VIX Futures ETNs and Their Derivatives

by

Sebastian A. Gehricke

A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy in Finance
at the
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin, New Zealand

October 2018
Abstract

This thesis studies the VIX futures exchange-traded notes (ETNs) (Chapters 2 and 3) and their derivatives (Chapter 4).

In Chapter 2, we examine the VIX futures ETN market’s tracking performance, price consistency and price discovery. The VIX futures ETN market has become one of the main avenues for volatility trading. We show that VIX futures ETNs do not track their indicative values perfectly and are mostly inconsistently priced, at the daily and intraday level. We find that the ETNs lead the underlying VIX futures in price discovery, although this relationship is time varying, which could explain the ETNs poor tracking performance. Between the different ETNs there is no clear overall leader in price discovery; again, these dynamics are time varying.

In Chapter 3, we develop a model for the VXX, the most actively traded VIX futures ETN, using Duffie, Pan, and Singleton’s (2000) affine jump diffusion, where the volatility process has jumps and a stochastic long-term mean. We calibrate the model parameters using the VIX term structure data and show that our model provides the theoretical link between the VIX, VIX futures and the VXX. Our model can be used for pricing VIX futures, the VXX and other short-term VIX futures exchange-traded products (ETPs). Our model could be extended to price options on the VXX and other short-term VIX futures ETPs.

Lastly, in Chapter 4 we document and analyze the empirical characteristics of the VXX options market, providing useful information for developing a realistic VXX
option pricing model. We extend the methodology developed by Zhang and Xiang (2008) in order to study the term structure and time series of the VXX option implied volatility curves. The implied volatility curve is quantified through three factors; the level, slope and curvature. After quantifying the implied volatility curves of the VXX options market, we show that they are not usually a smirk, as for S&P 500 options, but rather an upward-sloping line with some convexity. As the option’s maturity increases usually the level (exact at-the-money implied volatility) increases, the slope decreases and curvature increases. The level and slope factors seem to mean-revert, while the curvature factor does not follow a easily identified pattern.
Acknowledgements

Firstly, I would like to thank my supervisor Professor Jin E. Zhang for his inspiration, careful guidance, valuable experience and endless support. He inspired my passion for research first when he was my supervisor during the research component of my Master’s degree. He showed me the opportunities awaiting me if I just worked hard and persevered. Thanks to his guidance I was able to raise my first child, consult for several businesses, teach many courses and finish this dissertation without ever being lost or overwhelmed. He provided continuous support, which led to a very enjoyable PhD process. Thank you, Jin, for guiding me on this journey; I will never forget what you have done for me and my family.

Secondly, I would like to thank my colleagues in the Department of Accounting and Finance at the University of Otago including but not limited to Professor Timothy Falcon Crack; Associate Professor Ivan Diaz-Rainey; Dr. Xing Han, Professor David Lont; Professor I M Premachandra; seminar and conference participants for their helpful comments particularly, Associate Professor Jedrzej Bialkowski, Ilnara Gafiatullina, Dr. Jose Da Fonseca, Professor Bart Frijns and Dr. Xinfeng Ruan; our PhD group meeting participants Dr. Fang Zhen, Nhu Nguyen, Tian Yue, Jianhui Li and Pakorn Aschakulporn; Dr. Jorge Lopez for his career and research support; Dr. Anella Munro for providing an academic visitation at the Reserve Bank of New Zealand; and our proof editor, Marianne Lown.

I sincerely acknowledge the generous support from the University of Otago Doc-
toral Scholarship, which has been essential in supporting me to finish the PhD program smoothly.

I am so grateful to my parents and sister for their unconditional love and supporting me financially and emotionally throughout my studies and in starting my young family. Mama und Papa ihr habt keine Ahnung wie viel ihr mir geholfen habt. Ich werde stetig versuchen zu so einem Elternteil zu werden wie ihr es seid. Ich werde nie eure harte Arbeit und alles was ihr für mich geopfert habt vergessen. Danke für alles, ich liebe euch.

Lastly, I want to thank you, my dearest Catherine, for your endless, unwavering love and your strong, insightful and practical support over the last three years. You have awakened a creativity and joy of life in me that I have never known before. We have achieved so much together and I could not have done any of this without you. You are the most amazing mother to our beautiful little girl, and I could not imagine a better life. I love you and Aurelia more than anything in this world.

Dear Catherine, I dedicate this PhD and the rest of my life to you. Will you do me the honor of being my wife?
Contents

Abstract i

Acknowledgements iii

1 Introduction 1
  1.1 VIX futures ETPs ............................................. 2
  1.2 Literature setting ............................................. 3
  1.3 PhD thesis structure .......................................... 5
  1.4 Contribution of this PhD thesis ............................ 8

2 The VIX Futures ETN Market 11
  2.1 Introduction .................................................. 11
  2.2 Data .......................................................... 18
  2.3 Tracking performance ........................................ 19
    2.3.1 Daily ..................................................... 20
    2.3.2 Intraday ................................................. 25
  2.4 Consistency ................................................... 27
    2.4.1 Daily ..................................................... 29
    2.4.2 Intraday ................................................. 32
  2.5 Price discovery ............................................... 37
    2.5.1 Price discovery between VIX futures ETNs and VIX futures . 38
    2.5.2 Price discovery between VIX futures ETNs ..................... 43
### 3 Modeling VXX under Jump Diffusion with Stochastic Long-Term Mean

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>53</td>
</tr>
<tr>
<td>3.2</td>
<td>VIX futures indices</td>
<td>60</td>
</tr>
<tr>
<td>3.3</td>
<td>Model</td>
<td>60</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Model dynamics</td>
<td>60</td>
</tr>
<tr>
<td>3.3.2</td>
<td>VIX term structure</td>
<td>63</td>
</tr>
<tr>
<td>3.3.3</td>
<td>VIX futures</td>
<td>64</td>
</tr>
<tr>
<td>3.3.4</td>
<td>Idealistic model</td>
<td>66</td>
</tr>
<tr>
<td>3.3.5</td>
<td>Realistic model</td>
<td>69</td>
</tr>
<tr>
<td>3.4</td>
<td>Calibration</td>
<td>70</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Method</td>
<td>70</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Results</td>
<td>71</td>
</tr>
<tr>
<td>3.5</td>
<td>Model fit</td>
<td>77</td>
</tr>
<tr>
<td>3.5.1</td>
<td>VXX model fit</td>
<td>77</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Short-term VIX futures ETPs model fit</td>
<td>79</td>
</tr>
<tr>
<td>3.6</td>
<td>Conclusions</td>
<td>81</td>
</tr>
<tr>
<td>3.7</td>
<td>Appendix</td>
<td>85</td>
</tr>
<tr>
<td>3.7.1</td>
<td>Central moments of $\theta_s$</td>
<td>85</td>
</tr>
<tr>
<td>3.7.2</td>
<td>Central moments of $V_s$</td>
<td>86</td>
</tr>
<tr>
<td>3.7.3</td>
<td>VIX index proof</td>
<td>89</td>
</tr>
<tr>
<td>3.7.4</td>
<td>Partial derivatives for change in log short-term VIX futures</td>
<td>91</td>
</tr>
<tr>
<td>3.7.5</td>
<td>Floating $\theta$ model dynamics and VIX formula</td>
<td>92</td>
</tr>
</tbody>
</table>
4 The Implied Volatility Smirk in the VXX Options Market 93

4.1 Introduction .................................................. 93

4.2 Methodology .................................................. 99
  4.2.1 Implied forward price and ATM IV ................. 99
  4.2.2 Moneyness of options ................................. 99
  4.2.3 Quantifying the IV curve ............................ 100
  4.2.4 Risk-neutral moments ................................. 101

4.3 Data .......................................................... 102

4.4 Empirical Results .......................................... 103
  4.4.1 Quantified IV curve .................................... 103
  4.4.2 Constant maturity quantified IV curve ............ 112
  4.4.3 VXX option pricing model implications .......... 122

4.5 Conclusions ................................................. 122

5 Conclusion ...................................................... 125
List of Figures

2-1  Market share by ETP type ........................................... 12
2-2  VXX total dollar trading volume and market capitalization . 14
2-3  Daily relative tracking error: Short-term ETNs ................ 23
2-4  Daily relative tracking error: Mid-term ETNs ................. 24
2-5  Daily relative tracking error: Dynamic ETNs .................. 24
2-6  Intraday tracking performance: Short-term ETNs ............. 27
2-7  Intraday tracking performance: Mid-term ETNs ............... 28
2-8  Daily difference in IF: Short-term ETNs ....................... 32
2-9  Daily difference in IF: Mid-term ETNs ......................... 33
2-10 Intraday RMSD and R-squared: Short-term ETNs .......... 35
2-11 Intraday RMSD and R-squared: Mid-term ETNs ............. 36
2-12 Time variation in VIX futures ETN and VIX futures Granger causality ratios: Short-term ETNs ....................... 42
2-13 Time variation in VIX futures ETN and VIX futures Granger causality ratios: Mid-term ETNs .............................. 43
2-14 Time variation in VIX futures ETN pair Granger causality ratios: Short-term ETNs ................................. 47
2-15 Time variation in VIX futures ETN pair Granger causality ratios: Mid-term ETNs ..................................... 48
3-1  Market share by maturity target ................................. 55
<table>
<thead>
<tr>
<th>Page</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-2</td>
<td>Daily $V_t$ and $\theta_t$ - full model.</td>
</tr>
<tr>
<td>3-3</td>
<td>Daily MSE.</td>
</tr>
<tr>
<td>3-4</td>
<td>Model implied vs market VXX price.</td>
</tr>
<tr>
<td>3-5</td>
<td>Realistic model - full VIX term structure.</td>
</tr>
<tr>
<td>3-6</td>
<td>Realistic model - two point VIX term structure.</td>
</tr>
<tr>
<td>3-7</td>
<td>Realistic model - three point VIX term structure.</td>
</tr>
<tr>
<td>4-1</td>
<td>Option Volume and Open Interest.</td>
</tr>
<tr>
<td>4-2</td>
<td>IV against moneyness on 27 July 2011.</td>
</tr>
<tr>
<td>4-3</td>
<td>IV against moneyness on 2 August 2013.</td>
</tr>
<tr>
<td>4-4</td>
<td>IV against moneyness on 27 May 2015.</td>
</tr>
<tr>
<td>4-5</td>
<td>IV against moneyness on 29 July 2011: with restraint.</td>
</tr>
<tr>
<td>4-6</td>
<td>IV against moneyness on 2 August 2013: with restraint.</td>
</tr>
<tr>
<td>4-7</td>
<td>IV against moneyness on 27 May 2015: with restraint.</td>
</tr>
<tr>
<td>4-8</td>
<td>IV curves from mean factors.</td>
</tr>
<tr>
<td>4-9</td>
<td>Term structure of mean interpolated factors.</td>
</tr>
<tr>
<td>4-10</td>
<td>Time series of Interpolated ATM IV and forward prices.</td>
</tr>
<tr>
<td>4-11</td>
<td>Time series of interpolated IV factors.</td>
</tr>
<tr>
<td>4-12</td>
<td>Time series of Interpolated coefficients: constrained.</td>
</tr>
<tr>
<td>4-13</td>
<td>IV curves from mean interpolated coefficients.</td>
</tr>
</tbody>
</table>
List of Tables

1.1 Thesis Chapters ......................................................... 6

2.1 VIX futures ETN market summary. ................................. 15
2.2 Potential unhedged profit of ETN issuers. .................... 16
2.3 Daily tracking performance. ........................................... 22
2.4 Intraday tracking performance. ..................................... 26
2.5 Daily consistency. ....................................................... 31
2.6 Intraday consistency. ................................................... 34
2.7 Granger causality between VIX futures ETNs and VIX futures. 41
2.8 Granger causality of ETN Pairs. .................................... 45

3.1 Summary statistics of the daily returns for the SPX, VIX, VXX and XIV. ......................................................... 56
3.2 Calibrated parameters for Zhang, Zhen, Sun, and Zhao (2017) sample ................................................................. 72
3.3 Calibrated parameters. .................................................... 73
3.4 Model implied VXX and market VXX .............................. 78

4.1 Summary of SPX, SPY, VIX and VIX futures ETP option markets. ................................................................. 95
4.2 Summary of the VXX option market activity. .................. 104
Chapter 1

Introduction

From February 2012 until December 2014, I was completing my Master of Business in Finance at the University of Otago. I had Professor Jin Zhang as the lecturer of teaching Mathematical Finance during the course work part of the degree. I found the course very interesting and ended up being one of the top students. Professor Zhang and I met several times after the course and together we decided he would supervise my master’s thesis. I first encountered VIX futures exchange-traded products (ETPs) when I read the seminal paper on these products, Whaley (2013). I was captivated by the abnormal structure and complexity of this market. The most popular VIX futures ETP, the VXX, had lost over 99.6% of its value since inception, and so we set out to create a model that could explain this, which became the focus of my master’s thesis.

The process of my master’s thesis research went very smoothly and I was able to get our paper accepted for presentation by the 2014 Auckland Finance Meeting (60% rejection rate). With the joy from and success of my first research project, I decided an academic career was the correct path for me. A couple of months after applying, I was offered the University of Otago Doctoral Scholarship, which I swiftly accepted and then embarked on my PhD journey in May 2015.

Professor Zhang and I quickly decided that continuing to study the VIX futures
ETP market would allow us to make significant contributions, as the literature in this area was quickly growing in popularity along with the market’s size and trading activity. Specifically, in Chapter 2 we study the price discovery relationship between the VIX futures exchange-traded notes (ETNs) and the underlying futures, and the ETNs tracking performance and pricing consistency. Then in Chapter 3, we extend the model of my master’s thesis no published in a high ranking journal, Gehricke and Zhang (2018), and empirically test its pricing performance. Finally, in Chapter 4 we quantify and analyze the time series and term structure of the implied volatility curve of the VXX options market.

1.1 VIX futures ETPs

The VIX futures ETP market is a relatively new market for trading volatility exposure on a stock exchange and has become very popular.

VIX futures ETPs were first introduced in January 2009, immediately following the inception of the indices they track. The majority of the ETPs track the S&P 500 VIX short-term futures total return index (SPVXSTR) and its mid-term counterpart (SPVXMTR). The short-term (mid-term) index tracks a daily rolling position in the nearest and second nearest (fourth-, fifth-, sixth-, seventh- and eighth-nearest) VIX futures contracts with the goal of a constant one-month (five-month) maturity futures position.

The ETPs’ underlying, the VIX futures market, is somewhat more matured as the first futures began trading in 2004 and were followed by VIX options in 2006. The trading volume of the futures and options had been steadily increasing since inception as the instruments became more popular for both hedging and speculation. When the VIX futures ETPs were introduced, in 2009, the trading of the underlying futures really took off, likely due to the issuers and some of the traders of the ETPs seeking to hedge their exposure. The ETP market has quickly grown to a total market size of
over 4 billion USD and is very active with daily dollar trading volumes often exceeding multiples of the market capitalization.

In 2010, the first options on VIX futures ETPs began trading and have grown in activity and size ever since. These options and their dynamics are also examined in Chapter 3 of this PhD thesis.

1.2 Literature setting

Markets for volatility trading have grown in popularity in recent history and so has the literature on their performance, relationship to other markets and pricing. This thesis contributes to each strand of this literature with regard to the VIX futures ETP market.

The underperformance of the long-exposure VIX futures ETPs (VXX, TVIX etc) has been well documented (Whaley, 2013; Eraker and Wu, 2017; Bordonado et al., 2017). Gehricke and Zhang (2018) show that this underperformance is driven by the negative roll yield, which is generated by the rebalancing mechanism of the underlying VIX futures position. The roll yield is in turn related to the market price of variance risk. We confirm the conclusions of Gehricke and Zhang (2018), with an extended model, in Chapter 3. Eraker and Wu (2017) similarly find that the underperformance is driven by the negative variance risk premium of risk-adverse investors. Further, it has been shown that, contrary to early marketing material, VIX futures ETPs are not useful for diversification (Alexander, Korovilas, and Kapraun, 2016; Deng, McCann, and Wang, 2012; Hancock, 2013). A rational motivation for investors to trade these products is short-term hedging or speculation, which are both supported by the trading activity of these products.

The market micro-structure relationship between the VIX index, VIX futures and VIX futures ETPs has also been explored. Shu and Zhang (2012) and Frijns, Tourani-Rad, and Webb (2016) show that there is no lead-lag relationship between VIX futures
Chapter 1: Introduction

and the VIX index at the daily level, but that VIX futures clearly lead the VIX index intradaily. Bollen, O’Neill, and Whaley (2017) show that the VXX leads the futures in intraday price-discover, which is confirmed in Chapter 2 and extended to all of the VIX futures ETNs. The Bondarenko (2013) study examines the relationship between three pairs of VIX futures ETNs and ETFs, showing in its tables, but not analyzing, that ETNs lead the ETFs. In Chapter 2, we also study the tracking performance and consistency of the VIX futures ETN market.

There is also a vast literature focusing on the pricing of VIX futures (Zhang and Zhu, 2006; Zhang, Shu, and Brenner, 2010; Lu and Zhu, 2010; Dupoyet, Daigler, and Chen, 2011; Zhu and Lian, 2012; Huskaj and Nossman, 2013) and VIX options (Wang and Daigler, 2011; Chung, Tsai, Wang, and Weng, 2011; Cont and Kokholm, 2013; Lian and Zhu, 2013; Bardgett, Gourier, and Leippold, 2018; Papanicolaou and Sircar, 2014). However, the literature on pricing VIX futures ETPs is rather sparse (Gehricke and Zhang, 2018, Eraker and Wu, 2017 and our Chapter 3), and developing a model for pricing options on the ETPs has only been attempted once in the literature by Bao, Li, and Gong (2012). The authors price VXX options by assuming a stochastic process for the VXX directly. This is not the ideal way to define the dynamics in order to understand the pricing of the options, as it ignores the underlying relationships of the VXX with the VIX futures and S&P 500 options markets. This motivates our research in Chapter 4, where we study the empirical dynamics of the VXX option market providing a starting place for a more realistic option pricing model. Chapter 4 is also related to the vast literature on quantifying and analyzing option implied volatility in other markets.¹

1.3 PhD thesis structure

Chapters 2, 3 and 4 of this thesis comprise three independent but highly-related papers. A summary of the papers can be found in Table 1.1, along with the contributions made by the candidate.

After deciding to further study the VIX futures ETN market during my PhD and reading the papers by Shu and Zhang (2012) and Frijns, Tourani-Rad, and Webb (2016), which studied the daily- and intradaily-level price discovery relationship between the VIX index and its futures, respectively, I quickly became aware of the lack of micro-structure research on the VIX futures ETN market. Frijns, Tourani-Rad, and Webb (2016) show that, at the intraday level, VIX futures clearly led the VIX index, although the strength of the relationship fluctuates and this fluctuation was driven by market conditions. The suggested reason for the informational dominance of the futures was that they are easily traded, whereas replicating the VIX index involves a very vast and complicated S&P 500 options portfolio; therefore information would be absorbed faster by the futures market. This made me question whether the same phenomenon existed between the futures and the ETNs, as they are also much easier to trade than the futures, since they trade like stocks. Also, several papers had talked about the influence of the VIX futures ETP issuers hedging activity on the futures prices, namely Alexander and Korovilas (2013), Mixon and Onur (2015) and Fernandez-Perez, Frijns, Tourani-Rad, and Webb (2015).

In our empirical study, *The VIX Futures ETN Market*, Professor Zhang and I found that the VIX futures ETNs lead the VIX futures in price discovery and that the strength of the relationship is time varying. We also document findings on the tracking performance and pricing consistency of the ETNs at the daily- and intradaily-level. The paper was presented at the 2016 Auckland Finance Meeting (roughly 60% rejection rate), and is currently under revision and will be resubmitted for publication with a high quality journal.
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Thesis Chapters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The VIX futures ETN market</td>
</tr>
<tr>
<td></td>
<td>Undertook all data collection and processing, methodology design and implementation, results analysis and writing. Coauthor provided comments on research question design, methodology and placement of our paper in the literature. Submitted for publication and to be presented at the 2018 Auckland Derivatives Conference and accepted for presentation at the 2018 New Zealand Finance Conference and for inclusion in the 2018 New Zealand Finance Colloquium and for submission to a journal.</td>
</tr>
<tr>
<td>2</td>
<td>Modeling VXX under jump diffusion with stochastic long-term mean and stochastic long-term volatility</td>
</tr>
<tr>
<td></td>
<td>Undertook formula derivation and testing and processing, model testing and processing. Coauthor provided guidance in model design and formula derivation. He also participated in analyzing the calibration results.</td>
</tr>
<tr>
<td>3</td>
<td>The implied volatility smirk in the VXX options market</td>
</tr>
<tr>
<td></td>
<td>Undertook all data collection and processing, methodology development and implementation, analysis and writing. Coauthor provided guidance in finding the topic and identifying the key literature.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 1.1: Thesis Chapters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 1: Introduction</td>
</tr>
<tr>
<td>Table 1.1: Thesis Chapters</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Upon completing Chapter 2, the paper derived from my master’s thesis (Gehricke and Zhang, 2018) had been accepted for publication by the *Journal of Futures Markets*. That paper also provided me an expenses-paid academic visitation to the Reserve Bank of New Zealand to present my research. This confirmed our confidence that the research was important, and so in the next paper (Chapter 3) we wanted to improve and empirically test our VXX model. We extended the model developed in Gehricke and Zhang (2018) by including jumps in the volatility process and by making the long-run mean of the volatility a mean-reverting process itself.

Including jumps in the volatility process, as in Duffie, Pan, and Singleton (2000), is important in capturing the volatility smile (Bardgett et al., 2018) and is better for pricing short-term VIX futures (Lin, 2007). Making the long-run mean of volatility itself mean-reverting allows for more realistic transient changes in the VIX and VIX futures term structures (Zhang and Huang, 2010; Zhang, Shu, and Brenner, 2010; Zhang, Zhen, Sun, and Zhao, 2017). We calibrated the new VXX model to the VIX term-structure data and found that it outperforms the nested models, including that of Gehricke and Zhang (2018), in fitting the VIX term structure. We further show that the model fits the VXX, and other short-term VIX futures ETPs very well. The model could therefore be extended to price VXX options in future research. In November 2017, I finished a joint paper with my supervisor, *Modeling VXX Under Jump Diffusion with Stochastic Long-term Mean*, presented it at the 2018 New Zealand Finance Colloquium and it will be presented at the 2018 New Zealand Finance Meeting.

The SPX, VIX and VXX options markets are all intertwined in their risk exposure and therefore developing a model which prices all three consistently is a major goal of mine. We have already created a model that works well for the VIX term structure and VIX futures ETPs (Chapter 3). Next we plan to model the options written on the VXX. In order to do this, we need to document and understand the dynamics of
this options market, which is the purpose of Chapter 4 in this thesis.

In Chapter 4 we quantify, document and analyze the dynamics of the implied volatility curve of VXX options. We quantify the curve through the level, slope and curvature factors. We analyze the time series and term structure of these, giving us a good idea of how the options are priced. In April 2018, I finished a paper jointly with my supervisor, *The Implied Volatility Smirk in the VXX Options Market*, which has been presented at the 2018 Derivative Markets Conference and will be presented at the 2018 New Zealand Finance Meeting. It is now under review by a good journal for publication.

The three working papers discussed in this section are as follows:


2. **Sebastian A., Gehricke** and Jin E. Zhang, 2017, Modeling VXX under Jump Diffusion with Stochastic Long-term Mean, Presented at the 2018 New Zealand Finance Colloquium and accepted for presentation at the 2018 New Zealand Finance Meeting [Chapter 3]


### 1.4 Contribution of this PhD thesis

Chapter 2 contributes to the literature by showing that all VIX futures ETNs clearly lead the underlying futures contracts while accounting for the structure of the un-
derlying portfolio of the VIX futures indices.\(^2\) We also provide the first analysis of the tracking performance and pricing consistency, daily and intradaily, of the VIX futures ETN market.

In Chapter 3, we provide a new model for the VXX and other short-term VIX futures ETPs, which captures the link between the S&P 500 index, the VIX index, VIX futures and the VXX. We calibrate the model and show that it outperforms the model of Gehricke and Zhang (2018) and other nested models in fitting the VIX term structure. Our model performs well in fitting the VXX and other short-term VIX futures ETPs. The model could be extended to price VXX and other VIX futures ETP options.

Lastly, in Chapter 4 we provide the first study on the dynamics of the VXX option market. We extend the methodology developed by Zhang and Xiang (2008) to quantify the shape of the VXX option implied volatility curve. We study the time series and term structure of the implied volatility curve, documenting empirical characteristics which will be useful for future development of a VXX option pricing model.

\(^2\)Bollen, O’Neill, and Whaley (2017) show, consistently with our results, that the VXX leads VIX futures in price discovery, but they do not consider any other ETNs and use a different approach.
Chapter 2

The VIX Futures ETN Market

This Chapter is joint work with Jin E. Zhang. An earlier version was presented at the 2016 Auckland Finance Meeting, 16-18 December 2016, AUT, Auckland, New Zealand.

2.1 Introduction

The VIX futures ETN market has become one of the most popular destinations for speculating on and/or hedging volatility risk.\(^1\) In this Chapter we examine the tracking performance, price consistency and price discovery dynamics of this young but well-developed market. The VIX futures ETNs provide easy access to volatility exposure for anyone, as they trade on a stock exchange, but they are complex debt instruments which are yet to be fully understood by sophisticated market participants, academics and retail investors. The previous literature has done some preliminary work on the tracking performance of this market at the daily level (Whaley, 2013), but no effort has been spent on the consistency of the prices of the ETNs that track

---

\(^1\)An ETN is a non-securitized debt obligation, like a zero coupon bond, that has a final redemption value based on the value of some underlying benchmark. The VIX futures ETN market has consistently made up about 60% of the market capitalization and almost 75% of the trading volume of the VIX futures ETP market, since the launch of VIX future exchange-traded funds (ETFs), as can be observed in Figure 2-1.
the same underlying VIX futures index. The price discovery dynamics of VIX futures ETN market has been a more popular research topic, but has not been studied for its entirety. We find that VIX futures ETNs do not track their benchmarks as well as investors may expect and are not priced very consistently to each other, at the daily or intradaily level. We also show that this market leads the underlying VIX futures market, which it is meant to track. Lastly, we show that different ETNs seem to lead in price discovery at different times.

“We are concerned that end-investors may not be clear about the types of products that they are investing in” Jennifer Grancio, managing director of iShares told the Financial Times (Makan, 2012). There are many valid concerns about ETNs, as they do not restrict the issuer to holding the underlying assets, are not void of counterparty risk, come with built-in early redemption and early close out options and new share creation is fully at the discretion of the issuer. These characteristics can lead to huge deviations of market prices from fair value. This was extremely obvious on 21 February 2012 when Credit Suisse halted issuance of new shares in the TVIX ETN,
which subsequently led to the TVIX trading at a tracking error of over 85% relative to its indicative value. This type of discrepancy and the complex and likely overlooked characteristics of the VIX futures ETNs motivate our main research question. Are VIX futures ETN investors getting the volatility exposure they are expecting? To answer this question we first investigate the tracking performance and consistency of the ETNs. We show convincing evidence that the ETNs do not track their benchmarks very well and are not priced all that consistently to each other. It is important for investors to know that these products are cheap and easy alternatives, but are by no means perfect replications of the underlying VIX futures positions.

Despite this imperfection the VIX futures ETN market continues to grow, becoming a very popular market for investors seeking volatility exposure. The total market cap for the VIX futures ETN market, as of 31 March 2016, was $US 2.6 billion and it has a average daily dollar trading volume of $US 1.6 billion, making it a substantially large and active market, especially for its age. In figure 2-2 we show this growth in trading activity graphically and table 2.1 provides a summary of the VIX futures ETNs we study in this Chapter. In the figure we can see that often the dollar trading volume spikes to many multiples of the market capitalization, even in the moving average depiction, the exploration of this phenomena is left for future research. The table shows the severe under (out) performance by long (short) exposure ETNs, which is in line with the findings in the literature (Whaley, 2013; Eraker and Wu, 2017; Bordonado, Molnár, and Samdal, 2017). Previously the variance swap market has been the primary market for trading volatility, but now the VIX futures ETP market is comparable in size (Dew-Becker, Giglio, Le, and Rodriguez, 2017) and is traded more throughout the day. The growth of the VIX futures ETN market has led to huge potential profits by the issuers, as shown in table 2.2.² With this market growth comes another question, is the VIX futures ETN market following, leading or

²The table shows the maximum profits the issuers could have made if they did not hedge their exposure.
simultaneously moving with the VIX futures market? In an open letter to investors on 29 June 2013, Mark Wiedman, global head of iShares, offered a declaration explaining why ETFs are often trading at a discount or premium to their Net Asset Value (NAV) (Wiedman, 2013).³ He says that “More and more, ETFs are becoming the true market”, and also states “The ETF price can become the true price for that market, and the underlying assets may eventually catch up with any gap...” (Wiedman, 2013). Although, the letter is in regard to ETFs it is synonymously relevant to ETNs. In this Chapter we show that the VIX futures ETN market is actually leading the VIX futures market in price discovery, or in similar terms to Mr. Wiedman we could say that it has become the true market.

Figure 2-2: VXX total dollar trading volume and market capitalization
In the first diagram we plot the 10-day moving average of the total daily dollar trading volume of the VIX futures ETN market. In the second diagram we plot the 10-day moving average of the daily total market capitalization of the VIX futures ETN market.

Shu and Zhang (2012) show that neither the VIX futures or VIX index leads in

³The indicative value for ETNs is synonymous with the NAV for ETFs.
<table>
<thead>
<tr>
<th>ETN Ticker</th>
<th>Issuer</th>
<th>Inception</th>
<th>Leverage Multiplier</th>
<th>ST or MT index</th>
<th>Fee</th>
<th>Mean Daily Return</th>
<th>St. Dev. Return</th>
<th>HPR</th>
<th>Market Cap ($000)</th>
<th>Average Daily Trading Volume ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VXX</td>
<td>Barclays</td>
<td>29/01/2009</td>
<td>1</td>
<td>ST</td>
<td>0.89%</td>
<td>-0.18%</td>
<td>4.07%</td>
<td>-97.06%</td>
<td>1,130,420.00</td>
<td>1,092,338.97</td>
</tr>
<tr>
<td>VXZ</td>
<td>Barclays</td>
<td>29/01/2009</td>
<td>1</td>
<td>MT</td>
<td>0.89%</td>
<td>-0.11%</td>
<td>2.03%</td>
<td>-82.96%</td>
<td>41,362.40</td>
<td>17,475.04</td>
</tr>
<tr>
<td>XIV</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>-1</td>
<td>ST</td>
<td>1.35%</td>
<td>0.14%</td>
<td>4.07%</td>
<td>113.27%</td>
<td>898,783.00</td>
<td>357,253.87</td>
</tr>
<tr>
<td>ZIV</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>-1</td>
<td>MT</td>
<td>1.35%</td>
<td>0.08%</td>
<td>2.01%</td>
<td>189.75%</td>
<td>90,002.70</td>
<td>2,138.99</td>
</tr>
<tr>
<td>VIIX</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>1</td>
<td>ST</td>
<td>0.89%</td>
<td>-0.19%</td>
<td>4.11%</td>
<td>-97.00%</td>
<td>7,645.40</td>
<td>6,465.01</td>
</tr>
<tr>
<td>VIIZ</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>1</td>
<td>MT</td>
<td>0.89%</td>
<td>-0.07%</td>
<td>1.94%</td>
<td>-82.75%</td>
<td>607.6</td>
<td>179.72</td>
</tr>
<tr>
<td>TVIX</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>2</td>
<td>ST</td>
<td>1.65%</td>
<td>-0.43%</td>
<td>7.74%</td>
<td>-99.99%</td>
<td>456,023.00</td>
<td>91,119.51</td>
</tr>
<tr>
<td>TVIZ</td>
<td>Credit Suisse</td>
<td>29/11/2010</td>
<td>2</td>
<td>MT</td>
<td>1.65%</td>
<td>-0.23%</td>
<td>4.04%</td>
<td>-98.39%</td>
<td>816.5</td>
<td>205.62</td>
</tr>
<tr>
<td>XXV</td>
<td>Barclays</td>
<td>16/07/2010</td>
<td>-1</td>
<td>ST</td>
<td>0.89%</td>
<td>0.02%</td>
<td>0.69%</td>
<td>14.64%</td>
<td>684.3</td>
<td>1,634.20</td>
</tr>
<tr>
<td>XVZ</td>
<td>Barclays</td>
<td>17/08/2011</td>
<td>Dynamic ST &amp; MT</td>
<td>-0.04%</td>
<td>1.10%</td>
<td>-47.18%</td>
<td>9,409.40</td>
<td>1,550.64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>XVIX</td>
<td>UBS</td>
<td>30/11/2010</td>
<td>Dynamic ST &amp; MT</td>
<td>0.05%</td>
<td>0.77%</td>
<td>-36.41%</td>
<td>11,842.50</td>
<td>284.22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CVOL</td>
<td>Citi</td>
<td>17/11/2010</td>
<td>Dynamic CVOLTR</td>
<td>1.15%</td>
<td>0.35%</td>
<td>6.61%</td>
<td>-99.95%</td>
<td>2,453.50</td>
<td>248.10</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.1: VIX futures ETN market summary.**

This table shows a summary for the VIX futures ETNs that are still traded today. It shows the issuer, inception date, leverage with respect to the underlying index, whether the ETN tracks a short-term or mid-term index as the underlying and its investor fee. We also report the mean daily return, standard deviation of daily returns, the holding period return (HPR) from 20th December 2010 to the 31st March 2016 and the market capitalization as of the 31st March 2016, for each of the ETNs.
Chapter 2: The VIX Futures ETN Market

<table>
<thead>
<tr>
<th>ETN ticker</th>
<th>Issuer</th>
<th>Potential Profit</th>
<th>Total Potential Profit for Issuer</th>
</tr>
</thead>
<tbody>
<tr>
<td>VXX</td>
<td>Barclays Capital iPath</td>
<td>5,822,950,802.10</td>
<td></td>
</tr>
<tr>
<td>VXZ</td>
<td></td>
<td>340,418,370.00</td>
<td></td>
</tr>
<tr>
<td>XXV</td>
<td></td>
<td>−16,228,510.00</td>
<td></td>
</tr>
<tr>
<td>XVZ</td>
<td></td>
<td>164,649,462.10</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>−6,311,790,124.20</td>
<td></td>
</tr>
<tr>
<td>XIV</td>
<td>Credit Suisse Velocity Shares</td>
<td>−672,953,998.60</td>
<td></td>
</tr>
<tr>
<td>ZIV</td>
<td></td>
<td>−22,395,643.30</td>
<td></td>
</tr>
<tr>
<td>VXIX</td>
<td></td>
<td>68,457,263.80</td>
<td></td>
</tr>
<tr>
<td>VIIZ</td>
<td></td>
<td>9,325,272.00</td>
<td></td>
</tr>
<tr>
<td>TVIX</td>
<td></td>
<td>1,771,864,922.10</td>
<td></td>
</tr>
<tr>
<td>TVIZ</td>
<td></td>
<td>13,179,007.20</td>
<td></td>
</tr>
<tr>
<td>XVIX</td>
<td>UBS E-TRACS</td>
<td>−8,071,436.30</td>
<td>−8,071,436.30</td>
</tr>
<tr>
<td>CVOL</td>
<td>Citi Bank C-TRACS</td>
<td>658,298,500.00</td>
<td>658,298,500.00</td>
</tr>
</tbody>
</table>

Table 2.2: Potential un-hedged profit of ETN issuers.

This table shows the potential profit to the ETN issuer, if they did not hedge their ETN exposures other than through inverse ETNs, for each ETN and each issuer’s total. The potential profit is calculated using the change in shares outstanding multiplied by the closing indicative value, each day. Then summing this across the whole sample, adding the initial number of shares issued times the initial indicative value and subtracting the number of shares outstanding times the closing indicative value on the 31st March 2016.

price discovery, using daily data. However, Frijns, Tourani-Rad, and Webb (2016) use intradaily data and find strong evidence for a one-way Granger causality from VIX futures to the VIX index, suggesting that price discovery happens first in the futures market. Bollen, O’Neill, and Whaley (2017) show, consistently with our results, that the VXX leads VIX futures in price discovery, but do not consider any other ETNs. Chen and Tsai (2017) confirm, using the Hasbrouck (1995) information share and Gonzalo and Granger (1995) common factor component weight price discovery measure, that VIX futures dominate the VIX index in price and that they dominate more when the VIX futures basis is high and around macroeconomic news announcements. The authors also have a small section on the price discovery relationship between VIX futures and VIX futures ETPs. They claim that VIX futures also dominate the ETPs in price discovery, contrary to the results of Bollen, O’Neill, and Whaley (2017) and ours.4 Bordonado, Molnár, and Samdal (2017) study the price discovery between

---

4The difference in the findings of Chen and Tsai (2017) could be explained by several differences in methodology. The authors use a less granular sampling frequency, in order to avoid micro-structure effects, but price discovery is a micro-structure effect and as stated by Bollen, O’Neill, and Whaley (2017) the lead/lag relationship will be short lived as it is an arbitrage relation. The authors use just
three pairs of the most traded VIX futures ETFs and ETNs, for non-leveraged and leveraged types, and their results show that the ETNs lead the ETFs.\footnote{One of the pairs they study includes the TVIX, which we show has behaved very differently to the other ETNs since an issuance halt event in 2012. This event has caused the results of our study to differ from the norm, whenever the TVIX ETN is considered. Their results for the two pairs not including the TVIX are more reliable and show that VIX futures ETNs are leading the ETFs.}

Combining our result with those of the previous literature, we can conclude that the VIX futures ETN market has become the primary market for trading VIX exposure, which could also explain why the ETNs do not track their benchmark well contemporaneous. This dominance of VIX futures ETNs is likely a result of the ease of access to trading in this market compared with the other volatility derivatives, as well as the hedging activity of the ETN issuers.\footnote{Alexander and Korovilas (2013), Mixon and Onur (2015) and Fernandez-Perez, Frijns, Tourani-Rad, and Webb (2015) show the impact of the issuers hedging activity on VIX futures prices.}

Lastly, we investigate the lead-lag relationship between different ETNs tracking the same underlying index. Since the ETNs are not consistently priced, maybe some of them are dominating the price discovery process. We find that there is no clear leading ETN overall, but that there are times where one ETN leads another. This finding is consistent with Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad (2018), who study the price discovery dynamics between the VXX and XIV ETNs. They find that the price discovery of the VXX (relative to the XIV) increases with greater institutional ownership and on those days where the level of the VIX is high. We confirm that the relationship between the VXX and XIV, as well as other ETN pairs, is time varying and is a two-way lead-lag relationship on average.

Several other branches of literature have evolved around the VIX index and its derivatives in recent years. One example, due to the negative correlation between the VIX and S&P 500 index, the usefulness of VIX derivatives as hedging or diversification
tools (Daigler and Rossi, 2006; Szado, 2009; Chen, Chung, and Ho, 2011; Alexander, Korovilas, and Kapraun, 2016; Hancock, 2013; Bordonado, Molnár, and Samdal, 2017). Another very popular branch is that of modeling the VIX and its derivatives. VIX futures price modeling has been done extensively and ever more precisely in recent studies (Zhang and Zhu, 2006; Zhang, Shu, and Brenner, 2010). Mencía and Sentana (2013) empirically test the performance of several models for pricing VIX derivatives. Gehricke and Zhang (2018) model the VXX using the Heston (1993) framework and show that the negative returns of the VXX ETN are due to the roll yield of the underlying futures position. We extend their model in Chapter 3 confirming their conclusions and showing the usefulness of the model empirically.

The rest of the Chapter is organized as follows. In the next section we will describe the data used in this study. Then in section 2.3 we study the tracking performance of the VIX futures ETN prices. Section 2.4 presents the analysis of the consistency of the different ETN prices, relative to each other. In section 2.5 we explore the price discovery relationship between the VIX futures ETNs and VIX futures as well as the price discovery relationship between the different ETN pairs. Lastly in section 2.6 we offer our conclusions and discussions.

2.2 Data

In this study we use both daily and intradaily data for VIX futures ETNs and their indicative values. We do not include those ETNs which have been fully recalled or become stale due to the specification of their indicative methodologies. We exclude these ETNs because understanding their price dynamics is no longer of relevant importance.

All daily data are obtained from the Bloomberg Professional service. The one-second interval intraday quote and 15-second interval intraday indicative value data are taken from the Thomson and Reuters Tick History (TRTH), via SIRCA. Our
sample is from 20 December 2010 until 31 March 2016. However some ETNs’ inception is after the start date, so they are included from their inception date onward. There are also some limitations to the data availability of the intraday indicative values for some of the ETNs, therefore the samples for these ETNs will end earlier than the others.\footnote{Specifically the intraday indicative value data for the Credit Suisse issued ETNs are only available continuously and reliably until mid 2014.}

When we study the daily tracking performance, we use closing prices and indicative values. We also study the intraday tracking performance and the intraday price discovery relationship between the VIX futures and the VIX futures ETNs. For these analyses we sample the mid-quote price at the same 15-second intervals at which the indicative values are reported.\footnote{We choose to use the 15-second interval data as this is the smallest frequency for which the indicative value is reported. The empirical high-frequency volatility estimation/forecasting literature often uses frequencies of one to five minutes to avoid micro-structure effects (Andersen, Bollerslev, and Diebold, 2007a; Andersen, Bollerslev, and Dobrev, 2007b; Andersen, Bollerslev, and Huang, 2011; amongst others). However, Bollen et al. (2017) argue that price discovery is an micro-structure effect as the lead/lag relationship will be short-lived because it is a arbitrage relation, the shorter frequency is common in the price-discovery literature.} We also remove any days when the Chicago futures exchange (CFE) was closed or closed early, as the intraday indicative value is stale on those days.

When we study the intraday consistency and the price discovery relationships between the ETN pairs we use the one-second interval sampled mid-quote data. For this analysis we do not need to sample the one second data at a 15 second frequency, as all of the ETN quotes have data available at the one-second frequency (unlike the indicative values). We also do not need to remove data for CFE holidays for this analysis, as we do not use indicative value data.

\section{2.3 Tracking performance}

Whaley (2013) and Eraker and Wu (2017) study the tracking performance of VIX futures ETPs relative to the underlying VIX futures index, at the daily level, and
conclude that they track the indices fairly well. In this section we, however, study how
well the VIX futures ETNs track their indicative values, at the daily- and intradaily-
level. The indicative value represents the value an investor of the ETN would receive
at maturity or upon early redemption. Therefore we believe the ETNs should track
their indicative values rather than the index.\footnote{For the readers convenience we have summarized the indicative value methodologies for each of the ETNs in the appendix, section 2.7} Aroskar and Ogden (2012) investigate
the tracking performance of a general sample of ETNs (VIX futures ETNs excluded)
relative to their indicative value and underlying index and find that the ETNs track
their indicative values better than their underlying indices; therefore these seem to be
the better benchmark empirically. They also find varying tracking performance across
different asset class ETNs, further motivating our investigation into VIX futures ETN
tracking performance.

2.3.1 Daily

We first study the tracking performance of the VIX futures ETNs at the daily-level
using daily close prices and closing indicative values. To study the daily tracking
performance of the ETNs we use some of the measures of tracking performance that
are common in the literature (Aroskar and Ogden (2012); Whaley (2013)).

First we estimate the following tracking performance regression for each ETN over
the full sample:

\[
R_t^{MV} = \alpha + \beta R_t^{IV} + \epsilon, \tag{2.1}
\]

where $R_t^{MV}$ is the daily log return calculated using the closing price of the ETN and
$R_t^{IV}$ is the daily log return of the closing indicative value of the ETN on day $t$.\footnote{We do not run this regression in the levels of the data because the levels are not stationary, which would result in spurious regression results.}
root mean squared difference (RMSD) and the student-t test statistic and Wilcoxon test statistic for the null hypothesis of zero mean difference in the log returns of the market prices and indicative values. In addition, we estimate all of these tracking performance measures in the levels, of closing price and indicative values, as well.

All of the measures mentioned above are reported in table 2.3. We can see from the statistics reported in panel A that about half of the tracking regressions have high R-squared values (over 80%), an $\alpha$ not significantly different to zero suggesting that those ETNs track their indicative values reasonably well. However, all of the ETNs exhibit a $\beta$ coefficient significantly different to one, suggesting that the tracking performance for the ETNs, at the daily level, is not great. Looking at the join hypothesis test, testing $\alpha = 0$ and $\beta = 1$, we can see that the only ETN for which the null hypothesis is not rejected is the VXZ. The table also shows that the mean difference between ETN indicative and market value returns is zero on average, with neither the student t-test or the non-parametric Wilcoxon test rejecting the null hypothesis of zero mean difference. However, turning to the MAD and RMSD measures we can see that these are non-zero for all of the ETNs, providing further evidence that the ETNs tracking performance is not great, at the daily level. Further, the VIIZ, TVIX, TVIZ, XVZ and XVIX ETNs exhibit tracking regression R-squared values range from 52.79% to 75.26% which is quite low and the CVOL ETN tracks its indicative value returns very poorly with an R-squared value of 17.18%. The CVOL ETN again exhibits the largest tracking error based on the MAD and RMSD measures. The discrepancy in tracking performance, as shown by the R-squared values, may be explained by the trading activity in the markets, it seems that the more liquid ETNs have slightly larger R-squared values. Also, the dynamic strategy ETNs have poorer tracking performance possibly due to the complexity of their benchmarks.

In panel B of table 2.3 we can see the tracking performance measures in terms of levels (rather than returns). The results for the tracking performance in levels further
tracking regressions (defined in equation (2.1)), the F-statistic for the joint test of

This table summarizes the tracking performance of the VIX futures ETN market, at the daily

<table>
<thead>
<tr>
<th>ETN Ticket</th>
<th>MD</th>
<th>T-stat</th>
<th>W-stat</th>
<th>MAD</th>
<th>RMSD</th>
<th>α</th>
<th>β</th>
<th>F − statistic</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A:</td>
<td>Returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VXX</td>
<td>0.00%</td>
<td>0.00</td>
<td>5094.0</td>
<td>0.80%</td>
<td>1.23%</td>
<td>0.00</td>
<td>0.85***</td>
<td>24.9***</td>
<td>90.70%</td>
</tr>
<tr>
<td>VXX</td>
<td>0.00%</td>
<td>0.01</td>
<td>−79.0</td>
<td>0.51%</td>
<td>0.76%</td>
<td>0.00</td>
<td>0.94**</td>
<td>2.1</td>
<td>84.98%</td>
</tr>
<tr>
<td>XIV</td>
<td>0.00%</td>
<td>−0.01</td>
<td>−5193.5</td>
<td>0.88%</td>
<td>1.42%</td>
<td>0.00</td>
<td>0.84***</td>
<td>21.52***</td>
<td>86.63%</td>
</tr>
<tr>
<td>ZIV</td>
<td>0.00%</td>
<td>−0.16</td>
<td>6130.5</td>
<td>0.65%</td>
<td>0.94%</td>
<td>0.00</td>
<td>0.91***</td>
<td>3.88***</td>
<td>82.57%</td>
</tr>
<tr>
<td>VIX</td>
<td>0.00%</td>
<td>0.04</td>
<td>419.5</td>
<td>0.89%</td>
<td>1.34%</td>
<td>0.00</td>
<td>0.86***</td>
<td>22.43***</td>
<td>90.72%</td>
</tr>
<tr>
<td>VIIZ</td>
<td>0.00%</td>
<td>−0.02</td>
<td>−1540.5</td>
<td>0.88%</td>
<td>1.34%</td>
<td>0.00</td>
<td>0.68***</td>
<td>20.55***</td>
<td>52.79%</td>
</tr>
<tr>
<td>TVIX</td>
<td>0.00%</td>
<td>0.03</td>
<td>5558.5</td>
<td>2.00%</td>
<td>3.17%</td>
<td>0.00</td>
<td>0.79***</td>
<td>17.58***</td>
<td>75.26%</td>
</tr>
<tr>
<td>TVIZ</td>
<td>0.00%</td>
<td>0.05</td>
<td>−5334.0</td>
<td>1.61%</td>
<td>2.32%</td>
<td>0.00</td>
<td>0.84***</td>
<td>8.04***</td>
<td>71.86%</td>
</tr>
<tr>
<td>XXV</td>
<td>−0.01%</td>
<td>−0.53</td>
<td>−1423.5</td>
<td>0.15%</td>
<td>0.24%</td>
<td>0.00</td>
<td>0.90***</td>
<td>15.98***</td>
<td>93.24%</td>
</tr>
<tr>
<td>XVZ</td>
<td>0.00%</td>
<td>−0.01</td>
<td>−4147.0</td>
<td>0.48%</td>
<td>0.73%</td>
<td>0.00</td>
<td>0.75***</td>
<td>14.31***</td>
<td>59.17%</td>
</tr>
<tr>
<td>XVIX</td>
<td>−0.01%</td>
<td>−0.23</td>
<td>−4217.5</td>
<td>0.55%</td>
<td>0.77%</td>
<td>0.00</td>
<td>0.61***</td>
<td>37.42***</td>
<td>49.68%</td>
</tr>
<tr>
<td>CVOL</td>
<td>0.00%</td>
<td>0.00</td>
<td>9470.5</td>
<td>4.78%</td>
<td>9.29%</td>
<td>−0.01</td>
<td>0.38***</td>
<td>47.38***</td>
<td>17.18%</td>
</tr>
<tr>
<td>Panel B:</td>
<td>Levels</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VXX</td>
<td>0.28</td>
<td>0.84</td>
<td>44144.0</td>
<td>4.92</td>
<td>14.13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VXX</td>
<td>−0.05</td>
<td>−8.78***</td>
<td>−185151.0***</td>
<td>0.16</td>
<td>0.27</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XIV</td>
<td>0.00</td>
<td>0.50</td>
<td>−3557.0***</td>
<td>0.15</td>
<td>0.26</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ZIV</td>
<td>0.02</td>
<td>4.20***</td>
<td>66697.0***</td>
<td>0.14</td>
<td>0.21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VIX</td>
<td>−0.03</td>
<td>−0.31</td>
<td>25034.0</td>
<td>1.65</td>
<td>3.56</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VIIZ</td>
<td>−0.02</td>
<td>−0.71</td>
<td>−12158.0</td>
<td>0.30</td>
<td>0.73</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TVIX</td>
<td>70.70</td>
<td>4.81***</td>
<td>237674.0***</td>
<td>195.20</td>
<td>543.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TVIZ</td>
<td>−0.41</td>
<td>−2.72***</td>
<td>−53262.0***</td>
<td>2.36</td>
<td>5.41</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XXV</td>
<td>−0.01</td>
<td>−3.99***</td>
<td>−36569.0***</td>
<td>0.04</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XVZ</td>
<td>−0.02</td>
<td>−2.73***</td>
<td>−62718.0***</td>
<td>0.14</td>
<td>0.22</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XVIX</td>
<td>−0.02</td>
<td>−4.63***</td>
<td>−56844.5***</td>
<td>0.08</td>
<td>0.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CVOL</td>
<td>0.19</td>
<td>1.74</td>
<td>58181.0***</td>
<td>1.39</td>
<td>4.06</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3: Daily tracking performance.

This table summarizes the tracking performance of the VIX futures ETN market, at the daily level. The table displays the mean difference (MD) between the closing price and indicative value, student-t test statistic for the null hypothesis of zero mean difference (T-stat), Wilcoxon test statistic for the null hypothesis that the difference follows a symmetric distribution around zero (W-stat), mean absolute difference (MAD), root mean squared difference (RMSD), α and β coefficients of the tracking regressions (defined in equation (2.1)), the F-statistic for the joint test of α = 0 and β = 1 and the R-squared values. The β coefficient is tested against 1 and significance indicated. Panel A shows the results for log returns of the closing price and indicative values, where as panel B displays the results for the levels.

support the evidence of panel A, that none of the ETNs track their indicative values very well, not even the vastly popular and extremely liquid VXX and XIV ETNs. Although, none of the ETNs seem to track their indicative values well, there is vast variation in the tracking performance of the different ETNs.

The MD, MAD and RMSD of the TVIX ETN, in levels, are orders of magnitude larger than those of the other ETNs, even though it is one of the most liquid ETNs (see table 2.1). As described in Whaley (2013), this is due to Credit Suisse’s halt of TVIX share issuance on the 21st February 2012. The long demand on the TVIX could not be met by newly issued notes, and the price skyrocketed to a maximum
relative tracking error of 89.43% on the 21st March 2012. Ever since this event the TVIX has traded at a higher premium than before.\textsuperscript{11}

Figures 2-3, 2-4 and 2-5 display the daily relative tracking error.\textsuperscript{12} From the figures we can see that all of the ETNs exhibit at least some days with large tracking errors over the sample, although the XXV ETN seems to track its indicative value the best. The XIV and TVIZ exhibit more tracking error than most of the other ETNs, apart from the TVIX, which exhibits a lot of tracking error ever since the share issuance halt event. The CVOL ETN, which has exhibited the worst tracking performance according to the estimates in table 2.3, actually seems to track its indicative value really well until the end of 2012, after which it exhibits huge relative tracking errors, this corresponded with a substantial decrease in the trading activity.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2-3.png}
\caption{Daily relative tracking error: Short-term ETNs}
\end{figure}

This figure depicts the daily relative tracking error between the closing price and indicative value for the short-term ETNs.

\textsuperscript{11}The average relative tracking error for the TVIX ETN was -0.08% before 21st of March 2012 and 4.06% thereafter (if we start the ‘after’ period in April 2012, then it is 3.57%).

\textsuperscript{12}The relative tracking error is calculated as the difference between the market value and indicative value as a proportion of the indicative value.
Figure 2-4: Daily relative tracking error: Mid-term ETNs
This figure depicts the daily relative tracking error between the closing price and indicative value for the mid-term ETNs.

Figure 2-5: Daily relative tracking error: Dynamic ETNs
This figure depicts the daily relative tracking error between the closing price and indicative value for the dynamic ETNs.
In summary, the tracking performance of the VIX futures ETN market as a whole, at the daily level, is not great. All of the ETNs exhibit some substantial tracking error at one point or another. However, the magnitude and timing of the deviation varies significantly for different ETNs. The imperfect tracking performance could possibly be explained by speculative trading, pushing the ETN price away from its indicative value, until the underlying index catches up or the speculators change their mind. Since the tracking performance is not great at the daily level one should expect that the ETN prices will fluctuate around the IV, however the mean difference are small so in the mid/long run the ETNs will still provide investors close to the expected exposure. Next, we will examine the tracking performance of the ETNs at the intradaily-level, as the daily results are subject to data aggregation and mistiming issues.

2.3.2 Intradel day

We again calculate the tracking performance measures, but focus on the intraday returns. The intraday indicative value of the ETNs is reported at a 15-second frequency; therefore we sample our one-second interval ETN quote data at the same 15-second intervals at which their indicative values are reported.\(^{13}\) We then calculate the same statistics as in the daily analysis, but throughout each day.

From table 2.4 we can see that the intraday MD between the mid-quote log return and the indicative value log return is minuscule and rarely significant over the whole sample. For the VXZ and TVIZ ETNs both the student-t test and Wilcoxon test are significant, while for the ZIV only the t-test and for the VIIZ only the Wilcoxon is significant and neither statistic shows significance for the other ETNs. However, looking at the proportion of daily student-t and Wilcoxon tests that are significant

\(^{13}\)The 15-second interval for the indicative value is not always on the minute. Therefore using one-second quote data and matching it in this way avoids any mismatches you may get if you sampled the ETN quotes at the 15 second frequency starting on the minute.
at the 5% level, only the VIIZ (0% and 0.50%, respectively) and TVIX (0.12% and 0.12%, respectively) really show any signs of poor tracking performance. Turning to the MAD and RMSD most of the ETNs show evidence of some tracking errors, while the TVIX seems to be tracking its indicative value the least. The mean relative tracking error is only large for the TVIX, while the other ETNs show some evidence of small tracking errors. When we turn to the mean R-squared values, all of the ETNs seem to track their indicative values poorly, at the intraday level. The highest average daily R-squared value is 1.14%, for the VXZ ETN.

<table>
<thead>
<tr>
<th></th>
<th>MD</th>
<th>T-stat</th>
<th>W-stat (000's)</th>
<th>% of Sign. Daily T-stats</th>
<th>% of Sign. Daily W-stats</th>
<th>MAD</th>
<th>RMSD</th>
<th>Mean RTE</th>
<th>Mean $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VXX</td>
<td>0.00%</td>
<td>0.83</td>
<td>989397.0</td>
<td>0.00%</td>
<td>0.08%</td>
<td>0.09%</td>
<td>0.09%</td>
<td>0.05%</td>
<td>0.68%</td>
</tr>
<tr>
<td>VXZ</td>
<td>0.00%</td>
<td>1.88*</td>
<td>1108540.7</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.03%</td>
<td>0.07%</td>
<td>0.04%</td>
<td>1.14%</td>
</tr>
<tr>
<td>XIV</td>
<td>0.00%</td>
<td>−0.62</td>
<td>−256699.5</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.09%</td>
<td>0.15%</td>
<td>−0.04%</td>
<td>0.12%</td>
</tr>
<tr>
<td>ZIV</td>
<td>0.00%</td>
<td>−2.01**</td>
<td>−167835.7</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.03%</td>
<td>0.07%</td>
<td>−0.01%</td>
<td>0.27%</td>
</tr>
<tr>
<td>VIIX</td>
<td>0.00%</td>
<td>0.84</td>
<td>322240.8</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.09%</td>
<td>0.15%</td>
<td>0.03%</td>
<td>0.13%</td>
</tr>
<tr>
<td>VIIZ</td>
<td>0.00%</td>
<td>1.14</td>
<td>342995.1*</td>
<td>0.00%</td>
<td>0.50%</td>
<td>0.03%</td>
<td>0.12%</td>
<td>0.02%</td>
<td>0.24%</td>
</tr>
<tr>
<td>TVIX</td>
<td>0.00%</td>
<td>0.76</td>
<td>243634.5</td>
<td>0.12%</td>
<td>0.12%</td>
<td>0.16%</td>
<td>0.29%</td>
<td>4.28%</td>
<td>0.18%</td>
</tr>
<tr>
<td>TVIZ</td>
<td>0.00%</td>
<td>1.88*</td>
<td>565153.4**</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.06%</td>
<td>0.16%</td>
<td>0.03%</td>
<td>0.22%</td>
</tr>
</tbody>
</table>

**Table 2.4: Intraday tracking performance.**

This table summarizes the tracking performance of the VIX futures ETN market, at the intraday level. The intraday mid-quote prices are sampled at 15-second frequency. The table displays the mean difference (MD) between the mid-quote log returns and indicative value log returns, student-t test statistic for the null hypothesis of zero mean difference (T-stat) and the Wilcoxon test statistic for the null hypothesis that the difference follows a symmetric distribution around zero (W-stat). Also displayed are the percentage of days where the two test statistics are significant at the 5% level. Lastly, the table shows the mean absolute difference (MAD), root mean squared difference (RMSD), the mean relative tracking error (mean RTE) and the mean R-squared of the tracking regressions which estimated intradaily each day.

The results in table 2.4 are confirmed in figures 2-6 and 2-7, which show that the daily R-squared value is almost always very low for all of the ETNs, suggesting very poor intraday tracking performance. However, looking at the daily mean relative tracking errors, they are also low for all of the ETNs other than the TVIX, most of the time. This suggests that although the ETNs do not track their indicative values very well at the intraday level, the errors are small in magnitude, with some exceptions. Again, we can see the effect of the TVIX share issuance halt event, as the magnitude
of the TVIX relative tracking errors increases profoundly thereafter. The small but frequent tracking errors of the ETNs could be explained by the fact that the ETN market tends to lead the futures market, as we show in section 2.5. Since the ETN incorporates the new information first, it will diverge from the price of the underlying futures position until the futures prices also respond to the same information.

![Figure 2-6: Intraday tracking performance: Short-term ETNs](image)

This figure depicts the daily average intraday relative tracking error (gray line on primary axis) and the daily R-squared value (black crosses on secondary axis) of the intraday tracking regression for the short-term VIX futures ETNs. The intraday mid-quote prices are sampled at 15-second frequency.

### 2.4 Consistency

In this section we investigate the consistency of the market prices of the VIX futures ETNs to each other. This is an important analysis, as many of the ETNs track the same underlying VIX futures index and should therefore be priced consistently. We have shown that the ETNs do not track their indicative values very well, but do they do this in a consistent way? Eraker and Wu (2017) do a very brief analysis of VIX futures ETN consistency, where they consider the difference in annual returns of a
synthetic VXX (VXX returns replicated through VIX futures prices) and both the XIV and TVIX ETN indicative values and market prices, at the daily level. They find that the difference is negligible and could be accounted for by fee differences. However, in our study we actually account for the differences in fees and indicative value methodology, study all relevant ETNs and delve into the intraday consistency.

We should not study the difference in ETN prices directly, as different ETNs often have varying specifications in their indicative value methodologies. Therefore, we will extract the implied excess return index performance factors (IF) of each ETN and compare these instead, effectively controlling for leverage and fee differences. To get the IF we make the index return the subject in the indicative value formula (presented in the appendix, section 2.7), of each ETN, for the index return. We then substitute the market values for the indicative values to get the IF.
2.4.1 Daily

We start with the analysis of the ETN price consistency at the daily-level, followed by a similar analysis at the intradaily-level in the next sub-section.

For the VXX and VXZ the daily IFs are given by

\[
IF^j_t = \left( \frac{MV^j_t}{MV^j_{t-1}} \frac{1}{(1 - 0.0089 \frac{d}{365})^d} \right) - TBR_t, \tag{2.2}
\]

where \(MV^j_t\) is the closing price of ETN \(j\) on day \(t\), \(d\) is the number of calendar days since the last business day and \(TBR_t\) is the Treasury Bill return on day \(t\).

The daily IF for the VIIX, VIIZ, XIV, ZIV, TVIX and TVIZ are given by

\[
IF^j_t = 1 + \frac{1}{L} \left( \frac{MV^j_t}{MV^j_{t-1}} \frac{1}{(1 - \frac{fee}{365})^d} - 1 - TBR_t \right), \tag{2.3}
\]

where \(MV^j_t\) is the closing price of the ETN \(j\), \(L\) is that leverage factor of the ETN and \(fee\) is the annual investor fee of that ETN.

Table 2.5 reports the MD, MAD, RMSD, the student-t and Wilcoxon test statistics for zero mean difference of the IF for all possible ETN pairs that track the same index.\(^{14}\) We also estimate the following regression:

\[
IF^j_t = \alpha + \beta IF^h_t + \epsilon_t, \tag{2.4}
\]

where \(IF^j_t\) is the IF of the first ETN in the pair and \(IF^h_t\) is the IF of the second ETN in the pair, on day \(t\). The higher the R-squared value for this regression, the more consistently priced that pair of ETNs is.

Naturally we can only compare the IF of ETNs with the same maturity underlying index; that is \(IF^{VXX}\) can only be compared with the IF of other ETNs that follow the short-term VIX futures index, be it the excess return or total return version.

\(^{14}\)Some of the ETNs are not traded on some days; therefore there is no data on their market value that day. These days are excluded from the sample for that pair.
In table 2.5 we can see that all of the MDs are very small, with the largest being -0.06% between the TVIX and XIV ETNs; the student-t and Wilcoxon test statistics reject the zero mean difference hypothesis for the XIV-VXX, XIV-VIIX and TVIX-XIV pairs, suggesting that the XIV ETN is priced inconsistently to its peers. In panel A of table 2.5 we can see that the MADs and RMSDs are much larger than the MDs. The short-term ETN pairs which are least consistent are TVIX-VXX, TVIX-VIIX and TVIX-XIV, with MAD (RMSD) values of 0.41% (0.83%), 0.48% (0.90%) and 0.46% (0.9%), respectively. This could be expected, as all these pairs include the problematic TVIX ETN which experiences major tracking errors due to the share issuance halt event, as discussed earlier. The R-squared values also point to the TVIX being the most inconsistently priced short-term ETN, as any pair with this ETN has a substantially lower R-squared value. Looking at the mid-term ETNs, in panel B, we can see that all of the MAD and RMSD values are higher than those of the short term ETNs. The R-squared values are also much further from 100%, with the ZIV-VXZ pair having the highest at 88.84%. All of the ETN pairs other than the VIIX-VXX pair seem to be inconsistently priced, according to at least one of the measures in the table.

Figures 2-8 and 2-9 plot the daily difference in IF of the short-term and mid-term ETN pairs, respectively, against time. The figures tell the same story as the summary statistics in table 2.5; that is, the short-term ETNs are priced more consistently than the mid-term ETNs, apart from the TVIX ETN, at the daily-level. We can also see in the figures that the consistency of the ETN prices is time-varying and that all of the pairs experience at least several days of inconsistency.

From the analysis above we can say that the mid-term ETNs are less consistently priced than the short-term ETNs and that the TVIX share issuance halt event has led to this ETN being priced inconsistently to its peers, at the daily-level. However, only the VIIX-VXX pair is priced consistently by all of the measures in table 2.5.
Table 2.5: Daily consistency.

This table shows the mean difference (MD) in implied index performance factors (IF) for each ETN pair, student-t test statistic (T-stat) for the null hypothesis of zero mean difference and Wilcoxon test statistic (W-stat) for the null hypothesis that the difference follows a symmetric distribution around zero. The table also shows the mean absolute difference (MAD), root mean squared difference (RMSD) and the R-squared value of regressing the first IF on the second IF of the pair.

and Figures 2-8 and 2-9, at the daily level. All of the other ETNs experience some inconsistency, although this varies across the different ETNs and through time. These findings relate to the findings of Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad (2018), who show that the VXX and XIV are not exact mirrors of each other, as institutional investors switch from long to short volatility exposure and vice versa, creating market imperfections.

Next, we will examine the consistency of the ETN market prices at the intradaily-level, as the daily results are subject to data aggregation and mistiming issues, as in the tracking performance section.
This figure plots the differences in implied excess return index performance factors (IF) for each short-term ETN pair, as defined in section 2.4.

2.4.2 Intraday

We again calculate the IF, but at the intraday level.\(^{15}\) We ignore overnight returns, therefore the intraday IF for the VXX and VXZ can be calculated as:

\[
IF_i^j = \left( \frac{MV_i^j}{MV_{i-1}^j} \right),
\]

where \(MV_i^j\) is the mid-quote price of the ETN at second \(i\). The intraday IF for the VIIX, VIIZ, XIV, ZIV, TVIX and TVIZ are given by

\(^{15}\)We remove any observations where both ETN IF are 1, as this represents stagnant prices and would bias some of our measures toward consistent pricing.
Figure 2-9: Daily difference in IF: Mid-term ETNs
This figure plots the differences in implied excess return index performance factors (IF) for each mid-term ETN pair, as defined in section 2.4.

\[ IF^j_i = 1 + \frac{1}{L} \left( \frac{MV^j_i}{MV^j_{i-1}} - 1 \right), \]  

(2.6)

where \( L \) is the leverage of that ETN with respect to the underlying index. The investor fees and interest on the underlying positions drop out when we look at the intraday IF as these are paid/earned at the end of the trading day and not intradaily.

Table 2.6 shows the summary statistics for the intraday consistency of the ETN pairs. Again the MD is very low for all of the ETN pairs and does not tell us much; however, it is statistically significant for almost all of the short-term ETNs and some of the mid-term ETNs using the Wilcoxon test statistic. Looking at the MAD and RMSD the only pairs which are inconsistent are the ones with the problematic TVIX
ETN or its mid-term sibling the TVIZ ETN. This is supported by the percentage of daily student-t statistics and Wilcoxon statistics that are significant at the 5% level. The mean R-squared value supports the notion that the TVIX is priced inconsistently to the other short-term ETNs, but does not support the same conclusion for the TVIZ ETN.

<table>
<thead>
<tr>
<th>ETN Pair</th>
<th>MD</th>
<th>T-stat</th>
<th>W-stat</th>
<th>% of Sign.</th>
<th>% of Sign.</th>
<th>MAD</th>
<th>RMSD</th>
<th>Mean R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>VIIX - VXX</td>
<td>0.0001%</td>
<td>1.34</td>
<td>0.0000%</td>
<td>0.08%</td>
<td>0.08%</td>
<td>0.02%</td>
<td>0.04%</td>
<td>58.30%</td>
</tr>
<tr>
<td>XIV - VXX</td>
<td>0.0101%</td>
<td>31.16</td>
<td>−1510747.8**</td>
<td>0.08%</td>
<td>0.53%</td>
<td>0.03%</td>
<td>0.04%</td>
<td>46.66%</td>
</tr>
<tr>
<td>TVIX - VXIX</td>
<td>−0.0003%</td>
<td>0.26</td>
<td>2932967.3**</td>
<td>2.33%</td>
<td>4.44%</td>
<td>0.10%</td>
<td>0.17%</td>
<td>25.87%</td>
</tr>
<tr>
<td>XIV - VIIX</td>
<td>0.0000%</td>
<td>0.56</td>
<td>−3328713.0***</td>
<td>0.00%</td>
<td>0.90%</td>
<td>0.02%</td>
<td>0.04%</td>
<td>47.97%</td>
</tr>
<tr>
<td>TVIX - XIV</td>
<td>−0.0004%</td>
<td>−1.00</td>
<td>7011326.1***</td>
<td>2.11%</td>
<td>4.37%</td>
<td>0.08%</td>
<td>0.13%</td>
<td>24.98%</td>
</tr>
<tr>
<td>TVIX - TVIX</td>
<td>−0.0004%</td>
<td>−2.11**</td>
<td>28055024.8***</td>
<td>1.73%</td>
<td>3.84%</td>
<td>0.09%</td>
<td>0.15%</td>
<td>19.59%</td>
</tr>
</tbody>
</table>

Table 2.6: Intraday consistency.
This table summarizes the consistency of the VIX futures ETN market, at the intraday level. The intraday mid-quote prices are sampled at one-second frequency. The table shows the mean difference (MD) between the implied index performance factors (IF) for each ETN pair, student-t test statistic (T-stat) for the null hypothesis of zero mean difference and Wilcoxon test statistic (W-stat) for the null hypothesis that the difference follows a symmetric distribution around zero. Also displayed are the percentage of days where the two test statistics are significant at the 5% level. The table also shows the mean absolute difference (MAD), root mean squared difference (RMSD) and the mean of the daily R-squared value of regressing the first IF on the second each day.

Figures 2-10 and 2-11 show the daily R-squared and RMSD values. Both of the figures show large time variation of the daily R-squared values and therefore in the consistency of the ETN prices, for all of the ETN pairs. Looking first at Figure 2-10 we can see that the TVIX is clearly inconsistently priced to the other short-term ETNs ever since the share issuance halt event, as the R-squared values drop significantly after this and is often almost 0%. We can also see that the consistency fluctuates a lot across different ETN pairs. Turning to Figure 2-11 we can see that all of the mid-term ETNs exhibit fairly large R-squared values most of the time, with
Chapter 2: The VIX Futures ETN Market

Figure 2-10: Intraday RMSD and R-squared: Short-term ETNs
This figure displays root mean squared difference (RMSD) between the implied index performance factors (IF) for each short-term ETN pair, each day and using intraday IF (grey line on the primary axis). The figure also shows the daily R-squared value of regressing the first intraday ETN IF on the second ETN IF each day. The intraday mid-quote prices are sampled at one-second frequency.
Figure 2-11: Intraday RMSD and R-squared: Mid-term ETNs
This figure displays root mean squared difference (RMSD) between the implied index performance factors (IF) for each mid-term ETN pair, each day and using intraday IF (grey line on the primary axis). The figure also shows the daily R-squared value of regressing the first intraday ETN IF on the second ETN IF each day. The intraday mid-quote prices are sampled at one-second frequency.
some small periods of very low values and the occasional day where the R-squared is almost 100%. Overall the figures suggest that the mid-term ETNs are consistently priced more often than the more heavily traded short-term ETNs.

The figures also show that the daily RMSD is usually very low, but does occasionally spike for a day. The mid-term ETN pairs experience more of these spikes than the short-term ETN pairs. Any ETN pair that includes the TVIZ ETN experiences the most spikes in RMSD, which likely drives the results of table 2.6, suggesting this ETN is not very consistently priced. Therefore, this ETN is not overly inconsistently priced, but experiences more extreme inconsistent days than the other mid-term ETNs.

From table 2.6 and Figures 2-10 and 2-11 we can say that, firstly, the TVIX ETN is inconsistently priced to other short-term ETNs, at the intraday level. Secondly, the mid-term ETNs are more often consistently priced than the short-term ETNs, but also experience more highly inconsistent days. There is a vast time variation in the consistency for all of the ETNs. The inconsistency in the short term ETNs could be explained by the speculative trading behaviour in the short-term ETNs forcing the prices away from their intrinsic values more often than for the mid-term ETNs. Speculation is likely to occur in those short-term ETNs that are more highly traded. The mid-term ETNs have larger differences but they are rare, this could be explained by some ETN specific shocks. The time-varying lead-lag relationship between different ETN pairs, that we show in the next section, are likely also contributing to the inconsistencies, as traders prefer some ETNs over others, sometimes.

### 2.5 Price discovery

We have found that most of the ETNs do not track their indicative values very well and are not always priced consistently, at the daily or intraday level. The poor tracking performance may be due to either the VIX futures or VIX futures ETN market leading the other in price discovery, rather than simultaneously adjusting to
new information. Similarly, the pricing inconsistency may be due to one or some ETNs leading the others in price discovery. In this section we explore the intraday price lead-lag between VIX futures ETNs and VIX futures, as well as between VIX futures ETN pairs. Our analysis of the lead-lag relationships will shed light on the price discovery dynamics of the entire VIX futures ETN market.

2.5.1 Price discovery between VIX futures ETNs and VIX futures

Frijns, Tourani-Rad, and Webb (2016) show that, at the intraday level, VIX futures lead the VIX index in price discovery. However, it may also be the case that the price discovery for VIX futures is being led by the ETN market, as the ETN market is much more liquid and accessible. Such dynamics may help to explain the poor tracking performance of the VIX futures ETN market. This hypothesis is motivated by Alexander and Korovilas (2013) and Dong (2016), who show that the large hedging demand by VIX futures ETN dealers influences VIX futures prices, using daily data and hedging demand proxy estimates. Frijns, Tourani-Rad, and Webb (2016) show that the micro structure of the VIX futures market has changed significantly from before the VIX futures ETN market inception to after. These results suggest that the VIX futures ETNs may be leading VIX futures; we now investigate this directly at the high frequency level. Bollen, O’Neill, and Whaley (2017) show that the VXX ETN leads VIX futures in price discovery; however, in this Chapter we consider all of the ETNs.

There are many VIX derivatives available to investors for trading volatility. One of the derivatives may be favored because of higher liquidity, fewer restraints (some funds are restricted from investing in options and/or futures) or lower trading costs. Whichever derivative is favored will likely reflect market information first, for example Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad (2018) show that the price
discovery for the VXX and XIV ETNs will happen in the ETN which is favored by institutional investors.

To access the lead-lag (price discovery) relationship between VIX futures and VIX futures ETNs we follow a similar methodology to Frijns, Tourani-Rad, and Webb (2016). We estimate the following VAR, at the intraday level each day and for each ETN:

\[
\ln M_V^i = \alpha_1 + \sum_{j=1}^{p+d} \phi_{1,j} \ln M_V^{i-j} + \sum_{j=1}^{p+d} \theta_{1,j} \ln I_V^{i-j} + \epsilon_{1,i+1} \\
\ln I_V^i = \alpha_2 + \sum_{j=1}^{p+d} \phi_{2,j} \ln M_V^{i-j} + \sum_{j=1}^{p+d} \theta_{2,j} \ln I_V^{i-j} + \epsilon_{2,i+1},
\]

(2.7)

where \( \alpha_1, \alpha_2, \phi_{1,j}, \phi_{2,j}, \theta_{1,j} \) and \( \theta_{2,j} \) are the estimated model coefficients, \( \ln M_V^i \) and \( \ln M_V^{i-j} \) are the log mid-quote prices at time \( i \) and \( i-j \), respectively, and \( I_V^i \) and \( I_V^{i-j} \) are the log indicative values at time \( i \) and \( i-j \), respectively. We use the intraday indicative value to study the price discovery relationship between the VIX futures ETNs and VIX futures because the indicative value will only change when the underlying VIX futures prices change, during the trading day. This also allows us to investigate the relationship between the ETN and all of its underlying futures prices simultaneously without having to create our own proxy VIX futures position. The drawback of this approach is that the most granular we can go is to a 15-second sampling frequency, as this is the frequency at which the indicative value is reported.

The optimal lag length, \( p \) in equation (2.7), is determined by minimizing the average daily Schwartz Information Criterion (SIC), up to 12 lags. If there is significant autocorrelation in the model residuals then more lags are added until this is resolved.\(^{16}\)

We then add \( d \) lags, where \( d \) is the maximum expected order of integration of the indicative value and mid-quote time series (always 1 in our analysis). Our methodol-

\(^{16}\)p ends up being equal to four lags for all of the ETNs, except the TVIX, which has an optimal lag length of five.
ogy for dealing with concerns of non-stationarity and cointegration comes from Toda and Yamamoto (1995). We use the log levels of the data regardless of whether they may or may not be stationary or cointegrated. Then we add \( d \) extra lags, as Toda and Yamamoto (1995) show that this will prevent any pretest bias when it comes to calculating the Granger causality statistics. When using their methodology to do a Wald-test, for calculating the Granger causality statistics, we do not include the extra \( d \) lag coefficients, they are treated as a exogenous variable.

In table 2.7 we report the results of the Granger causality tests. We report the mean of the daily Granger causality statistics over the sample period and the percentage of days where the Granger causality is significant, at traditional levels. We find that for all of the ETNs the mean Granger causality statistic from the market value to the indicative value is very large, ranging from 436.58 to 1032.16. For all ETNs we find that the percentage of significant daily Granger causal effects from the market value to the indicative value is very close to 100%, at traditional levels of significance. The average Granger causality statistic from the indicative value to the market value is much smaller, ranging from 5.76 to 13.31. The percentages of significant daily Granger causality statistics for this direction of the relationship are much lower, ranging from 1.68% (23.26%) to 18.60% (58.38%) of days at the 1% (10%) level of significance. From table 2.7 we conclude that all of the ETNs lead the VIX futures in price discovery. This is consistent with the finding of Bollen, O’Neill, and Whaley (2017), who show that the VXX leads VIX futures in price discovery.

We also investigate the time variation of the intraday price discovery relationship between VIX futures and VIX futures ETNs by adopting the methodology from Frijns, Tourani-Rad, and Webb (2016). We define the log Granger ratio as:

\[
\text{LogGrangerRatio}_{t,i} = \ln \left( \frac{GC_{t,i}^{MVtoIV}}{GC_{t,i}^{IVtoMV}} \right),
\]

where \( GC_{t,i}^{MVtoIV} \) is the Granger causality statistic from the VIX futures ETN \( i \) to
Table 2.7: Granger causality between VIX futures ETNs and VIX futures. This table presents the results of the price discovery analysis between VIX futures and VIX futures ETNs. The intraday mid-quote prices are sampled at 15-second frequency. The table reports the mean daily Granger causality statistics from (to) each ETN to (from) its indicative value. The table also reports the percentage of days that the Granger causality statistics are significant at the 1%, 5% and 10% levels of significance.

VIX futures on day $t$ and $GC_{t,i}^{IVtoMV}$ is the Granger causality statistic from the VIX futures to the VIX futures ETN i on day $t$.

We plot the 10-day moving average of the log Granger ratios for the short-term and mid-term VIX futures ETNs, in Figures 2-12 and 2-13 respectively. In Figure 2-12 we can see that the Granger causality lead-lag relationship between short-term VIX futures ETNs and VIX futures varies over time; however, the log Granger ratios stay within a band from three to seven on most days. This shows that the short-term VIX futures ETN market leads the VIX futures in price discovery throughout time, supporting the findings in table 2.7. The TVIX ETN log Granger ratio departs from the norm in the second half of 2012; this is likely caused by the halt of share creation.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Mean Granger Causality Statistic</th>
<th>% of Significant Daily Granger Causality statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\alpha = 0.01$ $\alpha = 0.05$ $\alpha = 0.10$</td>
</tr>
<tr>
<td>VXXIV</td>
<td>VXX</td>
<td>5.76</td>
<td>1.68% 14.87% 23.26%</td>
</tr>
<tr>
<td>VXX</td>
<td>VXXIV</td>
<td>1032.16</td>
<td>99.85% 99.92% 99.92%</td>
</tr>
<tr>
<td>VXZIV</td>
<td>VXZ</td>
<td>12.13</td>
<td>18.60% 48.86% 58.38%</td>
</tr>
<tr>
<td>VXZ</td>
<td>VXZIV</td>
<td>456.93</td>
<td>99.85% 99.92% 99.92%</td>
</tr>
<tr>
<td>XIVIV</td>
<td>XIV</td>
<td>6.62</td>
<td>3.71% 18.91% 29.54%</td>
</tr>
<tr>
<td>XIV</td>
<td>XIVIV</td>
<td>766.92</td>
<td>99.13% 99.75% 99.88%</td>
</tr>
<tr>
<td>ZIVIV</td>
<td>ZIV</td>
<td>10.23</td>
<td>11.50% 40.67% 49.32%</td>
</tr>
<tr>
<td>ZIV</td>
<td>ZIVIV</td>
<td>436.58</td>
<td>99.13% 99.63% 99.75%</td>
</tr>
<tr>
<td>VIIXIV</td>
<td>VIIX</td>
<td>6.88</td>
<td>3.58% 16.19% 24.47%</td>
</tr>
<tr>
<td>VIIX</td>
<td>VIIXIV</td>
<td>855.41</td>
<td>98.76% 99.51% 99.51%</td>
</tr>
<tr>
<td>VIIZIV</td>
<td>VIIZ</td>
<td>13.01</td>
<td>10.14% 32.88% 42.27%</td>
</tr>
<tr>
<td>VIIZ</td>
<td>VIIZIV</td>
<td>480.11</td>
<td>97.90% 99.01% 99.26%</td>
</tr>
<tr>
<td>TVIXIV</td>
<td>TVIX</td>
<td>13.31</td>
<td>18.42% 36.96% 44.50%</td>
</tr>
<tr>
<td>TVIX</td>
<td>TVIXIV</td>
<td>565.59</td>
<td>98.39% 99.38% 99.63%</td>
</tr>
<tr>
<td>TVIZIV</td>
<td>TVIZ</td>
<td>9.46</td>
<td>8.53% 33.00% 42.15%</td>
</tr>
<tr>
<td>TVIZ</td>
<td>TVIZIV</td>
<td>481.81</td>
<td>98.76% 99.51% 99.51%</td>
</tr>
</tbody>
</table>
by the issuer, which we discussed in section 2.3.1.

Figure 2-12: Time variation in VIX futures ETN and VIX futures Granger causality ratios: Short-term ETNs

This figure plots the 10-day moving average of the log Granger causality ratio, as defined in equation (2.8), for the short-term ETNs. The intraday mid-quote prices are sampled at 15-second frequency.

Figure 2-13 shows that the mid-term ETN log Granger ratios are always positive and usually between two and six. This shows that the price discovery is happening mostly in the VIX futures ETNs rather than the VIX futures.

Combining the evidence from table 2.7 and Figures 2-12 and 2-13 we conclude that there is a one-way Granger causality relationship from VIX futures ETNs to VIX futures. This tells us that the VIX futures ETN market leads the VIX futures market in price discovery. We can therefore confirm that the tail (VIX futures ETNs) is truly wagging the dog (VIX futures), as suggested by Alexander and Korovilas (2013) and shown for the VXX by Bollen, O’Neill, and Whaley (2017).

The results in this section could explain the poor tracking performance we have found at the daily and intraday levels (see section 2.3). Since the ETN market is
leading the futures market we would expect a contemporaneous tracking error.

Frijns, Tourani-Rad, and Webb (2016) show that price discovery for the VIX index and VIX futures is lead by the latter. Combining our results with those of Frijns, Tourani-Rad, and Webb (2016), we can conclude that price discovery for the VIX index, VIX futures and VIX futures ETNs happens in the VIX futures ETNs first, followed by VIX futures and then the VIX index.

### 2.5.2 Price discovery between VIX futures ETNs

We have investigated the intraday lead-lag relationship between VIX futures and VIX futures ETNs; we now turn to the price discovery relationship between VIX futures ETN pairs. If any of the ETNs are leading the other ETNs in price discovery, then this could explain the contemporaneous inconsistency among their market prices.
Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad (2018) study the price discovery relationship of one of these pairs, the VXX and XIV. They find that the price discovery between these two ETNs is time varying, which we confirm. However, this may be the case for other ETN pairs too. If there is strong one-way Granger causality in an ETN pair then price discovery happens in one ETN before the other.

To access the lead-lag relationship between VIX futures ETN pairs, at the intraday level, we estimate the following VAR each day and for each ETN pair:

\[
\begin{align*}
\ln MV_{1,i} &= \alpha_1 + \sum_{j=1}^{p+d} \phi_{1,j} \ln MV_{1,i-j} + \sum_{j=1}^{p+d} \theta_{1,j} \ln MV_{2,i-j} + \epsilon_{1,i+1} \\
\ln MV_{2,i} &= \alpha_2 + \sum_{j=1}^{p+d} \phi_{2,j} \ln MV_{2,i-j} + \sum_{j=1}^{p+d} \theta_{2,j} \ln MV_{1,i-j} + \epsilon_{2,i+1},
\end{align*}
\]  

(2.9)

where \(\alpha_1, \alpha_2, \phi_{1,j}, \phi_{2,j}, \theta_{1,j} \) and \(\theta_{2,j} \) are the estimated model coefficients, \(\ln MV_{1,i} \) and \(\ln MV_{1,i-j} \) are the log mid-quote prices of the first ETN at time \(i \) and \(i-j \), respectively, and \(\ln MV_{2,i} \) and \(\ln MV_{2,i-j} \) are the log mid-quote prices of the second ETN at time \(i \) and \(i-j \), respectively. We again eliminate any pretest bias by following the methodology outlined by Toda and Yamamoto (1995), as in section 2.5.1.

Table 2.8 reports the mean daily Granger causality statistics and percentage of daily Granger causality statistics that are significant in each direction for each ETN pair. We can see that the mean daily Granger causality statistics are large in both directions for all the ETN pairs, ranging from 122.24 to 568.98. We can also see that for all of the ETN pairs the percentage of significant daily Granger causality statistics is high in both directions, ranging from 70.08% (88.47%) to 99.92% (100.00%) at the 1% (10%) level of significance. Overall, this table shows that the lead-lag relationship for VIX futures ETN pairs seems to be a strong two-way relationship on average, with no clear leader in price-discovery. These results provide some evidence that traders do not seem to prefer a certain ETN to do their trading in, on average.
Table 2.8: Granger causality of ETN Pairs.
This table presents the results for the price discovery analysis between VIX futures ETN pairs. The intraday mid-quote prices are sampled at one-second frequency. The table reports the mean daily Granger causality statistics for each ETN pair and in each direction. The table also reports percentage of days that the causality statistics are significant at the 1%, 5% and 10% levels of significance. Panel A reports the results for the short-term ETN pairs and Panel B for the mid-term pairs.

We again want to investigate the time variation in the lead-lag relationship, as different ETNs may lead at different times, as Fernandez-Perez, Frijs, Gafiatullina, and Tourani-Rad (2018) show is the case for the VXX and XIV pair. As in equation (2.8) we define the log Granger ratio as
\[ \log \text{GC Ratio}_t = \ln \left( \frac{G_{t}^{MV_1 \to MV_2}}{G_{t}^{MV_2 \to MV_1}} \right), \quad (2.10) \]

where \( G_{t}^{MV_1 \to MV_2} \) is the Granger causality statistic for the causal effect from the first ETN to the second on day \( t \) and \( G_{t}^{MV_2 \to MV_1} \) is the Granger causality statistic for the causal effect from the second ETN to the first on day \( t \).

Figures 2-14 and 2-15 plot the time series of the 10-day moving average of the log Granger ratio for the short-term and mid-term ETN pairs, respectively. We can see that for all of the ETN pairs the log Granger ratio is time varying. The figure shows that most of the time there is a two-way lead-lag relationship between the ETNs, as the ratio fluctuates around zero. However, we can also see that there seem to be time periods where the relationship is stronger one-way or the other. This is especially the case for any pair that includes the XIV ETN, likely due to the trading behavior of institutional investors. Institutional investors tend to be better informed, and so when volatility is high they will be long in long exposure VIX futures ETNs, and this will result in the price discovery taking place in the long exposure ETNs, and vice versa (Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad, 2018). It would be interesting for future research to examine what drives these changes in the relationships, as in Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad (2018), for all of the ETN pairs. The time variation in the price discovery relationship between mid-term ETN pairs seems smaller, but can still be large at times.

2.6 Conclusion

In this Chapter we examine the tracking performance, price consistency and price discovery dynamics of the VIX futures ETN market.

None of the VIX futures ETNs we study track their indicative values very well, at the daily level, and experience at least some days with very large relative tracking
errors. At the intraday level, the ETNs do not track their indicative values well either, with frequent tracking errors. The intraday tracking errors are small in magnitude most of the time. There is vast variation in the magnitude and timing of the tracking errors across the different ETNs.

To investigate the consistency of the VIX futures ETN prices with respect to each other we calculated the index returns implied by the ETN market prices and compared these at the daily and intraday level. When examining the consistency of the ETN pairs at the daily and intraday level we find that all of the ETNs experience some inconsistency; however, this again varies across the ETNs and throughout time. Only the VIIX-VXX ETN pair is priced consistently at the daily level, but not at the intraday level. The mid-term ETNs are less consistently priced at the daily level, but when we delve into the intraday data the mid-term ETNs are priced consistently
Figure 2-15: Time variation in VIX futures ETN pair Granger causality ratios: Mid-term ETNs
This figure plots the 10-day moving average of the log Granger causality ratio, as defined in equation (2.10), for the mid-term ETNs. The intraday mid-quote prices are sampled at one-second frequency.

more often than the short-term ETNs. However, when the mid-term ETNs are priced inconsistently, the discrepancy is larger in magnitude.

We also investigated the lead-lag (price discovery) dynamics between the VIX futures and VIX futures ETNs, using intraday data. We find that the ETN market leads the futures market, confirming that the “tail is wagging the dog”, as Bollen, O’Neill, and Whaley (2017) show for the VXX. This also builds on the findings of Fernandez-Perez, Frijns, Tourani-Rad, and Webb (2015), who find that the microstructure of the VIX futures market changes dramatically after the introduction of VIX futures ETNs. Given the findings of Frijns et al. (2016) that VIX futures lead the VIX index, we conclude that VIX futures ETNs lead VIX futures which lead the VIX index. Although the ETNs lead futures throughout our sample, the magnitude of the relationship varies through time, with rare occasions of the relationship reversing.
briefly for some ETNs. Investigating what drives this time variation could be of
further interest, as Frijns, Tourani-Rad, and Webb (2016) do for the time variation
in the price discovery relationship between the VIX and VIX futures. The price
discovery relationships we have uncovered could also explain why the ETN prices do
not track their indicative values all that well contemporaneously, as they are actually
leading them.

When investigating the price discovery relationships of the ETNs with respect to
each other, we find a strong two-way lead-lag relationship between all ETN pairs,
indicating that neither ETN leads the other. Again, this price discovery relationship
between ETN pairs varies over time, fluctuating around an even two-way Granger
causality (log Granger ratio of zero). This finding of time variation in the leading ETN
is consistent with the findings of Fernandez-Perez, Frijns, Gafiatullina, and Tourani-
Rad (2018) for the VXX and XIV. It could also help explain the contemporaneous
inconsistency between the ETN prices. If one ETN is leading another, they cannot
be priced consistently.

Looking at both the daily and intraday data we can see the effect of the share
issuance halt event on the TVIX ETN, which has tracked its indicative value very
poorly ever since, trading with a large relative tracking error. The effect of this event
can also be seen when looking at the consistency of ETN prices, as the TVIX has
been priced most inconsistently to the other ETNs ever since. Since this event the
TVIX ETN has also experienced more days where it is led by the VIX futures and
other ETNs in price discovery. The effect of this event could be explained anecdotally
by investors’ fears of such an event being repeated; however, this fear does not seem
to have spilled over to any of the other ETNs.

The VIX futures ETN market is one of the most traded and most accessible
volatility derivatives markets, as well as the youngest and least explored in the lit-
erature. Therefore, fully understanding this market is of interest to unsophisticated
investors, institutional investors, policy makers and academics alike. The empirical characteristics outlined in this Chapter could also be used to create profitable trading strategies.

2.7 Appendix

The indicative value methodologies displayed in this section are summarized mathematical representations of the specifications in the respective ETN prospectuses.

The closing indicative values of the VXX and VXZ are defined as

\[
IV^j_t = IV^j_{t-1} \left(1 - \frac{0.0089}{365}\right)^d \left(\frac{TRI_t}{TRI_{t-1}}\right),
\]

(2.11)

where \(IV^j_t\) is the closing indicative value of the VXX or VXZ on day \(t\), \(TRI_t\) is the closing value of either the SPVXSTR index on day \(t\), when considering the VXX, or the SPVXMT index on day \(t\), when considering the VXZ, and \(d\) is the number of days since the last index business day.

The closing indicative value of the VIIX, VIIZ, XIV, ZIV, TIIX and TVIZ are defined as

\[
IV^j_t = IV^j_{t-1} \left(1 - \frac{fee}{365}\right)^d \left[1 + TBR_t + L \left(\frac{ERI_t}{ERI_{t-1}} - 1\right)\right],
\]

(2.12)

where \(IV^j_t\) is the closing indicative value of the VIIX, VIIZ, XIV, ZIV, TIIX or TVIZ on day \(t\) and \(ERI_t\) is the closing value of either the SPVXSER index on day \(t\), when considering the VIIX, XIV, or TVIX ETNs, and the SPVXMER index on day \(t\), when considering the VIIZ, ZIV or TVIZ. The \(TBR_t\) is Treasury Bill return on day \(t\) defined as

\[
TBR_t = \left(\frac{1}{1 - \frac{Tbill_{t-1} \cdot 91}{360}}\right)^{\frac{d}{360}} - 1,
\]

(2.13)
where $Tbill_{t-1}$ is the three-month Treasury Bill rate on the prior index business day reported by Bloomberg and $d$ is the number of calendar days since the prior index business day.

The indicative value of the XVIX ETN is defined as

\[
IV^j_t = IV^j_{t-1} \left[ 1 + TBR_t - 0.5 \left( \frac{SPVXSER_t}{SPVXSER_{t-1}} - 1 \right) \right. \\
+ \left( \frac{SPVXMER_t}{SPVXMER_{t-1}} - 1 \right) - \left( \frac{0.0085}{365} \right) \right]^d,
\]

(2.14)

where $IV^j_t$ is the closing indicative value of the XVIX on day $t$, $SPVXSER_t$ is the closing value of the SPVXSER index on day $t$ and $SPVXMER_t$ is the closing value of the SPVXMER index on day $t$.

The indicative value of the XXV ETN is defined as

\[
IV^j_t = IV^j_0 \left( 1 - \left( \frac{SPVXSER_t}{SPVXSER_0} - 1 \right) \right) + IA_{0,t} - FA_{0,t},
\]

(2.15)

where $IV^j_t$ is the closing indicative value of the XXV on day $t$, $SPVXSER_0$ is the value of the SPVXSER index at the time of inception of the XXV ETN, $IA_{0,t}$ is the interest accrued since the inception of the ETN defined as

\[
IA_{0,t} = \sum_{j=1}^{t} XXV_{j-1} \left( \frac{TBBR_j}{360} d_j \right),
\]

(2.16)

where $TBBR_j$ is the 28-day Tbill rate on day $j$. $FA_{0,t}$ in equation (2.15) is the accrued fees since the inception of the ETN, defined as

\[
FA_{0,t} = \sum_{i=1}^{t} XXV_{i-1} \left( \frac{0.0089}{365} \right).
\]

(2.17)

The CVOL ETN indicative value is calculated by
\[ IV^j_t = IV^j_{t-1} \times \left( \frac{CVOLTR_t}{CVOLTR_{t-1}} \right) \times \left( 1 - \frac{0.0115}{365} d \right) \], \quad (2.18)

where \( IV^j_t \) is the closing indicative value of the CVOL on day \( t \) and \( CVOLTR_t \) is the Citi Volatility Total Return index (CVOLTR) value on day \( t \). The CVOLTR index is an index created by Citigroup that mimics a dynamic position in the third and fourth nearest maturing VIX futures contracts as well as the S&P 500 index (SPX).\(^{17}\)

\(^{17}\)For more details on the calculation of the CVOLTR please refer to the CVOL prospectus available at http://www.c-tracksetns.com
Chapter 3

Modeling VXX under Jump Diffusion with Stochastic Long-Term Mean

This Chapter is joint work with Jin E. Zhang. It was presented at the 2018 New Zealand Finance Colloquium, 7-9 February 2018, Massey University, Palmerston North, New Zealand and has been accepted for presentation at the 2018 New Zealand Finance Meeting, 16-19 December 2018, AUT, Queenstown, New Zealand.

3.1 Introduction

In this Chapter we extend the VXX model of Gehricke and Zhang (2018), by including jumps and letting the long-term mean of volatility be mean-reverting, motivated by the vast literature on VIX derivative pricing. The VXX and other VIX futures ETPs have become very popular instruments for trading volatility since their inception. However, their complicated structure has led to puzzling under (over) performance for long (short) exposure ETPs. Their complicated structure has even led to some being discontinued after a jump in volatility led to un-precedented returns, yet they are still actively traded. When our model is calibrated to the VIX term structure it fits the VXX and other short-term VIX futures ETPs, which dominate the VIX
futures ETP market, well. Our model maintains the link between the S&P 500 index, VIX index, VIX futures and the priced ETP.

VIX index exposure first became accessible to investors in 2004, when VIX futures contracts were launched by the Chicago Board Options Exchange (CBOE), followed in 2006 by VIX options. More recently, since 2009, VIX futures ETPs have been heavily traded. The VXX was the first ETP tracking the short-term VIX futures index (SPVXSTR), which represents the return on a portfolio of VIX futures that is rebalanced to achieve an almost constant one-month maturity. Since 2009, the number of other VIX futures ETPs has been rapidly growing, but with first mover advantage the VXX has been the largest and most heavily traded VIX futures ETP throughout this period. In this chapter we model and fit the short-term VIX futures ETPs, while calibrating to the VIX term structure.

In figure 3-1, we can see that the market capitalization of VIX futures ETPs has grown to around $4 Billion and the average daily dollar trading volume is around $2 billion. On some days the ETPs are traded so heavily that the dollar trading volume is several multiples of the market capitalization, meaning the market can turn over several times in a day.

The under- (out-) performance of the short-term long- (short-) exposure VIX futures ETPs is well documented in the literature, and can be seen in table 3.1. Alexander and Korovilas (2013), Liu and Dash (2012) and Whaley (2013) suggest that the usually contango (upward-sloping) VIX futures term structure is the driver of the underperformance of the VXX. Gehricke and Zhang (2018) are the first to model the VXXs price while accounting for the dynamic relationships between the S&P 500 index, VIX index, VIX futures and the VXX. They show that the underperformance of the VXX relative to the VIX index is mainly due to the roll yield, which measures the effect of rebalancing from the nearest to the second nearest futures contract. We confirm this finding with our extended model. The roll yield will be negative (positive)
Chapter 3: Modeling VXX

Figure 3-1: Market share by maturity target

This figure has four panels. The top left panel shows the total market capitalization of all the VIX futures ETPs grouped by their target maturity. The bottom left panel shows the daily proportion of total market capitalization for each group. The top right panel shows the five day moving average of the daily dollar trading volume for each group. The last panel (bottom right) shows the five day moving average of the daily proportion of dollar trading volume for each group. ST represents the ETPs that track the short-term VIX futures indices, MT represents the ETPs that track the mid-term VIX futures indices, WEEK represents ETPs that provide exposure to shorter term VIX futures (weekly futures) and dynamic/hybrid represents those ETPs that are not linearly tracking one of the indices.

when the VIX futures term structure is in contango (backwardation). Gehricke and Zhang (2018) show that the negative roll yield is driven by the market price of variance risk, on aggregate. Their result is consistent with that of Eraker and Wu (2017), who show that the underperformance of the SPVXSTR index is driven by the variance risk premium. The market price of variance risk and the variance risk premium are two closely related concepts, which Zhang and Huang (2010) show are almost proportional to each other. Eraker and Wu (2017) show, in a consumption-based equilibrium setting, that the underlying driver of the negative variance risk premium is investor risk aversion.

The long-exposure ETPs were initially marketed as diversification tools for eq-
uity portfolios, due to the negative correlation between the VIX and the S&P 500. This motivated investors to take positions in these products, despite their terrible performance. However, several studies have shown that they are not useful for diversification (Alexander, Korovilas, and Kapraun, 2016; Deng, McCann, and Wang, 2012; Hancock, 2013). The reason why these products are not good for diversification is due to their underperformance relative to the VIX index, which is an empirical fact that stands in contrast to the common misconception that investing in the VXX is like investing in the VIX index. In October of 2017 Wells Fargo was ordered to pay remunerations of $3.4 million to investors because they were advising them to invest in VIX futures ETPs as hedging tools (Banerji, 2017).

Table 3.1: Summary statistics of the daily returns for the SPX, VIX, VXX and XIV. This table is an extended and updated version of the table found in Gehricke and Zhang (2018), with a longer sample size and a short exposure VIX futures ETP, the XIV, included. The table shows the summary statistics and correlations of the SPX, VIX, VXX and XIV returns from the 30 January 2009 (XIV was not launched until 29 November 2010) to the 24 April 2017. Here, \( r_D \) represents estimates using discrete daily returns and \( r_C \) represents estimates using continuously compounded daily returns. The annualised standard deviation is calculated by multiplying the standard deviation by \( \sqrt{252} \). The Holding Period Return (HPR) is the return from the first day to the last day of the sample. The Compound Annual Growth Rate (CAGR) is the constant yearly growth rate that would lead to the corresponding HPR, it is calculated by \( CAGR = (HPR + 1)^{\frac{T}{T}} - 1 \), where \( T \) is the length of the sample in years.

The short exposure VIX futures ETPs have been very profitable for investors,
especially during the market tranquility of 2017. Recently however, there have been several news releases (Jakab, 2018; Zuckermann and Fletcher, 2018; Burger, 2018) on the sudden collapse of the XIV ETN, February 2018, which tracked the inverse performance of the SPVXSTR index. This event was caused by a spike in the VIX which led to massive losses for the XIV, and further selling pressure which compounded this effect, providing more evidence that the negative returns to the SPVXSTR index (and ETPs which track it) in normal times are a premium for the risk of a spike in volatility.

A feasible motivation for investors to trade these products is short-term hedging against volatility spikes or speculation on the direction of volatility. Both of these motivations are in line with the high trading activity of these products. Another explanation could be the ease of access to this market; any investors can easily invest in VIX futures ETPs, as they are traded on stock exchanges. This is a dangerous setting, as these are highly complicated derivative instruments which are not yet fully understood by academics or practitioners, let alone retail investors. The VIX futures ETP market is also a way that some mutual funds, which are restricted from investing in traditional derivatives, can enter volatility positions.

In this Chapter our main focus is on developing a new model for the VXX, building on the work of Gehricke and Zhang (2018), which accounts for the relationship between the S&P 500, the VIX term structure, VIX futures and the VXX. We calibrate our model to the VIX term structure, which allows us to fit the VXX time series well. We compare the fit of our model with three simpler nested models, showing that our full model is better at fitting the short or full VIX term structure.\footnote{The first nested model is the Heston (1993) model, as used by Gehricke and Zhang (2018). The second nested model is the floating $\theta$ model, as presented in appendix A.6 equations (3.54), (3.55) and (3.56). The last nested model is equivalent to the full model, as presented in equations (3.3), (3.4) and (3.5), but where $\kappa = \kappa_V = \kappa_\theta$.}

We provide two different formulas for modeling the VXX and other short-term VIX futures ETPs, one idealistic (continuously rebalanced with constant maturity)
and one realistic (daily rebalanced with varying maturity) model. We show that the realistic model outperforms the idealistic in fitting the VXX time series. The model fit is best when calibrating to the first three-points of the VIX term structure, compared with calibrating to the full or first two point VIX term structure. This is intuitive, as the VXX and other short-term VIX futures ETPs, represent exposure to short-term volatility and should not be affected by market expectations on longer-term volatility. Our model can also be used to price other short term VIX futures ETPs. Short-term VIX futures ETPs dominate the VIX futures ETP market, as shown in figure 3-1. The figure shows that the short-term VIX futures ETPs consistently make up around 80% and almost 100% of the VIX futures ETP market size and trading volume, respectively.

This study contributes to the vast literature on volatility derivative pricing. Many papers have studied different model settings for pricing VIX futures (Zhang and Zhu, 2006; Zhang, Shu, and Brenner, 2010; Lu and Zhu, 2010; Dupoyet, Daigler, and Chen, 2011; Zhu and Lian, 2012; Huskaj and Nossman, 2013). Lin (2007) derives and studies the pricing performance of closed form VIX futures pricing formulas under several different affine dynamics. Although Lin (2007) does not test the out-of-sample pricing ability of our model, the study finds that out of the models tested the stochastic volatility with jumps in the volatility (SVVJ) performs best for short-term (< 60 days) VIX futures contracts, which is the model closest to ours. Developing models for VIX options has also been the focus of several articles (Wang and Daigler, 2011, Chung, Tsai, Wang, and Weng, 2011, Cont and Kokholm, 2013, Lian and Zhu, 2013, Bardgett, Gourier, and Leippold, 2018, Papanicolaou and Sircar, 2014). Luo and Zhang (2012) study the VIX term structure; that is, they calculate different maturity VIX indices and examine their properties. They also provide an affine model for the VIX term structure with jumps in the stock price and a stochastic long-term mean of volatility. They calibrate their model using a similar calibration method to that
in used in this study. Bao, Li, and Gong (2012) price VXX options by assuming an affine structure for the VXX and its volatility. This approach ignores the relationship between the SPX, VIX term structure, VIX futures and the VXX. Our model could be extended to price VXX options; while accounting for these relationships.

Several authors have also studied the daily and intra-daily price discovery dynamics between the VIX index and its futures (Shu and Zhang, 2012; Frijns, Tourani-Rad, and Webb, 2016; among others) concluding that, at the intraday level, VIX futures lead the VIX index. Bordonado, Molnár, and Samdal (2017) indirectly show that VIX futures ETNs lead VIX futures exchange-traded funds (ETFs). Bollen, O’Neill, and Whaley (2017) show that the VXX leads the VIX futures in price discovery and that VIX futures lead VIX options in price discovery. While, Chapter 2 of this dissertation shows that all of the VIX futures ETNs lead the VIX futures and that no single ETN leads the others consistently. Combining the results we could say that VIX futures ETNs lead VIX futures, which in turn lead the VIX index. This further highlights the importance of understanding and accurately pricing the VIX futures ETPs, as these products drive the market for volatility trading.

The rest of this Chapter is organized as follows. Section 3.2 summarizes the methodology for calculating the short-term VIX futures index that the short-term ETPs are tracking. Then section 3.3 outlines the model set-up and derives the formulas for the VIX index, VIX futures, the VXX and other short-term VIX futures ETPs. Next in section 3.4 we calibrate our model and nested models to the short and full VIX term structure and compare their fit. In section 3.5 we examine the fit to the VXX and other short-term VIX futures ETP time series, comparing the performance of the short and full VIX term structure calibration and the discrete and idealistic model. Finally, in section 3.6 we conclude.
3.2 VIX futures indices

The underlying indices of the short-term VIX futures ETPs are either the S&P 500 short-term VIX futures total return index (SPVXSTR) or the S&P 500 short-term VIX futures excess return index (SPVXSER). The SPVXSTR index is calculated as:

\[
SPVXSTR_t = SPVXSTR_{t-1}(1 + CDR_t + TBR_t),
\]

and the SPVXSER index is equal to this, but without the interest return on the underlying futures position, \( TBR_t \). The contract daily return \( (CDR_t) \) of the futures position is given by:

\[
CDR_t = \frac{w_{1,t-1}F_{t-1}^{T_1} + w_{2,t-1}F_{t-1}^{T_2}}{w_{1,t-1}F_{t-1}^{T_1} + w_{2,t-1}F_{t-1}^{T_2}} - 1,
\]

where \( F_{t}^{T_i} \) is the current price of the \( i \)-th maturing VIX futures contract and \( w_{i,t-1} \) is the weight of the position invested in the \( i \)-th maturing VIX futures contract on the preceding business day. The weights are calculated such that the futures position has a maturity of one month, which fluctuates around 30 days.\(^2\)

3.3 Model

3.3.1 Model dynamics

We extend the model of Gehricke and Zhang (2018), who use the Heston (1993) framework, by adding a jump component in the instantaneous variance process and letting the long-term mean level of the instantaneous variance itself be a stochastic mean-reverting process. For the ex-dividend stock price and its variance, under the

\(^2\)For further details on the calculation of the weights and indices please see Gehricke and Zhang (2018) and/or S&P Dow Jones Indices (2012).
risk-neutral probability measure, we adopt the following dynamics:

\[
dS_t = rS_t dt + \sqrt{V_t}S_t dB_{1,t} \\
dV_t = \kappa_V (\theta_t - V_t) dt + \sigma_V \sqrt{V_t} dB_{2,t} + y dN_t - \lambda E^Q[y] dt \\
d\theta_t = \kappa_\theta (\bar{\theta} - \theta_t) dt + \sigma_\theta \sqrt{\theta_t} dB_{3,t} ,
\]

where \( \theta_t \) is the effective long-term mean level of \( V_t \), which is the instantaneous variance of the SPX.\(^3\) Here, \( r \) is the risk free rate. \( \kappa_V \) and \( \kappa_\theta \) are the mean-reverting speeds of \( V_t \) and \( \theta_t \), respectively. The effective long-term mean of \( \theta_t \) is given by \( \bar{\theta} \). The volatility of the variance is given by \( \sigma_V \), while \( \sigma_\theta \) measures the volatility of \( \theta_t \). \( B_{1,t}, B_{2,t} \) and \( B_{3,t} \) are three standard Brownian motions that describe the diffusive randomness in \( S_t, V_t \) and \( \theta_t \), respectively. The Brownian motions \( B_{1,t} \) and \( B_{2,t} \) are correlated by a constant correlation coefficient \( \rho \), while \( B_{3,t} \) is independent of the other two. Also, \( dN_t \) is a Poisson process with arrival intensity \( \lambda \) and jump size \( y \). The jump size \( y \) can be any independently distributed random variable.

For the nested Heston model we usually impose the Feller’s condition \( \kappa_V \theta > \frac{1}{2} \sigma_V^2 \), which is adjusted due to the jump terms to be \[ \kappa_V \theta - \lambda E^Q[y] > \frac{1}{2} \sigma_V^2 . \] This insures that the instantaneous variance \( V_t \) will be positive for our model.

The jump component allows more flexibility in the modeling of the density of the instantaneous variance \( V_t \). Our jump term is compensated, which keeps the VIX formula and estimation simple compared with models that do not compensate the jump component (Luo and Zhang, 2012). Also, Lin (2007) shows that for VIX futures with maturity of less than 60 days the SVVJ (stochastic volatility with jumps in volatility) model outperforms the SV (stochastic volatility, i.e. Heston, 1993), SVJ (stochastic volatility with jumps in the stock price) and SVJJ (stochastic volatility with jumps in equity and volatility) models.

\(^3\)If jumps are included in the stock price process then \( V_t \) can be seen as the instantaneous squared VIX, as in Luo and Zhang (2012), and all following results are identical.
Our model differs from the standard SVVJ model as the long-term mean level of the instantaneous variance $\theta_t$ is stochastic and mean-reverting. Implementing a stochastic $\theta_t$ allows for more realistic transient changes in the VIX and VIX futures term structures (Zhang and Huang, 2010; Zhang, Shu, and Brenner, 2010; Zhang, Zhen, Sun, and Zhao, 2017). Bardgett, Gourier, and Leippold (2018) find that mean-reverting volatility and jumps in volatility are important in capturing the implied volatility curves’ shape in both the S&P 500 and VIX index options markets.

In order to derive formulas for the VIX index term structure and VIX futures prices, we will need the first two risk-neutral conditional central moments of $V_s$ and $\theta_s$, where $s > t$. These are provided in Lemma 1 below.

**Lemma 1.** The risk-neutral first and second central moments of $\theta_t$ and $V_t$ can be derived as:

$$E_t^Q[\theta_s] = e^{-\kappa\theta(s-t)}\theta_t + (1 - e^{-\kappa\theta(s-t)})\bar{\theta}, \quad (3.6)$$

$$E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] = \frac{\sigma_{\theta}^2}{\kappa_{\theta}}\left( e^{-\kappa\theta(s-t)} - e^{-2\kappa\theta(s-t)} \right) + \frac{\sigma_{\theta}^2}{2\kappa_{\theta}}(1 - 2e^{-\kappa\theta(s-t)} + e^{-2\kappa\theta(s-t)}) \quad (3.7)$$

$$E_t^Q[V_s] = e^{-\kappa V(s-t)}V_t + \frac{K_V}{K_V - \kappa_{\theta}}\left( e^{-\kappa\theta(s-t)} - e^{-\kappa V(s-t)} \right)\theta_t + \left( 1 - e^{-\kappa V(s-t)} - \frac{K_V}{K_V - \kappa_{\theta}}(e^{-\kappa\theta(s-t)} - e^{-\kappa V(s-t)}) \right)\bar{\theta} \quad (3.8)$$

$$E_t^Q[(V_s - E_t^Q[V_s])^2] = X + Y. \quad (3.9)$$

where

$$X = \frac{\kappa_{\theta}^2 \sigma_{\theta}^2}{(K_{V} - \kappa_{\theta})^2} \left[ \left( e^{-\kappa\theta(s-t)} - e^{-2\kappa\theta(s-t)} \right) + \frac{2e^{-\kappa\theta(s-t)}(1 - e^{-\kappa V(s-t)}) + e^{-\kappa\theta(s-t)} - e^{-\kappa V(s-t)}}{2K_{V} - \kappa_{\theta}} \right] \theta_t + \left( \frac{1 - e^{-\kappa\theta(s-t)}}{2\kappa_{\theta}} + \frac{2(1 - e^{-\kappa\theta(s-t)} - e^{-\kappa V(s-t)})}{K_{V} - \kappa_{\theta}} \right)\bar{\theta} \quad (3.10)$$

$$Y = \frac{\sigma_{\theta}^2}{K_{V}}(e^{-\kappa V(s-t)} - e^{-2\kappa V(s-t)})\theta_t + \frac{\lambda}{2\kappa_{V}}\left( 1 - e^{-\kappa V(s-t)} \right)E_t[y^2]$$
The proofs for the moments of $\theta_t$ and $V_t$ can be found in the appendix; sections 3.7.1 and 3.7.2, respectively.

**Remark 1.** The coefficients in front of $\theta_t$ and $\bar{\theta}$ in equation (3.6) can only take on values between 0 and 1 for any value of $s$ between 0 and $\infty$. The sum of the coefficients is equal to 1. Therefore we can say that $E_t^Q[\theta_s]$ is a weighted average of $\theta_t$ and $\bar{\theta}$, where the two coefficients mentioned are the weights.

**Remark 2.** The coefficients in front of $V_t$, $\theta_t$ and $\bar{\theta}$ in equation (3.8) can only take on values between 0 and 1 for any value of $s$ between 0 and $\infty$. The sum of the three coefficients is equal to 1. Therefore we can say that $E_t^Q[V_s]$ is a weighted average of $V_t$, $\theta_t$ and $\bar{\theta}$, where the three coefficients mentioned are the weights.

### 3.3.2 VIX term structure

The VIX index measures the market’s expectation of 30-day implied volatility. However, the VIX methodology can be applied to essentially any maturity, and the CBOE has recently started to report several S&P 500 implied volatility indices with different maturities. From the CBOE website we can get implied volatility time series for any maturity index.

Carr and Wu (2009) show that the VIX index is equivalent to the 30-day variance swap rate which is equal to the risk-neutral conditional expectation of variance over the next 30 days, when the stock price is modeled without jumps. Therefore the VIX squared with any maturity, $T_i$, is given by:

$$
\left( \frac{VIX_t^{T_i}}{100} \right)^2 = E_t^Q \left[ \frac{1}{\tau_i} \int_t^{t+\tau_i} V_s ds \right] = \frac{1}{\tau_i} \int_t^{t+\tau_i} E_t^Q [V_s] ds
$$

(3.12)
where $\tau_i$ is the time to maturity in years. We can interchange the expectation and integral (justified by Tonelli’s theorem) to get the second equality. Using lemma 1 we get the following proposition 1, below.

**Proposition 1.** The VIX index, under our model dynamics, is given by:

$$\frac{VIX_t}{100} = \sqrt{AV_t + B\theta_t + (1 - A - B)\bar{\theta}},$$

(3.13)

where

$$A = \frac{1 - e^{-\kappa_V\tau_0}}{\kappa_V\tau_0},$$

$$B = \frac{\kappa_V(1 - e^{-\kappa_\theta\tau_0})}{\kappa_\theta\tau_0(\kappa_V - \kappa_\theta)} - \frac{(1 - e^{-\kappa_V\tau_0})}{\tau_0(\kappa_V - \kappa_\theta)}.$$

and $\tau_0 = 30/365$ can be replaced by any maturity in order to model the VIX term structure.

The proof for this proposition is provided in the appendix, section 3.7.3.

**Remark 3.** The coefficients in front of $V_t$, $\theta_t$ and $\bar{\theta}$ in equation (3.13) can again only take on values between 0 and 1 for any value of $s$ between 0 and $\infty$ and the sum of the three coefficients is equal to 1. Therefore we can say that the VIX index is the square root of a weighted average of $V_t$, $\theta_t$ and $\bar{\theta}$, where the three coefficients mentioned are the weights.

### 3.3.3 VIX futures

Now that we have a formula for the VIX index, equation (3.13), we can derive a VIX futures price formula. The VIX futures price is given by the conditional risk-neutral expectation of the VIX index at the futures’ maturity $T$:

$$\frac{F_t^T}{100} = \mathcal{E}^Q\left(\frac{VIX_T}{100}\right)$$

(3.14)
Plugging in our formula for the VIX index from proposition 1 we get:

\[
\frac{F^T_t}{100} = E_t^Q \left[ \left( AV_T + B\theta_T + (1 - A - B)\bar{\theta} \right)^2 \right].
\] (3.15)

**Lemma 2.** The risk-neutral conditional expectation of the VIX squared, or the theoretical VIX squared futures price, is given by:

\[
E_t^Q \left[ \left( \frac{VIX_T}{100} \right)^2 \right] = AE_t^Q[V_T] + BE_t^Q[\theta_T] + (1 - A - B)\bar{\theta}
\]

\[
= A \left[ e^{-\kappa_V(T-t)}V_t + \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left( e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)} \right) \theta_t \right.
\]

\[
+ \left. \left( 1 - e^{-\kappa_V(T-t)} - \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left( e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)} \right) \right) \bar{\theta} \right]
\]

\[
+ B \left[ e^{-\kappa_\theta(T-t)}\theta_t + (1 - e^{-\kappa_\theta(T-t)})\bar{\theta} \right] + (1 - A - B)\bar{\theta}
\]

\[
= CV_t + D\theta_t + (1 - C - D)\bar{\theta},
\] (3.16)

where

\[
C = Ae^{-\kappa_V(T-t)},
\]

\[
D = A \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left( e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)} \right) + Be^{-\kappa_\theta(T-t)}.
\]

To approximate the expectation of the non-linear VIX equation, equation (3.15), we follow a similar methodology as Zhang, Shu, and Brenner (2010). We expand the square root form equation (3.15) using the two variable Taylor expansion near the points \( E_t^Q[V_T] \) and \( E_t^Q[\theta_T] \), which results in proposition 2 below.

**Proposition 2.** The VIX futures price can be approximated by:

\[
\frac{F^T_t}{100} = \left( CV_t + D\theta_t + (1 - C - D)\bar{\theta} \right)^\frac{1}{2} - \frac{1}{8} \left( CV_t + D\theta_t + (1 - C - D)\bar{\theta} \right)^{-\frac{3}{2}} A^2 \left( X|_{s=T} + Y|_{s=T} \right)
\]

\[
- \frac{1}{8} \left( CV_t + D\theta_t + (1 - C - D)\bar{\theta} \right)^{-\frac{3}{2}} B^2
\]

\[
\times \left[ \frac{\sigma_\theta^2}{\kappa_\theta} e^{-\kappa_\theta(T-t)} - e^{-2\kappa_\theta(T-t)} \right] + \frac{\sigma_\bar{\theta}^2}{2\kappa_\theta} \left( 1 - 2e^{-\kappa_\theta(T-t)} + e^{-2\kappa_\theta(T-t)} \right),
\] (3.17)
where $T$ is the maturity of the VIX futures contract.\footnote{An analytical formula for the VIX futures price can be found by using the technique developed by Duffie, Pan, and Singleton (2000) for affine jump diffusion models. A comparative study between our approximate formula and the analytical formula is left for further research.}

The proof for this proposition is given in the appendix, section 3.7.3.

As in Gehricke and Zhang (2018) we can further approximate the short-term, up to 60 days maturity, VIX futures price by removing the convexity adjustments in equation (3.17). This is justified as they do not have much impact at short maturities. Therefore the short-term VIX futures price can be approximated by:\footnote{Please note that the jump parameters have dropped out in our formula for the short-term VIX futures and the VXX models. For long-term VIX futures and the mid-term VIX futures ETPs the jump parameters could play an important role. Our setup with jump diffusion allows for these more general cases. The formulas for these will be presented in future research.}

$$\frac{F_t^T}{100} = \left( CV_t + D\theta_t + (1 - C - D)\bar{\theta} \right)^{\frac{1}{2}}. \tag{3.18}$$

### 3.3.4 Idealistic model

We term one model for the VXX the idealistic model and another the realistic model, and examine the fit of both. The idealistic model is presented here where the maturity of the SPVXSTR is assumed to be a constant 30 days and its futures position is rebalanced continuously. However, the weighted average maturity of the SPVXSTR’s underlying futures position actually fluctuates between 27 and 37 calendar days. The index is also rebalanced daily rather than continuously. The realistic model takes these factors into account by using the short-term VIX futures pricing formula, equation (3.18), in the SPVXSTR methodology with the fluctuating time to maturity and daily rebalancing. The realistic model will be presented in section 3.3.5.

We now derive the idealistic model for pricing the VXX and other short-term VIX futures ETPs assuming the underlying futures position rebalances continuously. To
do this we first take the natural logarithm of equation (3.18):

$$\ln \left( \frac{F_t^T}{100} \right) = \frac{1}{2} \ln \left( C V_t + D \theta_t + (1 - C - D) \bar{\theta} \right)$$  \hspace{1cm} (3.19)

The SPVXSTR index is rebalanced daily to maintain a VIX futures position with one-month maturity. The contract daily return ($CDR_t$) of the SPVXSTR’s underlying VIX futures position can therefore be modeled as the log return of the short-term VIX futures price, given by equation (3.19), with a maturity of 30 days, given by:

$$CDR_t = \Delta \ln \left. \left[ F_t^T \right]_{T=t+\tau_0} \right. = \ln F_{t+\tau_0}^{t+\Delta t} - \ln F_{t-1}^{t+\tau_0}$$

$$= \ln F_{t+\tau_0}^{t+\tau_0} - \ln F_{t-1}^{t+\tau_0} + RY_t$$

$$= \Delta \ln F_t^{t+\tau_0} + RY_t,$$  \hspace{1cm} (3.20)

where $\tau_0 = \frac{30}{365}$, $\Delta \ln F_t^{t+\tau_0}$ is the log return of the constant 30-day-to-maturity VIX futures price and $RY_t$ is the roll yield.

The SPVXSTR return, as shown in equation (3.1), consists of the $CDR_t$ and a risk-free return earned on the notional of the underlying VIX futures position. The VXX return is equal to the SPVXSTR return less an investor fee, accounting for this leads to the idealistic VXX model in proposition 3 below.

**Proposition 3.** We can model the log return of the VXX as follows:

$$\Delta \ln VXX_t = CDR_t + (r_t - 0.0089) \Delta t$$

$$= \ln F_t^{t+\tau_0} - \ln F_{t-1}^{t+\tau_0} + (r_t - 0.0089) \Delta t,$$

$$= \Delta \ln F_t^{t+\tau_0} + RY_t(r_t - 0.0089) \Delta t$$  \hspace{1cm} (3.21)

where $r$ is the risk free rate, the investor fee is 0.89% per annum, $F_t^{t+\tau_0}$ is the price of the 30-day-to-maturity VIX futures contract and $RY_t$ and $F_t^{t+\tau_0-1}$ is the price of
the VIX futures contract.\footnote{Empirically the market 30-day-to-maturity VIX futures price can be calculated by linear interpolation, as in Zhang, Shu, and Brenner (2010) and Gehricke and Zhang (2018).}

Our model for the VXX return presented in proposition 3, confirms the finding of Gehricke and Zhang (2018) that the underperformance of the VXX, relative to the constant 30-day-to-maturity VIX futures price is driven by the VXX’s roll yield. The constant 30-day-to-maturity VIX futures price tracks the VIX index’s returns almost identically. The difference here is the expressions for the roll yield and the 30-day-to-maturity VIX futures price.

The short-term VIX futures ETFs will try to match the leveraged daily return of the SPVXSTR, i.e. two times leveraged SPVXSTR for the UVXY, by holding a daily rebalanced replicating portfolio of VIX futures, swaps and money market instruments, where as the ETNs just promise to pay the final indicative value at maturity or upon early redemption.\footnote{An ETN is a non-securitized debt obligation, like a zero coupon bond, that has a indicative value based on the value of some underlying benchmark and contract specifications.} The effect of this is that both types of ETPs should track their indicative values fairly well.\footnote{Although, in Chapter 2 we show that they do not track their indicative values perfectly, especially at the intraday level.} Therefore if we can model one of the short-term ETPs, we can model them all.

Any non-dynamic VIX futures ETP that tracks either the total return or excess return short-term VIX futures index can be modeled by:

\[
\Delta \ln ETP_{i,t} = L_i \cdot CDR_t + (r_t - fee_i) \Delta t
\]

\[
= L_i \left( \Delta \ln F_{t+\tau}^i + RY_i \Delta t \right) + (r_t - fee_i) \Delta t \tag{3.22}
\]

where \( ETP_{i,t} \) is the price of the ETP \( i \) at time \( t \), \( fee_i \) is its investor fee (expense ratio) and \( L_i \) is its target leverage.

In figure 3-1 we can see that the short-term VIX futures ETPs make up about 80% of the total VIX futures ETP market capitalization and almost 100% of the total
daily dollar trading volume, and have done so historically. Therefore, our model is useful the most traded segment of the VIX futures ETP market.\footnote{To model the longer term VIX futures ETPs we would not recommend removing the convexity adjustment for the VIX futures formula, as in equation (3.18), but our methodology can be followed using the full approximation to arrive at a model for mid-term VIX futures ETPs.}

### 3.3.5 Realistic model

Our model in the previous section and that of Gehricke and Zhang (2018) assume that the underlying futures position of the SPVXSTR has a weighted maturity resultant from the rebalancing is always 30 days. Gehricke and Zhang (2018) show that the weighted maturity of the index can fluctuate between 27 and 37 days. This effect could lead to substantial errors when comparing the calibrated model implied and market VXX price time series. To account for these effects we model the VXX by plugging our formula for short-term VIX futures, equation (3.18), as shown in proposition 4 below.

**Proposition 4.** We can model the VXX discretely by using the model implied nearest and second nearest VIX futures time series, as follows:

\[
VXX_t = VXX_{t-1} \left( \frac{wF_{T1}^{imp,t} + (1 - w)F_{T2}^{imp,t}}{wF_{T1}^{imp,t-1} + (1 - w)F_{T2}^{imp,t-1}} + \frac{r_t - 0.0089}{365} \right), \tag{3.23}
\]

where \( w \) is the weight in the nearest maturity VIX futures contract at time \( t \), calculated using the SPVXSTR methodology. \( F_{T1}^{imp,t} \) and \( F_{T2}^{imp,t} \) are the model implied VIX futures prices using the short-term VIX futures formula, equation (3.18).

Again, we can extend this model for any short-term (non-dynamic) VIX futures ETP by accounting for the difference in fees and leverage as follows:

\[
ETP_{i,t} = ETP_{i,t-1} \left( 1 + L_i \left[ \frac{wF_{T1}^{imp,t} + (1 - w)F_{T2}^{imp,t}}{wF_{T1}^{imp,t-1} + (1 - w)F_{T2}^{imp,t-1}} - 1 \right] + \frac{r_t - fee}{365} \right). \tag{3.24}
\]
Chapter 3: Modeling VXX

3.4 Calibration

3.4.1 Method

We estimate the model parameters using the daily term structure of the VIX index. The VIX term structure data are obtained from the CBOE website. Every day we use all option expirations available, each representing a different maturity VIX index, for a daily sample from November 24, 2010 to June 22, 2017.\(^{10}\) We need to estimate the \(\kappa_V, \kappa_\theta\) and \(\bar{\theta}\) structural parameters as well as the daily \(V_t\) and \(\theta_t\) parameters. To do this we modify the efficient two-step iterative estimation of Christoffersen, Heston, and Jacobs (2009), also used by Luo and Zhang (2012) and Zhang, Zhen, Sun, and Zhao (2017), among others.

Since we have three structural parameters the two step procedure runs into some non-convergence issues, therefore we estimate the \(\bar{\theta}\) parameter as the mean of the daily \(\theta_t\) values from the floating \(\theta\) model.\(^{11}\) Our calibration procedure is then as follows:

**Step one:** We set our \(\bar{\theta}\) equal to the mean of the daily \(\theta_t\) values from the floating \(\theta\) model calibration. \(^{12}\) and as our sample is the longest available, this justifies our step one. **Step two:** We obtain the time series of \(\{V_t, \theta_t\}\) for \(t = 1, 2, \ldots T\) for a given parameter set \(\{\kappa_V, \kappa_\theta, \bar{\theta}\}\), by solving \(T\) minimizations of the daily sums of squared errors as follows:

\[
\{\hat{V}_t, \hat{\theta}_t\} = \arg \min \sum_{i=1}^{n_t} (VIX^{MKT}_{t,\tau_i} - VIX^{MI}_{t,\tau_i})^2, \quad t = 1, 2, \ldots T, \quad (3.26)
\]

\(^{10}\)We call the different maturity implied volatility calculated using the VIX methodology, different maturity VIX indices throughout this Chapter.

\(^{11}\)The floating \(\theta\) model dynamics and VIX term structure formula are outlined in appendix A.6. This model is calibrated using only steps two and three of the calibration procedure.

\(^{12}\)Taking the limit of the average of the daily \(\theta_t\) as the sample gets infinitely large results in:

\[
\lim_{s \to \infty} \frac{1}{s - t} \int_t^s E_t[\theta_u]du = \lim_{s \to \infty} \frac{1}{s - t} \int_t^s [e^{-\kappa_\theta(u-t)}\theta_t + (1 - e^{-\kappa_\theta(u-t)})\bar{\theta}]du = \bar{\theta}, \quad (3.25)
\]
where $VIX^{MKT}_{t,\tau_i}$ is the market value of the $\tau_i$ maturity VIX index and $VIX^{MI}_{t,\tau_i}$ is the implied $\tau_i$ maturity VIX index value using our model, equation (3.13), on day $t$. Here, $n_t$ is the number of maturities on day $t$ and $T$ is the total number of days in the sample.

**Step three:** We estimate the parameter set $\{\kappa_V, \kappa_\theta\}$, using the daily $\hat{V}_t, \hat{\theta}_t$, estimated from step two, by minimizing the total SSE as:

$$\{\kappa_V, \kappa_\theta\} = \arg \min_{\kappa_V, \kappa_\theta} \sum_{t=1}^{T} \sum_{i=1}^{n_t} (VIX^{MKT}_{t,\tau_i} - VIX^{MI}_{t,\tau_i})^2.$$ 

Steps two and three are then repeated until there is no further significant decrease in the error.

When calibrating any of the nested models (Heston (1993), floating $\theta$ or $\kappa = \kappa_V = \kappa_\theta$ models) step one is not necessary. When calibrating the Heston (1993) model we estimate $\kappa$ and $\bar{\theta}$ in step two and only estimate the daily $V_t$ in step three. We calibrate $\kappa$ in step two and the daily $V_t$ and $\theta_t$ in step three for the floating $\theta$ model. For the $\kappa = \kappa_V = \kappa_\theta$ nested model $\kappa$ and $\bar{\theta}$ are estimated in step two and the daily $V_t$ and $\theta_t$ are estimated in step three.

### 3.4.2 Results

We first verify our calibration procedure by estimating the parameters of the Heston (1993) and floating $\theta$ models over a sub sample, which is equivalent to the sample used by Zhang, Zhen, Sun, and Zhao (2017). This allows us to verify our calibration method. Table 3.2 shows the results of their calibration and ours, in panels A and B respectively. From the table we can see that our estimation is virtually identical to theirs. The slight difference could be explained by a difference in software or optimization algorithms used.

The main calibration results are presented in table 3.3. In panel A we present the
results of calibrating the Gehricke and Zhang (2018), floating $\theta$, $\kappa = \kappa_V = \kappa_\theta$ and full models using the full VIX term structure. The results using only the first two and first three maturities of the VIX term structure are presented in panels B and C, respectively. We can see in the table that the full model is able to fit the VIX term structure the best for any of the samples, as it has the lowest RMSE.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{V}_t$</th>
<th>Std($V_t$)</th>
<th>$\hat{\theta}_t$</th>
<th>Std ($\theta_t$)</th>
<th>VIX TS RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston</td>
<td>0.28</td>
<td>0.1651</td>
<td>0.0342</td>
<td>0.0264</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Floating $\theta$</td>
<td>1.91</td>
<td>–</td>
<td>0.0299</td>
<td>0.0289</td>
<td>0.0729</td>
<td>0.0253</td>
</tr>
<tr>
<td>Panel A: Zhang, Zhen, Sun, and Zhao (2017) calibration</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{V}_t$</th>
<th>Std($V_t$)</th>
<th>$\hat{\theta}_t$</th>
<th>Std ($\theta_t$)</th>
<th>VIX TS RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston</td>
<td>0.28</td>
<td>0.1653</td>
<td>0.0345</td>
<td>0.0266</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Floating $\theta$</td>
<td>1.91</td>
<td>–</td>
<td>0.0301</td>
<td>0.0291</td>
<td>0.0732</td>
<td>0.0254</td>
</tr>
<tr>
<td>Panel B: Our calibration</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: Calibrated parameters for Zhang, Zhen, Sun, and Zhao (2017) sample

This table shows the calibrated parameters to the Heston (1993) and the floating $\theta$ models over the sample period of Zhang, Zhen, Sun, and Zhao (2017). Panel A shows the parameters as estimated by Zhang, Zhen, Sun, and Zhao (2017) and Panel B shows our estimation of the parameters over the same sample.

Table 3.3 also shows that when we are using either of the short VIX term-structure samples, $\kappa_\theta$, for the full model, is essentially zero. However, when using the full VIX term structure $\kappa_\theta$ is 0.2406. From this evidence we can conclude that the mean reversion of $\theta_t$ is less important when modeling only the short VIX term structure, but is more important when we want to model the full VIX term structure.

Interestingly, no matter which model is used the mean daily $V_t$ and $\theta_t$ are quite close, apart from the mean $\theta_t$ for the $\kappa = \kappa_V = \kappa_\theta$ using the full VIX term structure. In figure 3-2 we present the daily time series of $V_t$ and $\theta_t$ for the full model calibrated to the two short and the full VIX term structures. $^{13}$ We can see in the figure that most of the time $\theta_t$ is above $V_t$; this is because the VIX term structure is usually in contango (upward sloping). However, there are days in the sample where $V_t$ spikes

$^{13}$We can see that $V_t$ is always positive even though the Feller’s condition is not satisfied ($\kappa_\theta \sim 0$), showing that the Feller’s condition is important theoretically, but not empirically, in this setting.
### Table 3.3: Calibrated parameters.

This table shows the estimated parameters using either the full VIX term structure, in Panel A, using only the first two points of the VIX term structure, in Panel B or using the first three points of the VIX term structure.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa_V$</th>
<th>$\kappa_\theta$</th>
<th>$\kappa$</th>
<th>$\bar{\theta}_t$</th>
<th>$V_t$</th>
<th>Std($V_t$)</th>
<th>$\theta_t$</th>
<th>Std($\theta_t$)</th>
<th>VIX TS RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Full sample and full VIX term structure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heston</td>
<td>0.3814</td>
<td>0.1287</td>
<td>0.0322</td>
<td>0.0244</td>
<td></td>
<td></td>
<td>0.068</td>
<td>0.0225</td>
<td>1.3193</td>
</tr>
<tr>
<td>Floating $\theta$</td>
<td>2.0341</td>
<td>0.0281</td>
<td>0.0263</td>
<td>0.068</td>
<td>0.0244</td>
<td>1.3193</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa = \kappa_V = \kappa_\theta$ model</td>
<td>0.5908</td>
<td>0.0176</td>
<td>0.0289</td>
<td>0.0248</td>
<td>0.1419</td>
<td>0.0500</td>
<td>0.0678</td>
<td>0.0262</td>
<td>0.5951</td>
</tr>
<tr>
<td>Full model</td>
<td>2.0969</td>
<td>0.2406</td>
<td>0.0680</td>
<td>0.0280</td>
<td>0.0260</td>
<td>0.0678</td>
<td>0.0262</td>
<td>0.5276</td>
<td></td>
</tr>
<tr>
<td><strong>Panel B: Full sample and first two VIX term structure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Floating $\theta$</td>
<td>10.0000</td>
<td>0.0269</td>
<td>0.0314</td>
<td>0.0406</td>
<td>0.0240</td>
<td>0.1766</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa = \kappa_V = \kappa_\theta$ model</td>
<td>4.2238</td>
<td>0.0567</td>
<td>0.0269</td>
<td>0.0289</td>
<td>0.0495</td>
<td>0.0399</td>
<td>0.0395</td>
<td>0.3193</td>
<td></td>
</tr>
<tr>
<td>Full model</td>
<td>12.0459</td>
<td>0.000013</td>
<td>0.0406</td>
<td>0.0266</td>
<td>0.0321</td>
<td>0.0236</td>
<td>0.1566</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Panel C: Full sample and first three VIX term structure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Floating $\theta$</td>
<td>7.8859</td>
<td>0.0266</td>
<td>0.0307</td>
<td>0.0446</td>
<td>0.0248</td>
<td>0.2731</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa = \kappa_V = \kappa_\theta$ model</td>
<td>3.1278</td>
<td>0.0644</td>
<td>0.0277</td>
<td>0.0278</td>
<td>0.0574</td>
<td>0.0398</td>
<td>0.3829</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full model</td>
<td>7.8863</td>
<td>0.000013</td>
<td>0.0446</td>
<td>0.0265</td>
<td>0.0306</td>
<td>0.0446</td>
<td>0.0234</td>
<td>0.2729</td>
<td></td>
</tr>
</tbody>
</table>
and is above $\theta_t$; these are the days where the VIX term structure is in backwardation (downward sloping). On the days where the VIX term structure is in backwardation there was likely some event/news causing short-term implied volatility to increase drastically, while the longer-term implied volatility does not change as much. When volatility spikes, the market seems to expect it to decrease again at some point in the future.

The Heston (1993) model used by Gehricke and Zhang (2018) to model the VXX, fits the VIX term structure the worst compared with the other models. This is because the long-term mean level of the instantaneous variance, $\theta$, is constant making it less flexible in modeling anomalies, such as days where the VIX term structure is in backwardation. Our model fits the VIX term structure well, whether it is in contango or backwardation because of the flexibility of $\theta_t$.

Figure 3-3 shows the daily mean squared error (MSE) of the full model calibration using the full and two shorter maturity VIX term structure samples. We can see that the MSE for the shorter maturity VIX term structure samples is usually lower, but especially so when volatility ($V_t$) goes very high. This is likely due to the model only needing to fit two or three-points of the VIX term structure by optimizing the two daily parameters. However, even with this flexibility the model did not fit the VIX spike, which was caused by the credit rating downgrade of the U.S. government in August 2011, very well. The calibration using the shorter VIX term structure samples is also better on and around this event. We can see in the figure that on most days the model is able to fit the VIX term structure well. It can be expected to have larger mean errors when trying to fit more points of the term structure using the same model.
Figure 3-2: Daily $V_t$ and $\theta_t$ - full model
This figure shows the daily estimates of $V_t$ and $\theta_t$ from calibrating the full model using the full, first two points or first three points of the VIX term structure, in order.
Figure 3-3: Daily MSE
This figure shows the daily estimates of the MSE from calibrating the full model using the full, first two and first three-points of the VIX term structure, in order.
3.5 Model fit

In this section we focus on how well our VIX term structure calibrated model can fit the VXX time series. We compare the fit of the calibrated model using either the continuous or discrete VXX model and calibrating to either the full, two-point or three-point short VIX term structure samples. We find that using the realistic model and calibrating to the three-point VIX term structure is best for modeling the VXX. Finally, we show that our model is also good for fitting the time series of other short-term VIX futures ETPs.

3.5.1 VXX model fit

We now estimate the model implied VXX time series using the VIX term structure calibrated parameters \( \{\kappa_V, \kappa_\theta, \bar{\theta}, V_t, \theta_t\} \), which are estimated as reported in section 3.4.2. To measure the model fit we estimate the RMSE in the levels and returns between the model implied and actual VXX time series. We also examine the fit of the first two moments of the returns of the VXX, the correlation between model implied and actual VXX returns and by graphing the implied and actual time series.

From table 3.4 and figure 3-4 we can see that the realistic model outperforms the idealistic model in fitting the VXX time series. The table shows that the VXX level RMSE for the realistic model is 416.93, 150.68 and 99.11 compared with 580.60, 345.02 and 653.13 for the idealistic model using the full, two-point short and three-point short VIX term structure samples, respectively. Conversely the idealistic model has a very slightly better but almost identical return RMSE. The VXX return RMSE, where the realistic models is 2.42%, 5.16% and 2.72% compared with 2.32%, 5.06% and 2.76% for the idealistic model, using the full, two-point short and three-point short VIX term structure samples, respectively. We also show that the mean VXX return implied by the model is a lot closer to the market value when using the realistic model, whether we use the full or short VIX term structure calibration. The standard
deviation of the implied VXX returns is barely different between the two models for but the realistic model’s value is slightly closer to that observed in the market. The correlation of market and model implied returns is also slightly higher for the realistic model, when calibrating to the first three points of the term structure, and slightly lower when calibrating to the full or two point VIX term structures. So the realistic model seems to perform better than the idealistic in fitting VXX prices.

### Table 3.4: Model implied VXX and market VXX

This table shows some summary statistics of the performance of our calibrated model in fitting the VXX time series. The model is calibrated using either the full, first two points or first three points of the VIX term structure. Then either the idealistic or realistic model is used to imply a VXX time series, which can be compared to the market VXX time series, presented in the first column. RMSE is the root mean squared error, which can be computed for the errors between the model implied and market VXX prices.

<table>
<thead>
<tr>
<th></th>
<th>Market</th>
<th>Full VIX term structure</th>
<th>First two VIX term structure</th>
<th>First three VIX term structure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Idealistic</td>
<td>Realistic</td>
<td>Idealistic</td>
</tr>
<tr>
<td>( \kappa_V )</td>
<td>2.097</td>
<td>12.046</td>
<td>7.886</td>
<td></td>
</tr>
<tr>
<td>( \kappa_\theta )</td>
<td>0.241</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.068</td>
<td>0.041</td>
<td>0.045</td>
<td></td>
</tr>
<tr>
<td>VIX TS Level RMSE</td>
<td>0.515</td>
<td>0.157</td>
<td>0.273</td>
<td></td>
</tr>
<tr>
<td>VXX Level RMSE</td>
<td>580.60</td>
<td>416.93</td>
<td>345.02</td>
<td>150.68</td>
</tr>
<tr>
<td>VXX mean Return</td>
<td>-0.32%</td>
<td>0.01%</td>
<td>-0.42%</td>
<td>0.02%</td>
</tr>
<tr>
<td>VXX Return std. dev.</td>
<td>3.97%</td>
<td>5.21%</td>
<td>5.01%</td>
<td>6.35%</td>
</tr>
<tr>
<td>Return correlation</td>
<td>0.89</td>
<td>0.88</td>
<td>0.61</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Both table 3.4 and figure 3-4 also show that the model implied VXX fits the market VXX best when using the realistic model calibrated to the three-point VIX term structure. From the table we can see that the VXX level RMSE for the realistic model is much lower when we use the three-point short (99.11) rather than the two-point short (150.68) or full (416.93) VIX term structure calibrations. Observing the VXX return RMSE we can see that the full (2.42%) is slightly better than the three point short (2.72%) VIX term structure calibration in fitting the returns of the VXX. However, the slight out-performance in fitting the returns of using the full VIX term-structure calibration is accompanied by a much worse performance in fitting the level. Turning to the return standard deviations we can see that the three-point calibration exhibits VXX return standard deviation much closer to the market value. Also, figure
This figure shows 6 graphs, each depicting the model implied and market VXX prices. The top three graphs display the model implied series using the idealistic model calibrated to the full (left), two-point (middle) and three-point (right) VIX term structure data. The bottom figures display the model implied series using realistic model. In the lower three graphs we also plot the replicated VXX time series using market VIX futures prices.

Figure 3-4 shows that the fit is much better over the sample using the three-point VIX term-structure calibration. The first three point VIX term structure calibration likely gives the best fit because longer term market expectations of volatility aren’t important for short-term VIX futures ETPs and the first two points of the VIX term structure may not have long enough maturity relative to the volatility expectations, that the short term VIX futures ETPs are exposed to.

Overall the table shows that the realistic model performs better than the idealistic model and calibrating to the first three points of the VIX term structure results is optimal for fitting the VXX market prices.

### 3.5.2 Short-term VIX futures ETPs model fit

We now briefly analyze the ability of our model to fit other short-term VIX futures ETPs. We examine the fit of the realistic model using either the shorter or full VIX
term structure calibrations. The other short-term VIX futures ETPs we try to model are the five most liquid, after the VXX, namely the XIV, SVXY, UVXY, TVIX and VIXY (in order of market size).

Figures 3-5, 3-6 and 3-7 show the model fit to the other ETPs using the two-point, three-point and full VIX term structure calibration, respectively. From the figures we can see that the two-point VIX term structure calibration works fairly well for fitting the long-exposure ETPs, namely UVXY, TVIX and VIXY. However, neither the two-point nor full VIX term structure calibration allows our realistic model to fit the short-exposure ETPs, namely XIV and SVXY, very well. When the model is calibrated using the three-point VIX term structure it fits all the short-term ETPs much better, as can see in figure 3-7.

The more volatile the underlying index the less likely a leveraged ETF is to achieve its target leverage that day, and it will often be over or under exposed. So, if the SVXY fails to achieve its target leverage of negative one it is to be expected that the model can not fit it well, as is the case for our model. The model seems to fit the UVXY, double-leverage, ETF quite well which could be due to long exposure leverage (two times) to be more replicable than the short exposure leverage (negative one times) of the SVXY. The model fits the leveraged ETNs, XIV and TVIX, well as they do not have this replication problem. ETNs should track their target exposure much closer, since they do not have to replicate a leveraged version of the underlying VIX futures position of the SPVXSTR, they just promise to pay such the leveraged amount upon redemption/maturity. In Chapter 2 we show that the VIX futures ETNs also do not track their indicative values perfectly and are often priced inconsistently to each other, so some error in modeling these is to be expected.

Overall we can say that our realistic VIX futures ETP model is useful in modeling any short-term VIX futures ETPs, which make up most of the market capitalization and trading of all VIX futures ETPs, as seen in figure 3-1.
Chapter 3: Modeling VXX

3.6 Conclusions

In this study we have created a new model for the VXX, which accounts for the relationship between the S&P 500, VIX index, VIX futures and the VXX. Our model extends that of Gehricke and Zhang (2018) by including jumps in the instantaneous variance ($V_t$) and making the long-run mean level of $V_t$ a stochastic mean-reverting process. We derive simple analytical formula for the VIX term structure, VIX futures prices, short-term (less than 60 days) VIX futures prices, the VXX and any other short-term VIX futures ETPs (which dominate the VIX futures ETP market). Our derived VXX model confirms the finding of Gehricke and Zhang (2018) that, theor-
Figure 3-6: Realistic model - two point VIX term structure
This figure shows the model implied, market and replicated prices of the 5 most liquid, after the VXX, short-term VIX futures ETPs. The model implied prices are calculated using the realistic model and the parameters from calibrating to the first two points of the VIX term structure.

ically, the roll yield is the main driver of the under-performance of the VXX relative to the VIX index.

We calibrate our theoretical model to the full VIX term structure as well as the first two and first three-points. We show that our model fits the VIX term structure better than the nested models, no matter which calibration portion of the term structure is used. Namely, the nested models are the model of Gehricke and Zhang (2018), its floating $\theta$ equivalent and our model with $\kappa_V = \kappa_\theta = \kappa$. We find that the mean reversion feature of the models contributes less when fitting the shorter VIX term structures, but is more important when fitting the full VIX term structure.
Chapter 3: Modeling VXX

Figure 3-7: Realistic model - three point VIX term structure
This figure shows the model implied, market and replicated prices of the 5 most liquid, after the VXX, short-term VIX futures ETPs. The model implied prices are calculated using the realistic model and the parameters from calibrating to the first three points of the VIX term structure.

We provide a realistic and idealistic model for the VXX and show that the realistic model outperforms in fitting the VXX time series, no matter whether we calibrate to the shorter or full VIX term structure. We find that our model performs best for fitting the levels and returns of the VXX when calibrating only to the first three-points of the VIX term structure. This is likely because longer term market expectations of volatility are irrelevant to the short-term ETP prices and using just the first two-points misses some relevant market information.

Lastly, we show that our model fits other short-term VIX futures ETP time series well, but fails to fit the SVXY ETF time series. This may be explained by the
difficulty for ETFs to achieve the target leverage over a holding period of more than one day and/or by market frictions in the VIX futures ETP market, which has become a topic of interest recently (Fernandez-Perez, Frijns, Gafiatullina, and Tourani-Rad, 2018).

Our model could be extended to the mid-term VIX futures ETPs, in which case we recommend not removing the convexity adjustment in the VIX futures price formula. It may also be useful for short-term forecasts of VXX and other short-term VIX futures ETP returns. Our model for the VXX could also be extended to price options written on the VXX and other short-term VIX futures ETPs. In Chapter 4 we document and analyze the dynamics of the VXX options market, so that we can extend our VXX model for VXX option pricing in the future.
3.7 Appendix

3.7.1 Central moments of $\theta_s$

In this section we derive the first and second central moments of $\theta_t$. Let us start with the first moment, to do this we define a function $f(\theta_t) = e^{\kappa \theta t} \theta_t$, then we get:

$$df(\theta_t) = d(e^{\kappa \theta t} \theta_t) = \kappa \theta e^{\kappa \theta t} \theta_t dt + \kappa \theta e^{\kappa \theta t} (\bar{\theta} - \theta_t) dt + e^{\kappa \theta t} \sigma \sqrt{\theta_t} dB_{3,t}$$

Taking the integral of both sides of equation (3.27) from $t$ to $s$ we get:

$$e^{\kappa \theta s} \theta_s - e^{\kappa \theta t} \theta_t = \kappa \bar{\theta} \int_t^s e^{\kappa \theta u} du + \sigma \int_t^s e^{\kappa \theta u} \sqrt{\theta_u} dB_{3,u}$$

Then taking the expectation of equation (3.28), conditional on the information at $t$, we get the first central moment of $\theta_s$ as:

$$E_t^Q[\theta_s] = \bar{\theta} + (\theta_t - \bar{\theta}) e^{-\kappa \theta (s-t)}$$

Using the definition of the second central moment of $\theta_s$ and plugging in equations (3.28) and (3.29) we get:

$$E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] = E_t^Q \left[ (\bar{\theta} + (\theta_t - \bar{\theta}) e^{-\kappa \theta (s-t)}) + \sigma \int_t^s e^{-\kappa \theta (s-u)} \sqrt{\theta_u} dB_{3,u} \right]^2$$

$$= E_t^Q \left[ \left( \sigma \int_t^s e^{-\kappa \theta (s-u)} \sqrt{\theta_u} dB_{3,u} \right)^2 \right],$$

(3.30)
then using Ito’s isometry and bringing the expectation inside the integral we get:

\[
E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] = \sigma^2 \int_t^s e^{-2\kappa\theta(s-u)} E_t^Q[\theta_u] du \quad (3.31)
\]

and substituting in \( E_t^Q[\theta_u] \), from equation (3.6) with \( s = u \), we get:

\[
E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] = \sigma^2 \left( \int_t^s e^{-2\kappa\theta(s-u)} [\tilde{\theta} + (\theta_t - \bar{\theta}) e^{-\kappa\theta(u-t)}] du \right)
\]

\[
= \sigma^2 \left( \int_t^s e^{-2\kappa\theta(s-u)} \tilde{\theta} du + \int_t^s e^{\kappa\theta u - 2\kappa\theta s + \kappa\theta t} \theta_t du - \int_t^s e^{\kappa\theta u - 2\kappa\theta s + \kappa\theta t} \bar{\theta} du \right)
\]

\[
= \frac{\sigma^2 \theta_t}{\kappa \theta} (e^{-\kappa\theta(s-t)} - e^{-2\kappa\theta(s-t)}) + \frac{\sigma^2 \theta_t}{2\kappa \theta} (1 - 2e^{-\kappa\theta(s-t)} + e^{-2\kappa\theta(s-t)})
\]

\[
= \sigma^2 \frac{\kappa \theta t}{\kappa \theta} (e^{-\kappa\theta(s-t)} - e^{-2\kappa\theta(s-t)}) + \frac{\sigma^2 \theta_t}{2\kappa \theta} (1 - 2e^{-\kappa\theta(s-t)} + e^{-2\kappa\theta(s-t)})
\]

\[
(3.32)
\]

### 3.7.2 Central moments of \( V_s \)

To find the risk-neutral expectation of instantaneous variance \( E_t^Q[V_S] \) we need to first derive a function for the future instantaneous variance \( V_s \). The dynamics of \( V_t \) can be rewritten as:

\[
dV_t = \kappa_V (\theta_t - V_t) dt + dM_t, \quad (3.33)
\]

where

\[
dM_t = \sigma_V \sqrt{V_t} dB_{2,t} + y dN_t - \lambda E_t^Q[y] dt, \quad (3.34)
\]

which is a martingale in the risk-neutral measure.

Let \( f(V_t) = e^{\kappa_V t} V_t \) then we get:

\[
\begin{align*}
\text{df}(V_t) &= d(e^{\kappa_V t} V_t) \\
&= \kappa_V e^{\kappa_V t} V_t dt + e^{\kappa_V t} \kappa_V (\theta_t - V_t) dt + e^{\kappa_V t} dM_t \\
&= e^{\kappa_V t} \kappa_V \theta_t dt + e^{\kappa_V t} dM_t.
\end{align*}
\]

\[
(3.35)
\]
Integrating both sides from $t$ to $s$ gives and solving for $s$ we get:

$$e^{\kappa V s} V_s - e^{\kappa V t} V_t = \int_t^s e^{\kappa V u} \kappa V \theta_u du + \int_t^s e^{\kappa V u} dM_u$$

$$V_s = V_t e^{-\kappa V (s-t)} + \int_t^s e^{-\kappa V (s-u)} \kappa V \theta_u du + \int_t^s e^{-\kappa V (s-u)} dM_u. \quad (3.36)$$

We can then take the conditional risk-neutral expectation of $V_s$, which yields:

$$E^Q_t[V_s] = V_t e^{-\kappa V (s-t)} + \int_t^s e^{-\kappa V (s-u)} \kappa V E^Q_t[\theta_u] du \quad (3.37)$$

We then substitute $E^Q_t[\theta_u]$, from equation (3.6) with $s = u$, into the expected future instantaneous variance, equation (3.37), which results in:

$$E^Q_t[V_s] = V_t e^{-\kappa V (s-t)} + \int_t^s \kappa V e^{-\kappa (s-u)} [\bar{\theta} + (\theta_t - \bar{\theta}) e^{-\kappa \theta (u-t)}] du$$

$$= V_t e^{-\kappa V (s-t)} + \bar{\theta} (1 - e^{-\kappa V (s-t)}) + \frac{\kappa V}{\kappa V - \kappa \theta} (\theta_t - \bar{\theta}) (e^{-\kappa \theta (s-t)} - e^{-\kappa V (s-t)})$$

$$= e^{-\kappa V (s-t)} V_t + \frac{\kappa V}{\kappa V - \kappa \theta} (e^{-\kappa \theta (s-t)} - e^{-\kappa V (s-t)}) \theta_t$$

$$+ \left(1 - e^{-\kappa V (s-t)} - \frac{\kappa V}{\kappa V - \kappa \theta} (e^{-\kappa \theta (s-t)} - e^{-\kappa V (s-t)})\right) \bar{\theta} \quad (3.38)$$

We can get the second central moment of $V_s$ using the definition of the second central moment and plugging in equations (3.36) and (3.38), resulting in:

$$E^Q_t[(V_s - E^Q_t[V_s])^2]$$

$$= E_t\left[\left(\int_t^s e^{-\kappa V (s-y)} \kappa V (\theta_y - E_t[\theta_y]) dy + \int_t^s e^{-\kappa V (s-u)} dM_u\right)^2\right]$$

$$= E_t\left[\left(\kappa V \sigma \theta \int_t^s e^{-\kappa V (s-y)} \int_t^y e^{-\kappa \theta (y-u)} \sqrt{\theta_u dB_{3,u}} dy + \int_t^s e^{-\kappa V (s-u)} dM_u\right)^2\right]$$

$$= E_t\left[\left(\kappa V \sigma \theta \int_t^s \int_t^u e^{-\kappa V (s-y)} e^{-\kappa \theta (y-u)} dy \sqrt{\theta_u dB_{3,u}} + \int_t^s e^{-\kappa V (s-u)} dM_u\right)^2\right]$$

$$= E_t\left[\left(\frac{\kappa V \sigma \theta}{\kappa V - \kappa \theta} \int_t^s (e^{-\kappa \theta (s-u)} - e^{-\kappa V (s-u)}) \sqrt{\theta_u dB_{3,u}} + \int_t^s e^{-\kappa V (s-u)} dM_u\right)^2\right]$$
where

\[
X = E_t \left[ \left( \frac{\kappa_V \sigma_{\theta}}{\kappa_V - \kappa_{\theta}} \right) \int_t^s \left( e^{-\kappa_{\theta}(s-u)} - e^{-\kappa_V(s-u)} \right) \sqrt{\theta_u} dB_{3,u} \right]^2 \\
+ E_t \left[ \frac{\kappa_V \sigma_{\theta}}{\kappa_V - \kappa_{\theta}} \int_t^s \left( e^{-\kappa_{\theta}(s-u)} - e^{-\kappa_V(s-u)} \right) \sqrt{\theta_u} dB_{3,u} \times \int_t^s e^{-\kappa_V(s-u)} dM_u \right] \\
+ E_t \left[ \left( \int_t^s e^{-\kappa_V(s-u)} dM_u \right)^2 \right] \\
= X + Y, \tag{3.39}
\]

and

\[
Y = E_t \left[ \left( \int_t^s e^{-\kappa_V(s-u)} dM_u \right)^2 \right] \\
= \int_t^s e^{-2\kappa_V(s-u)} \sigma_V^2 E_t^Q [V_u] (dB_{2,u})^2 + \int_t^s e^{-2\kappa_V(s-u)} E_t^Q (y^2) \lambda d \mu \\
= \int_t^s e^{-2\kappa_V(s-u)} \sigma_V^2 e^{-2\kappa_V(u-t)} V_d du
\]


\[+ \int_t^\infty e^{-2\kappa_V(s-u)} \sigma^2_V \frac{\kappa_V \theta_t}{\kappa_V - \kappa_\theta} \left( e^{-\kappa_\theta(u-t)} - e^{-\kappa_V(u-t)} \right) du \]

\[+ \frac{\lambda}{2\kappa_V} \left( 1 - e^{-2\kappa_V(s-t)} \right) E_t[y^2] \]

\[= \frac{\sigma^2_V}{\kappa_V} \left( e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)} \right) V_t \]

\[+ \frac{\kappa_V \sigma^2_V}{\kappa_V - \kappa_\theta} \left( \frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V} \right) \theta_t \]

\[+ \sigma^2_V \left( \frac{1 - e^{-\kappa_V(s-t)}}{2\kappa_V} - \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left[ \frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V} \right] \right) \bar{\theta} \]

\[+ \frac{\lambda}{2\kappa_V} \left( 1 - e^{-2\kappa_V(s-t)} \right) E_t[y^2] \]

\[= A V_t + B \theta_t + (1 - A - B) \bar{\theta}, \quad (3.42) \]

\[3.7.3 \ \text{VIX index proof} \]

The VIX squared index is equal to the risk-neutral conditional expectation of the variance over the next 30 days, using this and lemma 1 we get:

\[ \left( \frac{VIX_t}{100} \right)^2 = \frac{1}{\tau_0} \int_t^{t+\tau_0} E^Q_t[V_s] ds \]

\[= \frac{1}{\tau_0} \left[ V_t \int_t^{t+\tau_0} e^{-\kappa_V(u-t)} du + \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left[ \int_t^{t+\tau_0} e^{-\kappa_\theta(u-t)} du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)} du \right] \right. \]

\[+ \left. \bar{\theta} \left( \int_t^{t+\tau_0} du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)} du \right) - \frac{\kappa_V}{\kappa_V - \kappa_\theta} \left( \int_t^{t+\tau_0} e^{-\kappa_\theta(u-t)} du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)} du \right) \right] \]

\[= A V_t + B \theta_t + (1 - A - B) \bar{\theta}, \quad (3.42) \]

where

\[A = \frac{1 - e^{-\kappa_V \tau_0}}{\kappa_V \tau_0}, \]

\[B = \frac{\kappa_V (1 - e^{-\kappa_\theta \tau_0})}{\kappa_\theta \tau_0 (\kappa_V - \kappa_\theta)} - \frac{1 - e^{-\kappa_V \tau_0}}{\tau_0 (\kappa_V - \kappa_\theta)}. \]
Then taking the square root gives us the VIX formula presented in proposition 1.

A.4 VIX futures approximate formula

We expand the square root form equation (3.15) (VIX formula) using the two variable Taylor expansion near the points $E_t^Q[V_T]$ and $E_t^Q[\theta_T]$, which results in:

$$f(V_T, \theta_T) = a_0 + a_1(V_T - E_t^Q[V_T]) + a_2(\theta_T - E_t^Q[\theta_T])$$

$$+ \frac{1}{2} a_3(V_T - E_t^Q[V_T])^2 + \frac{1}{2} a_4(\theta_T - E_t^Q[\theta_T])^2$$

$$+ a_5(V_T - E_t^Q[V_T])(\theta_T - E_t^Q[\theta_T]),$$

(3.43)

where

$$a_0 = f(V, \theta)\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]},$$

$$a_1 = \frac{\partial f}{\partial V}\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]},$$

$$a_2 = \frac{\partial^2 f}{\partial \theta^2}\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]},$$

$$a_3 = \frac{\partial^2 f}{\partial V^2}\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]},$$

$$a_4 = \frac{\partial^2 f}{\partial V \partial \theta}\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]},$$

$$a_5 = \frac{\partial^2 f}{\partial \theta^2}\big|_{V=E_t^Q[V_T], \theta=E_t^Q[\theta_T]}. $$

$$f(V, \theta) = \left(A \, V + B \theta + (1 - A - B)\bar{\theta}\right)^{\frac{1}{2}}.$$

Plugging the partial differentials into equation (3.43) we get:

$$f(V_T, \theta_T) = \left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{\frac{1}{2}} + \frac{1}{2} E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]^{-\frac{1}{2}} A(V_T - E_t^Q[V_T])$$

$$+ \frac{1}{2} E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]^{-\frac{1}{2}} B(\theta_T - E_t^Q[\theta_T])$$

$$- \frac{1}{8} E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]^{-\frac{3}{2}} A^2(V_T - E_t^Q[V_T])^2$$

$$- \frac{1}{8} E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]^{-\frac{3}{2}} B^2(\theta_T - E_t^Q[\theta_T])^2$$
\[-\frac{1}{8} \left( E_t^Q \left[ \left( \frac{VIX}{100} \right)^2 \right] \right)^{-\frac{3}{2}} AB(V_T - E_t^Q[V_T]) (\theta_T - E_t^Q[\theta_T]), \] 

(3.44)

We then take the expectation of \( f(V_T, \theta_T) \) and substitute in the second central moments of \( V_T \) and \( \theta_T \), from lemma 1 where \( s = T \), to get:

\[
E_t^Q[f(V_T, \theta_T)] = \left( E_t^Q \left[ \left( \frac{VIX}{100} \right)^2 \right] \right)^{\frac{1}{2}} \left( E_t^Q \left[ \left( \frac{VIX}{100} \right)^2 \right] \right)^{-\frac{3}{2}} A^2 \left[ X|_{s=T} + Y|_{s=T} \right] \\
- \frac{1}{8} \left( E_t^Q \left[ \left( \frac{VIX}{100} \right)^2 \right] \right)^{-\frac{3}{2}} B^2 \\
\times \left[ \frac{\sigma^2_{\phi \theta t}}{\kappa_{\phi}} \left( e^{-\kappa_{\phi}(T-t)} - e^{-2\kappa_{\phi}(T-t)} \right) + \frac{\sigma^2_{\phi \theta}}{2\kappa_{\phi}} \left( 1 - 2e^{-\kappa_{\phi}(T-t)} + e^{-2\kappa_{\phi}(T-t)} \right) \right],
\]

(3.45)

then substituting in \( E_t^Q \left[ \left( \frac{VIX}{100} \right)^2 \right] \) from lemma 3.16 gives us proposition 2.

### 3.7.4 Partial derivatives for change in log short-term VIX futures

The partial derivative of the short-term VIX futures price with respect to \( t \) in equation (??) is derived as:

\[
\frac{\partial \ln F_t^T}{\partial t} = \frac{1}{2} \frac{\partial C}{\partial t} V_t + \frac{\partial D}{\partial t} \theta_t - \left( \frac{\partial C}{\partial t} + \frac{\partial D}{\partial t} \right) \bar{\theta} \\
= \frac{\kappa_V}{2 CV_t + D \theta_t + (1 - C - D) \bar{\theta}},
\]

(3.46)

where

\[
\frac{\partial D}{\partial t} = \frac{\kappa_V}{\kappa_V - \kappa_{\theta}} \left( A\kappa_{\theta} e^{-\kappa_{\theta}(T-t)} - \kappa_V C \right) + \kappa_{\theta} B e^{-\kappa_{\theta}(T-t)}. \]

(3.47)

(3.48)
The other partial derivatives in equation (??) are derived as:

\[
\begin{align*}
\frac{\partial \ln F^T_t}{\partial V_t} &= \frac{1}{2} \frac{C}{CV_t + D\theta_t + (1 - C - D)\bar{\theta}} \\
\frac{\partial^2 \ln F^T_t}{\partial V_t^2} &= -\frac{1}{2} \frac{C^2}{(CV_t + D\theta_t + (1 - C - D)\bar{\theta})^2} \\
\frac{\partial \ln F^T_t}{\partial \theta_t} &= \frac{1}{2} \frac{D}{CV_t + D\theta_t + (1 - C - D)\bar{\theta}} \\
\frac{\partial^2 \ln F^T_t}{\partial \theta_t^2} &= -\frac{1}{2} \frac{D^2}{(CV_t + D\theta_t + (1 - C - D)\bar{\theta})^2} \\
\frac{\partial^2 \ln F^T_t}{\partial V_t \partial \theta_t} &= -\frac{1}{2} \frac{CD}{(CV_t + D\theta_t + (1 - C - D)\bar{\theta})^2}
\end{align*}
\]

### 3.7.5 Floating \(\theta\) model dynamics and VIX formula

Under the floating \(\theta\) model the risk-neutral dynamics are given by:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu S_t dt + \sqrt{V_t} S_t dB_{1,t}, \\
\frac{dV_t}{V_t} &= \kappa (\theta_t - V_t) dt + \sigma_V \sqrt{V_t} dB_{2,t}, \\
\frac{d\theta_t}{\theta_t} &= dB_{3,t},
\end{align*}
\]

which results in the following VIX term structure formula:

\[
\frac{VIX_t}{100} = \sqrt{\mathbb{E}_t^Q \left[ \frac{1}{\tau_i} \int_t^{t+\tau_i} V_s ds \right]} = \sqrt{(1 - G)\theta_t + GV_t},
\]

where:

\[
G = \frac{(1 - e^{-\kappa\tau_i})}{\kappa \tau_i}
\]

where \(\tau_i = T_i/365\) and \(T_i\) is the days to maturity of any maturity VIX index in the term structure.
Chapter 4

The Implied Volatility Smirk in the VXX Options Market

This chapter is joint work with Jin E. Zhang. The paper has was presented at the 2018 Derivatives Markets conference, 9-10 August 2018, AUT, Auckland, New Zealand and has been accepted for presentation at the 2018 New Zealand Finance Meeting, 16-19 December 2018, AUT, Queenstown, New Zealand.

4.1 Introduction

In this chapter we use a simple method to quantify and analyze the implied volatility (IV) curve of the VXX options market. The VIX futures ETP market has quickly become one of the most popular destinations for volatility traders and investors, with the VXX leading the way, even with the products’ strange structure and behaviour. Quantifying the VXX’s IV curve provides the basis necessary for creating a VXX option pricing model founded on the market’s empirical dynamics. Understanding the dynamics of the VXX options market will help to determine the correct starting assumptions for a pricing model. The IV curve represents all the information of the market option prices and reflects the risk-neutral distribution of the underlying asset
returns over different horizons.

The market for trading volatility derivatives has developed swiftly over the past 15 years. The VIX index was revised in 2003 to track the IV of S&P 500 OTM options. Later, on 26 March 2004, the CBOE launched the first futures contracts on the VIX index providing the first access to VIX exposure, which was desirable for its possible hedging/diversification benefits. In 2006, VIX options started to trade. In 2009, Standard & Poor’s started calculating VIX futures indices, such as the S&P 500 VIX Short-Term Futures Total Return index (SPVXSTR). The VIX futures indices track a daily rebalanced position of VIX futures contracts to achieve an almost constant maturity. Shortly after, on 29 January 2009, Barclays Capital iPath launched the first VIX futures exchange-traded product (ETP), the VXX exchange-traded note (ETN).¹ Finally, on 28 May 2010, the VXX and VXZ options markets were launched by the CBOE. Now options markets exist for many of the popular VIX futures ETPs.

The VIX futures ETPs and their option markets have become very popular. The VXX is the most popular VIX futures ETP with an average market cap of about $1 billion and average daily dollar trading volume of $1.4 billion.² The VXX options market is the most prominent of the VIX futures ETP option markets and has grown to an average daily trading volume of 328,884 and average daily open interest of 3,097,297 contracts.³ In figure 4-1 we can see the growth of the VXX options market over our sample. The open interest in VXX options has grown consistently from 2010 to 2016, reaching about three million contracts recently, while the trading volume has hovered around 300,000 contracts per day since 2013. The other VIX futures ETP options have also grown in popularity, but their trading volume and open interest are still quite low, relatively. This is why our study focuses on the VXX option IV curve.

¹An ETN is a non-securitized debt obligation, similar to a zero-coupon bond, but with a redemption value that depends on the level of something else, i.e. the SPVXSTR index for the VXX ETN.
²Averages described here are taken over the last year of our sample.
³For a detailed comparison between S&P 500 index (SPX), S&P 500 index ETF (SPY), VIX and VIX futures ETP option markets, please refer to table 4.1.
### Table 4.1: Summary of SPX, SPY, VIX and VIX futures ETP option markets.

<table>
<thead>
<tr>
<th>Style</th>
<th>SPX options</th>
<th>SPY options</th>
<th>VIX options</th>
<th>VXX options</th>
<th>VXZ options</th>
<th>UVXY options</th>
<th>SVXY options</th>
<th>VIXM options</th>
<th>VIXY options</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expiration</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
<td>30 days before 3rd Friday</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
<td>3rd Friday</td>
</tr>
<tr>
<td>Settlement</td>
<td>Cash</td>
<td>Physical</td>
<td>Cash</td>
<td>Physical</td>
<td>Physical</td>
<td>Physical</td>
<td>Physical</td>
<td>Physical</td>
<td>Physical</td>
</tr>
<tr>
<td>Underlying</td>
<td>SPX</td>
<td>SPY</td>
<td>VIX futures with same maturity</td>
<td>VXX</td>
<td>VXZ</td>
<td>UVXY</td>
<td>SVXY</td>
<td>VIXM</td>
<td>VIXY</td>
</tr>
<tr>
<td>Multiplier</td>
<td>$100 \times$ index</td>
<td>$100 \times$ price</td>
<td>$100 \times$ index</td>
<td>$100 \times$ price</td>
<td>$100 \times$ price</td>
<td>$100 \times$ price</td>
<td>$100 \times$ price</td>
<td>$100 \times$ price</td>
<td>$100 \times$ price</td>
</tr>
<tr>
<td>Average Daily Option Volume</td>
<td>1,071,517</td>
<td>2,715,706</td>
<td>643,266</td>
<td>328,884</td>
<td>91,349</td>
<td>17,254</td>
<td>759</td>
<td>15,707</td>
<td></td>
</tr>
<tr>
<td>Average Daily Open Interest</td>
<td>12,388,390</td>
<td>19,429,209</td>
<td>6,785,566</td>
<td>3,097,297</td>
<td>7,304</td>
<td>810,048</td>
<td>223,809</td>
<td>759</td>
<td>15,707</td>
</tr>
<tr>
<td>Underlying Average Daily Volume (000,000's)</td>
<td>$24,641</td>
<td>-</td>
<td>$1,444</td>
<td>$6</td>
<td>$799</td>
<td>$257</td>
<td>$1</td>
<td>$40</td>
<td></td>
</tr>
<tr>
<td>Underlying Average Market Cap. (000,000's)</td>
<td>$176,045</td>
<td>-</td>
<td>$965</td>
<td>$53</td>
<td>$492</td>
<td>$476</td>
<td>$28</td>
<td>$126</td>
<td></td>
</tr>
</tbody>
</table>
Figure 4-1: Option Volume and Open Interest.
This figure shows the 10-day moving average of the daily trading volume and open interest of the VIX, VXX, VXZ, UVXY, SVXY, VIXM and VIXY option markets.

The VIX futures ETPs have recently been making headlines as a spike in volatility led to unprecedented losses in the inverse exposure ETPs, some of which were even terminated. One article states “The problem with ETFs is that many of them appeal to retail investors, but are really meant for institutions” (Dillian, 2018). However, even after the increased media attention on the complexity and risk in these products, trading activity in their options is picking up again as noted by a recent article by Bloomberg (Kawa, 2018). These products trade like stocks and are accessible to retail investors, but possess complexities that even academics and highly trained institutional investors do not yet fully understand. Providing some more insight on these products is of the utmost importance in order to avoid unexpected outcomes, often not even considered as a possibility for retail investors.
We use the method of Zhang and Xiang (2008) to quantify the VXX IV curve for every maturity each day by three factors; the level, slope and curvature. This allows us to summarize the often vast number of IV-moneyness (-strike price) data points with three numbers. We can then examine the dynamics of these factors to draw conclusions on how the VXX options market behaves. Zhang and Xiang (2008) also provide a link between the IV curve factors and the risk-neutral moments and demonstrate how this can be used to calibrate option pricing models. They develop this methodology and demonstrate its application for a very small sample, whereas we apply the methodology to the VXX options market and extend it by studying the term structure, time series and time series of the term structure of the quantified IV curve. Fajardo (2017) adds a torsion factor into the polynomial regression, which quantifies the IV curve, but this is a model based factor and we want to keep our quantification simple. Our method captures close to 100% of the variation of the daily VXX IV curves without this extra factor.

Although there is a growing literature on pricing volatility derivatives (Zhang and Zhu, 2006; Zhang, Shu, and Brenner, 2010; Lu and Zhu, 2010; Chung, Tsai, Wang, and Weng, 2011; Wang and Daigler, 2011; Zhu and Lian, 2012; Mencía and Sentana (2013); Huskaj and Nossman, 2013; Lian and Zhu, 2013; Bardgett, Gourier, and Leippold, 2018; Papanicolaou and Sircar, 2014; Eraker and Wu, 2017; Gehricke and Zhang, 2018; and many more) and their empirical dynamics (Shu and Zhang, 2012; Whaley, 2013; Bordonado, Molnár, and Samdal, 2017; Bollen, O’Neill, and Whaley, 2017; Chapter 2; and many more), there is only one published paper looking at a VIX futures ETP options market. Bao et al. (2012) provide the only study on VXX options proposing and horse-racing several models for pricing the contracts. The authors, however, ignore the underlying relationships of the VXX with the VIX futures, VIX index and S&P 500 index, which are essential to understanding the VXX options market fully.
Many studies have documented the S&P 500 option IV shape and/or its dynamics (Rubinstein, 1985; Rubinstein, 1994; Aït-Sahalia and Lo, 1998; Skiadopoulos, Hodges, and Clewlow, 2000; Cont, Da Fonseca, et al., 2002; Carr and Wu, 2003; Foresi and Wu, 2005; Garleanu, Pedersen, and Poteshman, 2009). Some authors have tried to explain the shape/dynamics of the IV curve through other market and economic factors (Pena, Rubio, and Serna, 1999; Pan, 2002; Dennis and Mayhew, 2002; Bollen and Whaley, 2004). The predictability power of option market IV for the underlying assets return has also been explored (Xing, Zhang, and Zhao, 2010; Cremers and Weinbaum, 2010; Conrad, Dittmar, and Ghysels, 2013, Lin and Lu, 2015).

This study contributes to the literature by being the first empirical study of the dynamics of the VXX options market. We document and provide a comprehensive study of the VXX option IV dynamics as a starting point for developing VXX option pricing models in the future.\footnote{We plan to extend the model from Chapter 3 to price VXX options, using the insights of this Chapter.} We show that the IV curve of the VXX is usually an upward-sloping line with some convexity. As the options maturity increases the at-the-money (ATM) IV increases, the IV curve’s slope decreases and it becomes more convex. Our quantification of the VXX’s IV curve performs well with an average r-squared value of 94.55%. The fit is best for shorter maturity options, with an average r-squared of 98.49% for less than and 83.65% for more than 180 days to maturity.

In the next section we describe the methodology used to quantify the IV curve and how the IV factors can be converted to the risk-neutral moments of the VXX. In section 4.3 we describe our sample data and cleaning procedure. Then in section 4.4 we present the results and describe the dynamics of the VXX’s IV. Lastly, in section 4.5 we conclude.
4.2 Methodology

4.2.1 Implied forward price and ATM IV

In this Chapter we employ the methodology developed by Zhang and Xiang (2008) in order to summarize the VXX option IV curve (IV as a function of option moneyness), every day and for each maturity. For this we first calculate the implied forward price based on the ATM call and put prices as follows:\textsuperscript{5}

\[ F_{t_i}^{T_i} = K_{i,t}^{ATM} + e^{r_{i,t} \tau_{i,t}} (c_{i,t}^{ATM} - p_{i,t}^{ATM}), \]  

where \( F_{t_i}^{T_i} \) is the implied forward price, \( K_{i,t}^{ATM} \) is the ATM strike price, \( r_{i,t} \) is the risk free rate, \( \tau_{i,t} \) is the annualized time to maturity, \( c_{i,t}^{ATM} \) is the ATM call option price and \( p_{i,t}^{ATM} \) is ATM put option price, for maturity \( T_i \) on day \( t \).

4.2.2 Moneyness of options

We use the implied forward price to measure the moneyness of an option as follows:

\[ \xi = \ln \left( \frac{K_{t_i}^{T_i}}{F_{t_i}^{T_i}} \right) \sigma \sqrt{\tau_i}, \]  

where \( K_{t_i}^{T_i} \) is the strike price we are calculating the moneyness for and \( F_{t_i}^{T_i} \) is the implied forward price for maturity \( T_i \) on day \( t \). \( \sigma \) is the average volatility of the underlying, which we proxy by the 30-day ATM IV.\textsuperscript{6} Lastly, \( \tau_i = (T_i - t)/365 \) is the annualized time to maturity of the given expiry option contracts.

\textsuperscript{5}ATM is defined as the strike price where the difference between call and option prices is the smallest. This is not exactly at the money and we will be providing an estimate of exactly ATM IV using equation 4.3, which we call the “exactly ATM IV”.

\textsuperscript{6}The 30 day ATM VXX IV is calculated by linearly interpolating the two nearest to 30 day maturity ATM implied volatilities as \( IV^\tau = IV^{\tau_1} w_1 + IV^{\tau_2} (1 - w_1) \).
4.2.3 Quantifying the IV curve

Having calculated the moneyness of the options, we can quantify the IV curve by fitting the regression:

\[ IV(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2, \quad (4.3) \]

where \( IV \) is the IV and \( \xi \) is the moneyness of the option.\(^7\) The regression is fitted separately each day and for each maturity. Here, the coefficients \( \hat{\alpha}_0, \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) are termed the intercept, unscaled slope and unscaled curvature, respectively. We estimate this quadratic function to the IV by minimizing the volume-weighted mean-squared error given by:

\[ VWMSE = \frac{\sum\xi Vol(\xi) \times [IV(\xi)_{MKT} - IV(\xi)_{MDL}]^2}{\sum\xi Vol(\xi)}, \quad (4.4) \]

where \( Vol(\xi) \) is the volume, \( IV(\xi)_{MKT} \) is the IV from market prices and \( IV(\xi)_{MDL} \) is the model IV, for the option with moneyness \( \xi \), on a particular day for a given maturity. When estimating the IV function we only use OTM options, as is industry practice.\(^8\) This means that when the strike price is above (below) the implied forward price, that is, \( K_{i,t} > F_{i,t} \) (\( K_{i,t} < F_{i,t} \)), we only use call (put) options in estimating the IV curve.

Efficiently estimating the parameters in equation (4.3) should allow us to describe the entire volatility smirk for a given maturity on a certain day with just three parameters. We then document these parameters across time and maturities in order to describe and explore the dynamics of the VXX options market. We will present the results of the regressions with and without a constraint forcing the line to go through

\(^7\)The IV is supplied by OptionMetrics Ivy DB and is calculated using the Cox et al. (1979) binomial tree model and a proprietary algorithm to speed up convergence.

\(^8\)This is because OTM options are more liquid and are more sensitive to pricing models.
the ATM IV point in section 4.4. We can also present the parameters in a dimensionless form as follows:

\[ IV(\xi) = \gamma_0(1 + \gamma_1 \xi + \gamma_2 \xi^2), \]  

(4.5)

where

\[ \gamma_0 = \alpha_0 \]
\[ \gamma_1 = \frac{\alpha_1}{\alpha_0} \]
\[ \gamma_2 = \frac{\alpha_2}{\alpha_0} \]

where \( \gamma_0 \) is the level, \( \gamma_1 \) is the slope and \( \gamma_2 \) is the curvature factor. We can interpret the level coefficient (\( \hat{\alpha}_0 = \gamma_0 \)) as the exact ATM IV where moneyness is actually equal to zero, which will be slightly different to the ATM IV available in the market data, whose moneyness is the closest to zero available.

### 4.2.4 Risk-neutral moments

Transforming the coefficients of the regressions (\( \alpha \)'s) to the dimensionless factors (\( \gamma \)'s), as above, allows us to calculate the moments of the risk-neutral distribution of the VXX, as in Zhang and Xiang (2008). They show that the risk-neutral standard deviation, skewness and excess kurtosis (\( \sigma, \lambda_1, \lambda_2 \)) are related to the level, slope and curvature (\( \gamma_0, \gamma_1 \) and \( \gamma_2 \)) through the following asymptotic expansions:

\[ \gamma_0 = \left(1 - \frac{\lambda_2}{24}\right)\sigma + \frac{\lambda_1}{4} \sigma^2 \sqrt{\tau} + O(\sigma^3 \tau), \]  

(4.6)

\[ \gamma_1 = \frac{\lambda_1}{6(1 - (\lambda_2/24))} \frac{\bar{\sigma}}{\sigma} + \frac{\lambda_2(1 - (\lambda_2/24)) - (\lambda_2^2/2)}{12(1 - (\lambda_2/24))^2} \bar{\sigma} \sqrt{\tau} + O(\sigma \bar{\sigma} \sqrt{\tau}), \]  

(4.7)

---

\(^9\)We constrain the regression to go through the ATM IV because then the fitted volatility curve gives the same price as the market for ATM options. This means that there is no arbitrage between the model price and market price for ATM options.
\[
\gamma_2 = \frac{\lambda_2 \tilde{\sigma}^2}{24 \sigma^2} \left(1 - \frac{\lambda_2}{24}\right)^2 + \frac{\lambda_1 \lambda_2 \tilde{\sigma}^2 \sqrt{\tau}}{96 \sigma} \left(1 - \frac{\lambda_2}{48}\right) \frac{1 - \left(\frac{\lambda_2}{24}\right)^2}{(1 - \left(\frac{\lambda_2}{24}\right))^3} + O(\tilde{\sigma}^2 \sqrt{\tau}), \tag{4.8}
\]

which we call the full approximate relationship between the VXX’s IV curve factors and its implied risk-neutral moments.

Zhang and Xiang (2008) further show that if we ignore the second and higher-order terms and taking \(\tilde{\sigma} = \gamma_0\), we have the following relationships:

\[
\gamma_0 \approx \left(1 - \frac{\lambda_2}{24}\right) \sigma, \quad \gamma_1 \approx \frac{1}{6} \lambda_1, \quad \gamma_2 \approx \frac{1}{24} \lambda_2 \left(1 - \frac{\lambda_2}{16}\right), \tag{4.9}
\]

which we call the simpler approximate relationship.

Then if we further assume that \(\gamma_2 \ll 1\), we get the simplest approximate relationships:

\[
\gamma_0 \approx \left(1 - \frac{\lambda_2}{24}\right) \sigma, \quad \gamma_1 \approx \frac{1}{6} \lambda_1, \quad \gamma_2 \approx \frac{1}{24} \lambda_2. \tag{4.10}
\]

These relationships can be used to approximate the risk-neutral moments of the VXX from the IV curve factors. Once one has the risk-neutral moments these can be used to calibrate VXX option pricing models, as Zhang and Xiang (2008) demonstrate for S&P 500 options.

### 4.3 Data

Our sample is from 1 June 2010 to 29 April 2016. The options data are sourced from OptionMetrics, a widely used and very reliable source. The VXX options are American style; therefore, the IV is computed using an algorithm based on the binomial tree model of Cox et al. (1979), by OptionMetrics. We obtain the Treasury yield data from the U.S. Department of the Treasury website.\(^{10}\)

We apply the following standard option data filters to the option data, following

\(^{10}\)If there are no yield data on a day where there are option data we use the previous days value.
previous work by Bakshi et al. (1997), Zhang and Xiang (2008) and the VIX index option data cleaning methodology.

- We remove option quotes where the open interest, bid price or IV is zero or missing.

- We remove option quotes with a maturity of less than six days.

In table 4.2 we summarize the trading activity of the VXX options market overall and by maturity category after cleaning the data, as above. In the table we can see that as the maturity of the options contracts increases the number of observations, mean number of strikes, mean daily trading volume and mean open interest all decrease substantially. Most of the trading in VXX options happens in the shorter maturity contracts, as is common for options markets. This may be due to the short term nature of underlying VXX investments, buy and hold yields huge losses (Whaley, 2013; Gehricke and Zhang, 2018) and the VXX is not a good diversification tool over the mid-long run (Hancock, 2013; Alexander and Korovilas, 2013; Bordonado, Molnár, and Samdal, 2017).

4.4 Empirical Results

4.4.1 Quantified IV curve

In this section we present and analyze of the dynamics of the quantified IV curve of the VXX options market, as well as those of the option implied VXX forward price.

Table 4.3 shows a summary of the implied VXX forward price, the quantified IV curve coefficients ($\alpha_0$, $\alpha_1$ and $\alpha_2$) and the proportion of curves for which they are significant, the quantified IV curve factors ($\gamma_0$, $\gamma_1$ and $\gamma_2$), the goodness of fit of the regressions (R-squared) and the trading volume. The summary statistics are provided
Table 4.2: Summary of the VXX option market activity.

<table>
<thead>
<tr>
<th>Maturity (days)</th>
<th>Number of strikes</th>
<th>Mean number of strikes</th>
<th>Median number of strikes</th>
<th>Mean volume</th>
<th>Median volume</th>
<th>Mean open interest</th>
<th>Median open interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 90</td>
<td>12</td>
<td>38</td>
<td>36</td>
<td>17,035</td>
<td>5,319</td>
<td>82,613</td>
<td>168,055</td>
</tr>
<tr>
<td>90 - 180</td>
<td>701</td>
<td>47</td>
<td>45</td>
<td>37,149</td>
<td>24,315</td>
<td>86,388</td>
<td>220,756</td>
</tr>
<tr>
<td>180 - 360</td>
<td>2,226</td>
<td>42</td>
<td>47</td>
<td>6,744</td>
<td>19,900</td>
<td>114,098</td>
<td>1,098</td>
</tr>
<tr>
<td>&gt; 360</td>
<td>3</td>
<td>43</td>
<td>47</td>
<td>22,664</td>
<td>1,935</td>
<td>73,547</td>
<td>1,131</td>
</tr>
</tbody>
</table>

This table shows the mean and median daily number of strikes, trading volume, and open interest for the VXX option market. The statistics are shown overall and for each maturity category. The statistics for the daily open interest or volume are calculated as the mean/median of the daily trading volume for each maturity category or overall, as these measures are not available for each maturity category.
overall and by maturity category. Table 4.4 shows the same statistics but for the regressions that are constrained to pass through the ATM IV.

In tables 4.3 and 4.4 we can see that the mean implied VXX forward price across the entire sample and for all maturities is 26.21. Examining the mean forward price by maturity category shows that the implied forward price decreases as the maturity increases, from 26.45 to 26.26, for less than 30- and more than 360-day maturities, respectively. Therefore, the term structure of the implied forward price is in backwardation (downward sloping), on average. Also, the variation (standard deviation) of the implied forward price is 9.95 overall and tends to increase as the maturity becomes longer.

The level coefficient ($\alpha_0 = \gamma_0$), which is an estimate of the exact ATM IV, is 0.6873 (0.6854) on average, for the un-constrained (constrained) regressions. The mean level monotonically increases from 0.6445 (0.6413) to 0.7243 (0.7187), for less than 30 and more than 360 days to maturity, respectively. Therefore, the term structure of the exact ATM IV is usually in contango. Its standard deviation is 0.1338 (0.1356) overall and decreases as maturity increases. This shows us that, on average, the long-term projections of VXX volatility by option traders are higher than the short term and that their long term volatility projections are more consistent throughout the sample. This would be consistent with VXX option traders believing that ATM VXX volatility mean-reverts to some long-run level. The level coefficient is significant at the 5% level for essentially 100% of the fitted IV curves. This can be expected, as the exact ATM implied volatility should never be zero or negative.

Looking at the slope factor we can see that, on average and over all maturity IV curves, the curves are upward sloping, as the overall mean $\gamma_1$ is positive for the unconstrained and constrained regressions. On average, as the maturity increases

---

11 The maturity categories are based on the days to maturity of the contracts; therefore, for some days there will be multiple maturities in one category.

12 $\alpha_0 = \gamma_0$ is an estimate of the exact ATM IV, whereas the market ATM IV is the IV where the call and put prices are closest, that is, the closest available strike price to ATM.
Table 4.3: Summary of IV function estimation.

This table shows summary statistics of the estimated implied volatility function:

\[
IV(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2
\]

where \( IV \) is the implied volatility and \( \xi \) is the moneyness of the option. The regression is fitted separately each day and for each maturity. To estimate we minimize the volume-weighted squared errors. Here, \( \hat{\alpha}_0 \), \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) are the unscaled level, slope and curvature coefficients, respectively. The mean, median and standard deviation values are calculated overall and by maturity category.\(^a\) The percentage of significant parameter estimates is the percentage of parameter estimates that are significant at the 5% level of significance over the entire sample, in each maturity category. The mean volume is calculated as the mean of the daily sum of the trading volume for each maturity, either overall or by the maturity category grouping. The volume in this table is different than in table 4.2 because it includes only the OTM options used to create the implied volatility curves. We also present the mean and standard deviation of the forward price overall and by maturity category.

\(^a\)There can be more than one expiry in the same maturity category on any given day.
### Chapter 4: The implied Volatility Smirk in the VXX Options Market

This table shows summary statistics of the estimated implied volatility function:

\[ IV(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2, \]

when it is forced to pass through the point at-the-money. Here, \( IV \) is the implied volatility and \( \xi \) is the moneyness of the option. The regression is fitted separately each day and for each maturity. To estimate we minimize the volume-weighted squared errors. Here, \( \hat{\alpha}_0, \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) are the unscaled level, slope and curvature coefficients, respectively. The mean, median and standard deviation values are calculated overall and by maturity category.

The percentage of significant parameter estimates is the percentage of parameter estimates that are significant at the 5% level of significance over the entire sample, in each maturity category. The mean daily trading volume is calculated as the mean of the daily sum of the trading volume for each maturity category. The volume in this table is different than in table 4.2 because it includes only the OTM options used to create the implied volatility curves. We also present the mean and standard deviation of the forward price for each maturity category.

\(^a\)There can be more than one expiry in the same maturity category on any given day.

<table>
<thead>
<tr>
<th>Overall</th>
<th>By Maturity (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 30</td>
<td>30 – 90</td>
</tr>
<tr>
<td>( \hat{\alpha}_0 )</td>
<td>0.6854</td>
</tr>
<tr>
<td>( \hat{\alpha}_1 )</td>
<td>0.0982</td>
</tr>
<tr>
<td>( \hat{\alpha}_2 )</td>
<td>0.0049</td>
</tr>
<tr>
<td>( \gamma_0 )</td>
<td>0.6854</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>0.1508</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>0.0077</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Overall</th>
<th>By Maturity (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td><em>Standard deviation</em></td>
</tr>
<tr>
<td>( \hat{\alpha}_0 )</td>
<td>0.1356</td>
</tr>
<tr>
<td>( \hat{\alpha}_1 )</td>
<td>0.0673</td>
</tr>
<tr>
<td>( \hat{\alpha}_2 )</td>
<td>0.0388</td>
</tr>
<tr>
<td>( \gamma_0 )</td>
<td>0.1356</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>0.1040</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>0.0532</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>% significant parameters</th>
<th>Mean R(^2)</th>
<th><em>Daily R-Squared</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\alpha}_0 )</td>
<td>92.67%</td>
<td>99.47%</td>
</tr>
<tr>
<td>( \hat{\alpha}_1 )</td>
<td>87.95%</td>
<td>73.94%</td>
</tr>
<tr>
<td>( \hat{\alpha}_2 )</td>
<td>86.78%</td>
<td>76.14%</td>
</tr>
</tbody>
</table>


\(^a\)The percentage of significant parameter estimates is the percentage of parameter estimates that are significant at the 5% level of significance over the entire sample, in each maturity category.
the slope becomes less steep and even turns downward sloping for maturities over 360 days. The un-constrained (constrained) slope, $\gamma_1$ goes from 0.2151 (0.2165) to -0.0029 (-0.0114), for less than 30 and more than 360 days to maturity curves, respectively. The term structure of the slope factor is, on average, in backwardation. The unscaled slope coefficient is highly significant with 99.44% of fitted IV curves, with less than 180 and 79.81% of fitted IV curves with over 180 days to maturity showing significant slope coefficients, at the 5% level.

The last quantified IV curve factor is the curvature, $\gamma_2$. We can see that, on average and for all maturities, it is positive, meaning the VXX IV curves are usually convex. However, it is also very small in magnitude, so the convexity is not very prominent. The overall average curvature factor is 0.0051 (0.0077) for the un-constrained (constrained) regression. The unscaled curvature coefficient is significant for 64.73% (67.95%) of the fitted IV curves overall. The proportion of quantified IV curves with significant curvature coefficients decreases slightly for longer-maturity categories. The magnitude of the mean curvature factor estimates increases with maturity, meaning that as maturity increases the IV curves tend to become more convex. The average curvature factor is 0.0054 (0.0060) for less than 30-day and 0.0240 (0.0366) for more than 360-day maturity curves.

Constraining the regressions to fit the ATM IV exactly results in a lower level, flatter and more convex quantified IV curves on average, overall and for most maturity categories.

The reason both the slope and curvature become less significant and the r-squared values become much lower for maturities over 180 days, as seen in tables 4.3 and 4.4, may be that traders’ opinions on volatility are less reliable resulting in less consistently shaped IV curves. The lower trading volume may also be indicative of less efficient/informative prices at longer maturities.

Figures 4-2, 4-3 and 4-4 show the IV curves, the trading volume of each contract
Figure 4-2: IV against moneyness on 27 July 2011.
This figure plots the IV of VXX options against the moneyness for different maturities as at the close of 27 July 2011. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 27 July 2011.
and the fitted line for the unconstrained regression on 27 July 2011, 2 August 2013 and 27 May 2015, respectively. Figures 4-5, 4-6 and 4-7 show the same information but for the constrained quantified IV curves. We can see good examples of the usually upward-sloping and almost linear curves at shorter maturities. As the matur-

Figure 4-3: IV against moneyness on 2 August 2013.
This figure plots the IV of VXX options against the moneyness for different maturities as at the close of 2 August 2013. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 2 August 2013.
Figure 4-4: IV against moneyness on 27 May 2015.
This figure plots the IV of VXX options against the moneyness for different maturities as at the close of 27 May 2015. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 27 May 2015.
rity increases the fitted lines become more convex, consistent with the mean results discussed above.

In figure 4-8 we show the average fitted IV curves, that is, the predicted curves resulting from the mean factors presented in tables 4.3 and 4.4. We can clearly see the pattern described above; as the maturity increases the IV curve's slope decreases and they become more convex. Most maturity average IV curves are upward-sloping lines with some convexity, but the longer than 360 days to maturity lines look like the IV smirk found in the S&P 500 options market. We can also see that when the regressions are constrained to cross the ATM IV point, they become more smirked (skewed to the left), this is most apparent in the line for maturities longer than 360 days.

4.4.2 Constant maturity quantified IV curve

So far we have been examining the term structure of the VXX implied forward price and IV curves using maturity categories. However, within each maturity category there will often be multiple curves on a given day. To confirm the findings above, we create constant maturity implied forward prices and IV curve factors. This allows us to precisely study the term structure and time series of the variables covering the same horizon of traders’ expectations.

To create constant maturity implied forward prices and IV curve factors, we interpolate/extrapolate them to several target maturities as follows:

\[
F^\tau = F^{\tau_1} w_1 + F^{\tau_2}(1 - w_1), \quad (4.11) \\
\gamma_0^\tau = \gamma_0^{\tau_1} w_1 + \gamma_0^{\tau_2}(1 - w_1), \quad (4.12) \\
\gamma_1^\tau = \gamma_1^{\tau_1} w_1 + \gamma_1^{\tau_2}(1 - w_1), \quad (4.13) \\
\gamma_2^\tau = \gamma_2^{\tau_1} w_1 + \gamma_2^{\tau_2}(1 - w_1), \quad (4.14)
\]
Figure 4-5: IV against moneyness on 29 July 2011: with restraint
This figure plots the IV of VXX options against the moneyness for different maturities as at the close on 29 July 2011. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 29 July 2011.
Figure 4-6: IV against moneyness on 2 August 2013: with restraint
This figure plots the IV of VXX options against the moneyness for different maturities as at the close on 2 August 2013. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 2 August 2013.
Figure 4-7: IV against moneyness on 27 May 2015: with restraint
This figure plots the IV of VXX options against the moneyness for different maturities as at the close on 27 May 2015. The circles are computed from market prices of out-of-the-money calls and puts. The solid line is the fitted quantified IV curve. The bar chart is the trading volume for the corresponding options traded on 27 May 2015.
where

\[ w = \frac{\tau - \tau_2}{\tau_1 - \tau_2}, \]

the superscript \( \tau \) denotes the desired maturity, \( \tau_1 \) is the closest (second closest) maturity to the target from below and \( \tau_2 \) is the closest (closest) maturity to the target from above (below), when interpolating (extrapolating). We interpolate when there is a maturity either side of the target and extrapolate when the available maturities are all shorter than the target maturity.

Table 4.5 presents the mean and standard deviation of the interpolated implied VXX forward prices and level, slope and curvature coefficients and factors, for both the un-constrained and constrained estimations. These are also presented graphically in figure 4-9. From the table we confirm the previous result that, on average, the implied VXX forward price is slightly decreasing as maturity increases. The average constant maturity forward price goes from 26.60 to 26.12, for the 30- and 360-day target maturity, respectively.

In table 4.5 we can also see that the exact ATM IV (level factor) term structure is
Unconstrained and constrained IV curve factors are presented in panels A, B and C, respectively. The statistics for the implied forward price, and the implied volatility curve factors by maturity. The statistics for the implied forward price, unconstrained and constrained IV curve factors are presented in panels A, B and C, respectively.

### Chapter 4: The implied Volatility Smirk in the VXX Options Market

<table>
<thead>
<tr>
<th>Maturity (days)</th>
<th>30</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>150</th>
<th>180</th>
<th>270</th>
<th>360</th>
</tr>
</thead>
</table>

#### Panel A: Implied Forward Price

| \( \alpha^0 \) | 0.6768 | 0.6954 | 0.7042 | 0.7096 | 0.7130 | 0.7148 | 0.7182 | 0.7194 |
| \( \alpha^1 \) | 0.1285 | 0.1212 | 0.1119 | 0.1028 | 0.0928 | 0.0823 | 0.0600 | 0.0417 |
| \( \alpha^2 \) | 0.0013 | -0.0011 | -0.0005 | 0.0002 | 0.0010 | 0.0020 | 0.0040 | 0.0059 |
| \( \gamma^0 \) | 0.6768 | 0.6954 | 0.7042 | 0.7096 | 0.7130 | 0.7148 | 0.7182 | 0.7194 |
| \( \gamma^1 \) | 0.1971 | 0.1785 | 0.1623 | 0.1479 | 0.1329 | 0.1177 | 0.0858 | 0.0602 |
| \( \gamma^2 \) | 0.0017 | -0.0018 | -0.0008 | 0.0001 | 0.0015 | 0.0030 | 0.0059 | 0.0085 |

#### Panel B: IV regression

| \( \alpha^0 \) | 0.1475 | 0.1199 | 0.1047 | 0.0955 | 0.0899 | 0.0853 | 0.0773 | 0.0744 |
| \( \alpha^1 \) | 0.0379 | 0.0388 | 0.0378 | 0.0380 | 0.0392 | 0.0406 | 0.0446 | 0.0501 |
| \( \alpha^2 \) | 0.0115 | 0.0115 | 0.0113 | 0.0127 | 0.0155 | 0.0202 | 0.0341 | 0.0512 |
| \( \gamma^0 \) | 0.1475 | 0.1199 | 0.1047 | 0.0955 | 0.0899 | 0.0853 | 0.0773 | 0.0744 |
| \( \gamma^1 \) | 0.0645 | 0.0597 | 0.0572 | 0.0569 | 0.0580 | 0.0601 | 0.0640 | 0.0702 |
| \( \gamma^2 \) | 0.0166 | 0.0158 | 0.0152 | 0.0166 | 0.0206 | 0.0263 | 0.0441 | 0.0664 |

#### Panel C: Constrained IV regression

| \( \alpha^0 \) | 0.1483 | 0.1211 | 0.1061 | 0.0966 | 0.0910 | 0.0865 | 0.0797 | 0.0784 |
| \( \alpha^1 \) | 0.0387 | 0.0393 | 0.0387 | 0.0393 | 0.0409 | 0.0430 | 0.0500 | 0.0601 |
| \( \alpha^2 \) | 0.0137 | 0.0138 | 0.0147 | 0.0181 | 0.0241 | 0.0332 | 0.0565 | 0.0842 |
| \( \gamma^0 \) | 0.1483 | 0.1211 | 0.1061 | 0.0966 | 0.0910 | 0.0865 | 0.0797 | 0.0784 |
| \( \gamma^1 \) | 0.0658 | 0.0605 | 0.0585 | 0.0588 | 0.0602 | 0.0629 | 0.0706 | 0.0831 |
| \( \gamma^2 \) | 0.0200 | 0.0189 | 0.0196 | 0.0230 | 0.0314 | 0.0428 | 0.0717 | 0.1069 |

**Table 4.5: Interpolated Term structure.**

This table presents the mean and standard deviation of the interpolated implied forward price and the implied volatility curve factors by maturity. The statistics for the implied forward price, unconstrained and constrained IV curve factors are presented in panels A, B and C, respectively.
Figure 4-9: Term structure of mean interpolated factors.
This figure shows the term structure of the mean interpolated factors (level, slope and curvature) and their one standard deviation bands. The unconstrained and constrained regression results are shown in the top and bottom row of plots, respectively.

usually in contango. This is likely because the probability of a VXX volatility spike becomes larger as the maturity increases, during normal times. The variation in the exact ATM IV also decreases as time to maturity increases. The table also shows that the term structure of the slope factor is in backwardation and the variation in the slope factor is similar for all maturities. Lastly, we can see that the curvature factor’s term structure is very flat around a value of zero with a very slight increase at longer maturities.

These results are consistent with our findings using the average values grouped by maturity categories in the previous section. However, we also want to study the time series of the ATM IV, forward prices and IV curve factors. We present the time series of 30- and 180-day constant maturity ATM IV and forward prices in figure 4-10. Then we present the time series of the 30- and 180-day constant maturity level, slope and curvature factors in figures 4-11 and 4-12 for the un-constrained and constrained estimations, respectively.\textsuperscript{13}

\textsuperscript{13}For the time series investigation of the interpolated ATM IV, forward price and the curve factors
Chapter 4: The implied Volatility Smirk in the VXX Options Market

Figure 4-10: Time series of Interpolated ATM IV and forward prices.
This figure presents the time series of the interpolated ATM IV and forward prices for the 30 day and 180 day maturities (left) and their differences (right).

From figure 4-10 we can see that the ATM IV varies throughout time in a mean-reverting fashion. Referring to the difference between the 180- and 30-day ATM IV we can see that most of the time its term structure is in backwardation, although there are times when it is in contango. Examining the time series of the 30- and 180-day implied forward prices we can see that they vary significantly often spiking very quickly. We can also see that the term structure of implied forward prices is usually almost flat, with some periods of strong contango and backwardation.

Turning to the time series of the IV curve factors in figures 4-11 and 4-12, we can firstly see that the exact ATM IV (level factor) is also mean-reverting with a usually contango term structure, with brief times of backwardation. Secondly, we can see that the slope factor also seems to mean-revert with a usually backwarded term structure. Lastly, looking at the curvature factor we can see that it is usually very small in

we only use the 30 and 180 day interpolations because the data are often scarce and options illiquid, at longer maturities.
Figure 4-11: Time series of interpolated IV factors.
This figure shows the time series of the 30 and 180 day interpolated constant maturity level, slope and curvature (left) factors and the difference between the 180 day and 30 day (right).

magnitude for the 30- or 180-day maturity. There are also days where the curvature becomes very negative or positive, resulting in abnormally concave or convex curves, respectively. Looking at the difference between the 30 and 180 day curvature factor, it is usually close to zero, indicating a flat term structure, with some spikes. Some of the spikes in the factors and flipping of the term-structure may be linked to real-world economic events or even related to economic and market condition indicators, this is left for future research.

Figure 4-13 shows the predicted IV curves using the mean of the interpolated
Figure 4-12: Time series of Interpolated coefficients: constrained. This figure shows the time series of the 30 and 180 day interpolated constant maturity level, slope and curvature (left) factors and the difference between the 180 day and 30 day (right), when the fitted curve is forced to cross the point of ATM IV.

... factors. We can see a similar picture as in figure 4-8; as maturity increases the slope decreases and the curves become more convex. The time series observations are consistent with what we found, on average, in prior discussions. Further studying what drives the time variation in the VXX’s implied forward price and IV curve factors and their term structures is of interest for future research.
Chapter 4: The implied Volatility Smirk in the VXX Options Market

4.4.3 VXX option pricing model implications

Using the results from section 4.4.1 and ?? we can make recommendation for the dynamics of a VXX option pricing model. Firstly, the model must have a upward sloping term structure of volatility. Using the conversion from IV curve factors to the moments of the risk-neutral distribution of the VXX, discussed in section ??, we can say that the model must also exhibit positive skewness, which decreases as maturity increases, due to the dynamics of the slope factor. Lastly, we show that kurtosis is of the risk-neutral distribution of the VXX should be very small, unless the maturity is very long, due to the dynamics of the curvature factor.

4.5 Conclusions

In this Chapter we document the empirical characteristics of the VXX options market as a starting place for developing an empirically grounded VXX option pricing model. We follow the methodology developed by Zhang and Xiang (2008) in order to quantify the IV curve of VXX options, through quadratic polynomial regressions. The IV curve is quantified through three factors - the level (exact ATM IV), slope and curvature - which we compute daily and for different maturities over a six year sample. We extend the methodology of Zhang and Xiang (2008) by estimating constant maturity
factors, which allows us to study the time-series and term structure dynamics of the
VXX IV more concisely. We quantify the IV curves with and without a constraint
that the curve has to pass through the ATM IV, resulting in very similar results.

We find that the implied VXX forward price term structure is usually in backwarda-
dation. We also show that the average exact ATM IV (level factor) increases with
maturity and estimates become less variable with longer maturities. Which could be
explained by traders expecting VXX ATM volatility to mean-revert, which we show
it does. The IV curves are also usually significantly upward sloping, although as the
maturity increases they become flatter and even downward sloping for the longest
maturities. The IV curves are slightly convex, on average, and become more convex
as the maturity increases.

Our quantification of the VXX IV summarizes all the information contained in
VXX option prices and should therefore be used when developing a VXX option
pricing model. The term-structure of the ATM volatility of the VXX should be
upward sloping. The risk-neutral distribution should be positively skewed, due to the
positive slope factor, and become less positively skewed as the maturity increases. The
risk-neutral distribution also should exhibit only minimal kurtosis as the curvature
factor is minuscule in magnitude, for maturities less that 180 days.

We study the time series of the short end of the term structure (less than 180
days). We show that the level and slope factors seem to mean-revert through time,
while the curvature does not follow an easily observed pattern. Although the level’s
(slope’s) term structure is usually in contango (backwardation), there are times when
it goes into backwardation (contango). The shorter maturity end of the curvature’s
term structure is usually almost flat, with some short-lived moments of backwardation
or contango.

Studying the drivers of the periodic shifts in the factors and their term structures
is left for future research. The quantified VXX IV factors could also be converted to
estimates of the VXX’s risk-neutral moments, which can then be used to calibrate new VXX option pricing models. We could potentially use the quantified VXX IV factors to predict the VXX returns and/or VXX option returns. The relationships between the SPX, VIX and VXX option implied volatility curves is also a topic of interest as understanding these would allow for more direction on developing a comprehensive option pricing model for all three markets. These extensions of the current work are left for future research.
Chapter 5

Conclusion

This thesis studies the VIX futures ETN market (Chapters 2 and 3) and its derivatives (Chapter 4). In Chapter 2, we find that VIX futures ETNs do not track their indicative values all that well and are usually inconsistently priced, using daily and intradaily data. Most importantly, we show that the VIX futures ETNs lead the VIX futures in price discovery most of the time. We also found that none of the ETNs is a clear leader of the others in price discovery overall, but there are times when one ETN leads another. Further exploring what drives the variation in the price discovery relationships between the VIX futures and ETNs, as well as between the ETN pairs, is left for future research.

In Chapter 3, we develop a new model for the VXX, using Duffie, Pan and Singleton’s (2000) affine jump diffusion. The model dynamics include jumps in the volatility process, and the long-term mean volatility is itself a stochastic mean-reverting process. We calibrate our model to the VIX term-structure data to show that it provides the theoretical link between the VIX index, VIX futures and the VXX. The model can be used to price VIX futures, the VXX and other short-term maturity VIX futures ETNs. The model could be extended to price options on the VXX and other short-term VIX futures ETPs.

Finally, in Chapter 4 we document the empirical characteristics of the VXX op-
tions market, which provides useful information for developing a VXX option pricing model. We use the methodology of Zhang and Xiang (2008) to quantify the implied volatility curves of VXX options. The implied volatility curve is quantified by the level, slope and curvature factors. We show that the VXX implied volatility curve is not usually a smirk, as is common for equity markets, but rather an upward-sloping line with some convexity. As the maturity of the options increases the level increases, slope decreases and curvature increases, on average. Throughout the sample the level and slope factors seem to mean-revert, while the curvature factor does not follow an easily identifiable pattern.
Bibliography


Alexander, Carol, and Dimitris Korovilas, 2013, Volatility exchange-traded notes: Curse or cure?, *Journal of Alternative Investments* 16, 52–70.


Chapter 5: Conclusion


Burger, Dani, 2018, This tiny hedge fund just made 8,600% On a VIX bet, Bloomberg.


Cont, Rama, José Da Fonseca, et al., 2002, Dynamics of implied volatility surfaces, Quantitative finance 2, 45–60.


Deng, Geng, Craig J McCann, and Olivia Wang, 2012, Are VIX futures ETPs effective hedges? Available at SSRN 2094624.


Dillian, Jared, 2018, Volatility funds worked as intended, that’s the problem, *Bloomberg*.

Dong, Xiaoyang Sean, 2016, Price Impact of ETP Demand on Underliers, *Available at SSRN 2788084*.


Fernandez-Perez, Adrian, Bart Frijns, Alireza Tourani-Rad, and Robert I Webb, 2015, Did the introduction of ETPs Change the intraday price dynamics of VIX futures? *Working Paper, Auckland University of Technology*.


Wiedman, Mark, 2013, Open letter to our investors, *iShares by Blackrock*.


Chapter 5: Conclusion


