidity term was the term that had the most effect—by inspecting it we can see that increasing the \(E/(1-\nu^2)\) factor in \(D\) by a factor of \(\alpha\) is roughly equivalent to increasing the thickness by a factor of \(\alpha^{1/3}\), and so the effects of changing them can be deduced from those of changing the thickness.

Then, if the incident wave spectrum for the region is known or can be measured, a strain density function can be calculated by combining the wave spectrum with strains calculated from the solution for the displacement obtained in this chapter. However, at the moment no wave spectra pertaining to waves beneath large ice sheets are available, so we can only make an arbitrary choice about when to use the frozen edge conditions and when to use the free edge conditions. Since the scattering by a lead that is 10%
refrozen seems the least affected by the edge conditions, it was decided (somewhat arbitrarily) that for leads thicker than that the frozen edge conditions would be used, and that the free edge conditions would be used for thinner leads.

### 8.2.3 Other Applications

This section serves to demonstrate the versatility of the method developed in this chapter by allowing for the three $h_j$ ($j = 0, 1, 2$) to be different, and also to let either or both of $h_0$ and $h_2$ to vanish, allowing us to also treat breakwater situations and floating bodies in the open ocean. We are also able to confirm some of the results in Chapter 7.

![Figure 8.7](image-url)

**Figure 8.7**: Further applications of the coupled Wiener-Hopf method. The scattering of normally incident waves by a strip of ice located between a 1-m-thick semi-infinite ice sheet on the left and a 2-m-thick semi-infinite ice sheet on the right. The strip is either 0.5 m thick (solid curves) or 2 m thick (dashed curves). The frozen edge conditions are applied at each end of the strip, which is either 15 m wide (a), or 100 m wide (b). The water depth is infinite.

Figure 8.7 shows the reflection produced when $h_0 = 1$ m, $h_2 = 2$ m and when $h_1$ is
either 0.5 m (solid curves) or 1.5 m (dashed curves). The former situation corresponds to that of a refrozen lead between two ice sheets of different thicknesses, and the latter situation is the same as that modelled by the double step in Figure 7.6. We can immediately say that the methods of this and of the previous chapter agree very well in this case.

Comparing the results for the two values of \( h_1 \), it can be seen that there is significantly more reflection when the central strip is thinner than when it is thicker. This is probably attributable to the increased thickness as the wave travels from the 0.5-m-thick region.

Figure 8.7 also shows an absence of the zeros in \(|R|\) compared to previous plots of this chapter when \( h_2 \) was the same as \( h_0 \). (In particular, compare the solid curves with the appropriate curves in Figures 8.6a and c.) Instead, as is predicted by equation (8.32) in Section 8.1.5, there are only successive nonzero minima. As in that case, these minima still become more numerous and closer together as the width is increased. This was also observed in Chapter 7.

Figures 8.8a and b also show results for when \( h_2 \neq h_0 \). In those figures \( h_0 = 1 \) m, \( h_1 = 2 \) m and \( h_2 \) is 3 m (solid curves), 1.5 m (dashed curves) or 0 m (chained curves). Again, the central strip is either 15 m wide (a) or 100 m wide (b).

As in Figure 8.7, there are no zeros in \(|R|\), only nonzero minima—one appearing in the plot for the larger width (b) that is not present in a. There are less of these minima due to the larger value of \( h_1 \) (2 m compared to 0.5 m).

The chained curve (\( h_2 = 0 \) curve) in Figure 8.8a confirms the breakwater results presented in Figure 7.11b. As in Figure 8.7, the two methods agree exactly. Surprisingly, though, the other two values of \( h_2 \) (1.5 and 3 m) in Figure 8.8a produce quite a lot more reflection. The same trend is shown in Figure b, with \(|R|\) being greater in general when \( h_2 = 3 \) m than for the other two thicknesses. However, the difference between the curves is not as pronounced in that figure as in Figure a. Going by Figure 8.7, we might have expected that a larger change in thickness moving from the middle region to the right hand one would have resulted in more scattering. However, it seems that when the free edge conditions are used, it is the overall difference between the two
outer thicknesses that has most effect.

Figure 8.8: Further applications of the coupled Wiener-Hopf method. Figures a and b show the scattering of normally incident waves by a 2-m-thick ice strip located between a 1-m-thick semi-infinite ice sheet on the left and either another semi-infinite ice sheet on the right with thickness of either 3 m (solid curves) or 1.5 m (dashed curves), or a semi-infinite region of open water (chained curves). Figures c and d show the scattering of normally incident waves by a 1-m-thick ice strip located between either two semi-infinite ice sheets with thicknesses of either 2 m (solid curves) or 0.5 m (dashed curves), or between two semi-infinite regions of open water (chained curves). The ice strips are either 15 m wide (a and c) or 100 m wide (b and d), and the free edge conditions are used. The water depth is infinite.

In Figures 8.8c and d, where $h_2$ and $h_0$ are equal again, these minima become zero once more. In that figure $h_1$ is kept at 1 m while $h_0 = h_2$ is varied, taking values of 2 m (solid curves), 0.5 m (dashed curves) and 0 m (chained curves). The chained curves
thus display the scattering by a strip of finite width floating in the open ocean and could be applicable to a floating airport or to a two-dimensional model of an ice floe (cf. Meylan, 1993). Tkacheva (2002) also solved this problem using the Wiener-Hopf method.

In both of figures $c$ and $d$, when the strip width is 15 m and 100 m respectively, there is significantly more reflection when $h_1 = 2$ m than when it is either 0.5 m or 0 m. There is very little difference in the amount of reflection between the latter two thicknesses, although the fine structure of the curves is different.

The structure of the $h_0 = 0.5$ m curves most resembles that of the 2 m curves, with the same number of zeros and maxima in each figure—with the exception of two right hand zeros in $d$ that are absent—they have merged to form a point of inflection similar to the one observed in Figure 8.4b. However, the maxima are not as high in the 0.5 m curves as in the 2 m ones, and landmark features have moved to the left.

In Figure 8.8c, the open water curve has moved even further to the left, and it has two fewer zeros. This is the result of those zeros also having merged as the thickness was reduced. The point of inflection that was formed by that process is no longer present, however. In figure $d$ also, the point of inflection in the 0.5 m curve has been "smoothed out" by the time the thickness has reduced to zero. The zeros and maxima that are present have again moved to the left, and the maxima have again reduced in height.
Chapter 9

Scattering by Multiple Irregularities

This chapter deals with the scattering by multiple features in a uniform ice sheet. Figure 9.1 illustrates the set-up when there are two ridges present, although by setting the thicknesses $h_{1j}(x)$ ($j = 1, 2$) to be constants less than or equal to $0.9h_0$ it could just as easily represent two leads in series.

Section 9.1 presents the solution methods used to solve the problem of two features. First the exact solution method is described in Section 9.1.1; then Section 9.1.2 describes both an adaptation of the NEW approximation given in Chapter 8 and another approximation which seeks to average out the effects of feature separation in the NEW approximation. (This last approximation will be called the serial approximation.)

Section 9.1 concludes with an introduction of some statistical distributions that will be referred to in Section 9.2, the results section. These distributions have been observed to describe properties of real ice fields, such as ridge sail heights and spacings.

To begin with Section 9.2 follows the structure of the multiple irregularities section by Williams and Squire (2004a), and starts by presenting results for the scattering at normal incidence by two pressure ridges or leads in Section 9.2.1, which mainly investigates the effect of feature separation on the scattering. The results are similar to results for two cracks presented in Chapter 8, but can be taken a little further in that we can also look at the effect of giving the two features different properties from each other. (All cracks are identical, so this couldn’t be done in the previous chapter.) This sec-
tion also examines the effect that including a keel makes on the scattering by two ridges.

These results for two features are also compared with the NEW approximation, which is able to produce results for different feature separations much more quickly than is possible if an exact solution is sought. (This was also done in Chapter 8.) The NEW two-ridge results are almost identical to the exact results, since the ridges generate very little evanescent wave energy compared to features like cracks and leads. The NEW results for leads (both open and refrozen) are not as accurate for small spacings, but the situation improves as the separation is increased; the exact and approximate curves generally coalescing when the separation is about three quarters of one wavelength.

Section 9.2.1 then seeks to average out the effect of separation by using a serial approximation, which can produce results even more quickly than the NEW approximation. Ridge separations are sampled from an observed type of distribution (log-normal in that case; cf. Section 9.1.5) and the average value of $|R|$ (calculated using the NEW method for two identical ridges) is calculated and compared with the serial result. The two curves are almost identical, although the variance of the results about the average is quite large.

This experiment is repeated in Section 9.2.2 for a larger number of identical ridges with similar success. It works just as well when the ridge sail eights are also drawn from a random distribution, or when large numbers of cracks or leads are used and separations and lead widths are randomly generated.

Having established that the serial approximation summarizes the general scattering properties of a series of features for normal incidence, we also attempt to use it to predict the effect of randomising the feature orientations. In addition, we combine the results for individual features in an attempt to model an ice field populated with cracks, pressure ridges and leads, in the manner of Williams and Squire (2004b). The scattering appears to be dominated by the effect of the leads.

Finally, Section 9.2.3 investigates the effect of changing the background ice thickness on scattering results when either one or 100 ridges or leads are present. (The serial approximation is used in the latter case.) A similar result for ridges was given by
Williams (2004), but since the leads seem to dominate the scattering by an ice field, results for them are also added to complete the chapter. These last results are presented with the intention of showing that the scattering changes enough with the background ice thickness that there is potential for it to be used to determine that thickness.

9.1 Solution Methods For Two Features

This section begins in Section 9.1.1 by describing the full solution for the problem of the scattering by two ridges or leads. Figure 9.1 illustrates the problem as well as introducing some additional parameters that we will use in this chapter.

Section 9.1.2 then presents a form of the NEW approximation discussed in Section 8.1.5 that has been adapted for this chapter. In order to average out the effect of the separation of the two features, we then present the serial approximation, which is derived by calculating the average of \( |T|^2 \), where \( T \) is the transmission coefficient obtained from the NEW formula. \( T \) is periodic with respect to the feature separation so its average is calculated from the integral of \( |T|^2 \) over one period.

Section 9.1.3 briefly outlines how the two approximate methods for two features may be extended when several are present, and Section 9.1.5 introduces some statistical distributions that will be required in the results section (Section 9.2).

9.1.1 Exact Solution

Figure 9.1 illustrates the problem to be addressed in this chapter. This problem could also be treated as a single region with thickness varying from \( x = 0 \) right up to \( x = a \), a la Chapter 6. However, the \( d_j \) functions in the kernel \( K(x, \xi) \) of the integral equation (6.2) would be identically zero from \( x = x_0 \) to \( x = x_1 \), and would not give us any extra information about \( \phi_r(x, 0) \). This would also increase the size of the matrix to be inverted when solving (6.2), slowing our results-gathering down unnecessarily. Therefore, we will treat the two features separately.

Figure 9.1 shows a wave of unit amplitude reaches the left hand feature, is partly reflected and partly transmitted. A wave of amplitude \( b_0' \) travels from the left hand feature to the right hand one, and it is also partially reflected and transmitted. The reflected wave has amplitude \( c_0' \) and part of it may be reflected again by the left hand
Figure 9.1: Schematic diagram showing the scattering by two ridges. The ridge on the left has left hand limit 0 and width $x_1$ and the ridge on the right has left hand and right hand limits of $x_2$ and $a$. $a$ is the total width of the variable region, and the separation of the two ridges is denoted by $\Delta x = x_2 - x_1$. The $b'_n$ and $c'_n$ ($n = -2, -1, 0, 1, 2 \ldots$) coefficients represent the infinity of wave modes generated at the left hand and right hand ridges respectively; if all but $b'_0$ and $c'_0$ are set to zero, the resulting reflection and transmission coefficients correspond to those given by the NEW approximation. The diagram can also represent the scattering by two leads if the thickness $h_{11}$ and $h_{12}$ are set to be constants less than or equal to $0.9h_0$.

There are also an infinity of damped and evanescent waves generated at each feature—with amplitudes $b'_n$ and $c'_n$ ($n = -2, -1, 1, 2 \ldots$). If the features are close enough together, these can also interact with the propagating waves and thus affect the overall values of $R$ and $T$.

The feature on the left has left hand limit 0 and width $x_1$, while the feature on the right has left hand and right hand limits of $x_2$ and $a$. $a$ is still the total width of the variable region, and the separation of the two ridges is denoted by $\Delta x = x_2 - x_1$. If $j = 1$ corresponds to the left hand feature and $j = 2$ corresponds to the right hand feature, the $j^{th}$ feature has thickness $h_{1j}(x)$, and corresponding plate operator

$$ L_{1j}(x, \partial_x) = L_0(\partial_x) + \sum_{r=1}^{4} d_{rj}(x) L_{1r}(\partial_x). $$

The $d_{rj}(x)$ are defined as in Section 2.2.

By applying Green's theorem and eliminating $\phi(x, 0)$ as in Chapter 4, we arrive at
a pair of coupled integral equations

\[
\frac{D(x)}{D_0} \phi_x(x, 0) = e^{i\alpha_0 x} \varphi_x(0) + \psi^T(x) P_0 + \psi^T(x - x_1) P_{x_1} \\
+ \sum_{n=-2}^{\infty} c_n e^{i\alpha_n (x_2 - x)} + \int_0^{x_1} K_1(x, \xi) \phi_x(\xi, 0) d\xi 
\text{ for } 0 < x < x_1, \quad (9.1a)
\]

\[
\frac{D(x)}{D_0} \phi_x(x, 0) = \psi^T(x - x_2) P_{x_2} + \psi^T(x - a) P_a \\
+ \sum_{n=-2}^{\infty} b_n e^{i\alpha_n (x - x_1)} + \int_{x_2}^{a} K_2(x, \xi) \phi_x(\xi, 0) d\xi 
\text{ for } x_2 < x < a, \quad (9.1b)
\]

where the kernels \( K_j \) are given by

\[
K_j(x, \xi) = \sum_{r=1}^{4} d_{jr}(\xi) \mathcal{L}_{1r}(\partial_\xi),
\]

\[
b_n = b'_n \varphi'_n(0) = \delta_n \varphi'_n(0) e^{-i\alpha_0 x_1} \\
+ i A_n (p^T(-\alpha_n) P_0 e^{i\alpha_n x_1} + p^T(-\alpha_n) P_{x_1} + \hat{\phi}_1(-\alpha_n) e^{i\alpha_n x_1}), \quad (9.2a)
\]

\[
c_n = c'_n \varphi'_n(0) = i A_n (p^T(\alpha_n) P_{x_2} + p^T(\alpha_n) P_a e^{i\alpha_n (a-x_2)} + \hat{\phi}_2(\alpha_n)), \quad (9.2b)
\]

and where the \( \hat{\phi}_j \) functions are defined in a similar way to \( \hat{\phi} \) in Section 4.2.1:

\[
\hat{\phi}_1(k) = \sum_{r=1}^{4} \mathcal{L}_{1r}(ik) \int_0^{x_1} d_{1r}(\xi) \phi_x(\xi, 0) e^{ik\xi} d\xi, \quad (9.3a)
\]

\[
\hat{\phi}_2(k) = \sum_{r=1}^{4} \mathcal{L}_{1r}(ik) \int_{x_2}^{a-x_2} d_{2r}(\xi) \phi_x(\xi, 0) e^{ik\xi} d\xi. \quad (9.3b)
\]

If we divide the intervals \((0, x_1)\) and \((x_2, a)\) into \( M_1 \) and \( M_2 \) subintervals in the same way that we did in Chapter 6, and then define vectors such that

\[
[\phi_{xj}]_m = \phi_x(x_{jm}, 0) \quad \text{for } 0 < m < M_j,
\]

where \( x_{1m} = mx_1/M_1 \) and \( x_{2m} = x_2 + m(a-x_2)/M_2 \), then the values of the transforms required in (9.2) can be approximated as follows (cf. \( \hat{\phi}(\pm \alpha_n) \) in equations 6.13 and 6.15):

\[
\hat{\phi}_j(\alpha_n) = F_j^+ \phi_{xj}, \quad \hat{\phi}_j(-\alpha_n) e^{i\alpha_n a_j} = F_j^- \phi_{xj}.
\]

Then we can write (9.2) in matrix-vector form as

\[
b = f_b + M_b \phi_{x1}, \quad (9.4a)
\]
\[
c = f_c + M_c \phi_{x2}, \quad (9.4b)
\]

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and (9.1) as

\[ M_{10} \phi_{z1} = f_{10} + E_1 (f_c + M_c \phi_{z2}), \]  
\[ M_{20} \phi_{z2} = f_{20} + E_2 (f_b + M_b \phi_{z1}). \]  

(9.5a)  
(9.5b)

In the above, the \( E_j \) matrices, which have elements

\[ [E_1]_{mn} = e^{i\alpha_n (x_2 - x_{1m})}, \quad [E_2]_{mn} = e^{i\alpha_n (x_{2m} - x_1)}, \]

contain all the dependence on the separation. Hence, if results for several separations were required, one could write the system (9.5) as

\[ M_{20} \phi_{z2} = f_{20} + E_2 \left( f_b + M_b M_{10}^{-1} (f_{10} + E_1 (f_c + M_c \phi_{z2})) \right), \]  

(9.6)

which would only necessitate inverting a \( M_2 \times M_2 \) matrix for each additional separation. (Alternatively, if \( M_1 < M_2 \), we could eliminate \( \phi_{z2} \) and then keep solving for \( \phi_{z1} \).) In general, however, it is faster to solve for both \( \phi_{zj} \) simultaneously.

### 9.1.2 NEW and Serial Approximations

When the separation of the two features is large enough that none of the damped or evanescent waves survive the journey between them, we can adapt the NEW approximation derived in Chapter 8 to give the following formulae:

\[ R = R_1 + \frac{R_2 T_1 T_1 e^{2i\alpha_0 x_2}}{1 + |R_1 R_2| e^{2i(\gamma + \alpha_0 x_2)}}, \]  
\[ T = \frac{T_1 T_2}{1 + |R_1 R_2| e^{2i(\gamma + \alpha_0 x_2)}}, \]

(9.7a)  
(9.7b)

where \( R_j \) and \( T_j \) are the reflection and transmission coefficients for the \( j^{th} \) feature, and \( 2\gamma = 2\text{Arg}[T_1] + \text{Arg}[R_2/R_1]. \) This is obtained by replacing \( k_0 \) with \( \alpha_0 \) and \( \alpha \) with \( x_2 \) in (8.30), and gives extremely fast results for multiple separations at once.

If we wish to average out the effect of feature separation on our results, then since equations (9.7) are periodic in \( x_2 \) we can calculate the average of \( |T|^2 \) over one period, giving

\[ |T_{av}|^2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{|T_1 T_2|^2 dt}{1 + |R_1 R_2|^2 + 2 \cos t} = \frac{|T_1 T_2|^2}{1 - |R_1 R_2|^2}. \]  

(9.8)

If the two ridges are identical, then

\[ |T_{av}|^2 = \frac{|T_1|^4}{1 - |R_1|^4} = \frac{|T_1|^2}{1 + |R_1|^2}. \]
The formula (9.8) is not very easy to extend to multiple features and therefore we approximate the denominator by 1 to give an approximation that we will call the serial approximation:

\[ |T_{av}| = |T_1 T_2|. \]  

(9.9)

This is just the product of the magnitudes of the individual transmission coefficients, which has an obvious extension to an arbitrary number of ridges.

Note that by neglecting the \( |R_1 R_2|^2 \) in the denominator of (9.8), (9.9) slightly overestimates the overall average transmission and consequently underestimates the average reflection (since we require that \( |R|^2 = 1 - |T|^2 \)).

### 9.1.3 Extension of Approximate Methods to Multiple Features

The serial approximation can be extended to account for \( N \) features by simply writing

\[ |T_{av}| = \prod_{j=1}^{N} |T_j|. \]  

(9.10)

The NEW approximation can also be extended to \( N \) features. This is done iteratively—first \( R \) and \( T \) for the first two features is calculated; these values are then used as \( R_1 \) and \( T_1 \), \( d \) is taken to be the left hand limit of the third feature, and new values for \( R \) and \( T \) are then calculated; this process is carried out \( N - 1 \) times to give the overall result.

### 9.1.4 Non-parallel Features

It should be noted that the present model is limited to dealing with parallel features. However, we would like to get some idea of how allowing for randomly oriented features might affect our results. Therefore, in Section 9.2 we will attempt to make some predictions about the effect of including randomly oriented features on the observed wave scattering. In the lack of another alternative, we will simply use the serial approximation (9.10) to do this.

Such an approach could present problems if the orientations permitted were too large—since \( T \to 0 \) for all features as the angle of incidence \( \theta \to \pi/2 \), only one
or two irregularities at higher angles could stop the transmission of waves completely. Therefore we will restrict ourselves to only using relatively smaller angles of orientation.

### 9.1.5 Statistical Distributions

In the following section, we investigate the average scattering when different properties, namely, ridge sail heights and separations, lead widths and separations, and feature orientations, are randomly sampled from appropriate distributions. Wadhams (1988) provides a summary of distributions describing various sea ice morphologies, and we will draw on his work where possible.

The distributions we will use fall into four categories—exponential, log-normal, power law and beta distributions. The probability density functions for the exponential and beta distributions are given respectively by

\[
\begin{align*}
    f_{pd}(X) &= \exp\left(-\frac{(X - X_0)}{\lambda}\right) / \lambda \quad \text{for } X_0 < X < \infty, \\
    f_{pd}(X) &= A(X - X_0)^\alpha (X - X_1)^\beta \quad \text{for } X_0 < X < X_1,
\end{align*}
\]

while if \( Y = \log(X - X_0) \), then the probability density function for the log-normal distribution is

\[
    f_{pd}(X) = \exp\left(-\frac{(Y - \mu)^2}{2\sigma^2}/(X - X_0) \sqrt{2\pi\sigma} \right) \quad \text{for } X_0 < X < \infty.
\]

Thus \( Y \) is normally distributed.

The probability density function for the power law distribution is not the usual one but instead is given in a piecewise fashion by

\[
    f_{pd}(X) = \begin{cases} 
        A(X/X_1)^{-n_1} & \text{for } X_0 < X < X_1, \\
        A(X/X_1)^{-n_2} & \text{for } X_1 < X < \infty.
    \end{cases}
\]

The distributions used when sampling ridge properties are illustrated in Figure 9.2. Sail height and spacing distributions have been observed to follow exponential and log-normal distributions respectively (Wadhams, 1988). In the absence of specified values for the required parameters, values were chosen to give sensible interquartile ranges. (These ranges are shown in Table 9.1, which lists the 99% interquartile range for each random variable \( X \), the interval that \( X \) has a 99% probability of being found in, and the expectation value \( E(X) \) for each variable.) For example, a ridge sail less than 10 cm
Figure 9.2: The statistical distributions used for sampling ridge properties. Figure a plots the probability density function (PDF) that is used for ridge sail height. It is an exponential PDF, of the form given in (9.11) using parameters $X_0 = 0.1 \text{ m}$ and $\lambda = 0.08 \text{ m}$. Figure b plots the ridge spacing PDF, which is log-normal and is given by (9.13) with $X_0 = 100 \text{ m}$, $\mu = 4.8$ and $\sigma = 0.8$. A beta type PDF is used to describe the orientations of all features (cracks and leads as well as ridges), and is shown in figure c. It is generated from (9.12) using $X_0 = -X_1 = \pi/4$ and $\alpha = \beta = 5$.

tall would scarcely be visible, and in 1-m-thick ice a sail of 50 cm would be about the upper limit—consequently, the minimum height was chosen to be $X_0 = 0.1 \text{ m}$, while $\lambda$ was chosen to be 0.08 m so that only 0.5% of ridge sails would be higher than 0.5 m.

Ridge spacings were taken to be predominantly between 110 m and 1.5 km, with an absolute lower limit of 100 m. These values were chosen to match the bounds on our lead spacings.

An observed distribution was not able to be located for ridge orientation, or for the orientation of any other feature, so we used a symmetric beta distribution for all fea-
tures (one with $\alpha = \beta$ in equation 9.12). As can be seen from Table 9.1, most features will be oriented at an angle less than or equal to $\pi/6$ to the incident wave. Although in practice feature orientations would probably be uniformly distributed from $-\pi/2$ to $\pi/2$, using such a range would probably predict total reflectance for all periods once random orientations were introduced, as $|R| \to 1$ as $\theta \to \pi/2$, and so it would just take one feature oriented in or near the direction of the incident wave to block it completely.

<table>
<thead>
<tr>
<th>$X$</th>
<th>99% IR</th>
<th>$E(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ridge Sail Height</td>
<td>$0.1 \text{ m} &lt; X &lt; 0.5 \text{ m}$</td>
<td>0.18 m</td>
</tr>
<tr>
<td>Ridge Separation</td>
<td>$110 \text{ m} &lt; X &lt; 1500 \text{ m}$</td>
<td>218 m</td>
</tr>
<tr>
<td>Feature Orientation</td>
<td>$-\pi/6 &lt; X &lt; \pi/6$</td>
<td>0</td>
</tr>
<tr>
<td>Lead Width</td>
<td>$5 \text{ m} &lt; X &lt; 675 \text{ m}$</td>
<td>47 m</td>
</tr>
<tr>
<td>Crack/Lead Separation</td>
<td>$101 \text{ m} &lt; X &lt; 1500 \text{ m}$</td>
<td>364 m</td>
</tr>
</tbody>
</table>

Table 9.1: Interquartile ranges and expectation values for all quantities randomly generated in this chapter. Given the random variable $X$ in the first column, the second column shows its 99% interquartile range, and the third its expectation value.

Such affects would be similar to Anderson localization of electrons in certain crystals (Anderson, 1958); only electrons within a relatively narrow band of energies can travel through each “cell” of a crystal, but electrons within this band are able to propagate through a perfectly ordered crystal. However, if the cells are all slightly different (in a random way), then an electron which can travel in one cell may not be able to enter the next cell, and it will be prevented from propagating further (i.e. it is “localized”). Theoretically, in a one-dimensional crystal, a single “rogue” cell could stop the transmission of the electron. (Of course, in three dimensions, it would take several).

However, since waves have actually been observed propagating in real ice fields, perfect uniform reflection would be nonsensical. As stated earlier, the intention of considering random orientations was simply to get an idea of what its effect might be, and it was thought that by only considering relatively low angles, where high-$\theta$ effects have not yet become apparent, we might be able to achieve that without blocking the incident waves completely. Each feature will then transmit a broad range of periods (as opposed to a narrow band as in the crystal example above), and wave energy will only gradually decrease as the wave travels through the ice field. In fact, exponential decay
is observed (Wadhams, Goodman, Cowan and Moore, 1988); this is predicted by the serial approximation (9.10), which can be rewritten as $|T_{av}| = \langle T \rangle^N = \exp (N \log \langle T \rangle)$, where $\langle T \rangle$ is the geometric mean of the transmission coefficients of each of the $N$ features.

Figure 9.3: The statistical distributions used for sampling lead properties. Figure a plots the power law type PDF (given by equation 9.14 with $X_0 = 5$ m, $X_1 = 100$ m, $n_1 = 1.5$ and $n_2 = 2.5$) that is used for lead width; an exponential PDF is used for cracks and lead separations and is shown in figure b. It is obtained from (9.11) by using the parameters $X_0 = 100$ m and $\lambda = 264$ m. (NB 1 hm, or one hectometre, is equivalent to 100 m.)

Figure 9.3 shows the distributions from which lead properties are sampled. Once again referring to Wadhams (1988), we use a piecewise PDF of the form (9.14) for lead widths, and another exponential distribution for lead spacings. Wadhams (1988) actually found that there was an excess of lead pairs with spacings outside the range 400 m to 1500 m (i.e. more than was predicted by an exponential distribution), but to avoid excessive complication we have persisted with the exponential distribution. We
assumed that 99.5% of spacings were between 100 m and 1500 m, choosing the lower limit mainly to be reasonably sure that the NEW approximation would still be valid, and the upper limit for simplicity.

We also used the same distribution for crack spacings as for lead spacings as no data were available for those.

It should finally be noted that all of the distributions used may easily be changed should more precise data become available.

9.2 Results

The primary purpose of this chapter is to present some results that indicate the potential for using the overall scattering by a given ice field to determine the average thickness of the undeformed section of the ice sheet. These are presented in Section 9.2.3 and are based on the assumption that the serial approximation provides a good description of an ice field’s average scattering—average over feature separation, and ridge and lead properties such as sail height and width.

Thus, earlier sections, in particular the second part of Section 9.2.1 and the beginning of Section 9.2.2, are to a large extent demonstrating the efficacy of the serial approximation at doing just that. First, however, we must check the limits of the NEW approximation, since the serial approximation is derived from assuming that that is valid (cf. equation 9.8). This is done in the first part of Section 9.2.1, which compares NEW results with exact results for two features.

The NEW two-ridge results are almost identical to the exact results, since the ridges generate very little evanescent wave energy compared to features like cracks and leads. The NEW results for leads (both open and refrozen) are not as accurate for small spacings, but the situation improves as the separation is increased, the exact and approximate curves generally coalescing when the separation is about three quarters of one wavelength. This is not surprising given that the NEW approximation is also known as the wide-spacing approximation (Newman, 1965; Evans, 1990).

Having established that the serial approximation summarizes the general scattering
properties of a series of features for normal incidence, in the latter part of Section 9.2.2
we also attempt to use it to predict the effect of randomising the feature orientations.
In addition, we combine the results for individual features in an attempt to model an
ice field populated with cracks, pressure ridges and leads, in the manner of Williams
and Squire (2004b). The scattering appears to be dominated by the effect of the leads.

9.2.1 Scattering by Two Features

This section investigates the scattering by two ridges or leads, also considering the
result of including ridge keel effects, or the effects of lead refreezing. It compares the
exact results with NEW approximate results, with the intention of establishing limits
on the latter, before proceeding to compare the serial approximation with the average
NEW results (averaging over the feature separation).

Figure 9.4 shows the scattering by two ridges without keels (a and b) and by two
open leads (c and d). The ridges have sails 0.5 m high and 2 m wide, while the leads
are 15 m wide. The left and right hand plots correspond to features 15 m and 100 m
apart respectively. Exact results are plotted as solid curves and NEW approximate
results as dashed curves.

We can compare the ridge results to Figure 6.7b, which showed the reflection by a
single ridge with the same parameters used in Figures 9.4a and b. The reflection by the
single ridge was fairly uncomplicated, decreasing monotonically with increasing period
from about 0.03 when the period was 2 s. The amount of reflection in the presence of
two ridges is of a similar order, but the patterns are more complex due to the presence
of zeros in $|R|$ in both Figure 9.4a and b. Since there is no difference between the ex­
act results and the NEW results, these zeros can be attributed to resonances between
the two features. As expected from Chapters 6 and 8, there are more zeros when the
separation is wider.

Moving to the results for open leads in Figures 9.4c and d, we can immediately see
far more reflection than that generated by two ridges. Another obvious difference is
the much greater number of zeros. However, referring to Figure 8.6d, we can see that
there were already a large number of zeros in $|R|$ for a single lead, and so from (9.7a)
these will always be zeros in the NEW results for $|R|$ when two ridges are present (note
that Figure 8.6 uses a period range of between 1 s and 20 s, whereas Figure 9.4 only

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plots $|R|$ for periods greater than 2 s. For the smaller separation, these account for just over half of the NEW zeros. However, the remaining zeros are extremely close to the single lead zeros. Most of the zeros in the exact curve have a corresponding zero in the NEW curve, although some may have moved due to interference from evanescent waves that have not decayed over the 15 m between the two ridges. Thus those can be attributed to the fact that the period they occur at is close to a period at which a single lead transmits a wave perfectly.

![Graphs showing scattering by two features](image)

Figure 9.4: The scattering by two features for separations of 15 m (a and c) and 100 m (b and d). The upper plots show exact results (solid curves) and NEW approximate results (dashed curves) for two identical ridges with sails 0.5 m high and 2 m wide, and lower plots show analogous results for two identical open leads that are 15 m wide. The incident waves are normally incident and the water depth is infinite. No keel effects are taken into account in the ridge results.

There are, however, two zeros which are not predicted by the NEW approximation, and they must therefore be attributed to additional interference by evanescent waves.
When the separation between the leads is 100 m, all the evanescent waves have decayed by the time they cross it, and so the exact and NEW curves are identical. Also due to the larger separation, there are more opportunities for longer waves to produce resonances between the two leads. The most noticeable one is at about 7 s, but there are one or two others in the lower period “comb”.

Figure 9.5: The scattering by two features for separations of 15 m (a and c) and 100 m (b and d). The upper plots show exact results (solid curves) and NEW approximate results (dashed curves) for two identical ridges with sails 0.5 m high and 2 m wide, and with keels 1.5 m deep and 6 m wide. The lower plots show analogous results for two identical refrozen leads that are 0.45 m thick and 15 m wide. (The frozen edge conditions are applied at each lead.) The incident waves are normally incident and the water depth is infinite.

Figure 9.4 confirms the observations made in Chapter 6 and Chapter 8 that large amounts of reflection correspond to large amounts of evanescent waves being generated. Hence the NEW approximation requires a larger distance between the leads than it needs between the ridges to converge to the exact results. Consequently, if we increased the reflection of each ridge by incorporating keel effects, we would expect the accuracy
of the NEW results to decrease; conversely, if we replaced the open leads with refrozen leads (that reflect less), then the NEW results would be more accurate for shorter separations.

These hypotheses are addressed in Figures 9.5 and 9.6. Figure 9.5 is analogous to Figure 9.4 but makes the changes to the ridges and leads used that are suggested above. Figure 9.6 makes further investigations into two-lead scattering, plotting $|R|$ (calculated by both exact and NEW methods) against separation (as a multiple of incident wavelength) for wave periods of 4.5 s and 6.5 s and different lead combinations.

First let us discuss the new two-ridge situation, as shown in Figures 9.5a and b. The first thing to note is that the overall reflection has approximately doubled, but the overall structures of the curves are basically the same as in Figures 9.4a and b. When the two ridges are 15 m apart, Figure a shows that including a keel does make the NEW ridge results more distinguishable from the exact ones, but not overly so. And again, the two methods produce identical results when the separation is 100 m. Consequently, we can say that even with keels allowed for, pressure ridges still don't generate much more in the way of evanescent waves.

Moving on to the scattering by two refrozen leads, as shown in Figures 9.5c and d, we can immediately see that there is about half the reflection that was produced by the two open leads, and much fewer zeros. However, one does not appear to see much improvement in the accuracy of the NEW approximation.

We can tell more about how the two types of lead compare from Figure 9.6. Figures a and b give results for two open leads, c and f give results for two refrozen leads, while e and d presents results for when there is one lead of each type.

Moving from top to bottom in either column, we can see a decrease in average reflection, a decrease in fine structure in the exact curves, and faster convergence of the NEW results. These observations also appear to hold true as we move from left to right, but the trends are slightly less noticeable. For the first and second rows, the NEW results take about three quarters of a wavelength to converge (slightly more and less respectively), but they take less than half a wavelength to converge for the bottom row. All three trends discussed can be attributed to less evanescent wave production.
by the refrozen leads.

One interesting observation from Figures 9.6 is the lack of zeros in $|R|$ when the two leads are different. This is a similar phenomenon to the one observed in Section 8.2.3, and could have been anticipated from equation (9.7b).

Figure 9.6: The scattering of normally incident waves by two 15-m-wide leads. Exact results are plotted as solid lines while NEW results are plotted as dashed lines, and they are plotted against $\gamma_0 \Delta x/2\pi$, the separation of the two leads divided by the incident wavelength. In Figures $a$ and $b$ both leads are open, in Figures $c$ and $d$ one is open and one is refrozen, and in Figures $e$ and $f$ both are refrozen. The refrozen leads are all 0.45 m thick and their edges are frozen to the man ice sheet. The left and right hand plots correspond to periods of 4.5 s and 6.5 s respectively, or equivalently, to wavelengths of 80 m and 103 m, and the water depth is infinite.

The previous three graphs have served to establish for what separations the NEW approximate is reasonable. They have shown that it converges for all types of features for separations greater than 100 m, with the open lead results being slowest to
converge. Hence in our PDF (9.11) for the lead and crack separations, shown in Figure 9.3b, we take the minimum separation $X_0$ to be 100 m. If it were shown that we needed to take a smaller value for $X_0$, when the NEW approximation was not appropriate, then in our sampling, reflection and transmission coefficients for lead pairs that were too close together could be calculated exactly, as done in Figures 9.4 and 9.5.

The latter two figures have shown that when the features in question are pressure ridges, the NEW approximation is accurate for far smaller separations than for leads. However, the ridge separation PDF (9.13), shown in Figure 9.2b, is also taken to have the same minimum separation for the sake of simplicity.

From here on, we assume that the NEW approximation does provide an accurate measure of reflection and transmission, and proceed to demonstrate that for two features, the serial approximation can provide a good measure of the scattering when the effects of separation are averaged. This is done in Figure 9.8 for the four features treated in this section—ridges without and with keels included (a and b), and open and refrozen leads (c and d). In that figure, separations are sampled randomly from the appropriate distributions, calculating $|R|$ each time, and calculating its rms (root mean square) value once this process is repeated a sufficient number of times (usually 100). Such a procedure is sometimes called taking an "ensemble average". The rms value is then plotted as a solid line, between the 5th and 95th percentiles which are plotted as dotted lines. For comparison, the serial approximation to $|R|$ is plotted as a dashed line. As can be seen there is a large amount of variation about the average, as different separations lead to their own unique patterns of constructive and destructive interference, which in turn results in their own patterns of maxima and minima in $|R|$. This will also be seen in future plots when several features are modelled.

It can be seen that for the two types of ridges, and for the refrozen leads the serial approximation is very close to the rms. We can see by comparing equation (9.9) to (9.8) that the error is greatest when there is a lot of reflection, and so it is expected that the fit should be better for the ridges than for the open leads. And indeed, for the open leads the serial approximation does differ from the rms NEW approximation over period ranges of high reflection, resulting in the former slightly overestimating the latter over those ranges. However, it is surprising that the serial should produce an overestimation when, if anything, the serial approximation was expected to underesti-
mate the average reflection. However, the formula (9.8) assumes that the separations are uniformly distributed over a scale of half a wavelength, which they clearly aren’t.

![Figure 9.7: Comparison of average results for the scattering by two identical features with serial approximate results. The features used are ridges without keels taken into account (a), ridges with keels (b), open leads (c) and refrozen leads (d). Each ridge has a sail 0.5 m high and 2 m wide, and if a keel is used it is 1.5 m deep and 6 m wide. The leads are both 15 m wide, and the refrozen lead is 0.45 m thick with the frozen edge conditions applied at each end. To produce each figure, feature separations are randomly sampled from the appropriate distribution (cf. Section 9.1.5) and NEW approximate results calculated and stored. This process is repeated 100 times, and the rms (root mean square) value of $|R|$ is calculated and plotted as a solid line. Such a procedure is sometimes called taking an “ensemble average”. The 5th and 95th percentiles are also calculated and are plotted as dotted lines. The dashed curve in each figure plots the serial approximation, which is independent of feature separation, and is intended to approximate the solid curve. The incident waves are normally incident and the water depth is infinite.]

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Even despite these discrepancies when there is a lot of reflection, the serial approximation still provides a remarkably good fit to the overall shape of the rms curves. However, the mean might not be the best measure of the average, as it is more effected by extreme values of $|R|$ than would another measure such as a median. Indeed, since $|R|$ is clearly bounded above by 1, in period ranges of high reflection all such extreme values would be small ones, and will consequently drag the mean down. Consequently, we shall use the median instead of the mean in future.

![Graphs](image)

Figure 9.8: Comparison of average results for the scattering by two identical features with serial approximate results. Same as Figure 9.7, but where the rms is replaced by the median as the measure of average used for the solid curves.

In addition, we now repeat Figure 9.7 using the median instead of the mean, to check that we still get similar agreement between the serial and the average results—and Figure 9.8 shows that we do, although the median results are a lot noisier than the rms results. Accordingly, we will assume that we will be able to use it in the following sections to describe the general scattering by a given number of features. First, though, we will check that it still gives a good approximation when the number of features are
increased and when other properties such as ridge and lead widths are also randomly sampled.

Before we do this, however, it is worth commenting on the zeros in both Figures 9.8c and d, the figures for the two types of lead. Those are the zeros that have persisted from the graphs for the single 15-m-wide leads (cf. Figure 8.6). Also note that the zeros in Figures 9.4 and 9.5 that were only due to the effect of separation are no longer present now that that effect has been averaged out. It will turn out that once the lead width has also been included in the averaging, those zeros that are particular to the 15-m-wide leads will no longer remain either.

9.2.2 Scattering by Many Features

This section aims to describe the general scattering by a given ice field when several different type of feature are included. This will be done primarily by using the serial approximation. Before we can do this, however, we must check that it still gives a good approximation to the average scattering behaviour when the number of features is increased and when other properties such as ridge and lead widths are also randomly sampled.

Since in the previous section the NEW results differed most from the serial results when two refrozen leads were used we will address the question of the validity of the serial approximation for larger numbers of features using that type of irregularity. This is done for five, ten, 50 and 100 refrozen leads in Figure 9.9. In that figure, if \( N \) leads are used \( N - 1 \) random separations are sampled from the exponential distribution generated by (9.11) with the parameters used in Figure 9.3b. \( R \) is then calculated using the NEW method, and the median and 90% interquartile interval for \(|R|\) are calculated after repeating 100 times. The median is used instead of the rms for the reasons discussed at the end of the previous section.

As can be seen from that figure, the serial approximation compares to the median extremely well in the period ranges where there is a lot of reflection. Unfortunately, this results in the serial approximation overestimating the average near the smallest maximum in \(|R|\) which occurs at around 11 s. The difference between the two curves increases as the number of leads increases. However, as can be seen from Figure 9.10c,
this becomes less important as the lead width is allowed to vary. And as commented at
the end of the previous section, the serial approximation gives a good fit to the general
shape of the average scattering curve.

Figure 9.9: Comparison of average results for the scattering by multiple identical open
leads with serial approximate results. Figures a, b, c and d are the analogous graphs to
Figure 9.8c, but where 5, 10, 50 and 100 leads are present respectively. Note that if
$N$ leads are present, $N - 1$ random separations are sampled, $R$ is calculated, and the
median and 90% interquartile interval for $|R|$ are calculated after repeating 100 times.

The main thing to notice about the general scattering apparent in Figure 9.9 is
that, as would have been expected, the amount of reflection increases as the number of
leads increases, although, as in Figure 9.8, zeros particular to a single open lead have
persisted into all the individual plots, regardless of the number of leads. This results
in an envelope that is moving farther and farther to the right, and with increasingly
sharp bands of perfect transmission around a few periods.
Figure 9.10: Comparison of average scattering by 100 features with serial approximation when features are no longer necessarily identical. The features used are cracks (a), ridges without keels taken into account (b) and open leads (c). In Figure a, only crack separations are sampled during the averaging process as all cracks are necessarily identical. However, in Figure b, in each repetition ridge sail heights as well as separations are randomly sampled, and in Figure c, lead widths and separations are both sampled. As in the previous graph, the median NEW value of $|R|$ is plotted as a solid line, and the 5th and 95th percentiles are plotted as the upper and lower dotted lines. And in Figure a, as in earlier graphs, the serial approximation is plotted as a dashed line. However, the extra variability in ridge and lead properties has introduced some variance in the serial results in Figures b and c. Consequently, in those figures the dashed lines represent the median value of the serial approximation, and its 5th and 95th percentiles are plotted as the two inner dotted lines.

Figure 9.10 checks the validity of the serial approximation when other feature properties that are able to be legitimately varied under the NEW approximation are included in the averaging. It deals with three cases—when there are (a) 100 cracks present, (b) 100 ridges (without keels) present, or (c) 100 open leads present. In the
case of the cracks, the only factor that can be varied is the separation, but for the
ridges the sail height can also be varied (its width is set as being four times the sail
height), and for the leads the width can be varied. In fact, the degree of refreezing
could also have been varied in the case of the leads, or the slope of the ridge sail, but
including those factors as well would have been quite testing computationally. And
more importantly, it would not have added much more to the thesis.

At any rate, the figure shows that the serial approximation still works quite well
even when these extra properties are included, although it does overestimate the aver­
age scattering by the ridges slightly, and similarly for the cracks near the maxima near
10 s. The deviation is not too great though, with the median results for the ridges at
least falling inside the interquartile range for the serial approximation.

Thus, in the following section, when we are considering the potential for using wave
action to determine average ice thickness, we will take the serial approximate results
to give us an idea about how much the general scattering by an ice field will vary as
its thickness is changed.

We will also use it to make some limited predictions about what the effect of feature
orientation might be, and also to investigate the general scattering behaviour by an
ice field populated with all four types of irregularities that have been discussed in this
chapter. This includes investigating the effect of the presence or absence of keels in
pressure ridges.

Figure 9.11 makes the first such set of predictions. It considers the same three
types of irregularities that were considered in Figure 9.10, but allows them to also be
randomly oriented. The most noticeable effect that this has can be seen in Figure a,
which corresponds to 100 randomly oriented cracks. Since the zero at 7.5 s moves as
the angle that the incident wave makes with the crack is changed (cf. Figure 5.4),
when the cracks are allowed to be randomly oriented the likelihood of a wave travelling
through a field of such cracks without any scattering diminishes greatly. Thus there is
only a minimum around 7.5 s. This effect was also seen in Figure 9.10c—when the lead
width was kept at 15 m as in Figure 9.9, there were about six wave periods that were
able to travel through a field of randomly separated open leads unreflected. However,
when the width was allowed to vary also, this no longer became very likely.
Figure 9.11: Predictions using the serial approximation about the effect of feature orientation on the scattering by 100 irregularities. The irregularities used are (a) cracks, (b) ridges without keels taken into account and (c) open leads. As well as feature orientation, ridge sail heights and lead widths are also averaged over. The solid curves are the median results produced by the serial approximation, and the dotted lines are the 5th and 95th percentiles.

Apart from the loss of this zero, however, and a slight decrease in the height of the maxima around 10 s, the inclusion of random orientations has not made much effect. Similarly, the scattering shown in Figures 9.11b and c has not changed significantly from the serial results in Figures 9.10b and c, with perhaps a slight decrease in the pressure ridge scattering.

The situation would probably be different if the distribution used for feature orientations (shown in Figure 9.2c) was changed so that a wider range of angles was permitted. However, that would then increase the chance of “rogue” irregularities having too great an effect on the scattering as $|R| \to 1$ as the angle of incidence $\theta \to \pi/2$. 

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Consequently, we are somewhat limited in the predictions we can make about feature orientations without exact results to confirm them.

Figure 9.12: Predictions using the serial approximation about the scattering by 100 pressure ridges when keels are included \( (a) \) and \( (b) \) and by 100 refrozen leads \( (c) \) and \( (d) \). In the ridge results, the sail heights were randomly sampled, the sail width and keel depth were taken to be four and three times the sail height respectively, and the keel width was taken to be three times the sail width to make the ridge neutrally buoyant. For the lead results, the width was randomly sampled. In the left hand plots, the features were kept parallel, but their orientation was randomized in the right hand plots. The refrozen leads are 0.45 m thick, the frozen edge conditions are applied, and the water is infinitely deep.

Figure 9.12 presents serial approximate results for the scattering by either 100 pressure ridges with keel effects included \( (a) \) and \( (b) \), or for 100 refrozen leads \( (c) \) and \( (d) \). The left hand plots illustrate the situation when the features are kept parallel and can thus be taken as being a reliable measure of the average scattering when feature separations and either ridge sail heights or lead widths are randomized. The right hand plots show the scattering when the features may be randomly oriented as well, and are thus depen-
dent on the distribution used for the orientations. Here, as in the previous figure, most of the features' normals are oriented at less than $\pi/6$ from the incident wave direction, and so again there is not much difference between Figures a and b (apart from slightly less reflection in b), and between c and d.

Figure 9.13: Predictions using the serial approximation about the scattering by an ice field populated by 33 cracks, 33 pressure ridges and 34 leads. Figure a shows results for when keel effects are not included, and when all the leads are open leads, while in Figure b keel effects are included. In Figure c, keel effects are still included, but half the leads are refrozen (0.45 m thick, frozen edge conditions applied). In all figures ridge sail heights and lead widths have been randomly sampled, but all features are parallel. The water depth is infinite.

One comment about the ridge scattering we can make is that, comparing Figure 9.12a with Figure 9.10b, the inclusion of the keels has increased the scattering by pressure ridges by about a factor of five. Another is that all our ridge properties are determined by the sail height—sail width, keel depth and keel width are respectively four, three and 12 times larger. Sail and keel slopes could also have been randomly
sampled, but that was thought to be overly complicated and of little extra interest, and indeed would have been extremely demanding computationally.

With regards to the scattering by the leads, comparing Figure 9.12c with Figure 9.10c, we can still say that refrozen leads produce less reflection than open leads, although the difference is no longer so apparent as it was for just two leads (cf. Figure 9.8). And like the open lead results, zeros in reflection have also been averaged out by randomising the lead widths (also cf. Figure 9.8).

Figure 9.13 shows the scattering by an ice field containing 33 cracks, 33 pressure ridges and 34 leads. In Figure a, the ridges have no keels and the leads are all open, while in Figure b keel effects are included. In Figure c, keel effects are still included, but half the leads are refrozen. All three graphs look extremely similar, and seem to be dominated by the effect of the leads. Comparing a with b, including ridge keels has hardly any effect at all, although there is a slight increase in reflection. Similarly, there is very little difference between b and c, so the overall scattering by an ice field seems relatively insensitive to the types of lead used.

9.2.3 Potential for Determination of Background Ice Thickness

This section uses the principal results of the previous one to investigate the difference that the value of the background ice thickness, $h_0$ makes on observed scattering results. If the results change quickly enough with it, then we have reason to hope that they can be inverted to provide us with a thickness estimate.

Figure 9.14 presents a similar graph to the one presented by Williams (2004), showing results for either one or 100 identical pressure ridges. The difference between our figure and that one is that here we have taken keel effects into account. This leads to a lot more reflection, but the general patterns are the same.

Figures 9.14a and c show $|R|$ plotted against wave period for ice thicknesses of 1 m (solid curves), 2 m (dashed curves), 5 m (chained curves) and 10 m (dashed curves) for one and 100 ridges respectively. Both patterns are fairly similar, showing a monotonic decrease in reflection as period increases, before a zero is reached followed by a slight
lifting off the period axis. There is then a small maxima before $|R|$ drops asymptotically to zero. In general there is less scattering for the larger thicknesses, as the pressure ridge in question becomes smaller in comparison to the size of $h_0$.

Figure 9.14: Potential for ice thickness determination using wave scattering. Figures a and b show exact results for a single pressure ridge, normal to the incident wave, with sail height 0.5 m, sail width 2 m, keel depth 1.5 m and keel width 6 m. Figures c and d show serial results for 100 identical parallel pressure ridges. The left hand graphs show $|R|$ plotted against period for background ice thicknesses of $h_0 = 1$ m (solid curves), 2 m (dashed curves), 5 m (chained curves) and 10 m (dotted curves). The right hand figures show $|R|$ plotted against $h_0$ for wave periods of 2 s (solid curves), 5 s (dashed curves), 10 s (chained curves) and 15 s (dotted curves). The water depth is infinite.

This is also borne out by Figures 9.14b and d, which plot $|R|$ against $h_0$ for four different wave periods. In both figures, the 2 s and 5 s curves both show a monotonic decrease as thickness increases but, to begin with, the 10 s curves increase to a maxima before they starts to drop. The 15 s curves have the same behaviour, but don’t reach their maxima in the range of periods plotted.
Thus, due to the slow variation of $|R|$ with thickness at large periods, it seems that for pressure ridges, there is more opportunity for differentiation between them at smaller periods. This is unfortunate, because, as discussed in Chapter 1, there is generally less wave amplitude at those periods.

Figure 9.15: Potential for ice thickness determination using wave scattering. Figure a shows exact results for a single open 15-m-wide lead, normal to the incident wave. Figure b shows serial results for 100 identical, parallel leads. Both graphs show $|R|$ plotted against the background ice thickness $h_0$ for wave periods of 2 s, 5 s, 10 s and 15 s. The water depth is infinite.

The previous section showed that the scattering by an ice field is dominated by leads rather than ridges, and so Figure 9.15 presents the equivalent results to Figures 9.14b and d for open leads.

Those results show rather different trends with increasing thickness and periods. Due to the increasing jump in thickness from 0 m to $h_0$, as $h_0$ increases, $|R| \to 1$. This
convergence is especially fast at smaller periods, with the result that when 100 leads are present $|R|$ is almost constantly 1 across the thickness range plotted. Disregarding the zeros which are still present because we have used identical leads, the 5 s and 10 s curves are also essentially constant.

The 15 s curve, then, is the one that shows the most potential for inversion, particularly for thicknesses less than about 5 m. This is promising as the ice in the polar ice sheets is generally less than about 3 m (Wadhams, 1995). And as stated earlier, longer waves are more prevalent beneath these ice sheets as the shorter waves tend to be filtered out by the pack ice near the ice edge.

Consequently, Figure 9.15 shows that the presence of leads lends itself to a greater potential for inverting a given set of scattering results to find the background ice thickness of the ice sheet that produced it.

However, the actual development of such a technique would be extremely difficult. In particular, actual data would be very noisy and in a one-dimensional ice sheet that was invariant in the second dimension would not have effects due to feature separations, lead widths, etc. averaged out. This last factor is not necessarily a disadvantage, as the various maxima and minima that one would find might give further opportunities for differentiating between different background thicknesses. And presumably despite the large variation of the reflection coefficient about its median, these variations would generally be distributed both above and below the average—implying that fitting the average curve to the curve corresponding to an ice field with a specific configuration might still be successful. However, to take account of noise, a lot more work would have to done to establish how sensitive the final estimate for the thickness is to measurement errors for example (in the scattering data, feature widths, heights and separations), and also to other potential sources of noise such as wind.

The final and not insignificant issue is the extension of the results to allow for variation in both dimensions of the ice sheet. At present, we can only calculate exact results when all irregularities are parallel and invariant in one direction, and so this is another limitation when coming to deal with real ice sheets. However, it has been suggested (by M.H. Meylan) that variations in the second dimension might destroy the interference patterns produced in two dimensions and have an averaging effect. Although this
proposition needs to be tested, it would imply that the average scattering we calculated earlier, that was well-modeled by the easily-computed serial approximation, would be close to the observed scattering by a real ice field.

In conclusion, using wave scattering to determine thickness has a lot of promise, but there needs to be a lot more research done to address some of the problems mentioned above. If such a thickness-measuring technique could be developed it would be an exciting prospect as a tool for monitoring the state of both the Arctic and Antarctic sea ice.
Chapter 10

Conclusions

This thesis has primarily been concerned with the application of Green's functions to various problems involving the scattering of ice-coupled flexural gravity waves by different two-dimensional irregularities such as cracks, pressure ridges, ramps and leads. After some introductory theory in the first four chapters, Chapters 5, 6, 7 and 8 dealt with the scattering by individual features of these types. Chapter 9 then examined the scattering by multiple irregularities, with the final intention of looking at the potential for using scattering by a given ice sheet populated with many such features as a tool in determining ice thickness.

The problem of a ramp, examined in Chapter 7 and primarily intended as a model for a transition region between sea ice and an ice shelf, is the most general shape of a single irregularity and combines both numerical and analytical techniques in its solution. The main analytical technique used is the Wiener-Hopf technique. Porter and Porter (2004) published an alternative solution to this problem that used a mild slope approximation and a variational technique. Both methods give similar results, which cross-validates both techniques.

The ramp solution extended the method developed by Williams and Squire (2004a) used to treat a pressure ridge in an otherwise uniform sheet of ice. That method was presented in Chapter 6, as were various results. Again, the paper by Porter and Porter can provide independent confirmation of some of these, as can the paper of Dixon and Squire (2001b). Other tests were performed on both the ridge and the ramp results that produced further confirmation. In particular, when the region of variable thickness was made constant, results could be confirmed by the method of Chapter 8. The
main limitation of the results is that submergence is neglected, although an idea to deal with that was presented in Appendix E.

The problem of a single crack has been dealt with by several authors (e.g. Marchenko, 1997; Squire and Dixon, 2000; Williams and Squire, 2002; Evans and Porter, 2003b), but its solution followed so straightforwardly from the “general solution” used in this thesis, that it is still included. The simplicity of the results produced also gave us a good opportunity to examine the general effects of parameters like wave period, angle of incidence and water depth, allowing other variables such as feature width and thickness to be focused on in later chapters.

Chapter 8 presented a mainly analytic solution to the scattering by a lead in an ice sheet, although the method can be applied equally as well to a problem like a strip of finite width floating in the open ocean. As well as being shown to agree with the results of earlier chapters, it was explicitly verified in Appendix F by a mode-matching method combined with the residue calculus technique. The method of Chapter 8 solved a pair of coupled Wiener-Hopf type equations, in a similar fashion to Tkacheva (2002), while the method used in the appendix is similar to that of Chung and Linton (2005).

Thus the general approach taken in this thesis can be adapted to a large variety of situations, and its good agreement with independent methods gives us considerable confidence in our single-feature results.

The chapter in this thesis devoted to multiple irregularities, Chapter 9, was more results-oriented. It was similar in its approach to papers by Williams and Squire (2004a; 2004b) and Williams (2004). Its initial objective is to establish the serial approximation derived in that chapter as a reliable guide to the average scattering of an ice sheet containing a large number of irregularities. In the process of doing this it can be seen that the scattering by leads dominates the scattering by cracks and ridges.

The chapter then briefly investigated the difference that the background ice thickness makes on the serial results. The variations obtained were large enough to indicate that there is definite potential for using wave scattering to determine average ice thickness. Obstacles to the development of a practical method of doing this are noise and considerations of two-dimensional variations in the ice—the former might come from
measurement errors and other sources like wind, and could probably be taken account of with the right statistical analysis; the latter could either be more difficult to account for, or they could simplify the transmitted spectra by causing the complicated interference patterns produced in one-dimensional ice sheets to be averaged out.

Consequently, more research needs to be done to address some of these problems/questions, and it should be done because if such a thickness-measuring technique could be developed it would be a powerful means of monitoring the state of both the Arctic and Antarctic sea ice. This would provide information which will become increasingly important in the years to come.
References


Shapiro, A. and Simpson, L. (1953). The effect of a broken ice field on water waves, 
*Eos Transactions of the American Geophysical Union* 34: 36–42.


Squire, V. (1984). A theoretical, laboratory, and field study of ice-coupled waves, 

Squire, V. and Allan, A. (1980). Propagation of flexural gravity waves in sea ice, in 
Pritchard, R.S. (ed.), *Sea Ice Processes and Models, Proceedings of the Arctic Ice 
Dynamics Joint Experiment*, University of Washington Press, Seattle, pp. 327–
338.


Stoker, J. (1957). *Water Waves. The Mathematical Theory with Applications*, Inter-
science, New York.

Tkacheva, L. (2001a). Scattering of surface waves by the edge of a floating elastic plate, 

Tkacheva, L. (2001b). Surface wave diffraction on a floating elastic plate, *Fluid Dy-
namics*.

Tkacheva, L. (2002). Diffraction of surface waves at a thin elastic floating plate, *Pro-
cedings of the 17th International Workshop on Water Waves and Floating Bodies*, 


Appendix A

Governing Equations for an Elastic Plate of Variable Thickness

There are a number of linear plate models in existence—the Euler-Bernoulli thin plate model, the Timoshenko-Mindlin model, which are the plate equivalents of the Euler-Bernoulli and Timoshenko beam theories, as well as models intermediate between them. Results in this thesis are generated by using the Euler-Bernoulli theory. In the following section, this choice is briefly justified, comparing it in the process to the Timoshenko-Mindlin model, before using a Lagrangian formulation to derive the plate's equations of motion in Section A.2. In addition, certain conditions must also be satisfied at any "edges" where a discontinuity in either the flexural rigidity or its derivative occurs—these are also presented in that section.

A.1 Plate Model

As waves that reach the central Arctic Ocean or the Antarctic ice sheets will have had to pass through either the pack ice of the Greenland Sea or the Marginal Ice Zone respectively, we will assume that they will have been attenuated enough that a linear (small displacement) plate theory will be adequate to model the deformations produced in the ice. Most models generally make the following assumptions:

1. The vertical displacement is approximately uniform throughout the thickness of the plate and is thus well-approximated by the displacement $\eta(x, y, t)$ of the plate's middle surface from its equilibrium position (Mindlin, 1951; Fung, 1965, pp 456–462).
2. Stresses normal to the plate surface are neglected (Fung, 1965) although, as pointed out by Mindlin (1951), it is only their average value that is neglected when integrating over the plate thickness.

3. Remaining quantities—stresses, strains, and displacements—vary linearly throughout the thickness of the plate (Mindlin, 1951; Fung, 1965).

Neglecting wind effects, the tractions on the ice sheets we are interested in will be minimal. If we assume the ice sheets are thin enough that this implies the tractions on all surfaces parallel to the sheets’ middle surface may also be neglected, then they would be sufficiently well described by the Euler-Bernoulli thin plate model. Using that theory, the kinetic and potential energies of the plate (per surface area) are respectively (Jaeger, 1964, pp 63–66)

\[ T(x, y, t) = \frac{1}{2} m(x, y) \eta^2(x, y, t), \]  
\[ V(x, y, t) = \frac{1}{2} D(x, y) (\nabla^{2} \eta(x, y, t))^2 \]
\[ + (1 - \nu) D(x, y) (\eta_{xy}(x, y, t) - \eta_{xx}(x, y, t) \eta_{yy}(x, y, t)) \]
\[ - Q(x, y, t) \eta(x, y, t), \]  
\[ (A.1b) \]

where \( m = \rho h \) and \( D = Eh^3/12(1-\nu^2) \), are the plate’s mass per unit area and flexural rigidity, which may vary over the extent of the plate; \( \rho, h, E, \) and \( \nu \) are respectively the plate’s density, thickness, Young’s modulus and Poisson’s ratio. The potential energy is the sum of the plate’s strain energy and the force potential of the conservative loading force (per unit area) \( Q(x, y, t) = P_a - P(x, y, t) \), which is the net external pressure on the plate. In the case of a floating plate this is the difference between the constant atmospheric pressure and the pressure exerted by the water beneath.

The Timoshenko-Mindlin plate model is valid for thicker plates as it no longer neglects tractions on surfaces parallel to the plate’s middle surface. Consequently squares of those terms also appear in the strain energy; additional terms corresponding to the squares of the angular velocities (in both the \( x-z \) and \( y-z \) planes) also appear in the kinetic energy, although neglecting rotational effects is less consequential than neglecting shear deformation (Mindlin, 1951). However, we will proceed to use the Euler-Bernoulli model, due to the relative thinness of the ice sheets in comparison with the lengths of the waves that are incident upon them (wavelengths in 1-m-thick ice range from 53 m for a period of 2 s to 612 m for a period of 20 s). The suitability of this choice of model is discussed further in Section 1.2.
A.2 Derivation of Plate Equation and Edge Conditions

Let the plate occupy the region $\Omega = \{(x, y) | -x_0 < x < x_0 \text{ and } y_0 < y < y_1\}$, where $x_0, y_0, y_1$ are arbitrary, and suppose that we are interested in its behaviour over an arbitrary time interval $\{t | t_0 < t < t_1\}$. Also let there be a crack at $x = 0$, as illustrated in Figure A.1, which shows a partial, constant-$y$ cross-section for the plate (the $y$ axis points out of the page). For future reference, the figure also provides the directions that positive values for the loading $Q$ and displacement $\eta$ correspond to, and also does the same for the bending moments $M^\pm$ and the vertical edge forces $S^\pm$ at the crack edges. The latter two quantities will turn out to feature in the natural boundary conditions that $\eta$ must satisfy on either side of the crack.

We will now proceed to derive those edge conditions, as well as the general equation of motion for the entire plate. Before doing so, however, it should be noted that the crack should actually have zero width, but the edges are drawn slightly apart in Figure A.1 to provide room for annotations. In fact, there may not even be a "crack" as such—the edges may be frozen together, in which case $\eta$ should satisfy a different set of conditions to those that are satisfied when the opposing edges are free to move independently. In both situations, however, we shall assume that there is no energy loss or gain at the crack. This means that the Lagrangian for the plate is simply defined

![Figure A.1: Schematic diagram showing the forces and moments accounted for in the thin plate model. $Q$ is the total pressure on the plate, the $S^\pm$ and the $M^\pm$ are the net vertical forces and bending moments on the edges of the indicated plates (acting in the directions shown). $\eta$ is the displacement of the middle of the plate below the $x$ axis.](image-url)
as the difference between its kinetic and potential energies, namely

$$
\mathcal{L}(\eta) = \int_{t_0}^{t_1} \int_\Omega (T(x, y, t) - V(x, y, t)) \, dx \, dy \, dt,
$$

(A.2)

where $T$ and $V$ are given by (A.1). According to Hamilton’s principle, the plate will move in such a way as to minimise the Lagrangian or, equivalently, by finding a non-trivial displacement that makes its first variation zero.

Porter and Porter (2004) derived a similar variational principle to (A.2), although they included the energy of the fluid in their Lagrangian, as well as just the plate energy. They also used a time-averaged Lagrangian, in which they assumed harmonic motion and averaged over one wave period.

Continuing with the application of Hamilton’s principle, the first variation of $\mathcal{L}$ is

$$
\delta \mathcal{L}(\eta) = \int_t^{t_1} \int_\Omega \left( m \frac{\partial^2 \eta}{\partial t^2} + Q \frac{\partial \eta}{\partial t} - D \nabla^2 \nabla^2 \frac{\partial \eta}{\partial x} \right) \, dx \, dy \, dt
$$

$$
- (1 - \nu) \int_t^{t_1} \int_\Omega D \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial x \partial y} \right) \, dx \, dy \, dt.
$$

(A.3)

Integrating (A.3) by parts gives

$$
\delta \mathcal{L}(\eta) = \int_\Omega \left[ \eta \frac{\partial \eta}{\partial x} \right]_{t_0}^{t_1} \, dx \, dy - \int_{t_0}^{t_1} \int_\Omega \left( m \frac{\partial \eta}{\partial t} + \mathcal{L}_{\text{plate}} \eta - Q \right) \, dx \, dy \, dt
$$

$$
- \int_{t_0}^{t_1} \int_{y_0}^{y_1} \left[ M_x \eta \frac{\partial \eta}{\partial x} + S_{x y} \eta \frac{\partial \eta}{\partial y} \right]_{x = 0}^{x_0} \, dy \, dt
$$

$$
+ \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left[ M_y \eta \frac{\partial \eta}{\partial y} + S_{x y} \eta \frac{\partial \eta}{\partial x} \right]_{x = 0}^{x_0} \, dx \, dt
$$

$$
+ 2 \int_{t_0}^{t_1} \left[ M_{x y} \eta \frac{\partial \eta}{\partial x} \right]_{y_0}^{y_1} \, dt - 2 \int_{t_0}^{t_1} \left[ M_{x y} \eta \frac{\partial \eta}{\partial y} \right]_{x_0}^{x_1} \, dt,
$$

(A.4)

where

$$
\mathcal{L}_{\text{plate}}(x, y, \partial_x, \partial_y) = \nabla^2 \nabla^2 (D(x, y) \nabla^2 \eta) + (1 - \nu) \left( 2D_{x y}^2 \frac{\partial^2 \eta}{\partial x \partial y} \right)
$$

$$
- \frac{\partial^2 \eta}{\partial x^2} \left( \frac{\partial^2 \eta}{\partial x \partial y} \right) - \frac{\partial^2 \eta}{\partial y^2} \left( \frac{\partial^2 \eta}{\partial y \partial x} \right),
$$

$$
M_x(x, y, \partial_x, \partial_y) = -D(x, y)(\nu \nabla^2 + (1 - \nu) \frac{\partial^2 \eta}{\partial x^2}),
$$

$$
S_{x y}(x, y, \partial_x, \partial_y) = -\frac{\partial \eta}{\partial x} M_x(x, y, \partial_x, \partial_y) - 2 \frac{\partial \eta}{\partial y} M_{x y}(x, y, \partial_x, \partial_y),
$$

$$
M_{x y}(x, y, \partial_x, \partial_y) = -(1 - \nu) D(x, y) \frac{\partial^2 \eta}{\partial x \partial y}.
$$
Since the $x_j$, $y_j$ and $t_j$ ($j = 0, 1$) are arbitrary, all but the second and the third integrals in (A.4) can be made to vanish simply by limiting the possible variations. This avoids having to put unreasonable restrictions on $\eta$. For example, requiring that $\delta \eta(x, y, t_0) = \delta \eta(x, y, t_1) = 0$ makes the first integral zero without requiring that $\eta$ be constant in time.

The second integral will vanish if $\eta$ satisfies the Euler-Bernoulli thin plate equation

$$m \dot{\eta}_{tt} + \mathcal{L}_{\text{plate}} \eta - Q = 0 \quad (A.5)$$

always and almost everywhere, leaving just the third to be considered. Before doing so, however, some simplifications in its integrand, and also in equation (A.5), can be obtained by observing that none of our irregularities vary at all in the $y$ direction, implying that we can take $D(x, y) = D(x)$. If we do this, $\mathcal{L}_{\text{plate}}$ becomes

$$\mathcal{L}_{\text{plate}}(x, \partial_x, \partial_y) = \nabla_{xy}^2 (D(x) \nabla_{xy}^2 \eta) - (1 - \nu)D'(x) \partial_y^2 \eta,$$

while $S_{xy}$ simplifies to

$$S_{xy}(x, \partial_x, \partial_y) = -\partial_x M_x(x, \partial_x, \partial_y) + 2D(x)(1 - \nu)\partial^3_{xy}\eta$$

$$= D(x) \mathcal{L}^+_{x}(\partial_x, \partial_y) \partial_x + D'(x) \mathcal{L}^-_{x}(\partial_x, \partial_y),$$

where $\mathcal{L}^\pm_x(\partial_x, \partial_y) = \nabla_{xy}^2 \pm (1 - \nu)\partial_y^2$. $M_x$ can also be written as $-D(x)\mathcal{L}^-_{x}(\partial_x, \partial_y)$, and it, $S_{xy}$ and $\mathcal{L}_{\text{plate}}$ have obvious simplifications when $D$ is a constant. Substitution of this simpler form of $\mathcal{L}_{\text{plate}}$, i.e. $D \nabla_{xy}^4$, back into (A.5) gives the Euler-Bernoulli equation in the form it is usually presented.

The third integral of (A.4) means it can also be made to vanish if

$$[M_x \delta \eta_x + S_{xy} \delta \eta_y]_{0^+}^{0^-} = 0, \quad (A.6)$$

for all $y, t$. Now, from Fung (1965, pp 456-463), the bending moments and vertical edge forces at the crack edges, as drawn in Figure A.1, are respectively given by

$$M^\pm(y, t) = M_x(0^\pm, \partial_x, \partial_y) \eta(0^\pm, y, t), \quad S^\pm(y, t) = S_{xy}(0^\pm, \partial_x, \partial_y) \eta(0^\pm, y, t).$$

Now, the $M^\pm$ are torques producing clockwise and anticlockwise rotations respectively, while the $S^\pm$ are respectively downward and upward forces. The variations in displacement $\delta \eta(0^\pm, y, t)$ are obviously the translations that the crack edges undergo when
subjected to those forces, while the variations in slope $\delta \eta_x(0\pm, y, t)$ are approximately the angles that the crack rotate through (in a clockwise direction) under the aforementioned bending moments. Consequently, (A.6) is demanding that no rotational or translational work be done at the crack edges—i.e. that no energy is lost or gained at the crack, as initially assumed.

If the two edges at $x = 0\pm$ are free, then the displacements and slopes on either sides of the crack are independent, and so their variations should also be independent. Thus, our energy conservation requirement can only be satisfied if the bending moments and edge forces vanish independently, i.e.

$$M^\pm(y, t) = 0, \quad S^\pm(y, t) = 0. \quad (A.7)$$

On the other hand, if the two edges are frozen together, then $\eta$ and $\eta_x$ and their variations $\delta \eta$ and $\delta \eta_x$ ought to be continuous. Accordingly, the work of rotation is $(M^+ - M^-)\delta \eta_x$, while the translational work is $(S^+ - S^-)\delta \eta$. These can only be zero for arbitrary variations if $M^+ = M^-$ and $S^+ = S^-$, or equivalently if the bending moments and edge forces on each side of the crack cancel each other out.

In conclusion, with the choice of suitable variations, any displacement function $\eta$ that satisfies the Euler-Bernoulli thin plate equation (A.5) and either satisfies (A.7) or is continuous and has continuous slope, and zero net bending moment and vertical edge force will make the left hand side of (A.4) vanish. Consequently, such an $\eta$ will minimise the Lagrangian (A.2) and so become a suitable candidate for the "actual" displacement. Other displacement functions which satisfy different edge conditions are possible, but the free edge and frozen edge conditions are most common.
Appendix B

Roots of the Dispersion Relation

The distribution of the roots of both the finite depth and the infinite depth dispersion relations have been published previously, by Fox and Squire (1990) and Squire and Dixon (2000) respectively. However, there are interesting exceptions to those rules, particularly for shorter periods. Although the periods at which these deviations from the usual behaviour occur are often small enough to make ice-coupled water waves physically impossible, for interest’s sake they are presented below. First, however, the properties of the shallow water roots are discussed, as they help to introduce the small-period behaviour of the finite depth roots. There is also a connection between the shallow water Green’s function and the Green’s function for an isolated plate (i.e., one that isn’t floating), which is explained in Section 5.1.2.

B.1 Shallow Water Roots

Recall that the finite depth dispersion relation for the left-hand sheet of ice is

\[ f_0(\kappa) = \frac{1}{\kappa \tanh \kappa H} - \Lambda_0(\kappa) = 0, \]

where \( \Lambda_0 = \kappa^4 + \varpi \). Since \( \tanh \kappa H \approx \kappa H \) for small \( H \), for shallow water we can obtain the approximate dispersion relation \( p_s(\kappa^2) = (\kappa^4 + \varpi)\kappa^2 H - 1 = 0 \), which is a cubic polynomial in \( \kappa^2 \).

Setting \( \varpi = 0 \) means \( p_s(\varpi) = H\varpi^3 - 1 = 0 \) when \( H^{1/3} \) is a cube root of unity. The roots usually show this kind of distribution—there is always one real root, \( \gamma_0^1 \), and two roots \( \gamma_2^1 \) and \( \gamma_2^2 \) in the left hand half of the complex plane. The latter two roots are usually a complex conjugate pair (if \( \text{Im}[\gamma_2^1] > 0 \)). However, by solving for when
Figure B.1: Small period behaviour of the roots of the shallow water dispersion relation for ice-coupled waves, \(f_0(\kappa) = -p_s(\kappa^2)/\kappa^2H\), where \(p_s(w) = (w^2 + \omega)wH - 1\). The roots \(\kappa = \pm \gamma_n (n = -2, -1, 0)\) of \(f_0\) are the square roots of the roots of the cubic \(p_s\).

The left hand column shows \(p_s\) plotted against \(w\) for a series of values of \(w\). Note that when \(\omega = \omega_s = -3/(2H)^{2/3}\), \(p_s(w)\) has a double root on the negative real axis which separates to form two simple roots for \(w < \omega_s\). The right hand column shows the locations of the roots of \(f_0\) in the complex plane for the same values of \(w\); arrows on two of these plots indicate the direction that the complex roots in the upper half plane move in as \(\omega\) becomes more negative. When \(\omega\) reaches \(\omega_s\), the complex roots in the upper half plane meet on the positive imaginary axis and, like the roots of \(p_s\), separate into two single roots as \(\omega\) becomes more negative (or equivalently, as period decreases and/or \(h_0\) increases). The value of the nondimensional water depth used is \(H = 1\).

\(p_s(w) = 0\), it can be shown that as \(\omega\) decreases towards \(\omega_s = -3 \times (2H)^{-2/3}\), they both move closer and closer towards the negative real axis. When \(\omega\) reaches \(\omega_s\), the complex conjugate pair form a double root at \(w = \omega_s = -(2H)^{-1/3}\), which then separates to form two simple roots that remain on the negative real axis for all \(\omega < \omega_s\). In that case we will assume \(\gamma_{-1}^2 > \gamma_{-2}^2\).
Figure B.1 shows the shape of \( p_\nu(w) \) for real \( w \) as the above transition takes place. As \( w \) becomes negative, \( p_\nu \) develops a local maximum which moves upwards as \( w \to w_\nu \), hitting the negative axis when it reaches it. It continues to move upwards as \( w \) decreases further, which produces two simple negative roots.

The behaviour of the roots of the actual dispersion relation is also shown in Figure B.1. Defining the \( \gamma_n \) themselves as having positive real parts (so that \( \gamma_{-2} = \gamma_{-1}^* \) is in the fourth quadrant for \( \omega > \omega_\nu \)), the two non-real roots \( \gamma_{-1} \) and \( -\gamma_{-2} \) approach the positive imaginary axis to form a pure imaginary double root as \( \omega \to \omega_\nu \); two simple imaginary roots are then produced as \( \omega \) becomes more negative (\( \gamma_1 \) being the root with smaller modulus). In addition, since \( f_0 \) is even, an equivalent process takes place in the lower half plane.

### B.2 Finite Depth Roots

The finite depth dispersion relation for the left-hand sheet of ice is

\[
f_0(\kappa) = \frac{1}{\kappa} \tan \kappa H - \Lambda_0(\kappa) = 0,
\]

where \( \Lambda_0 = \kappa^4 + \omega \). Since \( f_0 \) is even, the negative of any zero is also a zero, and so we will only investigate the location of those on the positive real axis or with imaginary part greater than zero.

For \( \kappa > 0 \), as \( \kappa \) increases from 0, \( \coth(\kappa H)/\kappa \) decreases from \( +\infty \), while \( \Lambda_0 \) increases from \( \omega \), so there will always be exactly one root \( \gamma_0 \) on the positive real axis. Additional roots may be found along the imaginary axis by studying the properties of

\[
p_H(w) = H^4 w \sin w \times f_0(iw/H) = (w^4 + \omega H^4)w \sin w + H^5 \cos w,
\]

where \( \omega = -i\kappa H \). Since \( p(n\pi) = (-1)^n H^5 \) \((n = 0, 1, 2, \ldots)\) and is continuous for all \( w \), it will always have at least one root \( w_n \) in each interval \( I_n = \pi \times (n-1, n) \) \((n = 1, 2, \ldots)\). Moreover, the roots \( w_n \) must satisfy the relation

\[
\tan w_n = -\frac{H^5}{(w_n^4 + \omega H^4)w_n}, \quad (B.1)
\]

so as \( w_n \) gets large, \( \tan w_n \to 0 \), or equivalently \( w_n \to n\pi^- \) (since for large \( w_n \) the right hand side of Equation B.1 will be negative, even if \( \omega < 0 \)). The error \( \varepsilon = n\pi - w_n \) can
also be estimated for large $w_n$ by expanding both sides of (B.1) as Taylor series and ignoring the terms of order greater $\varepsilon^2$. This gives

$$\varepsilon \approx \frac{(\tilde{\gamma}_n^4 + \omega)|\tilde{\gamma}_n|}{(\tilde{\gamma}_n^4 + \omega)^2|\tilde{\gamma}_n|^2 - (5\tilde{\gamma}_n^4 + \omega)/H}.$$  

(B.2)

where $\tilde{\gamma}_n = n\pi i/H$ are the zeros of $\tanh \kappa H$. For small values of $H$ or large values of $n$ it is clear that $\varepsilon$ is vanishing like $|\gamma_n|^{-5}$. Thus we can say that as $n \to \infty$, or as the water becomes shallower, the $w_n$ become closer and closer to $n\pi$, and consequently that the $\gamma_n$ also become closer and closer to the $\tilde{\gamma}_n$.

At this point we can say something about how the shallow water roots situation might arise from the finite depth one. As $H \to 0$, the imaginary zeros of $f_0(\kappa)$ are becoming closer and closer to the $\tilde{\gamma}_n$ ($n\pi i/H$, $n = 1, 2, \ldots$), which are themselves becoming infinite. If we suppose that in addition to the real root $\gamma_0$ and the imaginary ones that we know about, $f_0(\kappa)$ has two additional zeros $\gamma_{-1}$ and $\gamma_{-2}$, then the shallow water case could be reproduced in the limit. We might also anticipate that like their shallow-water counterparts, those roots will be complex for most periods, but may become additional imaginary roots for lower periods. It will be shown that this is indeed the case. First, however, we will prove their existence formally, and then enter into a more numerical discussion of their locations. Again, such a discussion is not particularly practical in that most of the phenomena described only occur at periods where the thin plate model might break down. Nevertheless, there are some exceptions, and the aforementioned phenomena are themselves quite fascinating.

Now, the number of zeros that a given function has in a particular region may be determined using the argument principle. Evans and Davies (1968) and Chung and Fox (2002a) applied that principle to effectively show that for a large enough choice of $N$, $p_H$ has $2N + 6$ zeros inside the square with corners $\pm(N + \pi/4) \times (1 + i)$. The $N$ real $w_n$ and their negatives, and $\pm w_0$, which are imaginary, account for $2N + 2$ of these roots, leaving four still to be located. Since it is easily shown that $w_0$ is a simple root ($f'_0(\gamma_0) < 0$), we have verified that $p_H$ must have two complex roots, $-w_{-2}$ in the first quadrant, and its complex conjugate $w_{-1}$, or else two additional real roots. In that case we will distinguish between them by choosing $w_{-1}$ to have a larger modulus than $w_{-2}$.
Figure B.2: Small period behaviour of the roots of the finite depth dispersion relation for ice-coupled waves, \( f_0(\kappa) = \coth \kappa H/\kappa - \Lambda_0(\kappa) \), where \( \Lambda_0(\kappa) = \kappa^4 + w \). In particular, this figure demonstrates the existence of double roots to the dispersion relation. The left hand plots show the behaviour of the function \( \tilde{p}_H(w) = p_H(w)/(w^5 + H^5) \) for real \( w \) for decreasing values of \( w \) (shorter periods and/or larger ice thicknesses). \( \tilde{p}_H \) has the same zeros as \( p_H(w) = H^4 w \sin w \times f_0(iw/H) \) but does not grow as quickly for large \( w \); if \( w_n \) is a root of \( p_H \) (or \( \tilde{p}_H \)), then \( \gamma_n = iw_n/H \) is a root of \( f_0 \). Note the formation of double zeros near \( w = -1.51 \) and between \(-60\) and \(-95\). The right hand plots show the locations of the \( w_n \) for the same values of \( w \). Arrows on two of those plots indicate the direction that the complex roots move in as \( w \) decreases. The formation of one double zero can be seen in them also as the two complex roots shown move onto the real axis and split to form a pair of simple roots; another double root is formed as one of the newly-formed single roots reaches one of the original real roots creating another pair of complex roots. The value of the nondimensional water depth used is \( H = 1 \).

Their locations can be narrowed down further by repeating the application of the argument principle, again following Chung and Fox (2002a), but this time to the triangle with vertices \( 0 \) and \((N + \pi/4) \times (1 \pm i)\) finding that \( p_H \) has \( N + 2 \) zeros inside that region. This implies that \( w_{-1} \) and \(-w_{-2} \) have arguments between \( \pm \pi/4 \). Since the \( \gamma_n \)
are related to the \( w_n \) by \( \gamma_n = iw_n/H \), \( \gamma_{-1} \) and \( \gamma_{-2} \) must be closer to the imaginary axis than they are to the real one.

Figure B.2 shows the behaviour of the \( w_n \) with decreasing period (or as \( \varpi \) becomes more negative) for \( H = 1 \), and also how the shape of the \( p_H \) curves evolve to produce this behaviour. At \( \varpi = -1.1 \), \( w_{-1} \) and \( -w_{-2} \) are still complex, and the first three real roots are extremely close to the zeros of \( \tan w \) at \( \pi, 2\pi \) and \( 3\pi \). However, \( p_H \) has a minimum only slightly above the real axis at about \( w = 0.28\pi \), which proceeds to become lower and lower as \( \varpi \) decreases past \(-1.3 \) and \(-1.51 \). As this happens, \( w_{-1} \) and \( -w_{-2} \) also move closer and closer to the real axis.

When \( \varpi \) reaches about \(-1.54 \), \( p_H \) touches the real axis and \( w_{-1} \) and \( -w_{-2} \) meet to form a double root which quickly separates to form two real roots as \( \varpi \) decreases further and the minimum in \( p_H \) drops below the real axis. As \( \varpi \) sinks past \(-8 \) and \(-60 \), this minimum deepens and become more symmetrical in the interval \( I_1 \). As this happens, \(-w_{-2} \) moves towards zero while \( w_{-1} \) moves towards \( w_1 \) at the other end of the interval. Note that at the same time the maximum which occurs at about \( w = 3\pi/4 \) when \( \varpi = -8 \) has also been moving to the right and moving closer to the real axis.

The maximum crosses the real line as \( \varpi \) decreases past \(-85 \), at the same time as \( w_{-1} \) meets \( w_1 \) to form another double root and lift off the real axis to produce the situation shown when \( \varpi = -95 \). This cycle of roots moving onto and off the real axis is repeated in every interval as the period becomes shorter and shorter.

The behaviour of the complex roots described and illustrated above for \( H = 1 \) is typical for all values of \( H \) less than \( H_1 \approx 2.40 \). As \( H \) increases towards that value, the two double roots in \( I_1 \) move closer and closer together until they meet and form a triple root. For \( H > H_1 \) there are no multiple roots in \( I_1 \).
Figure B.3: Demonstration of the existence of triple roots of the finite depth dispersion relation for ice-coupled waves $f_0(\kappa)$. $\tilde{p}_H(w)$ is defined in Figure B.2; its zeros $w_n$ are related to $\gamma_n$, the zeros of $f_0$, by $\gamma_n = iw_n/H$. The left hand plot shows the presence of a point of inflection in the graph of $p_H$ for real $w$ and for $w = 0.087$ at one of the real $w_n$. Consequently $\tilde{p}_H$ has a triple point at that point, and $f_0$ itself must have a triple root on the imaginary axis. The right hand plots show the location of the roots of $\tilde{p}_H$ for values of $\omega$ around 0.087, showing the two complex roots moving towards the real line, meeting one of the original real roots to form the triple root and immediately becoming complex again. The direction of their motion as $\omega$ decreases is indicated on the upper and lower figures. The value of the nondimensional water depth used is $H = 2.40$.

Figure B.3 shows the behaviour of the roots as $\omega$ decreases when $H = H_1$, confirming the formation of a triple root at that water depth. The left hand plot clearly shows the presence of a point of inflection when $\omega \approx 0.087$, while the series of plots on the right show the two complex roots merging with $w_1$ on the real axis before immediately becoming complex again. $w_{-1}$ and $-w_{-2}$ will become real again when $\omega \approx -6.35$, forming a double root in $I_2$ and from there on exhibiting the same pattern as observed
for $H = 1$.

In order to study this phenomenon of multiple roots further, and to describe the situation for larger water depths, let us attempt to quantify how the locations of the double roots, and the values of $w$ at which they occur, vary as $H$ changes. Suppose that an imaginary root does have multiplicity two. In that case we would have $f_0(\gamma_n)$ and $f'_0(\gamma_n) (n = -1, -2)$ both vanishing. If $w = -i\gamma_n$, $p_H(w) = p'_H(w) = 0$ and we can eliminate $w$ to give a relationship between $w$ and $H$ alone.

This is achieved by writing

$$-wH^4 = w^4 + \frac{H^5}{w \tan w} = \frac{(5w^4 - H^5) \tan w + w^5}{\tan w + w},$$

(B.3)

or

$$q_H(w) = 2w^4(1 - \cos 2w) - H^5(1 + \sin 2w/2w) = 0.$$  

(B.4)

The zeros of this function are plotted as a function of $H$ in Figure B.4a. The most striking feature of the curves is that each interval contains a certain extremal value of $H$ at which $dH/dw = 0$. Let $H_n$ be the extremal value in the $n^{th}$ interval. If $H > H_n$, then $p_H$ contains no double roots in either $I_n$ itself or any of the other intervals with indices smaller than $n$. If $H < H_n$, it has two zeros in $I_n$, one corresponding to when $w_1$ and $w_2$ move onto the real axis, and the other to when they move off it. If $H = H_n$ there is only one point in $I_n$ at which a double root forms. Differentiating (B.4) with respect to $w$, it can be seen that at such points $q_H'(w) = q_H(w) = 0$, and so the root $w$ is actually a double root of $q_H$ and is consequently a triple root of $p_H$. This was discussed above with reference to $H_1$.

The solid and chained lines in Figure B.4a are used to provide a correspondence between the two curves for each interval in Figure B.4b, which shows the values of $w$ at which each double root forms for a given $H$. The left hand pair of curves corresponds to $I_1$, the second pair from the left corresponds to $I_2$, and so on. Comparing the two graphs it can be seen that the root in an interval with smaller modulus always corresponds to a more positive value of $w$ (a larger period) than the root with larger modulus. Thus, the pattern observed in Figure B.2 can be said to be typical—the two

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Figure B.4: (a) The first and second double root of $p_H(w)$ (which correspond to simple roots of $q_H(w)$) in each interval $I_n$ are plotted against the nondimensional water depth $H$ as solid and chained lines respectively. ($w/\pi = n$ corresponds to the greater endpoint of $I_n$, and the first root is initially defined as the closest root to zero.) Note that for small $H$ there are always two double roots inside each interval but as $H$ increases these roots become closer and closer together before meeting and becoming complex. Points where the solid and chained lines meet in an interval correspond to triple roots of $p$ (points of inflexion for $p$); if the roots of $q_H(w)$ in a given interval have become complex then that interval contains no double roots. Also note that the first double root in the first interval is well approximated for values of $H$ less than about 0.5 by the dashed line, which corresponds to the point at which the complex shallow water roots become pure imaginary for that value of $H$. (b) plots the value of $w$ at which double roots occur. As in (a), solid lines correspond to the first double root in $I_n$, and chained lines to the second—note that the first root for a given $H$ always occurs at a greater value of $w$ than the second. The $n^{th}$ curve from the left corresponds to $I_n$, while the apex of each curve corresponds to the triple root in that interval. The triple root in $I_1$ occurs when $H = 2.40$ and $w = 0.08$ (a wave period of 2.64 s when $h_0 = 1$ m). The dashed line plots the value of $w$ at which the complex shallow water roots become imaginary—again for $H \lesssim 0.5$, it gives a good approximation to the point at which the first double root in the first interval occurs. For reference, dotted lines described by $w = 0$, $-0.35$, $-0.88$, and $-2.21$ (respectively from top) are also plotted. For 1-m-thick ice these correspond to periods of 1.9 s, 0.1 s, 0.01 s and 1 ms, which implies that we really only need to allow for double roots in the first interval, or equivalently when $H \lesssim 2.40$. 
complex roots move onto the real axis at one point in $I_n$, and then $w_{-1}$ moves to the right before meeting $w_n$ to form two more complex roots.

The dashed line in Figure B.4a is included for interest and compares the point at which the complex shallow water roots become imaginary with the first double root in $I_1$, and provides quite a good approximation for that root for $H$ less than about 0.5. Similarly, in Figure B.4b the dashed line corresponds to $\omega_8$ and also approximates the value of $\omega$ when the first double root forms for small water depths quite well.

The dotted lines in Figure B.4b are reference lines, which for 1-m-thick ice correspond to periods of 1.9 s ($\omega = 0$), 0.1 s ($\omega = -0.35$), 10 ms ($\omega = -0.88$) and 1 ms ($\omega = -2.21$). They are included to give an idea of when we need to be concerned about double and triple roots when solving the dispersion relation for physical wave periods.

The latter three periods are obviously far too small for the thin plate model for the ice to hold, and so for practical purposes there is only a very small range of water depths where we actually have to worry about the complex roots becoming imaginary. In fact, if we restricted ourselves to values of $\omega \gtrsim 0.087$, which for 1-m-thick ice corresponds to periods greater than about 2.64 s, we could say that we need never consider them.

If we do wish to calculate results for smaller periods, however, we would need to allow for multiple roots. Fortunately, though, we would generally only need to do this for a small range of values of $H$. If our minimum period was about 1.9 s, so that $\omega \geq 0$, multiple roots could only occur if $2.34 \leq H \leq H_1 \approx 2.40$. (This same range would also apply if we neglected the inertia term in the Euler-Bernoulli equation, so that $\mu = 0$ and $\omega = \lambda - \mu$ would be positive for all periods.) The lower limit for this interval would have to be decreased to about 1.50 if we required scattering results for periods down to 0.1 s. (This is the lower limit in most graphs presented in this thesis, although usually only infinite depth results are presented in which $H = 5$ is used if the infinite depth results are not calculated directly.)
B.3 Infinite Depth Roots

Letting \( H \to \infty \) in the finite depth dispersion relation (for real \( k \) and thus positive real \( \kappa \)) means that the dispersion relation becomes \( f_0(\kappa) = 1/\kappa - \Lambda_0 \), which has zeros when the polynomial \( p_\infty(\kappa) = -\kappa f_0(\kappa) = \kappa^5 + \omega \kappa - 1 \) does. The roots of this equation were presented for cases equivalent to \( \omega > 0 \) by Squire and Dixon (2000). Figure B.5 reproduces their figure.

![Schematic diagram showing the possible locations of the roots of the infinite depth dispersion relation \( 1/\kappa - \kappa^4 - \omega = 0 \) for positive values of the parameter \( \omega \). There is one positive real root \( \gamma_0 \) (indicated by a circle), and four complex roots (indicated by crosses). The complex roots consist of two complex conjugate pairs, \( \gamma_{-1} \) and \( \gamma_{-2} \) in the right hand half plane (\( \gamma_{-1} \) is in the first quadrant), and \( \gamma_1 \) and \( \gamma_2 \) in the left hand half plane (\( \gamma_1 \) is in the second quadrant). The shaded regions show the possible range of arguments that the complex roots may take for \( \omega > 0 \): \( \pi/4 < |\text{Arg}\gamma_j| \leq 2\pi/5 \) for \( j = -1, -2 \), and \( 3\pi/4 < |\text{Arg}\gamma_j| \leq 4\pi/5 \) for \( j = 1, 2 \). As \( \omega \) becomes negative \( \gamma_{-1} \) and \( \gamma_{-2} \) move towards the imaginary axis, while \( \gamma_1 \) and \( \gamma_2 \) move towards and eventually onto the negative real axis.](Image)

The bounds shown for the arguments of the complex roots may be deduced by considering the limits as \( \omega \to 0 \) and \( \omega \to \infty \). In the former case the roots satisfy
\[ \kappa^5 - 1 = 0; \text{ in the latter the transformed roots } \bar{\kappa} = \kappa/|\omega|^{1/4} \text{ satisfy } \bar{\kappa}^5 + \bar{\kappa} = \varepsilon, \]
where \( \varepsilon = |\omega|^{-5/4} \rightarrow 0. \) Thus, as \( \omega \) decreases from \( \infty \) to 0, \( \gamma_{-1} \) moves from the line \( \text{Arg}[^{'}] = \pi/4 \) to \( \text{Arg}[^{'}] = 2\pi/5, \) while \( \gamma_1 \) travels from the line \( \text{Arg}[^{'}] = 3\pi/4 \) to \( \text{Arg}[^{'}] = 4\pi/5. \) \( \gamma_0 \) is asymptotically zero for large \( \omega, \) and it increases to 1 as \( \omega \) decreases to 0. Also note that \( |\gamma_j| \rightarrow \infty \text{ for } j \neq 0 \text{ as } \omega \rightarrow \infty. \)

In this section, we demonstrate the behaviour of the infinite depth roots as \( \omega \) becomes more negative. In particular, we will show that the complex roots \( \gamma_1 \) and \( \gamma_2 \) actually move onto the negative real line when \( \omega \) becomes less than or equal to \( \omega_\infty = -5/4^{4/5} \approx -1.65, \) a proposition which follows easily by noting that \( p_\infty \) and its derivative \( p'_\infty(\kappa) = 5\kappa^4 + \omega \) are simultaneously zero when \( \kappa = \kappa_\infty = -1/4^{1/5} \) and \( \omega = \omega_\infty. \)

Figure B.6 plots the positions of the infinite depth roots for three different values of \( \omega: \) (a) \( \omega_\infty/2, \) (b) \( \omega_\infty \) and (c) \( 2\omega_\infty. \) It shows that a double root does indeed exist when \( \omega = \omega_\infty, \) and that it forms in the same way that the double roots of the shallow water and finite depth dispersion relations form on the imaginary axis. In this case \( \gamma_1 \) and \( \gamma_2 \) merge on the real axis, and separate again to form two simple zeros as \( \omega \) decreases. The more negative root, which we shall define as \( \gamma_2, \) becomes increasingly more negative as \( \omega \) decreases further, while the other root \( \gamma_1 \) increases towards 0. It also shows \( \gamma_0 \) getting larger, as do \( \gamma_{-1} \) and \( \gamma_{-2} \) as they move towards the imaginary axis.

The behaviour of all the roots as \( \omega \rightarrow -\infty \) may be shown formally by considering the same transformation that we used when considering their behaviour when \( \omega \rightarrow \infty. \) For large, negative \( \omega \) the roots satisfy \( \bar{\kappa}^5 - \bar{\kappa} = \varepsilon. \) Thus, in agreement with the results observed in Figure B.6, four roots become infinitely large—two, \( \gamma_0 \) and \( \gamma_2 \) travelling on the real axis towards \( \pm \infty, \) while another two, \( \gamma_{-1} \) and \( \gamma_{-2} \) travel asymptotically towards \( \pm \infty. \) \( \gamma_1 \) continues to move along the negative real axis towards 0, also in agreement with the figure.
Figure B.6: Small period behaviour of the roots of the infinite depth dispersion relation for ice-coupled waves. Demonstration of the existence of negative real roots to the infinite depth dispersion relation for ice-coupled waves. The figures show the location of the roots of the infinite depth dispersion relation when \( \omega \) takes values of (a) \( \omega = \omega_\infty / 2 \), (b) \( \omega = \omega_\infty \), and (c) \( \omega = 2\omega_\infty \). Note that the two roots in the left hand half plane move onto the real axis to form one double root as \( \omega \to \omega_\infty \), before separating to form two simple roots—one of which becomes more negative while the other moves towards zero. Their direction of motion as \( \omega \) becomes more negative is indicated with arrows on the left hand and right hand figures. The complex roots in the right hand half plane move asymptotically towards the imaginary axis as \( \omega \) decreases, with increasing modulus.

That \( \gamma_{-1} \) and \( \gamma_{-2} \) become closer and closer to the imaginary as \( \omega \to -\infty \) might have been expected from the behaviour of the finite depth roots for small periods, but the behaviour of \( \gamma_1 \) and \( \gamma_2 \) could not have been guessed by considering the finite depth dispersion relation alone. In particular, the loss in symmetry of the roots is quite unexpected. However, we can start to recover the lost roots in the left hand half plane by noting that \( \tanh \kappa H \to -1 \) for \( \Re[\kappa] < 0 \) as \( H \to \infty \), which means that they will become the solutions to \( q_\infty = -p_\infty(-\kappa) = \Lambda \kappa + 1 = 0 \), not the roots of \( p_\infty \) itself. The remaining roots of both \( p_\infty \) and \( q_\infty \) can be recovered from the finite depth situation by also considering the equation \( \Lambda \kappa \tanh \kappa H + 1 = 0 \). (The finite depth dispersion relation rearranges to \( \Lambda \kappa \tanh \kappa H - 1 = 0 \).) This equation still has infinitely many imaginary roots, but instead of having three roots in the left hand half plane it only has two,
which become the infinite depth roots $\gamma_1$ and $\gamma_2$ as $H \to \infty$. Moreover, these two roots also move onto the real axis when $\omega$ takes a certain negative value (depending on $H$), which explains the surprising behaviour of the aforementioned infinite depth roots.

Referring to equations (3.17) and (3.18), the presence of two additional real roots complicates the calculation of $g_0$ and $g_1$, as it introduces four additional real poles into those transforms as they are written. These poles could be dealt with in the same way as the poles at $k = \pm \alpha_0$ were. However, since values of $\omega < \omega_\infty$ only correspond to periods below about 2.1 ms (for 1-m-thick ice), it was not considered worthwhile to calculate $g$ for such values.
Appendix C

Computation of Infinite Depth Green’s Function

The normal incidence Green’s function for infinite depth has already been published (Squire and Dixon, 2000; Dixon and Squire, 2001b), the solution being expressed in terms of the auxiliary sine and cosine integrals, or equivalently in terms of the exponential integral (see Appendix C.2.3). Similarly, the finite depth Green’s function was derived by Evans and Porter (2003b) and was also presented in Section 3.1 of this thesis for both normal and oblique incidence. However, at present, the required derivatives \( g^{(n)}(x) \) of the infinite depth Green’s function for oblique incidence may only be calculated exactly when \( x = 0^\pm \). Formulae for these quantities were published by Williams and Squire (2002) and are also derived below in Appendix C.2.2. Although the aforementioned derivatives may be calculated when \( x \neq 0 \) by Fast Fourier Transform, as shown in Section 3.2, this is often not the most efficient approach, especially when the Green’s function is only required to be evaluated at a few points. The following two sections outline two approaches that are more suitable for such a scenario.

The first uses a partial fractions decomposition to reduce them (the required derivatives of \( g \), or more specifically of \( g_1 \)) into the sum of five simpler transforms which each satisfy linear inhomogeneous second order ordinary differential equations (ODEs). The inhomogeneous term is the zeroth order modified Bessel function of the second kind, \( K_0 \), which has a standard series representation (Abramowitz and Stegun, 1965, p 375). Using a method that is similar to both the Method of Undetermined Coefficients (Boyce and DiPrima, 1997, p164 ff.) and the Method of Frobenius (Boyce and DiPrima, 1997, p 262 ff.), we search for solutions that have similar expansions to that of \( K_0 \), and obtain simple recurrence relations for the series coefficients. The series obtained are convergent for all values of \( x \), with less terms needed in the expansion when \( x \) is smaller.
A second approach, which uses a contour integral to transform the inversion integral into one that may be calculated more easily by numerical quadrature, is more efficient for larger values of \(x\). It is described below in Section C.2, but first we will introduce the series expansion method.

Note that in both methods it is assumed in their derivations that the poles \(\alpha_n\) have imaginary part greater than zero. However, it will be seen that the final expressions for \(g\) retain the same form in the limits as any poles that need to be are allowed to return to the real line. Consequently, all results hold for any value of \(\omega\).

### C.1 Series Expansion

As stated in the introduction to this appendix, the series representation of \(g_1(x)\) is obtained by breaking it up into the sum of five simpler transforms which each satisfy a different ODE. Section C.1.1 presents this decomposition and derives the ODEs that they satisfy. Section C.1.2 solves these ODEs by proposing series solutions of a particular type, which are similar to the series expansion of the inhomogeneous term (a modified Bessel function of the second kind).

#### C.1.1 Ordinary Differential Equations

From (3.18), the Fourier transform of the \(2n\)-th derivative of \(g_1\) may be expanded by partial fractions as

\[
(-ik)^{(2n)} \hat{g}_1(k) = -\frac{1}{\pi i} \sum_{m=-2}^{2} A_m \gamma_m (i\alpha_m)^{2n+1} \hat{g}_1m(k),
\]

where the \(A_n\) were defined in Section 3.2 as \(A_n = -\gamma_n^2/\alpha_n(4\gamma_n^5 + 1)\), and

\[
\hat{g}_{1n}(k) = \frac{\pi}{\kappa(\alpha_n^2 - k^2)}.
\]

Multiplying this by \(\alpha_n^2 - k^2\) gives the Fourier transform of the ODE

\[
(\partial_x^2 + \alpha_n^2)g_{1n}(x) = K_0(l|x|),
\]
where $K_0$ is a zeroth order modified Bessel function of the second type (Abramowitz and Stegun, 1965). For $|\text{Arg}(w)| < \pi$, it has the expansion

$$
K_0(w) = -\left( \log(w/2) + \gamma_e \right) I_0(w) + \sum_{m=1}^{\infty} \chi_m \frac{(w/2)^{2m}}{(m!)^2}
$$

$$
= \sum_{m=0}^{\infty} a_m w^{2m} (d_m - \log w),
$$

where $\gamma_e$ is the Euler constant, $\chi_m = \sum_{r=1}^{m} 1/r$, and $I_0$ is the modified Bessel function of the first kind (also given by Abramowitz and Stegun, 1965):

$$
I_0(w) = \sum_{m=0}^{\infty} \frac{(w/2)^{2m}}{(m!)^2} = \sum_{m=0}^{\infty} a_m w^{2m}.
$$

The coefficients $a_m$ and $d_m$ are given by $a_m = 1/2^{2m}(m!)^2$ and if $\chi_0 = 0$, $d_m = \log 2 - \gamma_e + \chi_m$, so the above series converges for all $w$. Thus we anticipate that series representations for the $g_{1n}$ that satisfy (C.3) will also converge for all $x$. The $a_n$ and $d_n$ themselves may be calculated most efficiently by the recurrence relations $a_n = a_{n-1}/4m^2$ and $d_m = d_{m-1} + 1/m$, with $a_0 = 1$ and $d_0 = \log 2 - \gamma_e$.

The $g_{1n}$ should also satisfy the initial conditions

$$
g_{1n}(0) - q_n = g'_{1n}(0) = 0,
$$

where the $q_n$ can be evaluated analytically as shown in Section C.2.2 below.

In addition, we know that even derivatives of $g_1$ are even functions, and that odd derivatives are odd functions (since their Fourier transforms are). Consequently values of $g_1^{(n)}$ for negative $x$ can be generated in an obvious fashion from their values at $-x$, and so we will assume $x > 0$ from here on.

### C.1.2 Trial Series Solution

We will solve equations (C.3) by using a variation of the Method of Frobenius to obtain individual series expansions for each of the $g_{1n}$. Letting $w = lx$, $\alpha = \alpha_n/l$, and $J(w) = \ell^2 (g_{1n}(x) - q_n \cos \alpha_n x)$, $J$ should now satisfy

$$
J''(w) + \alpha^2 J(w) = K_0(w) \quad \text{for } w > 0,
$$

$$
J(0^+) = J'(0^+) = 0.
$$
We can see from their transforms that each of the \( g_n \) are continuous, even, and have continuous first derivatives, but that they have logarithmic singularities in their second derivatives, and so we propose a trial solution for (C.8) of the form

\[
J(w) = \sum_{m=0}^{\infty} (b_m + c_m \log w) w^{2m+2},
\]

which clearly satisfies (C.8b). This has first derivative

\[
J'(w) = \sum_{m=0}^{\infty} ((2m + 2)b_m + c_m + (2m + 2)c_m \log w) w^{2m+1},
\]

enabling odd derivatives of \( g_n \) to also be produced once the previous even one has been calculated. Proceeding to find \( J \), however, we now substitute our trial solution into the left hand side of (C.8a), giving

\[
J''(w) + \alpha^2 J = 2b_0 + 3c_0 + 2c_0 \log w
+ \sum_{m=1}^{\infty} ((2m + 2)(2m + 1)b_m + \alpha^2 b_{m-1} + (4m + 3)c_m) w^{2m}
+ \sum_{m=1}^{\infty} ((2m + 2)(2m + 1)c_m + \alpha^2 c_{m-1}) w^{2m} \log w.
\]

Consequently, \( J \) will satisfy (C.8) if our \( b_m \) and \( c_m \) satisfy the recurrence relations

\[ (2m + 2)(2m + 1)b_m + \alpha^2 b_{m-1} + (4m + 3)c_m = a_m d_m \quad \text{for } m \geq 1, \]  
\[ (2m + 2)(2m + 1)c_m + \alpha^2 c_{m-1} = -a_m \quad \text{for } m \geq 1, \]

and if \( 2c_0 = -a_0 \) and \( 2b_0 = a_0 d_0 - 3c_0 \) (i.e., if \( b_0 = (\log 2 + 3/2 - \gamma)/2 \) and \( c_0 = -1/2 \)). The latter relation (C.12b) implies that \( |c_m/c_{m-1}| \to 0 \) as \( m \to \infty \), and so the series involving those coefficients in (C.9) will converge for all \( w \). This also implies that the \( c_m \) themselves tend towards zero as \( m \to \infty \), and so we can also deduce from (C.12a) that the ratio \( |b_m/b_{m-1}| \) will do the same for large \( m \). Consequently, (C.9) will converge in entirety for all \( w \), although it will have a branch cut along the negative real axis due to the logarithmic factor.

**C.2 Contour Integration**

An alternative means of calculating \( g \) is by using contour integration to transform the inverse Fourier transform into an integral which is more easily evaluated numerically.
This technique is more efficient for larger values of \( x \), but it turns out that the resulting integral may actually be evaluated analytically when \( x = 0 \). Section C.2.1 explains the contour used and outlines the numerical method of approximation of the resulting integral for general \( x \) values, while Section C.2.2 derives the formula for that integral when \( x = 0 \).

Some additional simplifications can be obtained for normal incidence. We show that the integral for general \( x \) reduces to an expression involving the well-known and easily evaluated exponential integral function. This also acts as a further check on the oblique incidence results because the normal incidence results can be compared to those obtained by Dixon and Squire (2001b). This is done in Section C.2.3.

### C.2.1 Derivation and Numerical Approximation

For larger values of \( x \), when the series expansions for the \( g_{1n} \) take longer to converge, we would like to transform the integral

\[
g_{1n}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{i\kappa x} \, d\kappa}{\kappa(\alpha_n^2 - k^2)}
\]  

(C.13)

into one that is easier to evaluate numerically. To do this we close the contour in the upper half plane, allowing for the pole at \( k = \alpha_n \) and the branch cut along the line \( k = i \times [\mathbb{R}, \infty) \). Thus we can write

\[
g_{1n}(x) = g_{3n}(x) - \frac{\varepsilon_n \pi i}{2\alpha_n \gamma_n} e^{i\alpha_n x},
\]

(C.14)

where \( \varepsilon_n = \text{sgn}(\Re[\gamma_n]) \), and

\[
g_{3n}(x) = \int_{\mathbb{R}}^{i\infty} \frac{e^{i\kappa x} \, d\kappa}{\kappa(\alpha_n^2 - k^2)} = \int_{i\mathbb{R}}^{\infty} \frac{e^{-\kappa x} \, d\kappa}{\kappa(k^2 + \alpha_n^2)}.
\]

(C.15)

The \( \varepsilon_n \) results from defining the square root in \( \kappa \) as having a branch cut along the negative real axis. Consequently, \( \sqrt{\alpha_n^2 + \ell^2} = \varepsilon_n \gamma_n \) in the residue of \( g_{1n} \) at \( \alpha_n \). In addition, \( \kappa \) in the left hand integral in (C.15) is taken from the positive imaginary axis.

Now let us define the analogous sum to (C.1):

\[
g_3(x) = -\frac{1}{\pi} \sum_{n=-2}^{2} A_n \alpha_n \gamma_n g_{3n}(x) = -\frac{1}{\pi} \int_0^\infty \frac{ke^{-\kappa x} \, d\kappa}{\Lambda^2(k)k^2 + 1}.
\]

(C.16)
From (C.14), we now can write the relationship between \( g_1 \) and \( g_3 \):

\[
g_1(x) = g_3(x) + \frac{1}{2} \sum_{n=-2}^{2} A_n e^{i\alpha_n x}.
\]

(C.17)

Adding \( g_0 \) to \( g_1 \) then enables us to write \( g(x) = g_2(x) + g_3(x) \), where

\[
g_2(x) = i \sum_{n=-2}^{0} A_n e^{i\alpha_n x}.
\]

(C.18)

This confirms the results (3.20) and (3.21), which were obtained by taking the limit in (3.7) as \( H \to \infty \) directly.

\( g_3(x) \) as given by (C.16) is an integral that is more amenable to calculation by numerical quadrature than the inverse Fourier transform \( g_1 \). The integrand will never have any singularities on the real line and, because it is also \( O(e^{-kx}) \) as \( k \to \infty \), \( g_3 \) will be infinitely differentiable for \( x > 0 \). However, since the non-exponential factor is only \( O(k^{-9}) \) as \( k \to \infty \), its eighth derivative will be singular at the origin, as was mentioned in Section 3.1.1.

When \( \theta = 0 \), \( g_3 \) may be calculated analytically in terms of the exponential integral \( E_1 \) as described below in Section C.2.3. First, however, we will show how any derivative of \( g_3 \) may be efficiently calculated for oblique incidence.

Differentiating (C.16) \( n \) times gives

\[
g_3^{(n)}(x) = -\frac{(-1)^n}{\pi} \int_{t}^{\infty} \frac{k \kappa^n e^{-\kappa x} \, dk}{\Lambda^2(k)k^2 + 1},
\]

(C.19a)

\[
= -\frac{(-1)^n}{\pi} x^{8-n} e^{-lx} \int_{0}^{\infty} \frac{t^{1/2}(t + lx)^{n} (t + 2lx)^{1/2} e^{-t} \, dt}{t(t + 2lx)(t^2(t + 2lx)^2 + \omega x^4)^{1/2} + x^{10}},
\]

(C.19b)

where \( t = (\kappa - l)x \). This integral may now be found using generalised Gauss-Laguerre integration, using the weight function \( t^{1/2} e^{-t} \). The remainder of the integrand will now be relatively smooth, with the exception of the \( \sqrt{t + 2lx} \) term which has a branch cut on the line \( (-\infty, -2lx] \). However, we don't need to integrate over that singularity and the larger \( x \) gets the further it moves away from the region of integration. Hence, the integral will converge faster for larger values of \( x \). Also note that (C.19b) shows that \( g_3^{(n)} \) decays exponentially as \( x \to \infty \).
C.2.2 Exact Calculation of Derivatives at the Origin

Equation (C.15) when \( x = 0 \) is

\[
q'_n = q_n + \frac{i\varepsilon_n}{2\alpha_n\gamma_n} = g^{3\gamma}(0^+) = I(\alpha_n) + I(-\alpha_n),
\]

where

\[
I(\alpha) = \frac{1}{2\alpha \int_{\infty}^{0} \frac{d\kappa}{k(\kappa - i\alpha)}} = -\frac{1}{\alpha} \int_{0}^{1} \frac{dw}{(\alpha - i\beta)w^2 - (\alpha + i\beta)},
\]

and we have made the transformation \( \kappa = l(1 + w^2)/(1 - w^2) \). If we also define \( \beta \) and \( \gamma \) by the relations \( \beta^2 = (\alpha + i\beta)/(\alpha - i\beta) \) and \( \beta = \gamma/(\alpha - i\beta) = (\alpha + i\beta)/\gamma \) (so that \( \gamma^2 = \alpha^2 + l^2 \)), (C.21) becomes

\[
I(\alpha) = -\frac{1}{\alpha(\alpha - i\beta)} \int_{0}^{1} \frac{dw}{w^2 - \beta^2} = -\frac{1}{2\alpha \gamma} \left( \int_{0}^{1} \frac{dw}{w - \beta} - \int_{0}^{1} \frac{dw}{w + \beta} \right)
\]

\[
= \frac{1}{2\alpha \gamma} (\log(1 + \beta) - \log(1 - \beta) - \log(\gamma + \beta) + \log(-\beta)).
\]

To simplify (C.22), we choose \( \beta \) to be the square root from the upper half plane (which also implies that \( 1 + \beta \) is in the upper half plane, and that \( \log(-\beta) = \log \beta - i\pi \)). \( 1 - \beta \) will then be in the lower half-plane so the quotient \( (1 + \beta)/(1 - \beta) \) corresponds to an anti-clockwise rotation of \( 1 + \beta \). Potentially this could rotate \( 1 + \beta \) across the branch cut in the logarithm on the negative real axis, but it can be shown explicitly that with our choice of \( \beta \) the aforementioned quotient remains in the upper half plane. Thus, the normal logarithm rules for quotients hold, and so (C.22) may be written

\[
I(\alpha) = \frac{1}{2\alpha \gamma} (\log \frac{1 + \beta}{1 - \beta} - \pi i) = \frac{1}{2\alpha \gamma} (\log \frac{\gamma + \alpha + i\beta}{\gamma - \alpha - i\beta} - \pi i).
\]

Note that the above expression is even in \( \gamma \), so it doesn’t matter after all which root we choose for it. Thus,

\[
q'_n = \frac{1}{2\alpha_n \gamma_n} \left( \log \frac{\gamma_n + \alpha_n + i\beta}{\gamma_n - \alpha_n - i\beta} - \log \frac{\gamma_n - \alpha_n + i\beta}{\gamma_n + \alpha_n - i\beta} \right) = \frac{1}{2\alpha_n \gamma_n} \log \Theta_n,
\]

where \( \Theta_n = (\gamma_n + \alpha_n)/(\gamma_n - \alpha_n) \). The latter step is justified since if \( \Im[\alpha_n] > 0 \) the argument of the first logarithm will be in the first quadrant, while the argument of the second logarithm will be in the second quadrant. This means that we can now write

\[
g^{2\gamma}_n(0^+) = -\frac{1}{\pi} \sum_{n=-2}^{2} A_n \alpha_n \gamma_n q'_n = -\frac{1}{2\pi} \sum_{n=-2}^{2} A_n (i\alpha_n)^{2\gamma} \log \Theta_m.
\]

Now, for small \( l \), \( \alpha_n \sim \varepsilon_n \gamma_n (1 - l^2/(2\gamma_n + O(l^4))) \), where \( \varepsilon_n = \text{sgn}(\Im[\gamma_n]) \), so at first glance it seems like the above expression will become singular as \( l \to 0 \). However, this
is not the case. If we define $A'_n = A_n \alpha_n / \gamma_n = -\gamma_n^2 / (4\gamma_n^5 + 1)$, $A_n \sim \varepsilon_n A'_n + O(l^2)$ and it is easily shown that for $n = 0, 1, 2, 3$ they satisfy the rules

$$
\sum_{m=-2}^{2} A'_m \gamma_m^{2n} = 0 \quad \text{for } n = 0, \ldots, 3,
$$

(C.26)

which are similar to those presented by Squire and Dixon (2000). Thus as $l \to 0$,

$$
g_3^{(2n)}(0^+) \sim -\frac{1}{2\pi} \sum_{m=-2}^{2} A'_m (1 + O(l^2)) \times \left( \log \left( \frac{\gamma_n^2 (1 + O(l^2))}{4} \right) - \log \frac{l^2}{4} \right)
$$

$$
\to -\frac{1}{2\pi} \sum_{m=-2}^{2} A'_m (i\gamma_m)^{2n} \log \gamma_m^2.
$$

(C.27)

### C.2.3 Normal Incidence Simplification

An alternative expansion to (C.20) for the $g_{3n}$ is

$$
2\gamma_n^2 g_{3n}(x) = \int_{l}^{\infty} \left( \frac{2}{k} - \frac{1}{k + i\gamma_n} - \frac{1}{k - i\gamma_n} \right) e^{-\kappa x} d\kappa
$$

$$
= 2K_0 (lx) - \int_{l}^{\infty} \left( \frac{1}{k + i\gamma_n} + \frac{1}{k - i\gamma_n} \right) e^{-\kappa x} d\kappa.
$$

(C.28)

Using the rules (C.26) to eliminate the Bessel function $K_0$, we can thus write the $2n$-th derivative of $g_3$ as

$$
g_3^{(2n)}(x) = -\frac{1}{\pi} \sum_{m=-2}^{2} A'_m (i\alpha_m)^{2n} g_{3m}(x)
$$

$$
= \frac{1}{2\pi} \sum_{m=-2}^{2} A'_m (i\alpha_m)^{2n} \int_{l}^{\infty} \left( \frac{1}{k + i\gamma_n} + \frac{1}{k - i\gamma_n} \right) e^{-\kappa x} d\kappa.
$$

(C.29)

Now, if we let $\theta \to 0$, $l \to 0$, $\alpha_n^2 \to \gamma_n^2$, and $\kappa \to k$ on the real line, (C.29) becomes

$$
g_3^{(2n)}(x) = \frac{1}{2\pi} \sum_{m=-2}^{2} A'_m (i\gamma_m)^{2n} \left( e^{i\gamma_m x} E_1 (i\gamma_m x) + e^{-i\gamma_m x} E_1 (-i\gamma_m x) \right),
$$

(C.30)

where $E_1$ is the well-known exponential integral function (Abramowitz and Stegun, 1965, p 228 ff.) defined as follows

$$
E_1(w) = \int_{w}^{\infty} \frac{e^{-t}}{t} dt = -\gamma - \log w - \sum_{n=1}^{\infty} \frac{(-w)^n}{n n!} \quad \text{for } |\text{Arg}(w)| < \pi.
$$

(C.31)

Note that, since no imaginary solutions exist for the infinite depth dispersion relation, $|\text{Arg}[\pm i\gamma_n x]| < \pi$, so the above formulae are valid. Also note that the $g_3^{2n}$ will be
everywhere bounded, despite the logarithmic singularities in $E_1$. This also follows from the rules (C.26), allowing us to deduce that taking the limit in (C.30) as $x \to 0^+$ gives

$$g_3^{(2n)}(0^+) = -\frac{1}{2\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{2n} \left( \log(i\gamma_m) + \log(-i\gamma_m) \right)$$

$$= -\frac{1}{\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{2n} \log \gamma_m - 2i(A'_{-2}(i\gamma_1)^{2n} + A'_2(i\gamma_2)^{2n}) \quad (C.32a)$$

$$= -\frac{1}{2\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{2n} \log \gamma_m^2. \quad (C.32b)$$

which agrees with (C.27). Recalling that $A_n \to \text{sgn}(\Im\gamma_n)A'_n$ as $l \to 0$, adding $g_2^{(2n)}(0^+)$ to (C.32a) and using (C.26) to simplify the product, then gives the following result for $g^{(2n)}(0^+)$ itself:

$$g^{(2n)}(0^+) = -\frac{1}{\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{2n} \log \gamma_m - 2i(A'_{-2}(i\gamma_1)^{2n} + A'_2(i\gamma_2)^{2n}). \quad (C.33)$$

The above result is unstable as $\gamma_2$ becomes negative and so is only reliable for $\omega > \omega_\infty$. For a result that holds for all $\omega$, use (C.32b) and the ordinary definition of $g_2$.

Odd derivatives of $g_3$ may also be obtained by differentiating (C.30) directly, again making use of (C.31) and also (C.26) to eliminate any $1/x$ type singularities produced. Thus,

$$g_3^{(2n+1)}(x) = \frac{1}{2\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{2n+1} \left( e^{i\gamma_m x} E_1(i\gamma_m x) - e^{-i\gamma_m x} E_1(-i\gamma_m x) \right), \quad (C.34)$$

and so we can write the general $n^{th}$ ($0 \leq n \leq 7$) derivative of $g_3$ as

$$g_3^{(n)}(x) = \frac{1}{2\pi} \sum_{m=-2}^{2} A'_m(i\gamma_m)^{n} \left( e^{i\gamma_m x} E_1(i\gamma_m x) + e^{i(n\pi - \gamma_m x)} E_1(-i\gamma_m x) \right). \quad (C.35)$$

Recall that from (3.13) that we already know what values $g^{(2n+1)}$ will take at the origin so there is no need to take the limit in (C.34) as $x \to 0^+$. In addition, we can also check the above results numerically by letting $l \to 0$ and $\kappa \to k$ in (C.19a) and using generalised Gauss-Laguerre integration (with weight function $k^{n+1}e^{-kx}$).

A further confirmation of (C.35) can be obtained by checking the $n = 0$ result against the equivalent result of Dixon and Squire (2001b). Let us denote their nondimensional length variables as $x', z', \xi'$ and $\zeta'$, which equate to $x/a, z/a, \xi/a$ and $\zeta/a$.
respectively. In their expression (6) for the Green's function in that scheme, which we shall denote $G_{SD}$, the integral when $x' > \xi'$ and $\zeta', \zeta' \approx 0$ is obtained by doubling the real part of equation (14b). Doing so, and then differentiating with respect to $\zeta'$ followed by $x'$ gives the result

$$g(x - \xi) = \frac{1}{a^2} \partial_x^2 \partial_{\xi'} G_{SD}(x' - \xi', 0, 0) + iA_0 \cos \gamma_0 x$$

$$= i \sum_{m=-2}^{0} A'_m \varepsilon_m e^{i \gamma_m x} + \frac{1}{\pi} \sum_{m=-2}^{2} A'_m g_{\cos}(\varepsilon'_m \gamma_m x), \quad (C.36)$$

where $\varepsilon'_n = \text{sgn}(\Re[\gamma_n])$. The cosine term needs to be added to allow for the difference between taking the Cauchy Principal value of the integral (3.4) and letting the poles $\pm a_0$ approach the real line from the upper or lower half plane as we did, while $g_{\cos}$ is the auxiliary cosine function (Abramowitz and Stegun, 1965) which may be written as

$$g_{\cos}(w) = \int_0^\infty \frac{\cos t}{w + t} \, dt = \frac{1}{2} (e^{iw} E_1(iw) + e^{-iw} E_1(-iw)), \quad \Re[w] > 0. \quad (C.37)$$

Since the first summation in (C.36) is obviously $g_2(x)$, the second sum must be $g_3(x)$. In light of the definition of $g_{\cos}$, the result presented by Dixon and Squire (2001b) clearly agrees with (C.35).
 Appendix D

Energy Conservation Theorem

We wish to derive a simple relationship between the reflection and the transmission coefficients that will serve as a check to see whether the energy contained in the incoming wave is conserved as it is scattered by any irregularities it encounters in the ice sheet. Such a relationship was first derived by Evans and Davies (1968), although their formula for the intrinsic admittance was incorrect for oblique angles of incidence (the error was corrected by Fox and Squire, 1994). However, the derivation below is a variation of the one given by Balmforth and Craster (1999).

Let $\Omega = \{(x, z) \mid -\infty < x < \infty \ \& \ 0 < z < H\}$, and let $S_\Omega$ be the surface of the infinitely long (unit width) cuboid $\Omega \times [0, 1]$. From Stoker (1957), the (nondimensional) energy flux per period through $S_\Omega$ is proportional to

$$\mathcal{E} = \int_0^T \int_{S_\Omega} \Phi_t(x, y, z, t) \partial_n \Phi(x, y, z, t) dS dt$$

$$= \frac{\pi}{2i} \int_{S_\Omega} \left( \partial_n (\phi^*(x, z) e^{-i\nu y}) \phi(x, z) e^{i\nu y} - \partial_n (\phi(x, z) e^{i\nu y}) \phi^*(x, z) e^{-i\nu y} \right) dS, \quad (D.1)$$

where $\partial_n$ refers to the derivative with respect to the outward normal and $dS$ indicates that the integration over $S_\Omega$ is to be done with respect to surface area. We have also used our assumption (2.7) about the form of $\Phi$ to put $\mathcal{E}$ in terms of $\phi$ instead. It is easily seen that the integrals over the $\Omega \times \{1\}$ and the $\Omega \times \{0\}$ planes in (D.1) will cancel each other out, while the integrals over the remaining planes can be simplified into line integrals, giving an integral around the boundary of $\Omega$, $\partial \Omega$, with respect to arc length. Thus

$$\mathcal{E} = \frac{\pi}{2i} \oint_{\partial \Omega} (\partial_n \phi^*(x, z) \phi(x, z) - \partial_n \phi(x, z) \phi^*(x, z)) ds$$

$$= \frac{\pi}{2i} (\mathcal{E}_0 + \mathcal{E}_1), \quad (D.2)$$

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where $\mathcal{E}_0$ and $\mathcal{E}_1$ are given by

$$
\mathcal{E}_0 = \int_{-\infty}^{\infty} \left[ \phi_z^*(x,z)\phi(x,z) - \phi_z(x,z)\phi^*(x,z) \right] \frac{\partial}{\partial x} dx,
$$
(D.3a)

$$
\mathcal{E}_1 = \int_{0}^{H} \left[ \phi_z^*(x,z)\phi(x,z) - \phi_z(x,z)\phi^*(x,z) \right] \frac{\partial}{\partial z} dz.
$$
(D.3b)

A relationship between these two quantities can be obtained by applying Green's theorem to (D.2), and then applying (2.8a), which gives

$$
\mathcal{E}_0 + \mathcal{E}_1 = \iint_{\Omega} \left( \nabla_{xx} \phi^*(x,z)\phi(x,z) - \nabla_{zz} \phi(x,z)\phi^*(x,z) \right) dS
$$

$$
= \iint_{\Omega} \left( (\nabla_{zz} - l^2) \phi^*(x,z)\phi(x,z) - (\nabla_{xx} - l^2) \phi(x,z)\phi^*(x,z) \right) dS
$$

$$
= 0,
$$
(D.4)

implying that the net energy flux $\mathcal{E}$ through the surface of our cuboid is zero, i.e. any energy that enters it must also leave it or, alternatively, any energy that leaves it must somehow be replaced. We now proceed to use this fact to establish a simpler relationship between the reflection and transmission coefficients that can be checked more easily than (D.4) in its present form.

$\mathcal{E}_0$ may be simplified by substituting (2.8b) and (2.8d) into (D.3a) and integrating by parts to give

$$
\mathcal{E}_0 = \int_{-\infty}^{\infty} \left( \mathcal{L}(x, \partial_x)\phi_z(x,0)\phi_z^*(x,0) - \mathcal{L}(x, \partial_x)\phi_z^*(x,0)\phi_z(x,0) \right) dx
$$

$$
= \left[ \mathcal{E}(x, \partial_x)\phi_z^*(x,0) \right]_{x=-\infty}^{x=\infty} - \sum_{x_c \in X_c} \left[ \mathcal{E}(x, \partial_x)\phi_z^*(x,0) \right]_{x=x_c}^{x_c+},
$$
(D.5)

where the operator $\mathcal{E}(x, \partial_x)$ was defined in (4.12) as

$$
\mathcal{E}(x, \partial_x) = \mathcal{S}(x, \partial_x)\phi_z(x,0) - \phi_z(x,0)\mathcal{S}(x, \partial_x)
$$

$$
+ \phi_{xz}(x,0)\mathcal{M}(x, \partial_x) - \mathcal{M}(x, \partial_x)\phi_z(x,0)\partial_x,
$$
(D.6)

and $X_c$ is the set of points at which there are free edges or where either $D$ or $D'$ are discontinuous.

From (D.6), applying either set of edge conditions (2.12) or (2.13) at each point $x_c$ will make each term in the sum in (D.5) vanish. Indeed, any other set of energy conserving edge conditions would have the same effect. Using the asymptotic behaviour
of $\phi$ (cf. equation 4.4) in the terms in the limit as $x \to \pm \infty$, in addition to the edge conditions, $\phi_0$ reduces to

$$\phi_0 = 4i\alpha_0 D_0 \gamma_0^2 (1 - |R|^2) \phi'_0(0)^2 - 4i\alpha_0 D_2 \gamma_0^2 |T|^2 \phi'_0(0)^2. \quad (D.7)$$

In a similar way, $\phi_1$ can also be put in terms of $R$ and $T$ using (4.4). It can be written as

$$\phi_1 = 2i\alpha_0 (1 - |R|^2) I_{00} - 2i\alpha_0 (|T|^2) \dot{I}_{00}, \quad (D.8)$$

where

$$I_{mn} = \int_0^H \varphi_m(z) \varphi_n(z) dz, \quad \dot{I}_{mn} = \int_0^H \dot{\varphi}_m(z) \dot{\varphi}_n(z) dz.$$

Adjusting the rules presented by Evans and Porter (2003b), which were given earlier by Lawrie and Abrahams (1999) in an acoustics paper, the $I_{mn}$ and $\dot{I}_{mn}$ integrals can be written as

$$I_{mn} = -\frac{\delta_{mn}}{2\alpha_n A''_{nn}} - D_0 (\gamma^2_m + \gamma^2_n) \varphi'_m(0) \varphi'_n(0),$$

$$\dot{I}_{mn} = -\frac{\delta_{mn}}{2\alpha_n \dot{A}''_{nn}} - D_2 (\gamma^2_m + \gamma^2_n) \dot{\varphi}'_m(0) \dot{\varphi}'_n(0), \quad (D.9)$$

where the $A''_{nn} = -\Lambda_0^2 (\gamma_n) \gamma_n^2 / \alpha_n C_0 (\gamma_n)$ were given in Section 3.1, and the $\dot{A}''_{nn}$ are defined similarly as $\dot{A}''_{nn} = -\Lambda_0^2 (\gamma_n) \gamma_n^2 / \alpha_n C_2 (\gamma_n)$. ($C_j(\kappa) = H (\Lambda_0^2(\kappa) \kappa^2 - 1) + 5D_j \kappa^4 + \lambda - m_j \mu$.) Equations (D.9) may also be derived by taking the limits as $\kappa_n \to \gamma_n$ in (F.4a), or as $\kappa_n \to \dot{\gamma}_n$ in (F.4c).

Continuing, if the above rules (D.9) are used while combining (D.7) and (D.8), (D.4) becomes

$$0 = 2i\alpha_0 (1 - |R|^2) (I_{00} + 2D_0 \gamma_0^2 \varphi_0'(0)^2) - 2i\alpha_0 (|T|^2) (\dot{I}_{00} + 2D_2 \gamma_0^2 \varphi_0'(0)^2)$$

$$= \frac{1}{iA''_0} (1 - |R|^2) - \frac{1}{i\dot{A}''_0} |T|^2. \quad (D.10)$$

If we let $s = A''_0 / \dot{A}''_0$, then (D.10) simplifies to

$$|R|^2 + s |T|^2 = 1. \quad (D.11)$$

$s$ is called the intrinsic admittance, and allows for the fact that waves travelling beneath ice sheets of different properties have different energies associated with them. Allowing for that fact, (D.11) states that the energy flux inward with the incident wave must be balanced by the outward flux associated with the reflected and transmitted waves.
Also recall that in applying the edge conditions to (D.5) we have assumed there was no energy loss or gain at any discontinuities in the ice—if there was, (D.11) would have to be adjusted accordingly to allow for it.

Equation (D.11) provides an easily-implemented check that energy has been conserved by our solution. Although it is quite general (e.g. it does not distinguish between different sets of energy-conserving edge conditions) and it holding is no guarantee that \(|R|\) and \(|T|\) have actually been calculated correctly, nevertheless it does help check that they have not been calculated incorrectly and has proved extremely useful in locating programming and theoretical errors.
Appendix E

Proposed Solution Technique for a Pressure Ridge With a Submerged Keel

In this appendix we propose a solution method for the scattering by a pressure ridge that permits the keel of a pressure ridge to protrude into the water, as illustrated by Figures 6.1a and E.1. We model the ridge as a thin plate of variable thickness, and the solution method is a combination of the approaches of Dixon and Squire (2001b) and Williams and Squire (2004a). This is described in Section E.2. First however, we must adjust the equations that our velocity potential \( \Phi \) must satisfy slightly.

Figure E.1: Full geometry of a pressure ridge. The coordinate axes shown are displaced to the right to avoid clutter—the left hand limit of the ridge lies in the \( x = 0 \) plane, and the \( y \) axis points out of the page. Also, the ridge's width is \( a \), and its total thickness is given by \( h_1(x) \). However, in this appendix submergence is allowed for—the amount by which the keel protrudes into the water is given by the relation \( z = d(x) \). The ice is floating on sea water with a finite constant depth of \( H \).
E.1 Adjusted Equations and Boundary Conditions

This section is based on a personal communication with Porter (2005) on the effect of submergence, and presents the equations and boundary conditions that must be satisfied when submergence is allowed for. We still have the same relationship between $\Phi$ at the ice/water interface and the displacement $\eta$ as when there was no keel present

$$(L_{\text{plate}} + \rho_\omega g + m\partial_t^2)\eta_t - \rho_\omega \Phi_{tt} = 0,$$  

(E.1)

which comes from eliminating $P_t$ from the time-differentiated equations (2.1b) and (2.2). $\Phi$ must also still satisfy (2.1a), (2.1c), (2.1d), and (2.1f), and we retain the relationship $v = \nabla \Phi$. This equation should be applied at $z = d(x) + \eta(x, y, t)$, where $z = d(x)$ is the rest position of the bottom of the ridge (cf. Figure E.1).

However, the addition of the keel means that the kinematic condition (2.1e) must be adjusted slightly. Taking the total derivative of the equation of the surface with respect to time we obtain

$$\frac{dz}{dt} = d'(x) \frac{dx}{dt} + \eta_t + \eta_x \frac{dx}{dt} + \eta_y \frac{dy}{dt},$$  

(E.2)

or

$$\eta_t = \Phi_z - d'(x) \Phi_x - \eta_x \Phi_x - \eta_y \Phi_y.$$  

(E.3)

This condition can be approximated by neglecting nonlinear terms and applying it at $z = d$ instead of $z = d + \eta$ to give

$$\eta_t(x, y, t) = -s'(x) \partial_n \Phi(x, y, d, t),$$  

(E.4)

where

$$s(x) = \int_0^x \sqrt{1 + (d'(\xi)^2)} d\xi$$

is the arc length travelling along the keel profile (not including the lengths of vertical segments), $\partial_n = n^T \nabla$, and $n = (d'(x), -1)^T / s'(x)$ is the unit vector pointing normally into the keel. Writing the kinematic condition in terms of the normal derivative simplifies the application of Green's theorem in Section E.2.1. $n$ may be obtained by rotating the unit vector $s = (1, d'(x))^T / s'(x)$, which by inspection of Figure E.1 is a unit vector that is tangential to the keel profile.
Note that along segments of the keel where the ice is locally vertical, (E.4) becomes
\[ \partial_h \Phi(x, y, d, t) = - \lim_{s' \to -\infty} \eta(x, y, t)/s' = 0, \]
or
\[ \Phi_x(x, y, d, t) = 0. \quad (E.5) \]

Proceeding to nondimensionalize as in Section 2.2, and noting that we should also now apply (E.1) \( z = d \) instead of \( z = d + \eta \), we can rewrite the condition (2.8b) that \( \phi \) must satisfy at the surface as
\[ L(x, \partial_x)\chi(x) + \phi(x, d) = 0, \quad (E.6) \]
where \( \chi(x) = -s'(x)\partial_x\phi(x, d) \). Note that \( \chi \) reduces to \( \phi_z(x, 0) \) when submergence is neglected (i.e. when \( d(x) = d'(x) = 0 \)), reproducing (2.8b).

Along vertical sections of the keel (E.5) becomes
\[ \phi_x(x, d) = 0, \quad (E.7) \]
and at \( x = 0, x = a \), and at points within the ridge where \( D(x) \) and/or \( D'(x) \) are continuous the frozen edge conditions (2.12) should be applied, as described in the next section. In addition, \( \phi \) should also satisfy (2.8a), (2.8c) and (2.8d).

**E.1.1 Edge Conditions**

For a pressure ridge, the most appropriate set of edge conditions are the frozen edge conditions (2.4). As when submergence was neglected, these conditions must be applied at points \( x_c \in X_0^S \subset (0, a) \) where \( D(x) \) and/or \( D'(x) \) are discontinuous; now however, they must always be applied at \( x = 0, a \).

For notational convenience, let us extend \( \chi(x) \) outside the interval \((0, a)\) by defining it as simply being equal to \( \phi_z(x, 0) \). \( \eta \) is related to \( \chi \) by
\[ \eta_h = \Re \{ \chi(x)e^{i(y-\omega t)} \}, \]
so we can rewrite the nondimensional form of (2.4) by replacing \( \phi_z(x, 0) \) in (2.12) with \( \chi(x) \):
\[ \chi(x_c^+) = \chi(x_c^-), \quad (E.8a) \]
\[ \chi'(x_c^+) = \chi'(x_c^-), \quad (E.8b) \]
\[ M^+(x_c) - M^-(x_c) = S^+(x_c) - S^-(x_c) = 0, \quad (E.8c) \]
where, as in Section 4.1, the $M^\pm$ and $S^\pm$ are the bending moments and vertical edge forces edge forces. Now, however, they are given by

$$M^\pm(x) = \mathcal{M}(x^\pm, \partial_x)\chi(x), \quad S^\pm(x) = \mathcal{S}(x^\pm, \partial_x)\chi(x).$$

### E.2 Solution Method

Having stated the conditions that $\phi(x, z)$ must satisfy, we now proceed to find a solution. This is done by first using Green’s theorem to write $\phi$ in terms of the values it and its normal derivative take along the keel, and then by using a second Green’s function to eliminate $\partial_n\phi$ from the original expression for $\phi(x, z)$. That allows us to complete the solution by solving an integral equation in terms of $\phi$ alone.

#### E.2.1 Green’s Theorem

First let us deform the central contour shown in Figure 4.1 to travel around the now submerged keel; the left and right hand contours remain the same. Using the Green’s function presented in Chapter 3, applying Green’s theorem to each of the three regions, applying (2.8c) at $x = 0, a$, and combining the results gives us a similar expression to (4.34)

$$\psi(x, z) = \mathcal{E}(a^+, \partial_x; \phi)G_\zeta(x - a, z, 0) - \mathcal{E}(0^-, \partial_x; \phi)G_\zeta(x, z, 0) + I_\Gamma(x, z; \psi)
+ \mathcal{E}_0(0^-, \partial_x; \phi_0)G_\zeta(x, z, 0) - \mathcal{E}_0(a^+, \partial_x; \phi_0)G_\zeta(x - a, z, 0) \quad (E.9a)
= P_0^+ \mathcal{L}_{\text{edge}}G_\zeta(x - a, z, 0) - P_0^0 \mathcal{L}_{\text{edge}}G_\zeta(x, z, 0) + I_\Gamma(x, z; \phi), \quad (E.9b)$$

where

$$(P_\pm^\sigma)^T = \pm (p_0(x_0^\pm), p_1(x_0^\pm), M^\pm(x_0), S^\pm(x_0)), \quad I_\Gamma(x, z; \phi) = \int_{\Gamma} (\partial_n G\phi - G\partial_n \phi)ds,$$

and where the $p_j(x)$ have the same definitions as their original ones in Section 4.1 (recalling that $\chi(x) = \phi_0(x, 0)$ for $x$ outside the interval (0,a)): $p_0(x) = D(x)\chi(x)$, $p_1(x) = D(x)\chi'(x) - D'(x)\chi(x)$.

$\Gamma$ denotes the set of points along the interface between the ridge keel and the sea water, and (E.9b) follows from (E.9a) by noting firstly that $I_\Gamma$ is linear in its third argument, so $I_\Gamma(x, z; \psi) = I_\Gamma(x, z; \phi) - I_\Gamma(x, z; \phi_0)$; secondly that the last two terms of (E.9a) are equal to

$$I_0(x, z; \phi_0) = - \int_0^a (G_\zeta(x - \xi, z, 0)\phi_0(\xi, 0) - G(x - \xi, z, 0)\phi_0(\xi, 0))d\xi;$$
and thirdly that for \((x, z)\) in the fluid region
\[
\int_{\Omega_{\text{fluid}}} (\nabla_\xi^2 G\phi_0 - G\nabla_\xi^2 \phi_0) \, d\xi \, d\zeta = 0,
\]
so by applying Green's theorem to \(\Omega_{\text{keel}}\), the cross-section of the keel in the \(x-z\) plane, 
\(I_0(x, z; \phi_0) - I_1(x, z; \phi_0) = 0\).

Assuming that the only vertical segments of the keel are located at \(x = 0\) and \(x = a\), and applying the boundary conditions (E.7) on those segments, the integral \(I(x, z; \phi)\) is now split up into five different integrals: \(I(x, z; \phi) = \sum_{j=1}^{5} I_j(x, z)\), where the five \(I_j\) integrals are defined when \((x, z) \notin \Gamma\) as
\[
I_1(x, z) = \int_0^{d(0)} G_\xi(x, z, \zeta)\phi(0, \zeta) \, d\zeta = \text{sgn}(x) \sum_{n=-\infty}^{\infty} a'_{1,n} e^{i\alpha_n |x|} \varphi_n(z), \quad (E.10a)
\]
\[
I_2(x, z) = \int_0^{a} G_\xi(x - \xi, z, d) \phi(\xi, d) \, d\xi
\]
\[= \sum_{n=-\infty}^{\infty} \alpha_n A''_{n} \varphi_n(z) \int_0^{a} \text{sgn}(x - \xi) d'\xi(\phi(\xi, d)\varphi_n(d)e^{i\alpha_n |x-\xi|} \, d\xi, \quad (E.10b)
\]
\[
I_3(x, z) = \int_0^{a} G_\xi(x - \xi, z, d) \phi(\xi, d) \, d\xi
\]
\[= i \sum_{n=-\infty}^{\infty} A''_{n} \varphi_n(z) \int_0^{a} \phi(\xi, d) \varphi_n(d) e^{i\alpha_n |x-\xi|} \, d\xi, \quad (E.10c)
\]
\[
I_4(x, z) = \int_0^{a} G(x - \xi, z, d) \chi(\xi) \, d\xi
\]
\[= i \sum_{n=-\infty}^{\infty} A''_{n} \varphi_n(z) \int_0^{a} \chi(\xi) \varphi_n(d) e^{i\alpha_n |x-\xi|} \, d\xi, \quad (E.10d)
\]
\[
I_5(x, z) = \int_0^{d(a)} G_\xi(x - a, z, \zeta) \phi(a, \zeta) \, d\zeta
\]
\[= \text{sgn}(a - x) \sum_{n=-\infty}^{\infty} a''_{4,n} e^{i\alpha_n |x-a|} \varphi_n(z), \quad (E.10e)
\]
where \(A''_{n} = A_n A_0 (\gamma_n)^2\) and
\[
a'_{1,n} = A''_{n} \alpha_n \int_0^{d(0^+)} \phi(0, \zeta) \varphi_n(\zeta) \, d\zeta, \quad a''_{5,n} = A_n A''_{n} \alpha_n \int_0^{d(a^+)} \phi(a, \zeta) \varphi_n(\zeta) \, d\zeta.
\]

In particular, when \(x \notin (0, a)\), the three integrals \(I_2, I_3\) and \(I_4\) can also be written as eigenfunction expansions:
\[
I_j(x, z) = \begin{cases}
\sum_{n=-\infty}^{\infty} a'_{j,n} e^{i\alpha_n |x|} \varphi_n(z) & \text{for } x < 0, \\
\sum_{n=-\infty}^{\infty} a''_{j,n} e^{i\alpha_n (x-a)} \varphi_n(z) & \text{for } x > a,
\end{cases} \quad (E.11)
\]
where
\[ a_{2,n} = -\alpha_n A_n'' \int_{0}^{a} d' (\xi) \phi (\xi, d) \varphi_n (d) e^{i\alpha_n \xi} d\xi, \]
\[ a_{3,n} = i A_n'' \int_{0}^{a} \phi (\xi, d) \varphi'_n (d) e^{i\alpha_n \xi} d\xi, \]
\[ a_{4,n} = i A_n'' \int_{0}^{a} \chi (\xi) \varphi_n (d) e^{i\alpha_n \xi} d\xi, \]
and
\[ d_{2,n} = \alpha_n A_n'' \int_{0}^{a} d' (\xi) \phi (\xi, d) \varphi_n (d) e^{i\alpha_n (a-\xi)} d\xi, \]
\[ d_{3,n} = i A_n'' \int_{0}^{a} \phi (\xi, d) \varphi'_n (d) e^{i\alpha_n (a-\xi)} d\xi, \]
\[ d_{4,n} = i A_n'' \int_{0}^{a} \chi (\xi) \varphi_n (d) e^{i\alpha_n (a-\xi)} d\xi. \]

Scattering Coefficients

Using (E.9), (E.10) and (E.11), \( \phi (x, z) \) itself may be written as an eigenfunction expansion for \( x \notin (0, a) \):
\[ \phi (x, z) = \begin{cases} e^{i\alpha_n \xi} \varphi_0 (z) + \sum_{n=-\infty}^{\infty} a'_n e^{-i\alpha_n \xi} \varphi_n (z) & \text{for } x < 0, \\ \sum_{n=-\infty}^{\infty} d'_n e^{i\alpha_n (z-x)} \varphi_n (z) & \text{for } x > a, \end{cases} \]
where if \( A_n' = A_n'' \varphi_n (0) = -A_n \lambda_0 (\gamma_n), \)
\[ a'_n = -i A_n' P^T (\alpha_n) \left( P_0^- - P_a^+ e^{i\alpha_n a} \right) - a'_{1,n} + a'_{2,n} + a'_{3,n} + a'_{4,n} + a'_{5,n} e^{i\alpha_n a}, \]
\[ d'_n = e^{i\alpha_n a} \delta_n - i A_n' P^T (-\alpha_n) (P_0^- e^{i\alpha_n a} - P_a^+) \]
\[ + a'_{1,n} e^{i\alpha_n a} + d'_{2,n} + d'_{3,n} + d'_{4,n} - d'_{5,n}. \]

Equation (E.14) implies that our scattering coefficients are given by
\[ R = a'_0, \quad T = d'_0 / e^{i\alpha a}. \]

In addition, if we let \( a_n = a'_n \varphi_n (0) \) and \( d_n = d'_n \varphi'_n (0) \), we have the same eigenfunction expansions (4.38) and (4.42) for \( \phi_z (x, 0) \big|_{x \notin (0, a)} \) that we had when we assumed there was no submergence. These will help us when we come to apply the edge conditions (E.8) at \( x = 0 \) and \( x = a \).
E.2.2 Integral Equations

It can be seen from the formulae for the $a'_n$ and $d'_n$ coefficients that in order to determine $R$ and $T$ we must first calculate $\phi$ for $(x, z) \in \Gamma$, and also $\chi(x)$ over $(0, a)$. It will be shown that $\phi$ may be determined analytically on vertical segments of the keel, but that we must solve an integral equation proper to obtain $\phi(x, d)$ and $\chi(x)$ on the remainder of $\Gamma$.

Determination of $\phi$ Along Vertical Keel Segments

To find $\phi$ along those vertical segments, we first note that

$$\lim_{x \to x^\pm} G_x(x - x_c, z, \zeta) = \pm \frac{1}{2} \delta(z - \zeta).$$  \hfill (E.16)

This singular behaviour comes from the fundamental solution to the modified Helmholtz equation, $U(r) = -K_0(lr)/2\pi$. The fundamental solution is the part of the Green’s function that satisfies equation (3.1a) but neither of the other two equations in (3.1). From (C.5), $U$ has a logarithmic singularity when $x - \xi = z - \zeta = 0$; this logarithmic term may be written as a Fourier transform with respect to $z - \zeta$ (Meylan, 1993):

$$U_0(r) = \frac{1}{2\pi} \log(r) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|k|x} - \zeta |dk}{|k| e^{ik(z - \zeta)}}.$$ \hfill (E.17)

($U_0$ is the fundamental solution for normal incidence, i.e. when $\theta = 0$.) Equation (E.16) comes from differentiating the above equation and letting $x \to \xi$, and by noting that the rest of the Green’s function is continuous and odd with respect to $x - \xi$.

Substituting (E.16) into (E.9) permits us to say that

\begin{align*}
\frac{1}{2} \phi(0, z) &= \sum_{n = -2}^{\infty} (b'_n + \sigma'_{1,n} + (a'_5, n + c'_n) e^{i\alpha_{n}}) \varphi_n(z), \quad \text{(E.18a)} \\
\frac{1}{2} \phi(a, z) &= \sum_{n = -2}^{\infty} ((b'_n + a'_{1,n}) e^{i\alpha_{n}} + \sigma'_{5,n} + c'_n) \varphi_n(z), \quad \text{(E.18b)}
\end{align*}

where $\sigma'_{1,n} = \sum_{j=2}^{4} a'_{j,n}$, $\sigma'_{5,n} = \sum_{j=2}^{4} d'_{j,n}$ and

$$b'_n = \delta_{n,0} + i A'_n P^T(-\alpha_n) P_0^-, \quad c'_n = -i A'_n P^T(\alpha_n) P_a^+.$$

The above coefficients come from the eigenfunction expansions of the functions not inside the integral $I$ in (E.9). Consequently,

\begin{align*}
a'_{1,m} &= 2\alpha_m A'_m \sum_{n = -2}^{\infty} (b'_n + \sigma'_{1,n} + (a'_5, n + c'_n) e^{i\alpha_{n}}) I_{mn}(d(0^+)), \quad \text{(E.19a)} \\
a'_{5,m} &= 2\alpha_m A'_m \sum_{n = -2}^{\infty} ((b'_n + a'_{1,n}) e^{i\alpha_{n}} + \sigma'_{5,n} + c'_n) I_{mn}(d(a^-)), \quad \text{(E.19b)}
\end{align*}
where

\[ I_{mn}(z) = \int_{0}^{z} \varphi_m(\zeta)\varphi_n(\zeta)\,d\zeta. \]

Equations (E.19) may be written in matrix form as

\[ a' = M'_0(b' + D_a(c' + a'_z) + \sigma'_1), \quad (E.20a) \]
\[ a'_z = M'_5(D_a(b' + a'_1) + c' + \sigma'_4), \quad (E.20b) \]

where

\[ [M'_1]_{mn} = 2\alpha_m A''_m I_{mn}(d(0^+)), \quad [M'_5]_{mn} = 2\alpha_m A''_m I_{mn}(d(a^-)). \]

Equations (E.20) are essentially two sets of simultaneous equations for the vectors \( a' \) and \( a'_z \). Their dependence on each other may easily be eliminated, in the same way that the vectors \( b \) and \( c \) of (8.15) were decoupled. Ultimately this produces formulae for the \( a'_{1,n} \) and \( a'_{5,n} \) in terms of \( \phi(x, d) \) and \( \chi(x) \) for \( x \in (0, a) \); this dependence comes from the \( d'_{jn} \) and \( d''_{jn} \) coefficients \((j = 2, 3, 4)\).

**Determination of \( \phi \) and \( \chi \) Over Non-Vertical Keel Segments**

To find an expression for \( \phi(x, d) \) we must let \( z \to d(x) \) when \( x \in (0, a) \) in \( I_2, I_3, \) and \( I_4 \). The kernel of the integral \( I_4 \) is \( G(x - \xi, z, d) \), which has only a logarithmic singularity when \( x - \xi = z - \zeta = 0 \), and so is absolutely integrable regardless of where the point \((x, z)\) is. Thus \( I_4 \) is simply

\[ I_4(x, d(x)) = \int_{0}^{a} G(x - \xi, d(x), d(\xi))\chi(\xi)d\xi. \quad (E.21) \]

One approach to evaluating this integral is to write it as the sum

\[ I_4(x, d(x)) = i \sum_{n=-2}^{\infty} A''_n \varphi_n(d(x)) \int_{0}^{a} \chi(\xi)\varphi_n(d(\xi))e^{i\alpha_n|x-\xi|}d\xi, \quad (E.22) \]

and again approximate the terms in each integrand that do not depend on \( x \) in the manner of equation (6.5)—i.e. by dividing \((0, a)\) into \( M \) panels in the same way that we did in Section 6.1.1 and assuming that \( \varphi_n(d(\xi))\chi(\xi) \) is linear over each panel. If \( d(x) \) were a constant function, we could separate out the \( \varphi_n(d) \) term from each integral, and this would be quite efficient as the weights resulting from each mode could be combined into a single kernel matrix. In general, however, this would require us to construct individual matrices for each mode, which could become quite time-consuming as often (especially at large depths) a substantial number of modes is required to make a sum such as (E.22) converge.

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Alternatively, we can deal with the fundamental solution \( U(r) = -K_0(\lambda r)/2\pi \) separately, which has singular part \( \tilde{U}_0(r) = \log(r)I_0(\lambda r) \), by writing

\[
I_4(x, d(x)) = \int_0^a \tilde{U}_0(|x - \xi|) \chi(\xi) d\xi + \int_0^a V(x - \xi, d(x), d(\xi)) \chi(\xi) d\xi,
\]

where

\[
V(x - \xi, d(x), d(\xi)) = G(x - \xi, d(x), d(\xi)) - \frac{1}{2\pi} \log |x - \xi| I_0(l(x - \xi))
\]
is now relatively nonsingular. \( I_0 \) is the modified Bessel function of the first kind, which is given as a Taylor series in (C.6). In (E.23) we could have used \( U_0 \) instead of \( \tilde{U}_0 \), but for oblique incidence that would have left the kernel with a logarithmic singularity in its second derivative. (For normal incidence \( l = 0 \), so \( \tilde{U}_0 = U_0 \) since \( I_0(0) = 1 \).) Even just using the first four or five terms of the series (C.6) would help smooth \( V \)—this would give it either seven or nine bounded derivatives. Moreover, the extra polynomial terms are very easily dealt with using the result

\[
\frac{d}{dx} \left( \frac{x^n+1}{(n+1)^2} \right) = x^n \log |x|,
\]
especially if we use a piecewise linear interpolation for \( \chi \) as we did in Section 6.1.1.

The integrals \( I_2 \) and \( I_3 \) are slightly more tricky to deal with, as applying the normal derivative to the logarithmic function in the fundamental solution produces

\[
2\pi (d'(\xi) \partial_\xi - \partial_\xi) U_0(r) \bigg|_{\xi = d(\xi)} = \frac{d'(\xi)(\xi - x) + z - d(\xi)}{(\xi - x)^2 + (z - d(\xi))^2}
\]

\[
\sim \frac{1}{z - d(x)} \quad \text{as } \xi \to x.
\]

(E.24)

However, we can make some progress by first considering the case when \( d(x) = d \) is constant. From swapping \( x - \xi \) with \( z - \xi \) in (E.17), we know that

\[
\frac{1}{2\pi} \lim_{z \to d(x - \xi)} \frac{z - d}{(x - \xi)^2 + (z - d)^2} = -\lim_{z \to d} U_{0,\xi}(r) \bigg|_{\xi = d} = \frac{1}{2} \delta(x - \xi),
\]

(E.25)

where \( \varepsilon = z - d \). This is consistent with the Plemelj formulae (Roos, 1969, Section 4.6), which imply that for a H"older continuous function \( \psi(x) \),

\[
\psi(x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left( \int_0^a \frac{\psi(\xi) d\xi}{\xi - x - i\varepsilon} - \int_0^a \frac{\psi(\xi) d\xi}{\xi - x + i\varepsilon} \right)
\]

\[
= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_0^a \frac{\varepsilon \psi(\xi) d\xi}{(x - \xi)^2 + \varepsilon^2}.
\]

(E.26)
This result can be extended to when \( d(x) \) is not constant by subtracting a term that is similar to (E.25) from \( (d'(\xi)\partial_\xi - \partial_\xi)G(x - \xi, z, d(\xi)) \) to cancel the \( 1/(z - d(x)) \) singularity, and, assuming that \( \phi(x, d) \) is Hölder continuous (which may be confirmed a posteriori by the final integral equation E.28; cf. Roos, 1969, Section 4.1, for the definition of Hölder continuity), using (E.26) to write

\[
\lim_{z \to d(x)} \left( I_2(x, z) + I_3(x, z) \right) = \lim_{\varepsilon \to 0} \int_0^a U_0'(x - \xi, \varepsilon)\phi(\xi, d(\xi))d\xi
\]

\[
+ \lim_{z \to d(x)} \int_0^a G'(x - \xi, z, d(\xi))\phi(\xi, d(\xi))d\xi
\]

\[
= \frac{1}{2}\phi(x, d(x)) + \int_0^a G'(x - \xi, d(x), d(\xi))\phi(\xi, d(\xi))d\xi, \quad (E.27)
\]

where \( U_0'(x, z) = (z/2\pi)/(x^2 + z^2) \). This implies that

\[
G'(x - \xi, z, d(\xi)) = (d'(\xi)\partial_\xi - \partial_\xi)G(x - \xi, z, d(\xi)) - U'(x - \xi, z - d(x))
\]

is nonsingular when \( x = \xi \) and \( z = d(x) \). In fact since \( U_0'(x - \xi, 0) = 0 \) when \( x \neq \xi \), subtracting it is again to a large extent more of a formal allowance for the presence of the delta function than something that is actually required in practice. The value \( G' \) takes when \( x = \xi \) can be obtained by finding the limit \( G'(0^{\pm}, d(x), d(\xi)) \).

Using the results for the values the three \( I_j \) integrals \((j = 2, 3, 4)\) take as \( z \to d(x) \), we can substitute them into (E.9) to give

\[
\frac{1}{2}\phi(x, d(x)) = \sum_{n=-2}^{\infty} \left( (b'_{n} + a'_{1,n})e^{i\alpha_n x} + (c'_{n} + a'_{2,n})e^{i\alpha_n(a-x)} \right)\varphi_n(d(x))
\]

\[
+ \int_0^a G'(x - \xi, d(x), d(\xi))\phi(\xi, d(\xi))d\xi
\]

\[
+ \int_0^a \tilde{U}_0(l|x - \xi|) (x - \xi, d(x), d(\xi))\chi(\xi)d\xi
\]

\[
+ \int_0^a V (x - \xi, d(x), d(\xi))\chi(\xi)d\xi. \quad (E.28)
\]

This integral equation may be solved for \( \phi(x, d) \) in terms of \( \chi(x) \) by numerical quadrature as described in the following section. \( \chi \) itself can be written in terms of \( \phi(x, d) \) by applying the thin plate condition, as done in Section E.2.4.

### E.2.3 Quadrature Scheme

To solve (E.28) we must approximate the various integrands somehow. By approximating \( \chi(x) \) using (6.5), the first integral involving it may be done straightfor-
wardly using the series representation (C.6) of \( I_0 \). The other integrands, \( G'(x - \xi, d(x), d(\xi)) \phi(\xi, d(\xi)) \) and \( V(x - \xi, d(x), d(\xi))\chi(\xi) \), are relatively smooth and may also be approximated by piecewise linear functions (dividing \((0, a)\) into \(M\) panels again as in Section 6.1.1). These latter integrals could possibly even be done using Simpson’s rule, which uses piecewise cubic interpolation between each of the endpoints of each panel to approximate the integrands.

Either approach would allow us to write (E.28) as

\[
\frac{1}{2} \phi = E(b' + a'_j) + \tilde{E}(c' + a'_j) + G\chi + G'\phi, \tag{E.29}
\]

where \( E \) is the \((M+1) \times (N+3)\) matrix given in Section 7.1.2 (assuming that we have truncated the sums in equation E.28 at \( n = N \)), and \( \tilde{E} \) is an upside-down \( E \), i.e.

\[
[\tilde{E}]_{mn} = [-E]_{M+1-m,n} \quad \text{for } 0 \leq m \leq M+1.
\]

The matrices \( G \) and \( G' \), where \( G = U + V \), approximate the following integrals:

\[
[U\chi]_n \approx \int_0^a \tilde{U}_0(|x_n - \xi|)\chi(\xi)d\xi, \tag{E.30a}
\]

\[
[V\chi]_n \approx \int_0^a V(x - \xi, d(x), d(\xi))\chi(\xi)d\xi, \tag{E.30b}
\]

\[
[G'\phi]_n \approx \int_0^a G'(x_n - \xi, d(x_n), d(\xi))\phi(\xi, d(\xi))d\xi. \tag{E.30c}
\]

The \( a'_j \) vectors \((j = 1, 5, \text{ cf. equations E.19})\) still contain some dependence on \( \phi \) and \( \chi \) since they themselves are written in terms of the vectors \( a'_j \) and \( d'_j \) \((j = 2, 3, 4)\) given by (E.12) and (E.13). For constant \( d \), these may be calculated in the same way as the Fourier integrals in Section 6.1.2 were. However, if \( d \) is not constant, the \( \varphi_n(d(\xi)) \sim \cos|\gamma_n d(\xi)| \quad (n \geq 1) \) contributes additional oscillations, especially for large \( n \), and this will need to be taken into account somehow. This could be done by dividing \((0, a)\) into a further number of sufficiently small subintervals to ensure that the integrals in (E.12) and (E.13) are done accurately. This is not necessarily such a hardship, as the exponential terms \( \exp(-|\gamma_n\xi|) \) and \( \exp(-|\gamma_n(a - \xi)|) \quad (n \geq 1) \) decay more and more rapidly as \( n \) increases (i.e. as \( n \) increases, we only need to subdivide a smaller and smaller interval, assuming that the exponential terms are zero in the remainder of \((0, a)\)).

Noting now that equations (E.19) can be decoupled by writing

\[
a'_i = (I - M'_i D_a M'_5 D_a)^{-1} M'_1 g'_i, \tag{E.31}
\]

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where

\[ g'_i = b' + D_\alpha c' + \sigma'_i + D_\alpha M'_\delta (D_\alpha b' + c' + \sigma'_i), \]

we now have two linear relationships between the two \( \alpha'_j \) and the unknown vectors \( \phi \) and \( \chi \):

\[ 2\alpha'_j = f'_j + M'_j\phi + M'_j\chi \quad \text{for} \quad j = 1, 5, \quad (E.32) \]

omitting the details of the above vectors and matrices.

Consequently, \( \phi \) can be written

\[ M'_\phi \phi = f'_\phi + M'_\chi \chi, \quad (E.33) \]

where \( f'_\phi = E(2b' + f'_1) + \bar{E}(2c' + f'_5) \), and

\[ M_\phi = I - 2G' - EM'_\phi - \bar{E}M'_\phi, \quad M_\chi = 2G + EM'_\chi + \bar{E}M'_\chi. \]

The next section attempts to find another relationship between \( \chi \) and \( \phi \) by applying the thin plate condition, which would allow us to eliminate \( \chi \) from (E.33).

### E.2.4 Application of the Thin Plate Condition

Let \( m_{av} \) be the average (nondimensional) mass per unit area of the ice in the pressure ridge, and let \( D_{av} = m_{av}^3 \). Defining an operator \( L_{av}(\partial_x) = D_{av}(\partial_x^2 - l^2)^2 + \lambda - m_{av}\mu \), let us define the Green’s function for that operator as \( g_<(x - \xi) \) which satisfies

\[ L_{av}(\partial_\xi)g_<(x - \xi) = \delta(x - \xi). \quad (E.34) \]

\( g_<(x) \) can be written as an inverse Fourier transform:

\[ g<(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}dk}{D_{av}\kappa^4 + \lambda - m_{av}\mu} = \sum_{n=-\infty}^{0} \frac{e^{ik_{n}\kappa_{n}^4}}{4D_{av}\kappa_{n}^3}, \quad (E.35) \]

where \( \kappa_{-1} = i\kappa_0 \),

\[ \kappa_0 = \begin{cases} \sqrt{\omega_{av}}^1/4 e^{\pi/4} & \text{for} \quad m_{av}\mu \leq \lambda, \\ \omega_{av}^1/4 & \text{for} \quad m_{av}\mu > \lambda, \end{cases} \quad (E.36) \]

\( \omega_{av} = (m_{av}\mu - \lambda)/D_{av} \), and the \( k_n = \sqrt{\kappa_n^2 - l^2} \) are taken as either being positive or in the upper half-plane.
Now, multiplying (E.6) by $g_\prec$ and integrating from 0 to $a$ with respect to $\xi$ gives

$$
\int_0^a g_\prec(x-\xi) L(\xi, \partial_\xi) \chi(\xi) d\xi = \frac{D(x)}{D_{av}} \chi(x) + E_\prec(x) + \int_0^a K_\prec(x, \xi) \chi(\xi) d\xi
$$

where

$$
K_\prec(x, \xi) = \left( L(\xi, \partial_\xi) - \frac{D(\xi)}{D_{av}} L_{av}(\partial_\xi) \right) g_\prec(x-\xi)
$$

$$
= \sum_{j=1}^3 d_j(\xi) L_{1j}(\partial_\xi) g_\prec(x-\xi),
$$

(E.37)

and the edge terms $E_\prec(x)$ that result from integrating by parts are given by

$$
E_\prec(x) = [E(\xi, \partial_\xi) g_\prec(x-\xi)]_{\xi=a}^{\xi=0} - \sum_{x_e \in X_e^S} [E(\xi, \partial_\xi) g_\prec(x-\xi)]_{\xi=x_e^+}^{\xi=x_e^-}
$$

$$
= \psi_\prec^T(x) P_0^+ + \sum_{x_e \in X_e^S} \psi_\prec^T(x-x_e) P_{x_e} - \psi_\prec^T(x-a) P_a^-,
$$

(E.38)

where $\psi_\prec(x) = L_{\text{edge}}(\partial_\xi) g_\prec(x)$, and

$$
P_{x_e} = P_{x_e}^+ - P_{x_e}^- = (P_0(x_e), P_1(x_e), M(x_e), S(x_e))^T.
$$

As in Section 4.2.3, any unknown vectors are found by applying the frozen edge conditions, although in the presence of a submerged keel, these are given by (E.8) not (2.12).

The operators $L_{1j}$ in (E.38) are given by $L_{11}(\partial_\xi) = (\partial_\xi^2 - \ell^2) \partial_\xi$, $L_{12}(\partial_\xi) = L^-(\partial_\xi)$, and $L_{13}(\partial_\xi) = 1$, while the functions $d_j(x)$ are defined by

$$
d_1(x) = 2D'(x), \quad d_2(x) = D''(x),
$$

and

$$
d_3(x) = \lambda \left( 1 - \frac{D(x)}{D_{av}} \right) - \mu \left( m(x) - \frac{m_{av}}{D_{av}} D(x) \right).
$$

Equation (E.37) is an integral equation for $\chi$ in terms of $\phi(x, d)$. It is extremely similar to (6.2), and we will solve it in almost exactly the same way. By dividing the interval into $M$ panels and approximating the integrands as we did in Section 6.1.1, we can also write it as the matrix equation

$$
M\chi = -W_3\phi_1 - E_\prec,
$$

(E.40)

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where
\[ M = I + \sum_{j=1}^{3} W_j D_j, \quad [E_c]_n = E_c(x_n), \quad [D_j]_{nm} = d_j(x_n)\delta_{nm}, \]
and where the \( W_j \) approximate integral operators as follows:
\[
\int_0^a L_j(\partial_\xi)\gamma_c(x_n - \xi)v(\xi)d\xi \approx \sum_{m=0}^{M} [W_j]_{nm}v(x_m).
\]
Equation E.40 allows us to eliminate \( \chi \) from (E.33) and find \( \phi \), giving
\[
\phi = (M' + M^{-1}W_3)^{-1}(f' - M^{-1}E_c).
\] (E.41)

All that remains now is to apply the edge conditions, as described below.

**Application of the Edge Conditions**

As in Section 4.2.3, the edge conditions (E.8c) imply that for \( x_c \in X_0^c, M(x_c) \) and \( S(x_c) \) are both zero, while the two remaining constants \( P_0(x_c) \) and \( P_1(x_c) \) can be found by writing
\[
P_0(x_c) = (D(x^+_c) - D(x^-_c))\chi(x^+_c), \quad (E.42a)
\]
\[
P_1(x_c) = \left(D(x^+_c) - D(x^-_c)\right)\chi(x^+_c) - \left(D'(x^+_c) - D'(x^-_c)\right)\chi(x^+_c), \quad (E.42b)
\]
which are essentially equivalent to (4.44). As in Section 6.1.3, when \( D \) is discontinuous it is more convenient to rearrange (E.42b) into (6.19). (A similar expression to equation 6.18 may be obtained by differentiating equation E.37.) However, if \( D \) is continuous then \( P_0(x_c) = 0 \) again, and only (E.42b) needs to be applied to find \( P_1(x_c) \).

When \( x = 0 \) or \( a \), we eliminate eight unknowns by first writing \( P^+_a \) in terms of \( P^-_a \), and then \( P^-_a \) in terms of \( P^+_a \), as follows:
\[
D_0 P^+_0 = E_{\text{edge}}(0^+)P^-_0, \quad (E.43a)
\]
\[
D_0 P^-_a = E_{\text{edge}}(a^-)P^+_a, \quad (E.43b)
\]
where
\[
E_{\text{edge}}(x) = \begin{pmatrix} D(x)I_2 & 0 \\ 0 & D_0 I_2 \end{pmatrix} - D'(x)Z_{21}.
\]
and where $Z_{21}$ is defined by

$$[Z_{21}]_{mn} = \begin{cases} 1 & \text{if } m = 2 \text{ and } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The remaining eight unknowns in the vectors $P_0^-$ and $P_0^+$ can be found by first calculating the eigenfunction expansions (4.38) and (4.42) using equations (E.15), and then applying the definitions of each of their elements. This completes the solution, giving a final form for $\phi$ and $\chi$ on the keel outline. $R$ and $T$ can now be calculated to determine the amount of scattering that a pressure ridge with a submerged keel will produce.
Appendix F

Comparison of Residue Calculus Solution for a Lead with Wiener-Hopf Solution

In this appendix we apply mode-matching to the problem of three adjacent sheets of ice, and then simplify the resulting equations using residue calculus to reproduce the systems of equations (8.14), (8.8b) and (8.8c). Together these imply the system (8.15), and the remainder of the solution follows in exactly the same way as in Chapter 8 (application of the edge conditions and subsequent calculation of $R$ and $T$). Consequently, the results of Chapter 8 derived from Green's theorem coupled with the Wiener-Hopf technique are confirmed.

The solutions for the case when both or either of $h_0$ and $h_2$ are zero are easily obtained using the working below, with similar adjustments to those made in Section 8.1.3. However, when $h_1 = 0$ it is a little more difficult to reproduce our Wiener-Hopf results, and we were not able to do it. This is not to say that a residue calculus solution for an open lead is impossible, however, as one is given by Chung and Linton (2005); we are only saying that in this case it is more difficult to show analytically that the residue calculus solution and the Wiener-Hopf one are equivalent.
F.1 Derivation of Mode-Matching Equations

If we assume from the outset that \( \phi(x, z) \) can be expanded as in (8.34) as

\[
\phi(x, z) = \begin{cases} 
\phi_0(x, z) + \sum_{n=-2}^{\infty} a_n e^{-i\alpha_n x} \varphi_n(z) & \text{for } x < 0, \\
\sum_{n=-2}^{\infty} (b'_n e^{ik na} + c'_n e^{ik (a-x)}) \psi_n(z) & \text{for } 0 < x < a, \\
\sum_{n=-2}^{\infty} d'_n e^{-i\alpha_n (x-a)} \hat{\varphi}_n(z) & \text{for } x > a,
\end{cases}
\]  

(F.1)

where \( a_n = a'_n \varphi'_n(0) \), \( b_n = b'_n \psi'_n(0) \), \( c_n = c'_n \psi'_n(0) \) and \( d_n = d'_n \hat{\varphi}'_n(0) \). We know that such an expansion exists from equations (4.21), (4.31) and (4.25).

Applying the conditions (2.8c) at \( x = 0 \) gives

\[
\varphi_0(z) + \sum_{n=-2}^{\infty} a_n \varphi_n(z) = \sum_{n=-2}^{\infty} (b'_n + c'_n e^{ikna}) \psi_n(z),
\]  

(F.2a)

\[
i \alpha_0 \varphi_0(z) - i \sum_{n=-2}^{\infty} \alpha_n a_n \varphi_n(z) = i \sum_{n=-2}^{\infty} \alpha_n \varphi_n(z) = i \sum_{n=-2}^{\infty} \alpha_n (b'_n - c'_n e^{ikna}) \psi_n(z).
\]  

(F.2b)

The same conditions applied at \( x = a \) give

\[
\sum_{n=-2}^{\infty} d'_n \hat{\varphi}_n(z) = \sum_{n=-2}^{\infty} (b'_n e^{ikna} + c'_n) \psi_n(z),
\]  

(F.3a)

\[
i \sum_{n=-2}^{\infty} \alpha_n d'_n \hat{\varphi}_n(z) = i \sum_{n=-2}^{\infty} \alpha_n (b'_n e^{ikna} - c'_n) \psi_n(z).
\]  

(F.3b)

Now, by adjusting the rules of Lawrie and Abrahams (1999), we have the following integral rules:

\[
\int_0^H \varphi_n(z) \psi_m(z) \, dz = \left( \frac{f_0(\kappa_m)}{\gamma_n^2 - \kappa_m^2} - D_0(\kappa_m + \gamma_n^2) \right) \varphi'_m(0), \quad \text{(F.4a)}
\]

\[
\int_0^H \psi_n(z) \psi_m(z) \, dz = \left( \frac{C_1(\kappa_m)}{2\kappa_m^2} \delta_{mn} - D_1(\kappa_m^2 + \gamma_n^2) \right) \psi'_m(0), \quad \text{(F.4b)}
\]

\[
\int_0^H \hat{\varphi}_n(z) \psi_m(z) \, dz = \left( \frac{f_2(\kappa_m)}{\gamma_n^2 - \kappa_m^2} - D_2(\kappa_m^2 + \gamma_n^2) \right) \hat{\varphi}'_m(0). \quad \text{(F.4c)}
\]

The multiplier of the \( \delta_{mn} \) term in (F.4b) will be denoted \( C'_m = -f_1'(\kappa_m)/2\kappa_m \); incidentally this implies that \( 2i\kappa_m C'_m = 1/(iB_m) \). Note that that equation follows from either of (F.4a) or (F.4c) by taking the limit as \( f_j \to f_1 \) (\( j = 0 \) or \( 2 \)), and as \( \gamma_n \to \kappa_m \) or \( \gamma_n \to \kappa_n \); (F.4a) may be derived by first writing the left hand integral as

\[
\frac{1}{\gamma_n^2 - \kappa_m^2} \int_0^H (\varphi''(z) \psi_m(z) - \varphi_n(z) \psi_m'(z)) \, dz = \frac{\psi_m'(0) - \varphi'_m(0)}{\gamma_n^2 - \kappa_m^2},
\]
and then using the dispersion relations \( f_0(\gamma_m) = f_1(\kappa_m) = 0 \) to simplify it. Equation (F.4c) follows in a similar manner.

Hence multiplying (F.2) by \( \psi_m(z) \) and integrating with respect to \( z \) from 0 to \( H \) implies that

\[
\begin{align*}
  &i f_0(\kappa_m) \left( \frac{k_m \varphi'(0)}{\alpha_0^2 - k_m^2} + \sum_{n=-2}^{\infty} \frac{k_m a_n}{\alpha_n^2 - k_m^2} \right) + i k_m M^-(0) - i k_m P_0^-(0) f^+(\kappa_m) \\
  &= i k_m M^+(0) - i k_m P_0^+(0) f^+(\kappa_m) + i k_m C'_m (b_m + c_m e^{i k_m a}) , \\
  &i f_0(\kappa_m) \left( \frac{\alpha_0 \varphi'(0)}{\alpha_0^2 - k_m^2} - \sum_{n=-2}^{\infty} \frac{\alpha_n a_n}{\alpha_n^2 - k_m^2} \right) + S^-(0) - P_1^-(0) f^-(\kappa_m) \\
  &= S^+(0) - P_1^+(0) f^-(\kappa_m) + i k_m C'_m (b_m - c_m e^{i k_m a}) ,
\end{align*}
\]

which allows us to write

\[
\begin{align*}
  b_m &= i B_m P^T(-k_m) P_0 - B_m f_0(\kappa_m) \left( \frac{\varphi'(0)}{\alpha_0 - k_m} - \sum_{n=-2}^{\infty} \frac{a_n}{\alpha_n + k_m} \right) \\
  &= i B_m F_0(-k_m) + B_m f_0(\kappa_m) \sum_{n=-2}^{\infty} \frac{a_n}{\alpha_n + k_m} , \\
  &- B_m f_0(\kappa_m) \sum_{n=-2}^{\infty} \frac{a_n}{\alpha_n - k_m} = c_m e^{i k_m a} + i B_m P^T(k_m) P_0 - B_m f_0(\kappa_m) \frac{\varphi'(0)}{\alpha_0 + k_m} \\
  &= c_m e^{i k_m a} + i B_m F_0(k_m) .
\end{align*}
\]

Since \( \Psi^{-}(k) = -i \sum_{n=-2}^{\infty} a_n/(k - \alpha_n) \), (F.6a) is clearly equivalent to (8.8b), and (8.14a) is derived in the next section by using residue calculus to solve (F.6b) for the \( a_n \) in terms of the \( c_n \). Equation (F.6a) can also be obtained by substituting (F.1) into (4.32a) and using (F.4) to simplify the result.

The analogous set of equations to (F.6) follows from (F.3) and (F.4) in a similar fashion, and is

\[
\begin{align*}
  c_m &= i B_m P^T(k_m) P_a + B_m f_2(\kappa_m) \sum_{n=-2}^{\infty} \frac{d_n}{\alpha_n + k_m} , \\
  &- B_m f_2(\kappa_m) \sum_{n=-2}^{\infty} \frac{d_n}{\alpha_n - k_m} = b_m e^{i k_m a} + i B_m P^T(-k_m) P_a .
\end{align*}
\]

Again, since \( \Phi^+(k) = i \sum_{n=-2}^{\infty} d_n/(k + \alpha_n) \), (F.7a) is equivalent to (8.8c); like (F.6a), it can also be obtained by substituting (F.1) into (4.32b). And like (F.6b), (F.7b) can
be solved for the \( d_n \) with residue calculus to reproduce equations (8.14b). Working for the inversion of (F.6b) is given in the following section; the inversion of (F.7b) follows in exactly the same way and consequently is not presented.

### F.2 Application of Residue Calculus Method

To solve (F.6b), we look for numbers \( X_{nm} \) \((n, m \in \mathcal{N})\) which have the property that

\[
- \sum_{m=-2}^{\infty} \frac{X_{nm}B_m f_0(k_m)}{\alpha_r - k_m} = \delta_{nr}. \tag{F.8}
\]

Alternatively, we can look for functions \( \chi_n \) that can be expanded in the form

\[
\chi_n(k) = - \sum_{m=-2}^{\infty} \frac{X_{nm}B_m f_0(k_m)}{k - k_m} \quad \text{for } n \in \mathcal{N}, \tag{F.9}
\]

and that satisfy

\[
\chi_n(\alpha_n) = 1, \quad \chi_n(\alpha_r) = 0 \quad \text{if } r \neq n, \text{ for } r, n \in \mathcal{N}. \tag{F.10}
\]

For an expansion of the form (F.9) to exist, \( \chi_n \) must be meromorphic with poles when \( k = k_n \), and must decay at least as fast as \( O(1/k) \) as \( |k| \to \infty \). From our working in Chapter 8, a candidate function is

\[
\chi_n(k) = A_n K_0^+(\alpha_n)f_1(\gamma_n) \times \frac{K_0^-(k)}{k - \alpha_n}, \tag{F.11}
\]

which has residues at the desired poles of

\[
\text{Res}(\chi_n, k = k_m) = -X_{nm} B_m f_0(k_m) = A_n K_0^+(\alpha_n)f_1(\gamma_n) \times \frac{B_m f_0(k_m)}{K_0^+(k_m)(k_m - \alpha_n)}. \tag{F.12}
\]

Consequently, if we choose

\[
X_{nm} = \frac{A_n K_0^+(\alpha_n)f_1(\gamma_n)}{K_0^+(k_m)(\alpha_n - k_m)} = [M_\alpha]_{nm}, \tag{F.13}
\]

(F.8) is satisfied, and so multiplying (F.6b) by \( X_{nm} \) and summing over \( m \in \mathcal{N} \) produces (8.14a), as we were hoping.

The solution of (F.7b) follows in the same way, and so the final results for \( \Psi^- \) and \( \Phi^+ \) given by (8.13) follow; this produces the system (8.15) to solve for the \( b_n \) and \( c_n \), and the solution is completed by applying the edge conditions to find \( P_0 \) and \( P_\alpha \). \( R \) and \( T \) can then be found.