Modelling and Inference for
Electromagnetic Flow
Tomography

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Te Whare Wānanga o Otāgo

a thesis submitted for the degree of
Doctor of Philosophy
at the University of Otago, Dunedin,
New Zealand.

14 June 2019
Abstract

Electromagnetic flowmeters determine the bulk flow rate of an ohmic fluid in a pipe by measuring the voltage induced across the fluid by a transverse magnetic field. This thesis develops the theory of an electromagnetic flowmeter for groundwater aquifer applications. Electromagnetic flowmeters require slow, laminar flow for measurements of bulk flow to be accurate — even after calibration. In general, the measured voltage depends on the spatial distribution of the velocity of the fluid. Hence, determination of the velocity field is required in order to accurately measure the bulk flow rate in general flows. Accordingly, this thesis examines the inverse problem of electromagnetic flow tomography, which is the problem of determining the velocity field in a fluid from voltage measurements made at multiple locations. Electromagnetic flow tomography is a severely ill-posed linear inverse problem.

The relationship between the flow and the potential induced across the fluid is described by the flowmeter equation — a boundary value problem in Poisson’s equation, with the source due to the Faraday effect. A novel dipole-form of the flowmeter equation allows for analysis of spatial sensitivities. This boundary value problem is solved using Green’s functions, derived by the method of images for the geometry of pipe cross-section and half-space. Computational implementation of the forward map uses a finite element method discretisation and assumes idealised point electrodes to simulate measurements. Analysis of the measurement kernel reveals extreme sensitivity to flow near the electrode locations, with low sensitivity to the majority of flow away from the electrodes. The resulting non-uniqueness in inverting the forward map implies that assumptions must be made about the spatial flow profile in order to make estimates of the bulk flow.
This thesis examines a Bayesian formulation to this inverse problem, that includes a model for the forward map, and accounts for measurement noise and uncertainty in the velocity field. The Bayesian analysis of an inverse problem produces the posterior distribution, from which estimates of desired quantities may be calculated, along with uncertainties. In particular, prior modelling allows for exploring assumptions and representation of unknowns to determine potential biases. The resulting Bayesian model is a standard stochastic hierarchical model with hyperparameters to model uncertainties such as the smoothness of the flow profile, or other effects. The flow is modelled as a Gaussian Markov random field and the hyperparameters are modelled using a Jefferys prior.

The resulting model for the flow tomography inverse problem is a linear Gaussian model. Inference for this model is efficiently implemented using the recent marginal then conditional algorithm (Fox and Norton, 2016). That algorithm generates posterior samples by first using a Monte-Carlo Markov chain sampler for the low-dimensional marginal distribution over hyperparameters, then drawing from the full conditional distribution over the flow profile, which requires one solve of a linear equation. Posterior inference does not require the draw from the full Gaussian conditional, as moments of the Gaussian are available analytically. This method for computational Bayesian inference surpasses equivalent regularising methods in computational speed.

To the best of this author’s knowledge, this is the first time a Bayesian method has been used for analysing electromagnetic flow tomography.

Measurements of the bulk flow in a pipe are computed using simulated data generated from physically sensible phantom flow profiles. Various geometries and electrode placements are examined, with different shapes and scales of phantom flow. The effect of using the fluid dynamics no-slip condition and changing hyperparameter values is also explored. Not surprisingly, increased number of electrodes increases the spatial flow profile resolution and accuracy of bulk flow estimates. The flow profile reconstructions and bulk flow estimates are more accurate for flow profiles which could easily be interpolated from values near electrode locations. Additionally, it is shown that there is an implicit scale in the system — the standard deviation of
bulk flow correlated to the area of the pipe.

The use of invasive measurements for the purpose of measuring groundwater flow is also investigated. Analysis in this thesis shows that measuring groundwater flow presents significant difficulty; the resolution of bulk flow from a realistic signal-to-noise ratio is several orders of magnitude larger than the expected bulk flow rate in unconfined aquifers.
Acknowledgements

This work was funded as a part of the SfTI National Science Challenge 10’s Inverting Electromagnetics project. The team also contributed the design and estimated parameters of the mini-aquifer used in this work.

I would particularly like to thank Colin Fox, my supervisor. Conversations with him are always very enjoyable and entertaining and often leave me simultaneously feeling like I knew everything, yet nothing, but somehow my confidence and enthusiasm would always be restored.

Additionally, the entire Electronics Research Group at the University of Otago were invaluable. From the relaxing social chats and helpful advice, to our road trips to conferences and on-site locations, they make one feel very welcome.

Finally, a thank you to all to of my office mates and the people who regularly had lunch with me in the physics building common room. It is not an exaggeration to say that our insane, surreal conversations and joking around at lunch was often the highlight of my day. Completing a PhD would have been much harder without that outlet.
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Acronyms

1D  one dimensional.
2D  two dimensional.
3D  three dimensional.
ECT  electrical capacitance tomography.
EM  electromagnetic.
EMFT  electromagnetic flow tomography.
FEM  finite element method.
GMRF  Gaussian Markov random field.
 iid  independent and identically distributed.
MAP  maximum a posterior.
MCMC  Monte-Carlo Markov chain.
MTC  marginal, then conditional.
NSC  National Science Challenge.
PDE  partial differential equation.
SfTI  Science for Technology and Innovation.
SVD  singular value decomposition.
TSVD  truncated singular value decomposition.
Chapter 1

Introduction to Electromagnetic Flow Tomography

Electromagnetic (EM) flow measurement uses EM principles to measure the bulk flow rate of a fluid. When a conducting fluid moves in a magnetic field, the Faraday effect induces a current in the fluid, perpendicular to the field and the fluid’s velocity. In principle, the flow rate of the fluid can be determined by measuring the current-induced potential.

The simplest model of an EM flow measurement system is as depicted in Figure 1.1. Two point electrodes are immersed in a uniformly conductive fluid that moves with a constant velocity field, \( v \). An external, constant and uniform magnetic field, \( B \),

![Figure 1.1: A simple model of an EM flow measurement setup. The dark blue arrows represent the fluid’s velocity field and the orange arrows are the magnetic field. A potential difference is measured between two electrodes, represented as light blue squares.](image)

...
is applied. In the simplest configuration, the line connecting the two electrodes, the velocity field, and magnetic field are all orthogonal.

The potential difference induced in the fluid is theoretically given by the linear forward map

\[ V = k dB v, \]  

(1.1)

where \( d \) is the electrode separation and \( k \) is a proportionality constant. The inverse problem in this idealised case is well–posed as there is a well defined direct inverse given by division, i.e. \( v = \frac{V}{k dB} \). One can determine the proportionality constant analytically (see Section 3.1.3). However, it is not practical to analytically calculate \( k \) in real-world situations. The constant \( k \) has a non-trivial dependence on a multitude of features such as the electrode-fluid interface, the geometry of the system, flow profile (non-uniform \( v \) and \( B \)), and any deviation from this idealised model.

In general EM flow measurement is not as simple as in (1.1). The magnetic and velocity fields will vary spatially and the geometry of the system will also affect the measured potential. The voltage is actually given by the integral equation

\[ V = \int k(x)B(x)v(x)dx, \]  

(1.2)

over the space where \( v \) and \( B \) are non-zero, where \( x \) is the spatial coordinate and \( k(x) \) is a spatially-varying measurement kernel (see Section 3.2). In contrast to (1.1), equation (1.2) does not have a well defined inverse.

EM measurement of a general fluid flow is problematic due to the spatially dependant nature of (1.2). Different velocity fields can lead to the same measurement. In order to determine the bulk flow rate in a single direction, a 2D slice of the velocity field itself needs to be reconstructed. Therefore, imaging the velocity field through tomography is required for general flow measurement. Electromagnetic flow tomography (EMFT) is a technique of measuring bulk flow by imaging a 2D slice of the velocity field using multiple voltage measurements.

The forward map of EMFT must be understood in order for the inverse problem to be characterised and solved. The exact form of the forward map for a stationary system is found by solving the partial differential equation (PDE) for the potential, \( u \), given by

\[ \nabla^2 u = -\nabla \cdot (v \times B), \]  

(1.3)

subject to appropriate boundary conditions.
Equation (1.3) is known as the flowmeter equation and is analogous to Poisson’s equation (see Chapter 2). The source term (right hand side) is analogous to a dipolar charge distribution due to the divergence of $(v \times B)$. This means that potential difference measurements are more sensitive to sharp spatial changes in the velocity and magnetic fields than they are to constant fields, making bulk flow measurements difficult.

The flowmeter equation must be applied to some specific geometry with boundary conditions to be solved. Therefore, the flowmeter equation is a boundary value problem (BVP).

Boundaries introduce further complexities to EMFT. The dipolar form of $(v \times B)$ in (1.3) will cause a monopolar source build-up at any boundary. This includes electrode interfaces, that can, potentially, dominate any measurement (see Section 3.1.5). The majority of investigators avoid this build-up in their models by using the no-slip condition at boundaries from theoretical fluid dynamics. However, the no-slip condition is often improperly implemented in the numerical representation of flows (see Section 6.2.2) (Lehtikangas, Karhunen, and Vauhkonen, 2016; Kollár, Lucas, and Zhang, 2014).

Even with a no-slip condition, the measurement kernel $k(x)$ has singularities at the electrode locations (see Section 3.2.1). Singularities imply that the measurement is highly sensitive to the area near the electrodes. The dipolar source will lead to either a charge build-up (slip) or a sharp change in velocity (no-slip) at the electrode interfaces. Either case will generate a large potential, guaranteeing the potential measurement being determined almost entirely by the nearby source. This feature makes the measurements non-penetrative — i.e., insensitive to features away from electrodes.

Exacerbating the issue of local sensitivity, EM flow measurements suffer from electrochemical effects obfuscating the measurements. A DC magnetic field will lead to electrolytic build-up on the electrodes. AC magnetic fields are often used for this reason, making EMFT a stationary, time-averaged measurement system. However, electrochemical effects are still noticeable even with relatively non-reactive silver-silverchloride electrodes in an AC system (Filloux, 1973). Electrochemical effects are not considered here, only idealised electrodes.

The forward map used in this thesis is calculated using Green’s functions to solve the BVP, derived using the method of images, with a novel analysis of the measurement kernel. This is applied to the specific geometries of a circular pipe, a square pipe, as well as infinite and truncated half-spaces: see Chapter 3. The forward map is discretised
using a finite element method triangulation.

1.1 Electromagnetic flowmeters and electromagnetic flow tomography

![Image](image.png)

(a) EM flowmeter setup  
(b) EMFT setup

Figure 1.2: Cross-sectional depictions of an EM flowmeter and an EMFT system. The flowmeter has one potential difference measurement $V$, whereas the EMFT system has several electrode measurements $d$.

Many different techniques implementing EM fluid flow have been tried, including the measurement of ocean and river currents using undersea cables (See Section 2.2). The most prevalent and commercially viable measurement system is the electromagnetic flowmeter (Figure 1.2a). The EM flowmeter is the simplest application of the flowmeter equation; it measures the flow of a fluid confined in a circular pipe from a single voltage measurement. EM flowmeters make use of the trait that the voltage measurement is linearly correlated with the average velocity for axisymmetric velocity fields, as in (1.1) (Thürlemann, 1941). Bulk flow of axisymmetric flow can be accurately determined with an EM flowmeter after calibration against known flows. Axisymmetric flow occurs in slow, horizontal, laminar flow. However, this linear correlation is lost in more complicated flows, such as those with an up-stream disturbance (Williams, 1930). Thus, EM flowmeters find use in many industrial applications where long, straight sections of pipes and calibration against known flow rates are possible.

EMFT is a generalisation of the EM flowmeter for more complicated flows. The
bulk flow rate is determined by imaging a 2D cross-section of the velocity field by taking multiple voltage measurements (see Figure 1.2b). Where the previous issues of sensitivity and electrochemical effects are calibrated out of the system in an EM flowmeter, calibration of this kind is not useful when the flow profile is unknown — like in EMFT applications. The aforementioned issues must be explicitly accounted for if one wants accurate estimation of general flow. The added complexities in the model and computations are disproportionately large when compared to the improved flow rate accuracy. For this reason, EMFT systems have not rendered EM flowmeters obsolete in the commercial sector, despite being an active area of research for decades.

1.2 The inverse problem of electromagnetic flow tomography

The EMFT forward problem is ill-posed; there is a potentially infinite set of velocity fields consistent with any data set. This is exacerbated by the presence of noise in the measurement process. Noise comes from a number of sources including electrolytic effects, shot noise, and background radiation. This has the effect of increasing the space of flow states consistent with the data.

Methods for solving inverse problems can be categorised as deterministic or inferential. Both include extra information to restrict the potentially infinite set of states through a preferential weighting, but vary in approach.

Deterministic methods create a deterministic inverse map, which is often a regularised version of the direct inverse to the forward map. One common method is the regularised least squares, which favours smoother states (Engl, Hanke, and Neubauer, 1996). The specific state used as the solution to the inverse problem is chosen through a minimisation process. Deterministic inversion has historically been the more common method of solving inverse problems and all existent EMFT literature uses deterministic methods.

A black-box style implementation of a deterministic inversion is usually more computationally efficient than a similar inferential method. Additionally, deterministic methods are considered easier to understand given a standard training in physics, mathematics, or engineering which focuses on the direct problem and the notion of data-fitting by optimisation.

The output of a deterministic method is a single image with limited quantitative value. A single estimate is provided and includes errors only in terms of a single number:
the ‘magnitude’ of the noise (Watzenig and Fox, 2009). The selection of regularisation level is essentially based on whether or not the state “looks good” (Fox, Nicholls, and Tan, 2012). There are known quantitative problems, for examples the chosen state may not even lie on the support of physically possible states (Fox and Norton, 2016).

Deterministic methods are examined in Chapter 4 where EMFT is analysed using the truncated singular value decomposition method.

Errors in the measurement process are inherent in measurement systems. Explicitly modelling the error in the measurement process allows for the creation of a more accurate forward map. In contrast, interpreting error in terms of an idealised forward map can lead to substantial artefacts due to the ill-posed nature of the problem (Watzenig and Fox, 2009). Using a probabilistic model for the measurement errors and other uncertainties in the system lead to a probabilistic forward problem. Thus, the EMFT inverse problem becomes a problem of statistical inference over the unknown fluid flow.

The output of an inferential method is the probability distribution over the space of all possible states consistent with the data. Qualitative calculations of estimates and uncertainties can be performed over this distribution. Due to the ability to perform estimation, inferential methods are regarded as more general than deterministic methods.

A commonly stated issue with inferential methods — such as Bayesian formulations — when compared to deterministic methods is the large computational cost of sampling algorithms. Algorithms such as such as Monte-Carlo Markov chains (MCMC) can take a long time to approximate the target distribution. However, recent methods can surpass similar regularisation methods in this regard (Fox and Norton, 2016).

Single samples of the posterior from a Bayesian formulation are informative and can be better reconstructions than a regularised inversion (Fox, 2008). A small set of samples can provide a qualitative picture of the nature and scale of estimates and uncertainties (Watzenig and Fox, 2009).

Both deterministic and inferential inversion are explored in this thesis (Chapters 4 and 5, respectively). The main purpose of this work is to introduce the Bayesian formulation to the EMFT body of work. Hence, an inferential Bayesian formulation using MCMC sampling is the focus of this thesis.
1.2.1 Bayesian inference

The goal of any statistical inference is to recover optimal, unbiased knowledge of the true state, $f$, given some related noisy data, $d$. The posterior distribution, $\pi(f|d)$, describes this. Bayes’ rule,

$$
\pi(f|d) = \frac{\pi(d|f)\pi(f)}{\pi(d)}, \tag{1.4}
$$

provides the mechanism by which the posterior can be calculated. The prior density, $\pi(f)$, describes the information about $f$ before the inclusion of the data while the likelihood function, $\pi(d|f)$, contains the information of the measurement process by which data is gained. The denominator is a normalising constant and can be ignored for inference problems (von Toussaint, 2011).

Using Bayes’ theorem, the posterior can be explicitly constructed up to a normalising constant. The posterior is usually a very high dimensional object in computational implementations (dimensions of order $10^2$, up to $10^7$). Thus, visualisation and calculation using the posterior is difficult. Because the form of the posterior is known, the problem of estimating the likely true state, given some data becomes a problem of drawing samples of a high-dimensional, multivariate probability distribution. The estimates and uncertainties can then be found using Monte-Carlo integration.

Prior models

The Bayesian formulation of inverse problems has a large advantage over deterministic methods in that assumptions about the system can be explicitly included through the prior. Prior models can be easily constructed in a variety of representations. Representation and knowledge are linked and thus a representation should be chosen based on the type of information that one wishes to extract from the system. The goal of EMFT is to estimate the total flow rate of a fluid through a plane. Thus, a low-level representation of the unstructured mesh is used (see Section 3.3.2).

In order to obtain physically sensible flow profiles in the EMFT inverse problem, restrictions must be made to the space of possible velocity fields. The electrode measurements are insensitive to fluid flow far away from their location and therefore the state space must be restricted in order to infer flow distant to the electrode locations. An example of this is the axisymmetric assumption in EM flowmeters (Shercliff, 1962). A Bayesian formulation allows for these restrictions to be explicitly included into the prior model.

Flow is modelled as a random variable in this thesis, drawn from a relatively unin-
formative Gaussian Markov random field (GMRF). A locally connected linear GMRF is defined by drawing the value of the $i$th node as

$$f_i | f_{\partial_i} \sim N \left( |\partial_i|^{-1} \sum_{j \in \partial_i} f_j, \delta^{-1} |\partial_i|^{-1} \right),$$

(1.5)

where $\partial_i$ is the set of $f_j$'s neighbouring the $i$th node, $|\partial_i|$ area or length connecting the nodes (Fox and Norton, 2016). A GMRF prior favours a level of smoothness in the fluid’s velocity based on the parameter $\delta$.

**Hyperprior sampling**

The smoothness parameter and the level of noise in the measurement are hyperparameters; they are parameters which influence lower level models. The noise and the smoothness of a fluid are unknown and it is useful to account for their uncertainty in the prior model. This leads to EMFT being a hierarchical stochastic model, defined by

$$\vartheta \sim \pi(\vartheta),$$

(1.6)

$$f | \vartheta \sim N \left( 0, Q^{-1}(\vartheta) \right),$$

(1.7)

$$d | f, \vartheta \sim N \left( Af, \Sigma(\vartheta) \right),$$

(1.8)

where $\vartheta$ denotes the hyperparameters, $Q^{-1}$ is the GMRF’s variance, and $A$ is the forward map with measurement noise variance $\Sigma(\vartheta)$.

Bayes’ rule with unknown hyperparameters is:

$$\pi(f, \vartheta | d) = \frac{\pi(d | f, \vartheta) \pi(f | \vartheta) \pi(\vartheta)}{\pi(d)}. \quad (1.9)$$

The work in this thesis is intended to be applied in situations ranging from standard industrial EM flowmeter settings to measurements on the scale of nanovolt (see Section 1.3). An important feature to include in the prior model is scale invariance of the hyperparameters. Jeffreys prior is used as a hyperprior to achieve this (see Section 5.3.4).

**Marginal then conditional sampler**

The standard random walk MCMC sampling of the posterior $\pi(f, \vartheta | d)$ can be prohibitively expensive in hierarchical models due to the dependence (1.7) and (1.8) have on (1.6). This work uses the recent marginal then conditional (MTC) algorithm to resolve this problem (Fox and Norton, 2016).
The MTC algorithm speeds up the sampling calculations by making use of the fact that, for independent random variables,

$$\pi(f, \vartheta|d) = \pi(\vartheta|d)\pi(f|\vartheta, d), \quad (1.10)$$

where $\pi(\vartheta|d)$ is called the marginal posterior, calculated by the marginalisation integral

$$\pi(\vartheta|d) = \int \pi(f, \vartheta|d) df. \quad (1.11)$$

The sampling process can be resolved in the titular fashion: sampling from the marginal distribution, then using that to evaluate the full posterior.

In the linear Gaussian case, such as the EMFT model used in this thesis, sampling from $\pi(f|\vartheta, d)$ is equivalent to solving linear equations because the moments of a Gaussian can be found analytically. The MTC algorithm reduces an expensive MCMC over many dimensions into a cheap MCMC over a couple of dimensions, alongside solving linear equations. This process allows the MTC to exceed the computational efficiency of similar regularisation methods (Fox and Norton, 2016).

### 1.3 Mini-Aquifer

The work in this thesis is funded as a part of the SfTI NSC 10 Tranche 1: Inverting Electromagnetics project. The aim of the project is to develop a portable system for measuring groundwater flow using the principles of EM flowmeters. The mini-aquifer is an experiment created for the NSC 10 Tranche 1 in University of Canterbury’s School of Engineering as a prototype measurement system. It is intended to be a test-bed for the methods developed in this thesis in an easily controllable lab situation.

Cross-sectional diagrams of the mini-aquifer are shown in Figure 1.3. Geometrically this system is a large rectangular pipe with invasive electrodes. The main tank of the mini-aquifer is a 1.2m square perspex box filled with coarse beads to a depth of 1m. This tank contains the measurement plane through which the flow is measured.

On two opposite horizontal sides of the tank are uniform meshes of fine holes to allow the transport of water, but not the beads. These holes lead into chambers of the tank containing only water which are called the flow control due to the height differential of these tanks being the driving force of the water. The water levels are controlled by a pump from the low flow control tank to the high flow control tank. A stable flow rate is achieved when the heights of the flow control tanks are static.

The proposed magnetic field is generated by an electromagnet approximately 1m wide. A 17Hz driving current was chosen because this was the lowest frequency at
which a preexisting power supply could function. The lowest frequency was chosen with the intent that lower frequencies have a large penetration depth for the potential future real-world situation, whilst still minimising electrochemical effects.

Groundwater can vary in speed significantly, but a speed of $|v| = 10\text{mm/hr}$ was chosen as an approximate measure. The magnetic field strength is $|B| \approx 10\text{mT}$. Using the formula $V = dBv$ from the idealised half-space (Section 3.1.3) the voltage between two electrodes 1m apart is expected to be of order $V \approx 10\text{nV}$. This value is predicted to be well below the noise floor of the measurement system.

The mini-aquifer system is explored analytically in Section 3.1.3 and again in Chapter 7 using the MTC sampler.
1.4 Summary of preliminary conclusions

The Inverting Electromagnetics SfTI project was started with the notion that groundwater flow could easily be measured using Faraday’s law of induction, assuming the flow is a constant and infinite half-space. It was hypothesised that the bulk flow rate of groundwater through a region of space, $b$, could be easily determined with a simple measurement process. If two electrodes were driven underground and immersed in an aquifer, then the flow rate could easily be determined by a linearly correlated voltage measurement, after calculating the constant of proportionality. In the course of this thesis we have come to understand that this simple picture is not realistic. Indeed, many practical issues make it impractical and even misleading.

Here is a list of conclusions established in this work which correct and clarify this picture.

1. **The potential is linearly related to the bulk flow rate**

   Any spatially varying velocity field can be denoted $v(x) = bv_0(x)$, where $v_0(x)$ is a normalised function in the one-norm and $b$ is the bulk flow rate through a given region. The potential difference induced in a general flow is then proportional to the bulk flow rate $V = kb$, for a proportionality constant $k$. This linear relationship follows from the flowmeter equation BVP being linear.

   The proportionality constant, $k$, can be found analytically for point electrodes in idealised situations with simple geometries; see Section 3.1.3. In general however, voltages measured by electrodes have a non-linear relationship with the induced potential difference due to electrolytic effects and contact impedance.

2. **It is impossible to reliably determine $k$**

   The proportionality constant, $k$, has a complicated dependence on a multitude of things. One such dependence is on the shape of the velocity field $v_0(x)$. A section of flow can contribute positively or negatively to a measurement, dependent on the section’s location. Hence, different flow profiles, $v_0(x)$, can lead to the same measurement and even the sign of $k$ can be different for two fields with the same bulk flow rates, $b$.

   For point electrodes, $k$ is a singular function of $v_0(x)^1$ at the electrode locations. This means that the geometry of the electrode-fluid interface and surrounding region has significant, unpredictable effects on $k$.

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1Technically it is singular in $v \times B$. 

13
3. **Measurement sensitivity decreases quickly away from the electrodes**

The measurement kernel is singular at the electrode locations, but decreases rapidly at distance; see Section 3.2.1. This implies that the measurements are highly sensitive to \( v(x) \) near the electrodes, but insensitive to distant features of flow. Hence any measurement is dominated by the features neighbouring the electrodes and, in the case of a pipe EM flowmeter, does not contain much information about the central part of the flow.

In groundwater flow, “features neighbouring the electrodes” can include any small defects in the electrode; unknowns in the electrode-fluid interface, such as electrostatic buildup; or local features in the earth where the electrode is buried, such as compressed dirt from driving the electrode underground. Contact electrodes are not ideal as they exacerbate these issues.

4. **Sensitivity does not imply accuracy**

The singular nature of the measurement kernel means that the measurements are only sensitive to local features. The main body of flow will be distant to the electrodes, such as for the setups in Figure 1.2. Hence, a highly sensitive measurement — one with a large signal — implies significant features near the electrodes and will not necessarily correspond to an increase in accuracy when determining the bulk flow rate.

5. **One must determine \( v_0(x) \) in order to calculate the bulk flow rate of general flows**

Due to the measurement being dependent on the flow profile, one must find \( v_0(x) \) in order to calculate \( b \) for general flows. This is why EMFT is used in this thesis.

6. **Assumptions must be made about the flow profile**

There are an infinite number of flow profiles, \( v_0 \), and bulk flow rates, \( b \), consistent with any measurement — even if the measurements are perfectly error free. EMFT is an very ill-posed problem and the inevitable addition of error as noise exacerbates this problem. Assumptions must be made about the types of flow which are possible in order to both restrict the space of possible states consistent with the data, but also to infer information about flow far from the electrodes; the issue raised in Point 3. One simple assumption with a physical basis is to restrict to slowly varying, smooth flow (see Sections 2.3, 4.1, and 5.2.1 for three different approaches to this).
A Bayesian formulation is required to make quantitative statements of flow rate and uncertainties

Bayesian formulations allow one to track uncertainties in each component of the problem — the knowledge of \( v_0(x) \), measurement noise, geometry, etc. This provides quantified uncertainties of the flow rate and allows for more robust comparison of accuracy and precision of different measurement systems. A deterministic inverse problem is ill-posed, where error can significantly change the results. In contrast, Bayesian formulations of inverse problems are well-posed problems of statistical inference and are therefore preferential to deterministic formulations (see Chapter 5).

1.5 Organisation of thesis

Chapter 2 contains a summary of the history of EM flow measurement similar to that by Shercliff (1962), including the derivation of the underlying theory. It additionally contains a summary of literature up to contemporary times and analyses the different approaches with modern understanding of EM flow measurement. The chapter provides the history and context for the points in the previous section.

Chapter 3 contains the derivation of the forward map of EMFT using a Green’s function approach with the techniques of Stakgold and Holst (2011) as well as new analysis of the measurement kernel both for idealised cases and the pipe EMFT system. Additionally, Chapter 3 contains the description and tests of the mathematical and finite element based numerical models of the system used for solving the inverse problem.

Deterministic inversion of the pipe EMFT is explored in Chapter 4 using the methods of Fox et al. (2012) and Engl et al. (1996). I perform a truncated singular value decomposition (TSVD) in order to examine the properties of a deterministic inversion and provide a counterpoint to later chapters. The singular value decomposition (SVD) is used for the circular pipe EMFT for novel analysis of the forward map.

Chapter 5 introduces and explores the Bayesian formulation of EMFT which has never been done before. Section 5.1 contains the theory and principles behind Bayesian inference, based on the work of Robert (2007), von Toussaint (2011), and Watzenig and Fox (2009). The latter part of the chapter presents the specific implementation of the MTC sampler used in this thesis — adapted from the work of Fox and Norton (2016) — exploring specific considerations of pipe EMFT.
The results of the MTC applied to the circular pipe EMFT are contained within Chapter 6. A number of physically realisable flow fields used by Lehtikangas et al. (2016) and modelling decisions are explored. Chapter 7 applies findings from Chapter 6 to the square pipe EMFT in order to explore the mini-aquifer inverse problem.

Finally, Chapter 8 contains discussion and concluding remarks. This includes a summary of the advantages of using a Bayesian formulation, but also a discussion of specific issues with the EMFT implementation of the MTC.
Chapter 2

Context and History

EMFT is an inverse problem. One desires knowledge about the hidden state of the flow in a pipe which can only be gained indirectly through the measurement of an induced potential. The solving of an inverse problem is divided into three parts: the model of the state, the model of the forward map, and the method by which the inverse is obtained. The methods of solving this inverse problem differ in their respective approaches to these three categories. This work uses a Bayesian formulation for solving the EMFT inverse problem which will be fully covered in Chapter 5.

2.1 Flowmeter equation

All modern EM flow rate measurements of EM flowmeters and EMFT are based on the flowmeter equation, derived as follows.

Let there be an Ohmic fluid with conductance $\sigma$ travelling with velocity field $v$. An external magnetic field $B$ is applied to the system. The charged particles contained within the fluid experience a Lorentz force of

$$F = \rho(E + v \times B), \quad (2.1)$$

where $F$ is the force field and $\rho$ is the charge density of the fluid. Due to the Ohmic property of the fluid equation (2.1) can be rewritten as

$$j = \sigma(E + v \times B), \quad (2.2)$$

where $j$ is the current density vector. With stationary magnetic and electric fields it is possible to define a potential $u$ by

$$E = -\nabla u, \quad (2.3)$$
Figure 2.1: The basic idea of EM flow measurement. The dark blue arrows represent a fluid’s velocity field and the orange arrows represent a magnetic field. The potential is measured by way of a voltage between two electrodes, represented as light blue squares. As a BVP the boundary must be considered, depicted here by a section of a riverbed.

and due to Maxwell’s equations

\[ \nabla \times B = \mu j \implies \nabla \cdot j = 0, \]  

(2.4)

where \( \mu \) is the permeability of the fluid. Combining these with (2.2) we get the equation

\[ \nabla \cdot (\sigma \nabla u) = -\nabla \cdot (\sigma (v \times B)). \]  

(2.5)

Adding the assumption that the fluid has constant conductivity yields the flowmeter equation:

\[ \nabla^2 u = -\nabla \cdot (v \times B). \]  

(2.6)

The flowmeter equation is a partial differential equation relating the velocity of a fluid travelling through a magnetic field to the generated potential. EM flowmeters and EMFT systems aim to infer the velocity field through measuring the generated potential. This is done by taking a voltage measurement between electrodes with a portion of the velocity and magnetic fields between them.

A simple diagram of the basic idea is depicted in Figure 2.1. It can be seen from inspection of (2.6) that the generated potential will be maximised when the velocity and magnetic fields are at right angles. Additionally the potential gradient will be greatest when orthogonal to both of these fields.
The problem posed by the flowmeter equation is actually a BVP and requires the specification of the geometry and associated boundary values of the fields. This is shown in Figure 2.1 with a boundary similar to a riverbed. In order to infer what the velocity field of the fluid is from a potential difference measurement one must solve the flowmeter equation BVP. This will be covered in detail in Chapter 3.

2.2 Flowmeter history

EM flow measurement became possible in the 1800’s with the contributions of Michael Faraday and Hendrik Lorentz. They demonstrated the relationships between magnetic fields and moving charged particles allowing the measurement of both to be feasible. The earliest recorded attempt at measuring fluid flow using magnetic fields was by Faraday in 1832 when he reported to the Royal Society of London several magneto-electrical experiments. This included his unsuccessful attempt at measuring the flow of the Thames River through its interactions with the Earth’s magnetic field (Faraday, 1832).

Faraday’s reasoning for measuring the Thames flow was thus: the Earth has a magnetic field and water is a conductor due to charged particles contained within. From (2.1), a river in the Earth’s magnetic field will create a measurable potential difference due to the induced current from the Lorentz force. That voltage is proportional to the velocity of the water, i.e. the flow rate of the Thames river, because the Earth’s magnetic field and the fluid’s conductivity shouldn’t significantly change.

To test this hypothesis Faraday lowered two large electrodes into the Thames from the Waterloo Bridge. It was postulated by later researchers that Faraday’s measurements failed partially due to two of the major problems in EM flow measurement: electrochemical effects from using a DC field and thermoelectric effects of the materials involved (Shercliff, 1962). However Faraday failed to account for current flowing through the riverbed (Shercliff, 1962). It is clear with modern understanding of BVPs that Faraday disregarded the boundary effects and thus inaccurately modelled the system. This boundary would first be taken into account in Wollaston’s 1851 measurements of tidally induced currents in the English Channel (presented by Wollaston (1881)).

The experiences of Faraday and Wollaston are archetypal of early investigations into EM flow measurement. Technical difficulties of measuring the correct signal took decades to overcome. The larger problem is the complexity of the systems being mea-
sured. The velocity and conductivity are not constant in a channel of water as they are taken to be by Hughes (1969) for the Irish Channel. The channel is not separate from the rest of the world and can have complex geometries. Additionally, the voltage is not only dependent on the flow between the measurement points, as will be shown in the idealised situation in Section 3.1.3.

2.2.1 Oceanography

A significant portion of the literature of EM flow measurement lies within the field of oceanography. The measured potential difference is generated by the slowly changing ocean currents interacting with the static Earth’s magnetic field. A significant obstacle in measuring currents in oceanography is the prohibitively expensive cost of placing measurement equipment. However the prevalence of undersea electrical cables for power and telecommunication infrastructure means that there is preexisting measurement equipment spanning areas of interest, such as channels. There have been many investigations using undersea cables (Bowden, 1956; Hughes, 1969; Filloux, 1973; Robinson, 1976).

The above investigators limited themselves to finding correlations between the voltage measurements and other phenomena. This was due to the lack of modelling power and the prevalence of error such as from electrochemical effects. Correlation between the measured signal and the tides appear in all of the four mentioned papers. Bowden and Robinson also correlate the measurements with the regional winds.

These investigators also attempted to calculate the total flow. This calculation required significant assumptions about the shape of the velocity field and thus met with limited success. Robinson (1976) used the most sophisticated model of flow. He used a square-based, piece-wise constant, quasi-2D model attempting to account for spatial variation in the Irish Channel. Robinson reported both an additive and multiplicative difference between the predicted flow rate and the actual values obtained using other measurement systems. This difference was minor during large storm surges, but the error becomes overwhelming with smaller currents such as that driven by the wind. To quote Robinson (1976) “...it is clear that the cable predictions and the actual volume transport bear no obvious relation to one another for any of the cables”. He attributed this to the level of spatial variation in the velocity field.

Two other common assumptions used by investigators such as Hughes (1969) and Longuet-Higgins and Deacon (1949) were of homogeneity in the fluid travelling through an elliptical channel cross-section. Homogeneity is not true in many large bodies of
water due to changes in temperature and salinity levels in different regions and depths in the sea. The latter assumption will also cause error in the measurements due to the complex geometries of coasts and sea floors. Spatial variation of the velocity field such as that caused by complex geometries can lead to error in the measurements as Robinson found.

The works by Longuet-Higgins and Deacon (1949) and Guelke and Schoute-Vanneck (1947) are examples of oceanography measurements using specifically placed electrodes for the measurements. Guelke and Schoute-Vanneck (1947) designed electrode experiments to be placed within the body of water itself. It is also useful as an example of the problems in this field because Guelke and Schoute-Vanneck describe the setup in great detail. Notable was their attempt at reducing electrochemical effects at the measurement interface. These were minimised by using non-reactive silver-silver chloride electrodes and AC currents to prevent electrolic buildup. However, to quote from Filloux, “Even the best matched silver-silver chloride electrodes introduce variable electrochemical signals hard to maintain below a milivolt” (Filloux, 1973).

Discrepancies in the flow near the electrodes including those caused by the measurement system itself can have a large effect on the measurement (see Section 3.1). This effect was avoided by Longuet-Higgins and Deacon (1949) who aimed to cheaply measure the flow in the English Channel by placing the electrodes on the English mainland. The potential differences generated by tidal currents in the Earth’s magnetic field is of the order 0.05µV/m and can be significant on the scale of a country or ocean (Filloux, 1973). This potential difference doesn’t stop at the shore and penetrates into the land (Parkinson and Jones, 1979). Thus, Longuet-Higgins and Deacon (1949) managed to eliminate the problem of local sensitivity by removing the electrodes from the fluid flow itself.

**Other electromagnetic flow measurement**

There have been many other applications of the fundamental EM flow measurement theory. Other notable oceanographic applications were made by: Remenieras and Hermant (1954) — who applied the idea of the Williams flowmeter (see Section 2.3) by placing tubes close to the seabed and measuring the flow within — and Smith and Slepian (1917) who patented the “electromagnet ship’s log” — an on-boat device now known as a wall velocimeter.

A similar use of EM flow measurement on a ship under way was used by von Arx (1950). Von Arx created an EM flow measurement device consisting of electrodes on a
cable to be towed from the back of a boat as it travelled. Using the Earth’s magnetic field as a static reference allowed von Arx to take many shallow measurements of ocean currents relatively quickly. However von Arx stated in said paper that there were significant errors due to turbulence — particularly in the ship’s wake.

Another area where EM flow measurement was prevalent is medicine. Kolin (1936) and Wetterer (1937) independently proposed arterial flowmeters: a non-invasive, clip-on device for measuring the blood flow through an artery — similar in theory to the Williams flowmeter. While having a large body of academic work (Denison, Spencer, and Green, 1955; Westersten, Herrold, and Assali, 1960; Wyatt, 1968), these devices failed to gain traction in the medical community and no papers were published after the 1970s.

### 2.3 Electromagnetic flowmeter

![Figure 2.2: Cross-sectional slice of an EM flowmeter’s setup. A conductive fluid travels through a circular pipe. The velocity (blue dotted circles) points out of the page. An orthogonal magnetic field is applied in the $y$ direction (orange arrows). The generated potential difference is measured by two electrodes on the pipe wall, aligned in the $v \times B$ direction (teal squares).](image)

The first EM flowmeter in literature was by Williams (1930). Williams proposed an easily controllable lab situation of Faraday’s initial idea: an aqueous copper sulphate solution flowing in a straight, circular pipe. He measured the voltage generated by the flowing solution under the influence of an approximately 1T magnetic field. The
Figure 2.3: The “short curcuiting” current in a circular pipe cross-section. Source: Williams (1930) (with axes swapped).

Electrodes were placed on either side of the 1.075 cm wide glass pipe and the solution’s velocity was on the order of $0.1 \text{ms}^{-1}$. Williams measured voltages of the order of millivolts. These results matched closely with theoretical values he obtained using separation of variables to solve the PDE (2.6) by assuming the flow shape was

$$v_z = k(R^2 - r^2),$$

where $v_z$ is the $z$ component of the velocity field and is the only non-zero velocity component, $k$ is a scale constant, $R$ is the radius of the pipe, and $r$ is the radial spatial variable.

This setup (in Figure 2.2) is now recognisable as the standard setup for EM flowmeters used in industry.

Williams made many contributions to the EM flowmeter field. To start with, he noted that the no-slip condition of fluid dynamics applies as fluid approaches the edge of the pipe. Thus the contribution to the measurement from the flow would be much lower near the edge. He states that this will create a circulating current in the pipe as shown in Figure 2.3 (referred to as a “partial short circuit” by Shercliff (1962)). The consequence is that the measured voltage is less than the contribution from only considering the path directly between the electrodes. However, as argued by Williams and eventually demonstrated by Thürlemann, this effect makes the measurement of any axially symmetric flow equivalent to a measurement of constant flow with the same bulk flow rate (Thürlemann, 1941). This is all assuming that the pipe wall is non-conductive, as conductive pipes would have a full short circuit effect.
Williams also tried a similar experiment with a curved pipe and showed that the shape of the velocity field significantly changes the measurements (Williams, 1930). He also remarked that the observed behaviour would be quite different if the material was a much greater conductor. For example, the forces experienced by liquid metals due to these circular currents is much greater and there is also be a strong self-inductive effect (Shercliff, 1953).

2.3.1 Axisymmetric proof

Thürlemann analytically derived that the voltage measured between two point electrodes arranged perpendicular to $v$ and $B$ will be proportional to only the bulk flow rate for an axisymmetric flow field in a non-conductive circular pipe with a constant perpendicular magnetic field. This proof has been refined and shortened in subsequent papers such as by Kolin (1945). Presented here is an adaptation of the proof by Shercliff (1962), who derived the result for conductive pipe walls and then takes the limiting case to non-conductive walls.

Take a conducting circular pipe with conductance $\sigma_w$, internal radius $a$, and outer radius $b$. The pipe aligns with the basis for the cylindrical coordinate system $(r, \theta, z)$. An Ohmic fluid of conductance $\sigma_f$ travels in the $z$ direction and has axial symmetry about the $z$ axis. A constant magnetic field is present, aligned with the $x$ axis, and two point electrodes are located at the external radius of the pipe, on the $y$ axis. Let the pipe and fields extend far enough in $z$ so that the system can be approximated as a 2D system. The system is stationary and thus is governed by the flowmeter equation.
(2.6).

Using vector identities, (2.6) is rewritten as

\[ \nabla^2 u = B \cdot \nabla \times v - v \cdot \nabla \times B, \tag{2.8} \]

where the final term is negligible if the magnetic field is not affected by the induced currents. Because \( \nabla \times B = 0 \) in either a constant magnetic field or when the currents are perpendicular to the fluid flow, the flowmeter equation becomes

\[ \nabla^2 u = B \cdot \nabla \times v. \tag{2.9} \]

Separating the space into the pipe interior and pipe wall - denoted by the subscripts \( f \) and \( w \) - gives two differential equations

\[ \nabla^2 u_f = B \frac{\partial v}{\partial y} \quad \text{and} \quad \nabla^2 u_w = 0. \tag{2.10} \]

Equation (2.10) can be simplified further because of the flow's axial symmetry \( (v = v(r)) \) to obtain

\[ \nabla^2 u_f = B \frac{\partial v}{\partial r} \sin(\theta) \quad \text{and} \quad \nabla^2 u_w = 0. \tag{2.11} \]

Boundary conditions are required in order to solve (2.11). Upon examination of the setup, the first boundary condition is that no current flows out of the pipe. This yields the Neumann boundary condition

\[ \left. \frac{\partial}{\partial r} u_w \right|_{r=b} = 0, \tag{2.12} \]

where the vertical line means “evaluated at”.

The internal boundary conditions are more complicated. Firstly, current must be conserved. Secondly, there is a contact resistance from the sudden change of conductivity. Together these are expressed as:

\[ \left. \sigma_w \frac{\partial}{\partial r} u_w \right|_{r=a} - \left. \sigma_f \frac{\partial}{\partial r} u_f \right|_{r=a} = 0, \quad \text{and} \]

\[ \left. \left( u_f + \sigma_f \tau \frac{\partial}{\partial r} u_f \right) \right|_{r=a} - u_w \right|_{r=a} = 0, \tag{2.14} \]

where \( \tau \) is the contact resistance.

The solutions to the above equations are

\[ u_f = Z(r) \sin(\theta), \tag{2.15} \]

\[ u_w = \left( cr + \frac{d}{r} \right) \sin(\theta), \tag{2.16} \]
where \( c \) and \( d \) are constants and together with \( Z(r) \) satisfies

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} Z - rZ \right) = r^2 B \frac{\partial}{\partial r} v, \tag{2.17}
\]

\[
ca + \frac{d}{a} = Z(a) + \sigma_f \tau \left( \frac{\partial}{\partial r} Z \right)(a), \tag{2.18}
\]

\[
\sigma_w \left( c - \frac{d}{a^2} \right) = \sigma_f \left( \frac{\partial}{\partial r} Z \right)(a), \quad \text{and} \tag{2.19}
\]

\[
c = \frac{d}{b^2}. \tag{2.20}
\]

Integrating (2.17) and using the fluid’s no-slip condition at the boundary leads to

\[
a^2 \left( \frac{\partial}{\partial r} Z \right)(a) - aZ(a) = -Ba^2 \bar{v}, \tag{2.21}
\]

where \( \bar{v} \) denotes the mean velocity.

The average velocity can be determined from these simultaneous equations, however Shercliff instead looks at the flowmeter sensitivity. The sensitivity is defined as the difference between the voltage measured and that of a purely constant fluid velocity:

\[
S = \frac{V}{2bB \bar{v}} = \frac{2a^2}{(a^2 + b^2) + \frac{\sigma_w}{\sigma_f} \left( 1 + \frac{\sigma_f}{a} \right) (b^2 - a^2)}. \tag{2.22}
\]

Taking the limit of \( b \to a \) yields the result that \( S = 1 \), which can be viewed as changing the system to a non-conductive pipe with electrodes on the interior radius.

Thus, the measured voltage is proportional to the bulk flow rate for any homogeneously conductive fluid flowing axisymmetrically in a circular non-conductive pipe. Importantly, this voltage is linearly correlated to the bulk flow rate.

### 2.3.2 Shercliff’s contribution

In the field of electromagnetic flowmeters, the most significant contributions were made by J. A. Shercliff and his student M. K. Bevir. Shercliff brought together the flowmeter literature in his 1962 book “The Theory of Electromagnetic Flow-measurement”, including his own significant bibliography up to that point in time (Shercliff, 1962).

Shercliff breaks down the theory for many different types of EM flowmeters; square and circular, transverse and axial, to name just a few. The analysis clearly states assumptions, benefits, and practical problems involved for each. There are even large portions of the book devoted to cases where the self-inductive effects are non-negligible, Kolin-style arterial flowmeters, and other EM flow measurement systems such as Smith and Slepian’s EM velometer.
Shercliff mainly focuses on bulk flow rate measurements in pipes and establishes the theory for the modern commercial flowmeters; he only examines cases with single potential difference measurements. The inverse problem of tomography in more complex scenarios only became feasible with the rise of computers. To provide an insight into where Shercliff’s interest lies: he claims that a large advantage of an EM flowmeter is — to paraphrase — its blurriness. This blurring quality allows EM flowmeters to work with mixed materials and slurries such as “sewerage and blood” — (Shercliff, 1962). In contrast, modern interest in non-invasive measurement include trying to measure and characterise inhomogeneities and intrusions in a flow rather than blurring them out (Watzenig and Fox, 2009).

For the purpose of making a robust flowmeter, Shercliff investigated how deviations from the axial symmetry assumption affected the measurement accuracy. To begin with, he derived how turbulent flow affects a time-averaged measurement of voltage. This was done as follows.

Let \( u = \bar{u} + u' \), where \( \bar{u} \) denotes the time average flow and let \( u' \) be small perturbations due to turbulence. Then

\[
\nabla^2 \bar{u} = \nabla \cdot (\bar{v} \times \bar{B} + \bar{v}' \times \bar{B}') ,
\]

(2.23)

with the second term being negligible due to the lack of any correlation between velocity and magnetic field fluctuations. Thus, turbulent effects have very little effect on time averaged voltage measurements.

Other important observations and calculations include:

- Metal walls for any pipe creates an averaging effect of the velocity, allowing for measurements to be influenced by the mean flow more than fine features.

- Higher frequency AC magnetic fields increase the level of the self-inductance in the fluid. The fluid experiences the skin-effect at higher frequencies, where the magnetic field does not penetrate deep into the fluid. Shercliff states that these effects are insignificant if the quantity \( \sqrt{\mu \sigma \omega} \) is large compared to the length scales of the flowmeter, where \( \omega \) is the AC frequency.

- The effects of “end-shortening” is analysed. This is where the magnetic field decreases appreciably close enough up- or downstream to effect the measurements. To summarise his findings, end-shortening decreases the signal as it makes the approximation of the 2D flowmeter less valid.
2.3.3 The weight function

The theory summarised by Shercliff (1962) breaks down as soon as the flow is no longer a simple distribution. This was widely known even by Shercliff himself. He writes “[EM flowmeters] are liable to errors larger than $\pm 100$ per cent in the presence of upstream disturbances!” (Shercliff, 1962, p. 25). Two different shaped flows with the same bulk rate can lead to significantly different measurements. This is because the velocity of the fluid near an electrode contributes more than the velocity far away\(^1\).

For visualisation purposes of the effect of these disturbances, Shercliff proposed a spatially varying weight function, \(W\), which he defined in terms of the flowmeter sensitivity (Shercliff, 1954)

\[
S = \frac{V}{2bB\bar{v}} = \frac{\int v(x,y)W(x,y)dxdy}{\int v(x,y)dxdy}. \tag{2.24}
\]

For the non-conductive pipe setup from Section 2.3.1, the weight function was computed to be

\[
W = \frac{a^4 + a^2(a^2r^2\cos(2\theta))}{a^4 + 2a^2r^2\cos(2\theta) + r^4}, \tag{2.25}
\]

plotted in Figure 2.5.

\(^1\)More detail will be given to this in Chapter 3
With the tools provided by Shercliff to analyse and discuss flowmeter systems now available, more unorthodox EM flowmeter designs were proposed. Notably, Rummel and Ketelsen made the observation that nonuniform magnetic fields could improve flowmeter performance (Rummel and Ketelsen, 1966).

Shercliff’s once student, M. K. Bevir, later expanded upon the weight function (Bevir, 1970). Bevir re-framed the weight function in terms of electrical networks. He wrote the voltage as

\[ V = \int v \cdot W d\tau, \] \hspace{1cm} (2.26)

with \( W = B \times j \). In Bevir’s paper \( j \) is referred to as a “virtual current” because it is the equivalent of the current density generated if a unit current is run between the electrodes. In modern terms it can be seen as a kind of Green’s function or impulse response.

Bevir applies this weight function to many systems including velometers and arteries. However with the benefit of hindsight, a more important contribution was a definition used in the first paragraph of his paper. In saying “If the sensitivity is independent of the flow pattern, the meter will be called ‘ideal’.” Bevir shaped the discussion of following academic flowmeter research. He pointed out that ideal flowmeters are impossible with point electrodes. Subsequently, much of the literature was dedicated to finding the most uniform weight function using different arrangements of multiple electrodes and magnetic field shapes (Al-Khazraji and Baker, 1979; Bevir, O’Sullivan, and Wyatt, 1981; O’Sullivan, 1983; Teshima, Honda, and Tomita, 1995).

2.4 Modern electromagnetic flow measurement

Modern EM flow measurement can be categorised into two areas: commercial EM flowmeters and EMFT systems. Commercial flowmeters are easily available for purchase and are prevalent in industry but have not advanced theoretically from Shercliff’s axial symmetry work. EMFT is a field of study which uses multiple potential difference measurements to reconstruct a 2D image of the velocity field.

2.4.1 Commercial electromagnetic flowmeters

The standard setup of a commercial EM flowmeter is similar to the system in the axisymmetric proof above (as in Figure 2.2). It consists of a circular non-conducting pipe with electrodes on opposite horizontal sides of the interior wall. A pair of coils
are placed on the opposite vertical sides of the exterior wall in order to create an approximately spatially uniform field on the interior. They use alternating currents to drive the electromagnets in order to avoid electrolic effects on the surface of the electrodes.

The oscillating magnetic field inevitably induces an emf in the electrode circuit. However this is in quadrature to the useful measurement, which can be removed using an appropriate filter (Shercliff, 1962). EM flowmeters also use frequencies in the tens of hertz to minimise self inductance (Stewart, 2016).

The amplitude of the measured voltage is linearly proportional to the average flow velocity ($v$), the magnetic field ($B$), the conductivity (included in the constant $k$), and the distance between the electrodes ($2R$). These industrial systems must be calibrated for a specific fluid using known flow rates to find the proportionality constant $k$ in

$$V = 2kvBR,$$

assuming that $B$ are spatially uniform and $v$ is axially symmetric. Commercial flowmeters are purchased with a display and control panel built-in, creating a black-box style device (example in Figure 2.6a).

EM flowmeters have many well known issues. The assumption of axisymmetric velocity and constant magnetic field is not true in a flowmeter. The magnetic field is
generated by coils such as in Figure 2.6. At best, the generated field is approximately constant with some end-shortening. The axial symmetric flow assumption is only true if there is a long, straight section before the meter. Both of these are only minor problems if space for a long pipe section is readily available. Commercial EM flowmeters are sold as pipe sections with the length being 2+ times larger than the width which does minimise these issues. However, the flowmeters will fail if they cannot be placed in a length of straight pipe.

A large problem in this standard setup is that all factors must be constant over long time spans. If, for example, the magnetic field changes due to temperature variation in the coils, then the system will require a lengthy re-calibration (Stewart, 2016). Additionally, if the conductivity varies significantly in space then the measurement is impossible. Examples of common real world cases of this include a fluid with solids, bubbles, or oil in water flow. There is also the issue that the linear relationship is not completely accurate, as is discussed by (Shercliff, 1962) near the end of his book. Shercliff writes

“...this is only strictly true when the transverse field is absolutely uniform and end-shortening is absent. This is only imperfectly realised in practice.” — (Shercliff, 1962)

Despite this, commercial EM flowmeters still use the method described above which was originally proposed by Williams in 1930. This is mostly due to the robustness of the design. Shercliff explains that the measurements are noticeably not linearly correlated only in extreme cases where either the self inductance is noticeable or in flow profiles sufficiently divergent from axial symmetry.

This linear correlation is a major advantage over other types of flowmeters in conventional flowmetering situations because it “indicates the direction of flow unambiguously” (Shercliff, 1962). While empirical calibration is required, the linear correlation sidesteps any problems which may arrive from an imperfect installation or any minor real-world deviation from the idealised model. As will be explained in Chapter 3, the measurement is very sensitive to features closer to the electrodes. Therefore using contact electrodes will amplify any associated real-world error such as misplacement or even skin oil residue. In an EM flowmeter these model errors are included in the proportional constant \( k \). More sophisticated methods must account for these errors when the linear correlation no longer applies.

The assumption that fields are constant in \( z \) is not unique to EM flowmeters. Many
EMFT methods also use a 2D approximation to the system to simplify computation. The fields are also assumed to be constant in the $z$ direction of the pipe. Therefore, these EMFT systems must also have straight sections of pipe and have no advantage over EM flowmeters in this respect.

**Liquid metal**

One area where EM flowmeters excel over competing flow measurement devices is the measurement of liquid metals and other extreme fluids. A large fraction of the total EM flowmeter literature is from the 50s and 60s due to the development of nuclear power after the Second World War. The cooling and energy extraction requirements in nuclear reactors are extreme compared to conventional fluid systems, most notably in the compact fast reactors (Baker, 1977). Many reactor systems moved away from water to using liquid sodium as a coolant, which was known to occasionally solidify. Bismuth is another metal that was sometimes used by dissolving uranium in it to run the nuclear reaction. These conditions with radioactivity and high temperatures involved make traditional invasive control systems unreliable (Popper, Wiegand, and Glass, 1967).

EM flowmeters are ideal for these conditions. The lack of moving parts and the non-invasive nature meant that EM flowmeters are low maintenance. Temperature is also not much of a problem. As of the 60s, EM flowmeters had reportedly functioned with fluids over $800^\circ$C. The main concern is maintaining a stable magnetic field under such conditions (Shercliff, 1962).

Additionally, the inverse of the EM flowmeter is useful, where a current is pumped into a fluid under a magnetic field. If the system is highly conductive then this will create a force to propel the fluid into motion. Thus, the system is able to be changed into a non-invasive pump for these liquid metal systems with little alteration to the theory (Blake, 1957).

The use of EM flowmeters in nuclear reactors persists into modern times (Kondo and Takahashi, 2005).

**2.4.2 Electromagnetic flow tomography**

EMFT is an area of research which allows for general flow fields, not just axisymmetric ones. It is important to consider the spatial variation of flow for non-axisymmetric cases. This is especially important because there has not been any proposed EM
Figure 2.7: Cross-sectional slice of an EMFT system. A conductive fluid travels through a circular pipe. The velocity (blue dotted circles) points out of the page. An orthogonal magnetic field is applied in the $y$ direction (orange arrows). The generated potential difference is measured by a number of electrodes placed around the pipe wall (teal squares).

flowmeter design in the literature which is “ideal”. Thus, EMFT aims to increase the versatility and accuracy of EM flow measurement by reconstructing the whole velocity field from multiple potential difference measurements.

There are a few traits universal in EMFT systems:

1. More than one potential difference measurement is made. The number of electrodes in literature range from 4 to 32. To the best of this author’s knowledge, the electrodes are spaced equidistantly around the circumference in every case such as in Figure 2.7.

2. An assumption is still required for the shape of the velocity field. This is often a much weaker assumption than constant or axial symmetric flow.

3. A spatially varying model of how the fluid flow affects the potential at electrode locations is required (the forward map).

The first trait is for the simple reason that the more measurements taken, the more information gained about the system. More than one measurement is required to estimate the velocity field as a shape more complex than a constant flow (or axisymmetric flow). However, there are physical limitations to the number of electrodes which can be placed in a pipe. The physical size of the electrode is one such limitation, but also
electrodes may interfere with one another if placed too densely. It is also possible that this may break the assumptions of the model of the system.

The second trait is used because the inverse problem would be intractable without it. There are no necessary restrictions on the field $v$ in (2.6) and thus $v$ can be infinitely variable. This is problematic because EMFT seeks to map a set of 16 or so numbers to the most likely $v$ under noise, which is an ill-posed problem. A single measurement could be caused by an infinite set of possible velocities.

Restrictions must be applied to the velocity field to both narrow down the set of possible flow states consistent with a measurement, but also to ensure that flow profiles are realistic. One simple assumption is to restrict the space of states to only continuous functions. This is a real-world feature of viscous fluid flow (Kundu, Cohen, and Dowling, 2008). Additionally, the space of continuous functions is much smaller than the space of all functions.

Moreover, the electrodes are highly sensitive to neighbouring flow. Therefore the pipe wall measurements are non-penetrative. Assumptions about the shape or the level of smoothness of the velocity field are required in order to reconstruct realistic flows such as an axisymmetric laminar flow. This has the effect of both further restricting the space of states consistent with a measurement, but also forces some correlation between the electrodes’ measurements.

Every assumption restricts the space of possible flow profiles making each measurement more impactful and the problem closer to being well-posed. However, over-restricting is a problem. So too is using non-realistic restrictions. For example the assumption of a uniform velocity field allows for the unique determination of the flow with a single noiseless measurement. However the no-slip condition means that this does not exist in viscous flow, and is thus not realistic.

The third trait merely states that the forward problem is required in order to solve the inverse problem. This is highlighted here because a portion of the literature has the forward map contradict the assumptions of the flow field.

The forward map is itself intrinsically linked with the representation of the state of the system. An example of work which made use of this was by Horner, Mesch, and Trächtler (1996). Horner et. al. sought to match specific electrode arrangements with different flows by representing the velocity field as a sum of circular harmonics. The paper demonstrated that only certain harmonics are detectable by given arrangements of electrodes. For example, the third harmonic is not detectable with three equidistant electrodes using their arrangement. The aim of Horner et. al. was to introduce
this representation of the unknown flow and to show that increasing the number of electrodes also increases the accuracy of the bulk flow calculation of non-axisymmetric flows. They experimentally demonstrated this increase in accuracy with arrangements of up to 16 equidistant electrodes and in a pipe with two orthogonal Helmholtz coil pairs.

A more recent example of restricting the possible states was by Kollár et al. (2014). Kollár et al. limited their velocity field to 6th order polynomial functions. Both this work and Horner et al. (1996) analytically calculate the forward map using Green’s functions to solve (2.6).

Other methods such as that used by Lehtikangas et al. (2016) make a finite element representation of the flow and use a finite element method (FEM) solver for (2.6) to create the forward map. Lehtikangas et al. have weaker assumptions than any previous paper, modelling the fluid flow as a locally connected Gaussian Markov random field (GMRF). This is a weak condition which weighs possible flow states based on the 2-norm of the discrete Laplacian, favouring smoother states. The setup for their measurement system is the same as by Horner et al. (1996): a pipe with excitation from 2 Helmholtz coil pairs and measurement using 16 equidistant electrodes.

The method in this thesis also uses a finite element representation for the fluid’s flow, modelled as a GMRF. However, the forward map is constructed using a Green’s functions to solve the BVP (2.6).

Related fields

The EMFT setup is very similar to two other fields which also use EM principles to measure features of a material flowing through a pipe. These fields differ mostly by the measurement system used.

Inductive flow tomography (IFT) has the same goals as EMFT of reconstructing a conducting fluid flowing through a pipe, excited by an external magnetic field. The only difference is that the electrodes are replaced with magnetic pick-up coils to measure the magnetic field created by currents induced in the fluid (Yin, Peyton, Stefani, and Gerbeth, 2009).

The other field is electrical capacitance tomography (ECT) which measures the change of capacitance of the material in a pipe. ECT is used in many industrial situations for two reasons: it does not require the pipe to contain a fluid and is able to readily detect intrusions such as bubbles and solids Jaworski and Dyakowski (2001). The methods by which the inverse is obtained in ECT are more advanced than in
EMFT with the use of modern techniques such as Bayesian inference and machine learning (Watzenig and Fox, 2009; Xiao, Liu, Zhao, Li, and Huo, 2018).

Both of these systems use non-contact measurements and avoid the technical difficulties involving contact measurements (discussed in Chapter 3).

Additionally there are the more distant fields of the medical focused electrical impedance tomography (Cheney, Isaacson, and Newell, 1999) and the geophysical focused electrical resistance tomography (Dickin and Wang, 1996). These both measure the resistance or impedance between electrodes.

2.4.3 Inversion techniques

The largest point of difference between the work in this thesis and existing literature is the method by which the inverse problem is solved. The inversion techniques used in EMFT literature are old and out-dated. A number of different methods are used, however they are all deterministic methods. These methods have numerous problems as will be covered in Chapters 4 and 5.

Horner et al. (1996) and Kollár et al. (2014) both use what shall be referred to as truncated projection methods. Both papers take direct inverse from the potential measurements to reconstruct the velocity field in terms of either circular harmonics, or 6th order polynomial coefficients of the fluid flow. Both of these processes amount to creating a new truncated basis of the flow. The new basis vectors are passed through the forward map to create a measurement basis. Measurement data is projected onto this new basis and a direct inverse is calculated.

Truncated projection methods are very crude forms of regularisation. Regularisation weighs against higher frequency terms in solutions to an inverse problem, whereas these methods disallow higher frequency terms completely. In addition, the results are highly dependent on both the choice of basis and the point of truncation. The deterministic inversion method used in this thesis, the TSVD, is one of these methods and will be explored in Chapter 4.

Lehtikangas et al. (2016) use the more sophisticated maximum a posterior (MAP) estimate method to choose the flow state as the solution to the inverse problem. The problem is initially formulated in a Bayesian way; with a prior model of the velocity field, a measurement likelihood function, and the posterior probability distribution is formed. The state is then selected by a maximisation problem over the posterior probability distribution using a least squares calculation. This inversion method outputs a single qualitative flow state and is based on a mostly subjective choice of the
sometimes-called smoothness parameter $\delta$. Choices like this are required in deterministic methods and the relevant parameters are chosen because they make the chosen reconstruction “look good” (Fox and Norton, 2016).

Proper Bayesian formulations of the inverse problems and statistical sampling methods which exist in the similar field of ECT do not appear in the EMFT literature despite numerous advantages (see Chapter 5). In this respect, ECT lies on the forefront of inversion with papers recently published updating to newer sampling algorithms (Bardsley, Solonen, Haario, and Laine, 2014) or artificial intelligence deep learning techniques (Xiao et al., 2018).
Chapter 3

Modelling the Forward Map

Inverse problems seek to gain information about hidden quantities through measuring some related, observable properties. The system in inverse problems comprises of the hidden state $f$ (also called the image), the forward map $A$, and the measured data $d$. The forward map is the operator by which the state relates to the quantity being measured, becoming the data with the addition of a measurement model and noise $n$. A simplified diagram of the overall observation process by which data is obtained is depicted in Figure 3.1.\(^1\)

This chapter is divided into three parts. These are the mathematical model of the system, the measurement model, and the numerical approximation which can be entered into a computer.

\[ f \xrightarrow{A} d \]

Figure 3.1: System model of the measurement process. A state $f$ is passed through the forward map $A$, the output of which is combined with some additive noise $n$ to produce some data $d$.

\(^1\)The forward map is affected by the representation of $f$ and $d$. This will be covered in Chapter 4
3.1 Mathematical model

3.1.1 Flowmeter equation

Let an ohmic fluid with constant conductivity be contained within the region $\Omega$, surrounded by a perfectly insulating boundary $\partial \Omega$. The fluid travels with velocity $v$ under the influence of an externally generated magnetic field, $B$. The relationship between the velocity and magnetic fields and a measurable potential, $u$, is given by the flowmeter equation with the insulating (Neumann) boundary condition. Thus, the flowmeter equation becomes the BVP,

$$\nabla^2 u = -\nabla \cdot (v \times B), \quad \frac{\partial u}{\partial \hat{n}} \bigg|_{r=R} = 0,$$

where $\frac{\partial u}{\partial \hat{n}} = \nabla u \cdot \hat{n}$ and $\hat{n}$ is the outwards facing normal vector of the boundary. Equation (3.1) is a form of Poisson’s equation where, instead of electrostatic charge distribution as the source, the source is $-\nabla \cdot (v \times B)$. I use Greens functions to solve the distributional form of (3.1).

Distributions

The distributional form is defined by integration with test functions. A distribution generated by a locally integrable function $f(x)$ is written as

$$\langle f, \varphi \rangle_\Omega = \int_\Omega f(x) \varphi(x) dx,$$

for all $\varphi(x)$, where $\varphi(x)$ is a test function defined as an infinitely differentiable function with compact support. However, distributions can also be generated from pseudo-functions such as the Heaviside step function or the delta function (El Kinani and Oudadess, 2010).

A major advantage of using a distributional formulation of differential problems is that it allows the use of singularities and other idealised objects. The choice of test function effectively sets a resolution below which properties cannot be distinguished, set by the test function’s support.

Singularities arise from the use of idealised objects in a model, such as point electrodes. Idealised objects are very useful as they can significantly simplify the mathematics. An important distinction to make is that they do not exist in reality. What is important in the distributional formulation is the action of an idealised object when integrated with a test function. For example, a point charge and a small distribution
of charge can have the same effect on a measurement when integrated with a test function. Therefore, distributional formulations are more general and realistic than classical formulations.

### Green’s functions

Take a BVP of the form

\[
Lu(x) = f(x), \quad BC(u) = c,
\]  

(3.3)

where \( L \) is a linear partial differential operator, \( BC(u) = c \) is short-hand for the boundary conditions, and \( f(x) \) the source. The fundamental free-space solution to the PDE, \( h(x|\xi) \), is then defined by the equation

\[
Lh(x|\xi) = \delta(x - \xi),
\]

(3.4)

where the boundary is taken to be infinity. A Green’s function, \( g(x|\xi) \), is the solution to the equation

\[
Lg(x|\xi) = \delta(x - \xi), \quad BC(u) = c.
\]

(3.5)

For any self-adjoint \( L \), the solution to the BVP is then given by the convolution

\[
< Lu(x), g(x|\xi) >_\Omega = < f(x), g(x|\xi) >_\Omega  
\]

(3.6)

\[
u(x) = \int_\Omega g(x|\xi) f(\xi) d\xi - \oint_{\partial\Omega} \hat{n} \cdot J(u, g(x|\xi)) ds,
\]

(3.7)

where \( J(u, g(x|\xi)) \) is called the bilinear concomitant.

Applying this to (3.1) there are a few things to note. Firstly, the \( \nabla \) operator is self-adjoint, so (3.6) can be used. Secondly, the homogeneous Neumann boundary condition makes the \( J(u, g(x|\xi)) \) integral vanish. Thus, the solution to the flowmeter BVP is

\[
u(x) = - \int_\Omega g(x|\xi) \nabla \cdot (v(\xi) \times B(\xi)) d\xi.
\]

(3.8)

Thus, the problem of solving a differential equation (3.1) has been turned into solving an integral equation.

There are numerous advantages with solving the BVP using a Green’s function approach. To begin with, the integral operator is bounded and compact, whereas the differential operator is unbounded. Therefore the integral operator can be approximated to arbitrary accuracy with a finite representation, something that is impossible
with the differential operator. This is ideal for computational implementations because computers are finite machines (Stakgold and Holst, 2011).

The Green’s function can be constructed with the free-space fundamental solutions for simple geometries using the method of images. For homogeneous Neumann conditions for a circular region or infinite plane the Green’s function is

\[ g(x|\xi) = h(x|\xi) + h(x|\xi_r), \]  

(3.9)

where \( \xi_r \) is either \( \xi \) reflected about the plane, or in the case of a circular region \( \xi_r = (r_r, \theta, 0) \) with \( r_r = \frac{R^2}{r} \) and \( R \) the circle radius.

### 3.1.2 Dipolar sources

The purpose of a flowmeter is to make measurements of the bulk flow rate. To do this, a simpler map from velocity to potential is more useful than the convolution of a divergence of a cross product. For this purpose, equation (3.8) can be further simplified through integration by parts, moving the derivative to \( g(x|\xi) \):

\[ u(x) = \int_{\Omega} \nabla g(x|\xi) \cdot (v \times B) d\xi - \oint_{\partial\Omega} g(x|\xi)(v \times B) \cdot \hat{n} ds. \]  

(3.10)

Equation (3.10) can be described as two parts: a dipolar term, and a monopolar surface term. The function \( g(x|\xi) \) is sometimes called the monopolar Green’s function because it is generated from the source \( \delta(x - \xi) \). This corresponds with a monopolar charge with Poisson’s equation in electrostatics. Thus the second integral in (3.10) corresponds with a surface of source \( (v \times B) \cdot \hat{n} \) convolved with a monopolar Green’s function.

The first integral in (3.10) corresponds to the source \( (v \times B) \) convolved with the dipolar Green’s function \( \nabla g(x|\xi) \). The dipolar Green’s function is the solution to the BVP with a dipolar source pointed in an arbitrary direction \( a: a \cdot \nabla \delta(x - \xi) \). This is analogous to an electrostatic dipolar charge and in the case of the flowmeter equation, the dipoles are pointed in the direction \( (v \times B) \).

The expansion of (3.10) provides an insight into the flowmeter system. Each infinitesimal \( (v \times B) d\xi \) element can be viewed as an electrostatic dipole of strength \(||(v \times B)||\). Each positive charge in a dipolar distribution is effectively cancelled out by a neighbouring negative charge, excepting the boundary where one half of the dipole goes uncancelled. This leads to the surface monopole charge distribution. The effective cancellation means that the dipoles are harder to detect at distance, especially when the measurements are not aligned with the dipole. These traits are discussed in Section 42.
Figure 3.2: Depictions of the approximate potential (a) and the electric field (b) of an electrostatic dipole, generated by two close monopoles. Sources: https://en.wikipedia.org/wiki/File:Dipole_Contour.svg and https://commons.wikimedia.org/wiki/File:VFPT_dipole_electric_manylines.svg (Used under the Creative Commons Attribution-Share Alike 3.0 Unported licence)

3.1.4.

From here on, the velocity is assumed to be pointed in the $z$ direction and the magnetic field will be orthogonal to $z$. Denoting $v = (0, 0, v_z)$ and $B = (B_x, B_y, 0)$, (3.10) becomes

$$u(x) = \int_{\Omega} v_z \nabla g(x|\xi) \cdot (\hat{y}B_x - \hat{x}B_y) d\xi - \int_{\partial\Omega} v_z g(x|\xi)(\hat{y}B_x - \hat{x}B_y) \cdot \hat{n} ds.$$  \hfill (3.11)

From this point onward, let $B_x = 0$. The above equation is linear and therefore the work in this thesis easily extends into more complicated magnetic fields. Thus, (3.11) becomes

$$u(x) = -\int_{\Omega} v_z B_y \frac{\partial}{\partial x_1} g(x|\xi) d\xi + \int_{\partial\Omega} v_z B_y g(x|\xi) \hat{x} \cdot \hat{n} ds.$$  \hfill (3.12)

An important note: the vast majority of the EMFT literature disregards the surface integral. By referencing the no-slip condition from fluid dynamics one can set $v_z = 0$ at the boundary. This assumption is used in the mathematical model but not followed through with in the computational implementation (Lehtikangas et al. (2016); Kollár et al. (2014)), which is explored in Section 3.1.5 and Section 6.2.2.
3.1.3 Idealised aquifer model

The initial proposed model for the NSC outdoor aquifer measurement is an infinite unconfined aquifer. The geometry of which is a half-space divided by the ground air interface at \( y = 0 \). Below the ground there is a fluid flowing with constant velocity in the \( z \) direction and a constant magnetic field exists in the \( y \) direction. The direction of the potential gradient is therefore in the \( x \) direction. A potential difference is measured between two point electrodes placed \( \eta \) beneath the ground and separated by \( 2\epsilon \) in \( x \). Their locations are then \((\epsilon, -\eta, 0)\) and \((-\epsilon, -\eta, 0)\). This is shown in Figure 3.3.

![Figure 3.3: The idealised half-space model. A constant velocity field comes out of the page (blue dotted-circle) and a constant magnetic field is in the \( y \) direction (orange arrow). A voltage measurement is made between two point electrodes immersed in the half-space region \( \Omega \).](image)

The hypothesis was that the measurement would be proportional to the fluid’s velocity.

Suppose we have the above situation, but with \( B \) and \( v \) limited in \( x \) from \([-l, l]\). This can be modelled as a 2D system defined by the plane \( z = 0 \) because the fields are constant towards \( \pm \infty \) in \( z \). There is a Neuman boundary condition at \( y = 0 \) due to the lack of current travelling through the ground-air interface. This can then be solved using the method of images where the Green’s function is defined as

\[
g(x|\xi) = h(x|\xi) + h(x|\xi_r),
\]

(3.13)

with \( \xi_r = (\xi_1, -\xi_2) \). The solution to the flowmeter equation for a constant \( B_y \) and \( v_z \)
is the given by

\[ u(x) = -B_y v_z \int_{-l}^{l} \frac{\partial}{\partial x} g(x|\xi) \, d\xi_2 \, d\xi_1 + B_y v_z \int_{-l}^{l} g(x|\xi) \hat{x} \cdot \hat{y} \, d\xi_1 + B_y v_z \int_{-l}^{l} g(x|\xi) \, d\xi_2 \, d\xi_1 \]  \tag{3.14}

\[ u(x) = -B_y v_z \int_{-\infty}^{\infty} \frac{\partial}{\partial x} h(x|\xi) \, d\xi_2 \, d\xi_1. \]  \tag{3.15}

The surface integral is zero because the surface is orthogonal to the constant source vector. Additionally, the method of images effectively extends the integral to \(-\infty \) in \( y \).

The free-space fundamental solution has the property that \( h(x|\xi) \to 0 \) as \( ||x|| \to \infty \).

Taking a constant shift in variables \( y' = y + \eta \) is free due to the unitary Jacobian and the convergence to \( \pm \infty \). Therefore, the system can be represented as a 1D system by integrating out the \( y \) dimension.

Imposing the conditions that \( u(\pm l) = 0 \), the 1D Greens function is the distributional solution to the equation

\[ \frac{d^2}{dx^2} g(x|\xi) = \delta(x - \xi), \]  \tag{3.16}

which is in general

\[ g(x|\xi) = (x - \xi) H(x - \xi) + ax + b. \]  \tag{3.17}

Solving for the boundary conditions yields

\[ g(x|\xi) = \begin{cases} \frac{\xi - l}{2} (\xi + 1) & \text{if } x \leq \xi, \\ \frac{\xi - l}{2} (\xi + 1) + x - \xi & \text{if } x \geq \xi. \end{cases} \]  \tag{3.18}

The derivative of which is

\[ \frac{d}{dx} g(x|\xi) = H(x - \xi) + \frac{\xi - l}{2l}. \]  \tag{3.19}

The potential at point \( x \) is then given by

\[ u(x) = -B_y v_z \int_{-l}^{l} \frac{\partial}{\partial x} g(x|\xi) \, d\xi \]  \tag{3.20}

\[ = -B_y v_z \left( \int_{-l}^{x} \frac{\xi + l}{2l} \, d\xi + \int_{x}^{l} \frac{\xi - l}{2l} \, d\xi \right) \]  \tag{3.21}

\[ = B_y v_z x. \]  \tag{3.22}
To match the infinite unbounded aquifer case let $l \to \infty$, but the solution is independent of $l$ and is thus unchanged. The potential difference is

$$V = u(\epsilon) - u(-\epsilon) = 2\epsilon B_y v_z. \quad (3.23)$$

Therefore the voltage is directly proportional to the fluid’s velocity in this highly idealised situation, confirming the initial hypothesis.

The next level of approximation is using a limited magnetic field. Let the geometry be as above, but the magnetic field is now the rectangular top hat

$$B(\xi) = \begin{cases} 
B & \text{for } -b \geq \xi_1 \geq b \text{ and } 0 \geq \xi_2 \geq -c \\
0 & \text{otherwise},
\end{cases} \quad (3.24)$$

shown in Figure 3.4.

The potential is then

$$u(x) = -B_y v_z \int_{-b}^{b} \int_{-c}^{c} \frac{\partial}{\partial \xi_2} g(x|\xi) \, d\xi_2 \, d\xi_1 + B_y v_z \oint g(x|\xi) \hat{x} \cdot \hat{n} \, ds. \quad (3.25)$$

However, as $g(x|\xi) = g(\xi|x)$ for a self-adjoint differential operator, then $\frac{\partial}{\partial \xi_1} g(x|\xi) = -\frac{\partial}{\partial \xi_2} g(x|\xi)$. Thus,

$$u(x) = B_y v_z \int_{-b}^{b} \int_{-c}^{c} \frac{\partial}{\partial \xi_1} g(x|\xi) \, d\xi_2 \, d\xi_1 + B_y v_z \oint g(x|\xi) \hat{x} \cdot \hat{n} \, ds \quad (3.26)$$

$$= 2B_y v_z \int_{-c}^{c} \left[ g(x|\xi) \right]_{\xi_1=-b}^{\xi_1=b} \, d\xi_2, \quad (3.27)$$
where the Greens function is \(g(x|\xi) = (2\pi)^{-1}(\ln(||x - \xi||) - \ln(||x - \xi_r||))\) and \(\xi_r = (\xi_1, -\xi_2)\). This leads to the solution

\[
u(x) = \frac{B_y v_z}{\pi} \left( 2x_2 \ln \left( \frac{(x_1 + b)^2 + x_2^2}{(x_1 - b)^2 + x_2^2} \right) - (x_2 + c) \ln \left( \frac{(x_1 + b)^2 + (x_2 + c)^2}{(x_1 - b)^2 + (x_2 + c)^2} \right) \right.

\[
\left. - (x_2 - c) \ln \left( \frac{(x_1 + b)^2 + (x_2 - c)^2}{(x_1 - b)^2 + (x_2 - c)^2} \right) + (x_1 - b) \left( 4 \arctan \left( \frac{x_1 - b}{x_2} \right) \right) \right.

\[
\left. - 2 \arctan \left( \frac{x_1 - b}{x_2 + c} \right) - 2 \arctan \left( \frac{x_1 - b}{x_2 - c} \right) \right) - (x_1 + b) \left( 4 \arctan \left( \frac{x_1 + b}{x_2} \right) \right) \right) \right) \right)

(3.28)

The potential difference between the electrodes is then

\[
V = u(\epsilon) - u(-\epsilon) = 2u(\epsilon) > 2B_y v_z \epsilon,
\]

(3.29)

and is shown in Figure 3.5 for the case where \(b = 0.5\) m, \(c = 1\) m, and \(x_2 = -0.5\) m. These values are the approximate order of measurements from using the mini-aquifer setup. The one notable difference is the voltage being larger than the infinite half-space flow. This is due to the uncancelled dipolar charges on the boundary of the magnetic field leading to a monopolar surface charge. The contributions from the volume integral are less than in the infinite half-space case, but with the monopolar contribution the overall measurement is larger. Thus, limiting the field makes the measurement easier.
than the idealised half-space case.

The idealised mini-aquifer is similar to the previous case. For a square of length $l$, the potential at a point $x$ is given by

$$u(x) = -f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} g(x|\xi) \, d\xi_2 \, d\xi_1 + f \oint g(x|\xi) \hat{x} \cdot \hat{n} \, ds$$  \tag{3.30}$$

$$u(x) = xf - f \int_{-l}^{0} g(x|\xi) \, d\xi_2 \bigg|_{\xi_1 = -\frac{l}{2}} + f \int_{-l}^{0} g(x|\xi) \, d\xi_2 \bigg|_{\xi_1 = \frac{l}{2}},$$  \tag{3.31}$$

where the 2D free-space Green’s function is

$$g(x|\xi) = \frac{1}{2\pi} \ln \left( \sqrt{(x_1 - \xi_1)^2 + (x_1 - \xi_2)^2} \right).$$  \tag{3.32}$$

A single surface integral is resolved as follows

$$f \int_{-l}^{0} g(x|\xi) \, d\xi_2 = \frac{f}{2\pi} \int_{-l}^{0} \ln \left( \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2} \right) \, d\xi_2$$  \tag{3.33}$$

$$= \frac{f}{2\pi} \left[ (x_2 - \xi_2) \left( 1 - \frac{1}{2} \ln \left( (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 \right) \right) - (x_1 - \xi_1) \arctan \left( \frac{(x_2 - \xi_2)}{(x_1 - \xi_1)} \right) \right]_{\xi_2 = 0}.$$  

Choose $\xi_1 = \pm \frac{l}{2}$ with the sign depending on which surface is being evaluated and $x_2 = -\frac{l}{2}$. The integral then becomes

$$f \int_{-l}^{0} g(x|\xi) \, d\xi_2 =$$  \tag{3.34}$$

$$\frac{f}{2\pi} \left[ -l + \frac{l}{2} \ln \left( (x_1 + \frac{l}{2})^2 + \frac{l^2}{4} \right) + \left( x_1 + \frac{l}{2} \right) \arctan \left( \frac{l}{2(x_1 + \frac{l}{2})} \right) \right],$$

and the potential is

$$u(x) = xf +$$  \tag{3.35}$$

$$\frac{f}{2\pi} \left( \frac{l}{2} \ln \left( (x - \frac{l}{2})^2 + \frac{l^2}{4} \right) + (x - \frac{l}{2}) \arctan \left( \frac{l}{2(x - \frac{l}{2})} \right) - \left( x + \frac{l}{2} \right) \arctan \left( \frac{l}{2(x + \frac{l}{2})} \right) \right).$$  

Finally, the potential difference measurement between two electrodes at $(\epsilon, -\frac{l}{2})$ and $(-\epsilon, -\frac{l}{2})$ is given by

$$V = 2\epsilon f +$$  \tag{3.36}$$

$$\frac{f}{\pi} \left( \frac{l}{2} \ln \left( (\epsilon - \frac{l}{2})^2 + \frac{l^2}{4} \right) + (\epsilon - \frac{l}{2}) \arctan \left( \frac{l}{2(\epsilon - \frac{l}{2})} \right) - (\epsilon + \frac{l}{2}) \arctan \left( \frac{l}{2(\epsilon + \frac{l}{2})} \right) \right),$$  

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Figure 3.6: Normalised potential difference against separation distance between the electrodes. The red solid line is the idealised mini-aquifer measurement and the blue dotted line is the idealised unbounded aquifer measurement.

shown in Figure 3.6 for $l = 1$.

The idealised situation for the mini-aquifer is not the same as the unbounded case and has a greater signal due to the boundaries. The signal is even stronger than in the limited magnetic field case too due to the fact that the fields are effectively constant in the $y$ direction for the mini-aquifer due to the boundary. However, the idealised potential difference is proportional to the fluid velocity and is also approximately linear (see Figure 3.6). The linear proportionality is not surprising upon examining (3.30). This result follows directly from the assumption that $v_z$ is constant, allowing it to be taken outside the integrals.

### 3.1.4 3D pipe solution

Let $\Omega$ be a circular pipe and the basis aligned accordingly. The measurement plane is the plane normal to the $z$ axis, crossing the origin. Point electrodes are located where the boundary $\partial\Omega$ meets the measurement plane.

In three dimensions the free-space fundamental solution for Poisson’s equation is

$$h(x|\xi) = \frac{-1}{2\pi||x - \xi||}. \quad (3.37)$$

The Green’s function for a cylinder is then

$$g(x|\xi) = \frac{-1}{2\pi||x - \xi||} + \frac{-1}{2\pi||x - \xi_z||}, \quad (3.38)$$
where $\xi = (r, \theta, z)$ and $\xi_r = \left( \frac{R^2}{r}, \theta, z \right)$. From here on the rectilinear coordinates of $x$ and $\xi$ will be denoted by subscript numbers to avoid confusion (e.g. $y \to x_2$). Both vectors are from the same space, so the axes will remain denoted as the $x$, $y$, and $z$ axes. The derivative is given by

$$
\frac{\partial}{\partial x_1} g(x|\xi) = - \frac{\partial}{\partial x_1} \left( \frac{1}{2\pi||x - \xi||} + \frac{1}{2\pi||x - \xi_r||} \right) \\
= \frac{x_1 - \xi_1}{2\pi||x - \xi||^3} - \frac{1}{2\pi} \frac{\partial}{\partial x_1} \left( x_1^2 + x_2^2 + \frac{R^4}{r^2} - \frac{2R^2}{r}(x_1 \cos(\theta) + x_2 \sin(\theta)) \right)^{-\frac{1}{2}} \\
= \frac{x_1 - \xi_1}{2\pi||x - \xi||^3} + \frac{x_1 - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^3} \\
= \frac{R \cos(\phi) - x \cos(\theta)}{2\pi||x - \xi||^3} + \frac{R \cos(\phi) - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^3}. 
$$

The integral (3.11) can be evaluated to arbitrary accuracy by discretising and integrating with Gaussian quadrature.

The dipolar Green’s function in (3.40) decays like $r^{-2}$ in the $x$ direction, and goes like $r^{-3}$ in any other direction. This decay is very rapid when compared to the monopolar Green’s function in (3.38), which goes like $r^{-1}$ in all directions. This is due to the effect talked about in Section 3.1.2 where the positive and negative charges effectively cancel each other out when looking at a distance. The decay rate means that the flowmeter measurement system lacks the ability to penetrate deep into a flow. In other words: the measurements are almost entirely dependent on the flow around the electrode.

Figure 3.7: The pipe measurement plane. The domain, $\Omega$, is of a cross-sectional slice of a circular pipe with radius $R$. This is bounded by the pipe wall $\partial \Omega$. 

\[ \begin{array}{c}
\frac{\partial}{\partial x_1} g(x|\xi) = - \frac{\partial}{\partial x_1} \left( \frac{1}{2\pi||x - \xi||} + \frac{1}{2\pi||x - \xi_r||} \right) \\
= \frac{x_1 - \xi_1}{2\pi||x - \xi||^3} - \frac{1}{2\pi} \frac{\partial}{\partial x_1} \left( x_1^2 + x_2^2 + \frac{R^4}{r^2} - \frac{2R^2}{r}(x_1 \cos(\theta) + x_2 \sin(\theta)) \right)^{-\frac{1}{2}} \\
= \frac{x_1 - \xi_1}{2\pi||x - \xi||^3} + \frac{x_1 - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^3} \\
= \frac{R \cos(\phi) - x \cos(\theta)}{2\pi||x - \xi||^3} + \frac{R \cos(\phi) - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^3}. 
\end{array} \]
Length scales

The above results also provide qualitative measures for the 2D flowmeter approximation, where the fields are approximately constant in $z$. The 2D approximation is gained by integrating the constant sources to $\pm \infty$ along the $z$ axis to make the 3D Green’s functions into 2D ones. However, the contribution to measurements from the source decays rapidly. The majority contribution to this integral is in the neighbourhood of the measurement plane. Therefore the 2D approximation is valid if the fields are approximately constant over a chosen length, $a$, away from the measurement plane. The error of this approximation can be defined in general by the difference

$$
\left| \bar{f} \int_{-\infty}^{\infty} g(x|\xi) d\xi_3 - \int_{-a}^{a} g(x|\xi) f(\xi) d\xi_3 \right|, \quad (3.41)
$$

where $\bar{f}$ is the average of the source $f(\xi)$. The choice of $a$ is completely subjective, but it is possible to justify the choice using (3.38) and (3.40).

If the setup is exactly as above, then the potential generated by a delta source decays in the $z$ direction by $r^{-1}$ if it is on the boundary, or $r^{-3}$ if not. If the source lies away from the measurement plane then the measurement will be either approximately $z^{-1}$ or $z^{-3}$ weaker than one in the 2D plane, where $z$ is the distance from the plane. The measurable quantity of the electric field will decay like $z^{-2}$ as it is the gradient of the potential. Thus, surface charge sources will be approximately 4 times less measurable than those half the distance away.

In contrast to this, if there is no flow on the boundary, as the no-slip condition states, then the decay of the potential goes like $z^{-3}$ and the electric field like $z^{-4}$. Thus the potential from sources will be 8 times less than those half the distance away and the electric field will be 16 times less.

Another length scale argument comes from the homogeneous form of the flowmeter equation: $\nabla^2 u = 0$ over the region $z = [0, \infty)$. If we take a square pipe which has a plane of charge at $z = 0$ and insulating walls, then the solution to this problem can be found using separation of variables. The solution is the product of three functions

$$
X(x) = c_1 \cos(k\pi x) + c_2 \sin(k\pi x), \quad Y(y) = c_3 \cos(l\pi y) + c_4 \sin(l\pi y), \quad (3.42)
$$

$$
Z(z) = c_5 \exp\left( -\sqrt{\frac{(k\pi)^2}{R^2} + \frac{(l\pi)^2}{R^2} z} \right), \quad (3.43)
$$

for $k, l = 1, 2, 3, \ldots$, where $R$ is the width of the square pipe and the $c$s are constants. Ignoring $X$ and $Y$, the slowest rate of decay of the exponential is when $k = l = 1$. This
reaches the 5% mark at \( z = \frac{3R}{\sqrt{2\pi}} \approx 0.68R \). While this is for a square pipe as opposed to a circular one, the cases are very similar. The circular pipe involves Bessel functions and does not readily provide as clean results.

To conclude and provide the subjective cut-offs used in this thesis: if there is no flow on the boundary, then the 2D flowmeter approximation is valid if the fluid flow and magnetic fields are approximately constant up to \( R \) away. If there is boundary flow, then this should be increased to \( 2R \) to be safe. In practical applications it is often the case that the magnetic field is less likely to meet this criterion due to the difficulty and cost of creating constant magnetic fields over large areas. The velocity can meet it by extending the straight pipe length before and after the measurement plane. From 2.3.2, if it is the magnetic field that does not meet this criterion then one would experience end-shortening which has the effect of decreasing the measured signal (Shercliff, 1962).

Problems

The full 3D model of a pipe flowmeter may allow for a more accurate model of the system, however there is a decided increase in complexity moving from 2D to 3D. In Section 3.3 the discretisation of the current continuous theory will be discussed. What is relevant here are the two following points.

In 2D, the number of discrete elements are about 1000 whereas the 3D case goes up to the tens of thousands. This leads to the computational operations involving square matrices with size of the order \( 10,000^2 \times 10,000^2 \). This severely increases the computational time required to calculate solutions for the inverse problem.

As this work involves finding the bulk flow rate of a fluid in a pipe, this discretisation of the 3D system will require additional procedures in order to: a) allow for an easy cross-sectional integral to find the bulk flow rate; b) guarantee conservation of flow between all discrete cells; and c) implement some fluid dynamic properties, such as incompressibility. These added considerations would require a large increase in the complexity of both the mathematical and numerical models of this system.

A further point is that the 3D model will necessarily require the addition of another boundary (in the \( z \) direction) because computers are finite. This raises the question of how to implement that boundary, what conditions to use, etc. The magnetic field would also have to be modelled more realistically than “approximately constant in \( z \) near the measurement plane”.

For all of these reasons, it was decided that the 3D model of the flowmeter has small
payoff compared to the excessive increase in work required to implement it. The main contribution in this work is the Bayesian formulation, not the forward map. Thus, the 3D model is examined only as a way to further understand the approximations and consideration required for the 2D model.

3.1.5 2D solution

The 2D model has the 3D model’s space restricted to the measurement plane as displayed in Figure 3.7. The free-space fundamental solution of Poisson’s equation in 2D is

\[ h(x|\xi) = \frac{1}{2\pi} \ln(|x - \xi|), \]  

(3.44)

and thus the Green’s function for this circular pipe is

\[ g(x|\xi) = \frac{1}{2\pi} \left( \ln(|x - \xi|) + \ln(|x - \xi_r|) \right), \]  

(3.45)

where \( \xi = (r, \theta) \in \Omega, \xi_r = (r_r, \theta) \notin \Omega \) with \( r_r = \frac{R^2}{r} \), and the measurement location \( x = (R, \phi) \in \partial\Omega \). Similar to (3.40), this differentiated is

\[ \frac{\partial}{\partial x_1} g(x|\xi) = \frac{1}{2\pi} \frac{\partial}{\partial x_1} \left( \ln(|x - \xi|) + \ln(|x - \xi_r|) \right) \]  

(3.46)

\[ = \frac{x_1 - \xi_1}{2\pi||x - \xi||^2} + \frac{x_1 - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^2}. \]  

(3.47)

As \( x = (R, \phi) \)

\[ \frac{\partial}{\partial x_1} g(x|\xi) = \frac{R \cos(\phi) - r \cos(\theta)}{2\pi||x - \xi||^2} + \frac{R \cos(\phi) - \frac{R^2}{r} \cos(\theta)}{2\pi||x - \xi_r||^2}. \]  

(3.48)

Equation (3.11) can once again be approximated with arbitrary accuracy on a discretisation using Gaussian quadrature. However care must be taken in practice when setting up the boundary integral as the measurements are also on the boundary and thus it is an integration over singularities. These are integrable \( \ln(|x|) \) type singularities, but can cause problems when using quadrature.

For example, the integral of \( \ln(|x|) \) in one dimension from 0 to 1 can be evaluated analytically to yield the value of -1 and can be approximated well with Gaussian quadrature. In contrast, the integral from -1 to 1 does not work for any odd order quadrature and takes longer to converge. Different levels of Gaussian quadrature approximating this over different intervals are presented in Table 3.1.

This is an extreme example as the quadrature points in the second case land on the singularity. However, it does serve the purpose of showing that one must be careful.
Table 3.1: Gaussian quadrature of integrable singularities

| Gaussian quadrature | $\int_0^1 \ln(|x|)dx = -1$ | $\int_{-1}^1 \ln(|x|)dx = -2$ | $\int_{-0.5}^1 \ln(|x|)dx = -1.85$ |
|---------------------|-----------------------------|-----------------------------|-------------------------------|
| 1 point             | -0.69                       | $-\infty$                   | -2.08                         |
| 2 points            | -0.90                       | -1.40                       | -1.46                         |
| 3 points            | -0.94                       | $-\infty$                   | -2.20                         |
| 4 points            | -0.97                       | -1.51                       | -3.19                         |

when setting up discretisation and numerical integration over singularities. Notably in the third column on Table 3.1 which shows no sign of convergence. This further highlights that one should not naively numerically integrate across an integrable singularity. If possible, singularities should be analytically integrated.

**Square pipe**

The BVP for the square pipe case can also be solved using the method of images. Let the setup be the same as with the circular pipe above, only with $\Omega$ being a square domain with sides aligning with the rectilinear coordinate system and the centre of the square at the origin. Each insulating wall can be treated as an infinite plane with a Neumann boundary condition.

The Green’s function is the fundamental free-space solution, plus an equal image source reflected on the opposite side of the plane. If the plane is located $R/2$ from the origin in the $x$ direction and aligned such that the normal of the plane is also in the $x$ direction, then the Green’s function is given by

$$ g(x|\xi) = h(x|\xi) + h(x|\xi_r), $$

(3.49)

where $\xi_r = (R - \xi_1, \xi_2, \xi_3)$.

If there is a similar plane located at $x_1 = -R/2$, then that reflection applies to the previous image source too. This continues forever, thus the solution is an infinite series and must be either truncated using a scaling argument or the convergent sum must be evaluated.

The square pipe consists of two orthogonal sets of planes located $R/2$ distance away from the origin in the $x$ and $y$ directions respectively. Therefore the Green’s function
for a source at $\xi$ is given by

$$g(x|\xi) = \frac{1}{2\pi} \left( \ln(||x - \xi||) + \sum_{l} \sum_{k} \ln(||x - \xi_{l,k}||) \right),$$

where $l, k \in \mathbb{Z} \setminus \{0\}$ and $\xi_{i,j} = ((-1)^i \xi_1 + lR, (-1)^k \xi_2 + kR)$. I truncate this sum using the length scale arguments from Section 3.1.4 and only use the first two layers of reflections. The derivative of this is trivial.

**Approximations in the mathematical model**

For convenience’s sake, here is a summary of all the approximations used in the mathematical model.

**Flowmeter equation:**

- **Ohmic fluid:** The flowmeter equation requires that the fluid is an Ohmic resistor. Many common fluids are Ohmic until the potential difference induces electrolysis Shercliff (1962).

- **Constant conductivity:** The assumption of constant conductivity in a fluid simplifies the differential equation $\nabla \cdot (\sigma \nabla u) = -\nabla \cdot (\sigma v \times B)$ into an easier Poisson-style equation. This assumption is valid for liquid flow of a single type. If solids, bubbles, or different unmixed-fluids are introduced, this assumption breaks down.

- **Pseudo-static system:** The system is assumed to be unchanging in time. While some use the word “stationary” to describe this, stationary is often used to describe constant oscillatory systems like those describe by the Helmholtz equation with radiative effects due to the changing fields. There are no oscillations in this system\(^2\), but the system does have a moving fluid. Thus, I use the term “pseudo-static”. The system is technically static, as the abstract mathematical representation used to denote the velocity is static.

- **Non-magnetic fluid:** The fluid must not be a magnetic fluid. Work exists without this assumption in the field of magneto-hydrodynamics.

- **Low self-induction:** While not necessary for the flowmeter equation, it is often assumed that the magnetic field generated by the induced current is negligible. This is an appropriate assumption for the majority of aqueous solutions. E.g. Williams’ with a copper sulphate solution (Williams, 1930).

\(^2\)or rather, the oscillations in the magnetic field are time-averaged out in practical implementations.
2D flowmeter model:

- **Constant in $z$:** The 2D projection requires the various fields to be approximately constant in $z$ for a reasonable distance from the measurement plane. The magnetic field falling off early results in end-shortening and lower signals (Shercliff, 1962), while the velocity field changing near the plane due to upstream disturbances can ruin the signal (Williams, 1930). I have chosen the “reasonable distance” to be $2R$.

- **Point electrodes:** The electrodes are assumed to be infinitesimal points with no contact impedance and no current draw. This is not a good assumption. It is effectively a way to conceptually pick out the potential at a single point. Real contact electrodes draw current, introduce an impedance into the potential measurement, and are highly sensitive to geometry; including skin oil and minute defects in the electrode itself. However, for the sake of understanding the EMFT system and the inverse problem point electrodes are a fine assumption. A warning for the real-world implementation of this mathematical model — results will likely differ due to this.\(^3\)

- **Electrically insulated pipe:** The pipe does not allow any current to flow through the boundary. Many pipes are made with insulating materials, so this assumption has a physical basis\(^4\). This assumption is compatible with point electrodes, however not with real contact electrodes as they will draw current. A more complicated boundary must be used if one wishes to model electrodes properly.

Another assumption that is often used in the literature is the “no-slip” condition from fluid dynamics. This assumption further simplifies the solution to the flowmeter equation by eliminating the boundary integral in (3.11) by setting $v|_{\partial \Omega} = 0$. The effect of doing this can change the measurement significantly. In the majority of this work I avoid this assumption for a number of reasons.

Firstly, the range of the boundary layer can vary based on the fluid’s viscosity and the level of pressure in a system (Ferziger and Peric, 2012). While the no-slip condition does hold in all (but super-) fluids, the so-called boundary layer due to this property can vary in scale significantly (Kundu et al., 2008). The discrete representation used to

\(^3\)As a side note, Shercliff investigated so-called long electrodes, designed to conform more with the “constant in $z$” assumption, as small electrodes don’t.

\(^4\)The theory exists for conducting pipes in (Shercliff, 1962).
simulate the flow may not be fine enough to capture the boundary layer, particularly in more complex flows fields.

The boundary layer approximation also assumes that the flow is slow and laminar (Kundu et al., 2008). This contradicts some of the flow fields of interest presented in work such as (Lehtikangas et al., 2016). Flows used in that paper such as “flow after a pipe elbow” and “solids-in-water flows in inclined pipe” would have a degree of turbulence and pressure difference making it difficult to predict the scope of the discrete elements required to capture the boundary layer.

An inconsistency can be found in many papers that make this assumption, such as in (Lehtikangas et al., 2016) and (Kollár et al., 2014). The flow states chosen as the solutions to the inverse problem presented in these works do not conform to this assumption by having a non-zero flow at the boundary. Effectively, the predicted flow states lie outside of the assumed space of possible states. Care was not taken to numerically implement this assumption. This will be further explored in Chapter 6.

Another reason is that of scale. Previously I wrote about the dipolar terms decaying like $\frac{1}{r^2}$ or $\frac{1}{r^3}$ in 3D and the monopolar term in the surface density decays like the slower $\frac{1}{r}$. A surface source will contribute the same as a dipolar source half as distant to an electrode at best. This property is compounded by the fact that the electrodes are on the edge of the pipe. Overall this results in the potential contribution of the surface integral being an entire order of magnitude larger, as shown in Figure 3.8.

Aside: boundary element method

Green’s function in more complicated geometries where the free-space fundamental solution is known can be computed using the boundary element method. While I won’t go into detail here because the method of images is sufficient, a description of this method is presented in (Ang, 2008).

### 3.2 Measurement model

As previously mentioned, the measurements used in this thesis are point electrodes placed on the boundary of the pipe. However, it is impossible to measure the potential directly and potential differences must be measured. The measurement in standard EM flowmeters is the difference between a single pair of electrodes at locations $x_a$ and
Figure 3.8: Scatter plots of circular pipe measurement locations from two measurement sets, both generated from the same phantom flow profile. The colour map displays the measured potential of the corresponding electrode. Figure 3.8a includes the boundary term in the forward map, whereas 3.8b does not. There is almost an order of magnitude difference the corresponding measurements. These figures are generated by methods described in Section 3.3.

$x_b$ and can be described as such

\begin{align}
d &= V = u(x_a) - u(x_b) \\
&= - \int_{\Omega} B_y(\xi)v_z(\xi) \left( \frac{\partial}{\partial x_1} g(x_a|\xi) - \frac{\partial}{\partial x_1} g(x_b|\xi) \right) d\xi \tag{3.52} \\
&= \int_{\Omega} f(\xi)k(x,\xi) d\xi, \tag{3.53}
\end{align}

where $d$ is the measurement data (in this case: volts, $V$), and $k(x,\xi)$ is the measurement kernel — it is the functional which maps the source $f(\xi) = B_y(\xi)v_z(\xi)$ to a real number. The kernel is analogous to Shercliff and Bevir’s weight function $W$, only the magnetic field is not included in the kernel, but in the source. I have also excluded the surface integral partly to match (Bevir, 1970), but mostly for brevity’s sake as the process trivially extends.

Equation (3.51) is not the full picture as the measurement process has associated measurement noise. Noise can arise from a number of sources including shot noise, background radiation, fluctuations in the system, etc. Even assuming that there are no biases in the measurement it is assumed that there will be unpreventable noise. This is taken to be independent and identically distributed (iid) additive noise. The
noise is assumed to be Gaussian distributed with the standard deviation $\gamma^{-1}$, where $\gamma$ is the precision. EMFT noise is likely to be Gaussian distributed in reality due to the central limit theorem and the time averaging involved in the measurement.

The full measurement process of (3.51) is, therefore,

$$d = u(x_a) - u(x_b) + n$$

$$= - \int_\Omega f(\xi) \left( \frac{\partial}{\partial x_1} g(x_a|\xi) - \frac{\partial}{\partial x_1} g(x_b|\xi) \right) d\xi + n, \quad (3.55)$$

where the additive noise is $n \sim \text{iid } N(0, \gamma^{-1})$.

In general, the measurement process is a linear map

$$d = M(u(x)) + n, \quad (3.56)$$

where $M$ is a linear function and $n \sim \text{iid } N(0, \gamma^{-1}I)$, where $I$ is the identity matrix.

The measurement process used in the majority of this thesis is a generalisation of (3.54) to more electrodes. Because only differences of potentials can be measured, the rank of the data is one less than the number of electrodes used. One might choose one electrode to be a reference, but it was found that this can bias the likely flow states towards those with no flow near said electrode.

The rank of the measurement data is instead reduced by subtracting the average potential from each individual measurement. This matches reality as the potential can be determined up to a constant. Thus, the measurement process can be described by

$$d_m = u(x_m) - \frac{1}{N} \sum_{n=1}^{N} u(x_n) + n_m$$

$$= - \int_\Omega f(\xi) \left( \frac{\partial}{\partial x_1} g(x_m|\xi) - \frac{1}{N} \sum_{n=1}^{N} \frac{\partial}{\partial x_1} g(x_n|\xi) \right) d\xi$$

$$+ \oint_{\partial\Omega} f(\xi) \left( g(x_m|\xi) - \frac{1}{N} \sum_{n=1}^{N} g(x_n|\xi) \right) \hat{x} \cdot \hat{n} ds + n_m, \quad (3.58)$$

where $d_m$ denotes the measurement made at point $x_m$, and $n_m$ the associated iid Gaussian noise.

### 3.2.1 Measurement kernels and sensitivity

This section explores the measurement kernel properties in pipe EMFT. I consider only the volume integral from (3.11) for comparison with the weight function, but also because of the difficulty of viewing the boundary kernel alongside the volume one.
Figure 3.9: Measurement kernel of 2-electrode circular flowmeter. Figure 3.9a is truncated at 10 and 3.9b is truncated at 100.

Figure 3.9 is a plot of the measurement kernel from (3.51). This is of the circular 2-electrode flowmeter like that presented by Shercliff and Bevir (Figure 2.5). Figure 3.9a is the natural log of the kernel truncated at +10, whereas Figure 3.9b is the linear plot of the same function truncated at +100. There are two singularities at the electrode positions where the kernel tends towards infinity.

As stated earlier, these figures are equivalent to the weight function in Figure 2.5 and can be viewed as the amount of contribution that the source has on the measurement. Clearly from Figure 3.9b the measurement system is insensitive in the extreme to anything away from the electrodes themselves. This measurement would be a hopeless endeavour in a standard EM flowmeter without the assumption of axisymmetric flow.

Extending the weight function-style plot beyond 2-electrodes is somewhat troublesome for two reasons. Firstly, the dipolar nature of the system means that any measurement off the x-axis will have some negative contributions; visualisation with log plots is difficult in this respect. Secondly and more importantly, the measurement goes from one number to a vector of differences — either voltage pairs or the measurement from (3.57).

To demonstrate the behaviour of the kernel from (3.57) a 3-electrode system is shown in Figure 3.10. The potential at each electrode has the average potential of all three subtracted from it. Figures 3.10a-3.10c are plots of the three individual kernels for each electrode measurement, truncated at ±100. The relevant electrode in the plot is the one with the positive values surrounding it. The sign of the kernel
indicates whether source in a given region will contribute either positively or negatively to the measurement. Thus, it follows that any single measurement which contains both positives and negatives in the kernel will have a lower signal than that of the standard flowmeter setup in Figure 3.9 for any unidirectional flow. It is the combination of these weaker measurements that can allow for EM flow tomography to exist as each measurement has a very different kernel.

In order to display the proportional contribution which any region of source has on the overall measurement I define a quantity which shall be referred to as the absolute kernel by

$$k_{\text{abs}}(x_1,\ldots,N|\xi) = \sum_{n=1}^{N} |k(x_n|\xi)|,$$  \hspace{1cm} (3.59)
where $N$ is the total number of electrodes, and $|k(x_n|\xi)|$ is the absolute magnitude of an individual measurement kernel. This can be viewed as either the sensitivity of the whole measurement, or as related to the spatially varying level of distinguishability of the source. The absolute kernel is plotted for the 3-electrode system in Figure 3.10d. As can be seen, the dominant parts are the regions near the point electrodes due to the singularities.

Figure 3.10 is a very powerful example for why one must restrict the possible shapes of the flow field, as was discussed in Section 2.4. If the fields have some restrictions, such as denoting the velocity's function like Williams did in his initial investigation, then the problem of the overwhelming difference in sensitivity between the space surrounding the electrodes and the centre of the pipe is mitigated. That is, information about the field near the edge can imply information about the field in the centre.

Figure 3.11 shows the absolute kernel for 7, 8, 12, and 16-electrode systems, with the 16-electrode arrangement in Figure 3.11d displaying the natural log of the absolute kernel. No matter how many electrodes are placed, the kernel only shows high sensitivity near the boundary, with negligible value in the centre of the domain. However, the more electrodes there are, the more coverage the sensitive regions have on the outer section of the pipe. Thus, I conclude unsurprisingly that the more electrodes used the more accurate the tomography is. This trend is limited by practical considerations of electrode size and the interference real electrodes could have with each other.

It is worth noting that the measurement kernel used in (3.57) is not how real-world measurements would be made. Multi-electrode systems make voltage measurements from the electrode pairs. The respective measurement kernels would look more like those in Figure (3.12) than in the absolute kernels. However, the absolute kernel generated from these would look very similar to those above.

For reference, the equivalent plots for a square EM flowmeter are in Figure 3.13. It appears that the placement of the electrodes on the flat would be less effective than placing them on the corners from this model. Hence, I would recommend a diamond arrangement over this square one, where the $x$ and $y$ axes are from corner to corner.

The kernel of the boundary integral term will not be examined here. Firstly for plotting reasons as it is hard to visualise along with the interior kernel. But more importantly, it would not add anything to the argument made in this section: that the system is only sensitive to the region near the electrodes and not the interior of the pipe. This is trivial, as the boundary integral only exists on the boundary by definition.
3.3 Numerical model

The mathematical model is now complete and must be discretised in order to be implemented on a computer. The reason that a computational implementation is required is the difficulty of analytically solving the mathematical problem without taking advantage of the speed of a computer. This is more significant in inverse problems as often many solutions must be generated to either choose the regularisation parameter or effectively sample a distribution (see Chapters 4 and 5).

Additionally, it is clear that information is most likely going to be lost when discretising — when approximating a potentially infinite dimensional system as a finite
Figure 3.12: Measurement kernel of 2-electrode circular flowmeter with off-axis measurements. This is more like the measurement kernels for real-world multi-electrode flowmeters; they make voltage measurements between electrode pairs.

However there is freedom in how one discretises the system, and therefore freedom of choice in exactly what information is lost. Thus, it is essential to consider just what information is important to the problem.

The discretisation of a mathematical model like a BVP has two aspects: the representation of the functions and the operators. However, both the operator and functions do not both need to be discretised. An example of this is approximating a function by a finite Fourier series where the action of differentiation and integration operators can be represented exactly.

The main concerns with the EMFT numerical model are:

1. **Convergence:** That the finite approximation is equivalent to the continuous case in the appropriate limit. Analysis of convergence is also useful to show how close the discrete is to the “true” system, because the numerical model will not be working in the limit. Tests of convergence are in Section 3.3.4.

2. **Numerical artefacts:** Features of the computed solution that are from the numerical implementation, not the mathematical model. One common cause of numerical artefacts can be imposed structures in the numerical approximation, such as a regular square grid. Therefore an unstructured grid is used in this work (Section 3.3.2).

---

5 There are exceptions to this, such as a Gaussian which can be completely described with just two numbers in one dimension.
Figure 3.13: Measurement kernels for 2- and 4-electrode square flowmeters of different arrangements.

3. **Visualisation:** A more minor note, but it is useful to think about how parts of the solution will be accessed and visualised. The 2D geometry is easily viewable.

Stability is a concern in iterative solvers, however the methods in this thesis are not and thus stability is not a concern.

### 3.3.1 Finite element representation

The numerical model of the forward map is only required to calculate the integral (3.12). The structures from a finite element method are a natural way to do this. We resume from Section 3.1.5 and I will reprint the relevant things for convenience.

With $\Omega$ as the internal domain of a pipe and $\partial \Omega$ as the boundary, let $\mathcal{K}$ be a simply-connected, unstructured triangulation on $\Omega$. The source term in the BVP is $f(\xi) = v_z(\xi)B_y(\xi)$, which from here on shall be collectively referred to as the “flow”. Let $f_\mathcal{K} \approx f$ be a piece-wise linear approximation of $f$, where $f_\mathcal{K} = \sum_i f_i \psi_i \in \mathcal{K}$ with the constants $f_i$ and $\psi_i$ is the $i$th basis functions of $\mathcal{K}$ — defined by setting the $i$th node
to 1 and the rest to 0.

The values of $f_i$ can be found by projecting the true $f$ on to the basis. This is done by taking the inner product defined by

$$f_i = \langle f, \psi_i \rangle = \frac{\int_{\Omega} f(\xi)\psi_i(\xi)d\xi}{\int_{\Omega} (\psi_i(\xi))^2d\xi}.$$  \hfill (3.60)

However, for a fine enough triangulation/discretisation this can be accurately approximated as merely setting the nodal values $f_i$ to the same as the function $f$ at the node locations.

Collecting all the $f_i$'s and measurement potentials $u(x_m)$'s into vectors, which shall henceforth be denoted $f$ and $u$ respectively, allows the solution integral (3.11) to be written as

$$u = A_u f,$$  \hfill (3.61)

where $u = \{u(x_m)\}$, $f = \{f_i\}$, and

$$A_u = \left\{ \int_{\Omega} \psi_i \frac{\partial}{\partial x_1} g(x_m|\xi) d\xi - \int_{\partial\Omega} \psi_i g(x_m|\xi) \hat{x} \cdot \hat{n} ds \right\}.$$  \hfill (3.62)

The elements of $A_u$ can be resolved by one’s choice of numerical integration. I approximate the first integral with Gaussian quadrature. The surface integral will be done analytically for the circular pipe due to the difficulty of numerical integration of singularities (from Section 3.1.5). The surface integral will be calculated using quadrature for the square pipe because it is easier to control the locations of the nodes and electrodes due to the rational-sized boundary.

The full measurement process of the measured data vector, $d$, can be written as

$$d = M A_u f = Af,$$  \hfill (3.63)

where $A = MA_u$ is the full forward map and $M$ is the matrix describing the linear map from (3.56).

### 3.3.2 Meshing

The choice in number of elements used for the triangulation mesh is a trade-off between accuracy and computational cost in the finite element approximation of the solution. The numerical forward map is for use in EMFT imaging. The aim of imaging techniques is to have the discretisation sufficiently fine such that the measurement errors are larger than discretisation errors.
The largest source of discretisation error here is the Gaussian quadrature approximation of the integral. Gaussian quadrature of order \( n \) is exact in integrating up to \( 2n+1 \) order polynomials in one dimension. Because the logarithm in the measurement kernel gets smoother the further away from the singularity, the most extreme case of error will be in the elements neighbouring the singularity. Denoting \( \epsilon \) as the distance from a measurement location, in one dimension the series expansion about \( \epsilon \) is

\[
\ln(\epsilon + x) = \ln(\epsilon) - \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k \epsilon^k},
\]

which converges for \(|x| < \epsilon\). The sign alternates for subsequent terms and the function is locally integrable thus convergent. Therefore the discretisation error is capped by the \( s = 2n+2 \) order polynomial. Thus, if the 1D element has width \( 2a \) with \( a \leq \epsilon \), the error bound is

\[
\text{error} < \left| \int_{-a}^{a} \frac{x^s}{s\epsilon^s} \, dx \right| = 2 \frac{a^{s+1}}{s(s+1)\epsilon^s}.
\]

If \( a = \epsilon \), then this becomes

\[
\text{error} < 2 \frac{a}{s(s+1)} = \frac{a}{(n+1)(2n+3)}.
\]

For \( n = 1 \) (midpoint rule) this is \( \frac{a}{10} \), and \( n = 2 \) this is \( \frac{a}{2!} \). The actual problem is two-dimensional and has the functional form of \( \frac{1}{x} \). However, this can be considered as the 2D analogy of a natural logarithm. Thus the situations are similar to the one-dimensional case above. The integrals themselves are not just of the kernels, but the kernel multiplied by a linear function. However this will not add much to the quadrature error as this merely increases the order of polynomial by one.

Equation (3.65) serves to show that the error increases the closer to the measurement the element is. Therefore, decreasing the size of the mesh elements near the electrodes is a practical method to decrease error without making the computation overly expensive.

The two main meshes for the circular flowmeter used in this paper were one with 1213 nodes for the phantom to generate data, and one with 815 nodes for the inversion. Figure 3.14 contains some example meshes constructed such that the edge elements are finer than the central ones to minimise error whilst maintaining computational efficiency.
3.3.3 Calculating the forward map entries

A single element of \( A_u \) is given by the integrals in (3.62). The integrals are dependent on the basis functions, which need to be determined. Firstly we return to the notation of \( x = (x, y, z) \) to avoid confusion when dealing with different points. For a single triangle on \( K \), one part of the basis function \( \psi_i \) is given by the plane passing through three points:

\[
\begin{align*}
  p_i &= (x_i, y_i, 1) \\
  p_j &= (x_j, y_j, 0) \\
  p_k &= (x_k, y_k, 0),
\end{align*}
\]

defined by the equation

\[
  z = ax + by + c,
\]

for constants \( a, b, \) and \( c \). Solving the corresponding set of equations yields:

\[
\begin{align*}
  a &= \frac{y_j - y_k}{\det(p)} , \\
  b &= \frac{x_k - x_j}{\det(p)} , \\
  c &= \frac{x_j y_k - x_k y_j}{\det(p)}, \\
  \det(p) &= x_i y_j + x_k y_i + x_j y_k - x_j y_i - x_i y_k - x_k y_j.
\end{align*}
\]

The integral over that triangle is then

\[
\int_{\Delta_{ijk}} (a \xi_1 + b \xi_2 + c) \frac{\partial g}{\partial x_1} d\xi,
\]

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where $\Delta_{ijk}$ denotes the triangle defined by the above points. Thus the $(i, l)$th element of the internal integral component of the map $A_u$, denoted $A_{\text{vol}}$, is the sum of the integrals over the adjacent triangles:

$$A_{\text{vol}}^{il}(x_m) = \sum_{\text{Adj.}\Delta_{ijk}} \int (a\xi_1 + b\xi_2 + c) \frac{\partial g}{\partial x_1} d\xi_i.$$ (3.75)

This is evaluated using 1st order quadrature, as higher orders did not produce any noticeable improvements for the meshes used in this thesis.

### The surface integral

The second integral can be evaluated analytically. Let the $j$th triangle in $K$ lie on the boundary of the mesh. Then $z_j$ is the plane defined by $z_j = a_j\xi_1 + b_j\xi_2 + c_j$ as above. The line, $C$, connecting the two nodes of the triangle which lie on $\partial\Omega$ is defined by $\xi_2 = d_j\xi_1 + e_j$. Thus the value of $z_j$ on this line is:

$$z_j|_C = (a_j + bjd_j)\xi_1 + (bje_j + c_j)$$ (3.76)

$$= \alpha_j\xi_1 + \beta_j$$ (3.77)

The integral along that line from start to finish ($\xi_s$ to $\xi_f$) is then

$$\frac{\hat{n} \cdot \hat{x}}{\pi} \int_{\xi_{s1}}^{\xi_{f1}} (\alpha_j\xi_1 + \beta_j) \ln \left( \sqrt{(x-\xi_1)^2 + (y-d_j\xi_1 - e_j)^2} \right) d\xi_1
\begin{align*}
= \frac{\hat{n} \cdot \hat{x}}{4\pi \gamma^2_j} & \left[ (\alpha_j(\gamma_j^2\xi^2 + \gamma_j e_j - 2\delta^2_j) + 2\gamma_j\beta_j(\gamma_j\xi - \delta_j)) \ln(\gamma_j\xi^2 - 2\delta_j\xi + \epsilon_j) \\
+ 4(\alpha_j\delta_j + \beta_j\gamma_j)\sqrt{\gamma_j\epsilon_j - \delta_j^2} \arctan \left( \frac{\gamma_j\xi - \delta_j}{\sqrt{\gamma_j\epsilon_j - \delta_j^2}} \right) - \gamma_j\xi(\alpha_j\gamma_j\xi + 2\alpha_j\delta_j + 4\beta_j\gamma_j) \right]^{\xi_{f1}}_{\xi_{s1}},
\end{align*}
$$ (3.78)

where

$$d_i = \frac{\xi_{f2} - \xi_{s2}}{\xi_{f1} - \xi_{s1}},$$ (3.79)

$$e_i = \frac{\xi_{s2}\xi_{f1} - \xi_{f2}\xi_{s1}}{\xi_{f1} - \xi_{s1}},$$ (3.80)

$$\alpha_j = a_j + b_jd_j,$$ (3.81)

$$\beta_j = bje_j + c_j,$$ (3.82)
\[ \gamma_j = 1 + d_j^2 , \quad \text{(3.83)} \]
\[ \delta_j = x + d_j (y - e_j) , \quad \text{(3.84)} \]
\[ \epsilon_j = x^2 + (y - e_j)^2 , \quad \text{and} \quad \text{(3.85)} \]
\[ \hat{n} \cdot \hat{x} = \frac{\xi_{f2} - \xi_{s2}}{\sqrt{(\xi_{f1} - \xi_{s1})^2 + (\xi_{f2} - \xi_{s2})^2}} , \quad \text{(3.86)} \]

making use of the fact that \( \alpha_j \xi_1 + \beta_j = \frac{\xi_1 - \xi_{f1}}{\xi_{s1} - \xi_{f1}} \). This result is gained by noticing that the integral for the image source has \( x_{\text{im}} = (R^2 \xi)/ (\gamma_j \xi^2 - 2d_j e_j \xi + e^2)^{1/2} \). Since \( (\gamma_j \xi^2 - 2d_j e_j \xi + e^2)^{1/2} \approx R \), the image integral simplifies to the same integral as the source integral and thus the full integral can be approximated by multiplying the source integral by two (already presented above).

However, the rectilinear line integral as presented above is not a bijective map to the circumference of the circle. Thus an additional setup is required. The standard way to integrate around a circle is to integrate the polar angle \( \phi \) from 0 to \( 2\pi \).

The change in variables into rectilinear for a general function \( f(\phi) \) is as follows:

\[ \mathcal{f} f(\phi) Rd\phi = - \int_{-R}^{R} \frac{f(\phi(x)|_{x_2>0})}{\sqrt{R^2 - x_1^2}} dx_1 + \int_{R}^{-R} \frac{f(\phi(x)|_{x_2<0})}{\sqrt{R^2 - x_1^2}} dx_1 , \quad \text{(3.87)} \]
as \( d\phi = -dx_1/x_2 \) and \( x_2 = \pm \sqrt{R^2 - x_1^2} \).

### 3.3.4 Check vs. analytical solutions

In order to make sure the simulation is working as intended and the discretisation is consistent, comparisons with analytical solutions can be performed. Firstly use a unit delta source, \( \delta(\xi' - \xi) \), away from the boundary to test the volume integral. The analytical solution for the potential around the boundary is

\[ u(x) = \frac{\partial g(x|\xi)}{\partial x_1} \bigg|_{\xi = \xi'} . \quad \text{(3.88)} \]

The projection of a delta function on the global basis functions is

\[ \psi_i = \sum_{\text{Adj. } \Delta s_{\Delta jk}} \int (a_{ijk} \xi_1 + b_{ijk} \xi_2 + c_{ijk}) \delta(\xi' - \xi) d\xi \]
\[ = (a_{ijk} \xi'_1 + b_{ijk} \xi'_2 + c_{ijk}) , \quad \text{(3.89)} \]
with the sum collapsing down to only the triangle containing the delta function. The majority of \( \psi_i \)'s will be zero.

If the delta function is not near the edge of the domain, then the potential around the boundary from the numerical model is

\[
u_{\text{approx}}(x) = \sum_{i=1}^{n} \epsilon_i \int_{\Omega} \varphi_i \frac{\partial g}{\partial x_1} d\xi, \quad (3.91)
\]
evaluated using the previous quadrature rule. The convergence is calculated using the square difference of the analytical and numerical solutions, a.k.a., the \( L_2 \)-norm.

\[
\text{error} = |u(x) - u_{\text{approx}}(x)|_{L_2} = \oint_{\partial \Omega} (u(x) - u_{\text{approx}}(x))^2 R d\phi. \quad (3.92)
\]

This is plotted in Figure 3.15 for different maximum mesh characteristic lengths. The linear best fit of Figure 3.15b has the gradient of \(-2.5\) and thus the convergence is approximately a decaying exponential with the power of \(-2.5\) in characteristic length.

Then the boundary can also be checked by projecting the delta function onto the basis and removing the volume integral terms from the matrix for comparison. However better yet is to compare the computed and analytical solutions for a constant flow. The boundary integral in (3.11) is

\[
-\frac{1}{2\pi} \int_0^{2\pi} \ln \left( 2R^2(1 - \cos(\theta - \phi)) \right) R d\theta \\
= -\frac{1}{2\pi} \left[ -\theta \cos(\phi) - 2 \sin(\phi) \log(\sin((\theta - \phi)/2)) + \sin(\theta) \left( \log(2R^2(1 - \cos(\theta - \phi))) - 1 \right) \right]_{\theta=0}^{2\pi} \\
= \cos(\phi). \quad (3.94)
\]

Plotting the square difference (3.92) for the boundary against the characteristic length is presented in Figure 3.16. The convergence between the analytical solution and the boundary integral follows a power law with exponent \( \approx 4.5 \). Thus it converges very fast.

**Important reminder**

Once again, the convergence of the above situations is important, but in general the solutions converging in the \( L_2 \)-norm is not what is important. All that matters is that the approximate solution converges to the continuous solution in the distributional
Figure 3.15: Convergence of maximum mesh characteristic size for an approximate delta function located at the centre of the pipe.

Figure 3.16: Convergence of maximum mesh characteristic size for the boundary integral with constant flow

sense as decreasing characteristic length is taken to 0:

\[ \lim_{\text{approx}, \varphi} = \lim_{u, \varphi}, \]

for all test functions \( \varphi \).
Chapter 4

Deterministic Inversion

4.1 Deterministic inversion theory

Inverse problems can be defined in terms of the forward problem,

\[ d = A(f) + n. \]  (4.1)

Equation 4.1 is sometimes referred to as the observation or measurement process and is depicted as in Figure 4.1. The unknown that we wish to know about, \( f \), is called the image, or state of the system depending on the context. The space of all \( f \) is the image- or state-space. This is related to the measured data, \( d \), by the forward map, \( A(f) \), with the addition of some additive noise, \( n \).

The inverse problem is to predict the original state of the system given the noisy data and the knowledge of the forward problem. A method for solving an inverse problem is called deterministic if the reconstructed image is always the same under the given method. The output of a deterministic method is a single qualitatively chosen image.

Figure 4.1 is often used to describe the measurement process, with a heavily emphasised \( A \). This is where “the physics” is contained. It is a depiction of the causality

\[ f \rightarrow A \rightarrow d \]

Figure 4.1: System model of the measurement process, replicated of Figure 3.1.
as happens in reality; there is the object $f$, it is operated on by $A$, and some noise inevitably happens to produce $d$. A common but naive assumption is that this conceptually maps directly to a discretised inverse problem. Many deterministic inversion methods make the mistake of focusing on the object $A$ alone as what dictates the behaviour of the inverse problem. Representation of the image and data spaces have an equally important role in the inverse problem. Representation and knowledge are linked and thus the representation will influence the type of knowledge gained from the inverse problem (Watzenig and Fox, 2009).

The diagram in Figure 4.1 is thus misleading when considering the inverse problem — despite being causally true in reality, where the state and the data exist independent of any chosen representation. When modelling the measurement process, the structures used to represent $f$ and $d$ influence the results. Thus the chart conceptually depicting the inverse problem should be like Figure 4.2 for displaying the causality of the model used to find an estimate of $f$. The values $\delta$ and $\gamma$ abstractly represent the choice of model and representation.

### 4.1.1 Ill-posed and ill-conditioned

The majority of inverse problems are either ill-posed or ill-conditioned (Fox et al., 2012).

The forward map is called ill-posed if the forward problem:

1. has data outside the range of $A$,
2. has non-unique images for given data, or
3. Arbitrarily small changes in the data can lead to arbitrarily big changes in the image.

If the problem is well-posed, then the important issue is how relative error in the data, $\Delta d$, can propagate to relative error in the image, $\Delta f$. This is described by the condition number through

$$\frac{||\Delta f||}{||f||} \leq \text{cond}(A) \frac{||\Delta d||}{||d||}, \quad (4.2)$$

for the relevant inner product $|| \cdot ||$, where the condition number is

$$\text{cond}(A) = ||A|| ||A^{-1}||. \quad (4.3)$$

If the condition number is large, then the problem is called ill-conditioned.

There will always be error in the data due to the addition of noise in the forward problem. Additionally, the forward map often meets condition 2 due to limited data. Any single data set generated by $(4.1)$ could come from a potentially infinite set of possible images. Deterministic methods of solving inverse problems create a weighting function over all possible images consistent with the data and choose the most likely. The method of weighting and choosing a specific image is what differentiates the different deterministic methods.

EMFT is an ill-posed problem because the state-space is infinite with a finite data-space.

### 4.1.2 Regularisation

Regularisation is a common form of deterministic inverse (Engl et al., 1996). It is a process by which the image chosen to be the solution to the inverse problem is found by weighing against images with higher frequency terms and sharp changes in the image. There are three relevant terms in regularisation schemes. Firstly, the standard least-square style data misfit function $||d - Af||$. This is the difference between the measured data and noiseless data expected from the reconstruction. Next is a default solution $f^\infty$ which is useful if something is known about the family of solutions that one seeks to obtain. Finally, a linear operator $L$ used to select for specific traits. This operator is normally a discrete approximation of a derivative operator, or the identity matrix.

The most well-known regularisation scheme is Tikhonov regularisation, in which the reconstructed image is computed by

$$\tilde{f}_\lambda = \arg\min \left\{ \chi^2 ||L(f - f^\infty)||^2_2 + ||d - Af||^2_2 \right\}. \quad (4.4)$$
The reconstructed image is found by balancing the misfit of the data due to noise and the norm of the $L$ operating on the difference of the reconstruction from the default solution. As $L$ is often a derivative operator, the term $\|L(f - f^\infty)\|^2_2$ is small for smooth deviations from the default solution. The factor which dictates the level of preferential selection of smoothly varying images is $\lambda$: the regularisation parameter.

The solution to (4.4) is found by solving the equation

$$\left(\lambda^2 L^T L + A^T A\right) f = \lambda^2 L^T L f^\infty + A^T d. \quad (4.5)$$

Equation (4.5) is a new preferential equation to solve for the image, finding the direct inverse of a regularised forward map. The solutions can be examined by looking at the eigenvector decomposition of $(\lambda^2 L^T L + A^T A)$. If this is non-singular, then there exists a unique decomposition. The higher frequency eigenvectors change as $\lambda$ increases and their eigenvalues increase. This makes the higher frequency eigenvectors not contribute as much to the regularised solution calculated with the inverse of $(\lambda^2 L^T L + A^T A)$. Thus, the effect of uncorrelated noise is also lessened.

Tikhonov regularisation is a well regarded regularisation technique and therefore this is a good place to make two important points about deterministic regularisation. Firstly, the choice of the default solution is subjective. More importantly, not choosing one is no different than choosing 0 to be the default solution. This goes back to the discussion around Figure 4.2, that representation and knowledge are linked. For example, “not choosing” a default solution in a linear scale is akin to choosing it to be at $-\infty$ in logarithmic representation.

Secondly, the choice of the regularisation parameter $\lambda$ is qualitative, not quantitative. Regularisation solutions have been called “eye candy” as they are chosen based on how one expects the solutions to look. This is acceptable if the information one wants from the unknown image is also qualitative, such as in a deblurring inverse problem — particularly if the image contains text. However, the output of regularisation contains no quantitative data such as confidence in the solution and calculations using the solution can vary wildly from the true system.

The most common way that $\lambda$ is chosen is through solving (4.5) for a range of values of $\lambda$, then plotting on a log-log plot the solution semi-norm $\|L(f_\lambda - f^\infty)\|^2_2$ against the data misfit norm $\|d - Af_\lambda\|^2_2$. This produces the “L-curve”: a plot with a kink in the middle, much like the letter L. The selection criterion for the chosen reconstruction amounts to a vague guide of “[the value of $\lambda$ which] lies slightly to the right of the position of the largest upwards-pointing curvature.” — (Fox et al., 2012)
In general, a deterministic method requires one to solve a system of equations such as (4.5) multiple times in order to compare visually to find the “best” reconstruction. Despite computational efficiency often being stated as the main advantage of solving inverse problems using regularisation, it is still computationally intensive and requires many linear equation solve steps with large matrices in order to only gain a single qualitatively useful image.

4.1.3 Singular value decomposition

The forward map, and consequently deterministic methods, can be further analysed through the singular value decomposition (SVD). The SVD is an extension of an eigenvector decomposition for non-square matrices such as the forward map $A$. The SVD is defined as

$$A = \sum_{k=1}^{r} u_k \sigma_k w_k^T = U^T SW,$$

(4.6)

where the right and left singular vectors, $w_k$ and $u_k$, are defined by

$$A^T A w_k = \sigma_k^2 w_k,$$

(4.7)

and

$$A A^T u_k = \sigma_k^2 u_k,$$

(4.8)

with $S$ being a diagonal matrix containing the singular values $\sigma_k$ (Fox et al., 2012). The vectors $w_k$ lie within the image-space and $u_k$ in the data-space. Both vectors are connected by the shared singular value $\sigma_k$ with the relationship

$$A w_k = \sigma_k u_k,$$

(4.9)

and

$$A^T u_k = \sigma_k w_k.$$

(4.10)

The indexing in (4.6) goes up to $r = \min\{n, m\}$ as all singular values above that must be zero due to the difference in dimension of the respective spaces. It is a common convention to order the decomposition in monotonically decreasing order. The matrices $A^T A$ and $A A^T$ truncated to $r = \min\{n, m\}$ are very close to singular matrices in inverse problems such as EMFT. Therefore the singular values quickly become very small.

The SVD is useful for analysing deterministic inversion methods. The relationship between (4.9) and (4.10) means that the singular vectors can be considered the sensitivities of the numerical model\textsuperscript{1}.

\textsuperscript{1}This only weakly corresponds to sensitivities in the real-world system because the SVD is highly dependent on representation.
The simplest deterministic method is the direct inverse of the forward map in (4.1). This can clearly be seen as a poor choice by using the SVD. The generated solution image \( \tilde{f} \) is given by

\[
\tilde{f} = A^{-1}(d + n),
\]

\[
= f + \sum_{k=1}^{r} \frac{w_k^T u_k n}{\sigma_k}.
\]

Hence the direct inverse is dominated by high frequency, small \( \sigma_k \) singular vectors due to the noise. Equation (4.12) alone shows why regularisation methods seek to weigh images against the higher frequency terms.

**Truncated singular value decomposition**

A simple form of regularisation is the truncated singular value decomposition (TSVD), wherein the measured data is projected onto a truncated space. The truncation is made by removing all of the dimensions in the directions of singular vectors with lower singular value than a cutoff point. This cutoff point is often chosen by a signal to noise ratio. Thus, the reconstructed image is given by

\[
\tilde{f} = \sum_{k=1}^{s} \frac{w_k^T u_k d}{\sigma_k},
\]

where \( s < r \) is the cutoff point. The TSVD is a similar method of solving the EMFT inverse problem to the truncated projection methods by Horner et al. (1996) and Kollár et al. (2014).

**Regularisation and the singular value decomposition**

The singular value decomposition can be generalised by tempering it with the semi-norm as in Tikhonov regularisation. For (4.5) this is done by diagonalising the matrix \((\lambda^2 L^T L + A^T A)\). If \( L \) is a discrete derivative operator, then the generalised SVD favours smoother vectors, decreasing the singular values of the more rapidly changing vectors.

The solution to Tikhonov regularisation from (4.5) with \( L \) as the identity matrix is given by

\[
\tilde{f} = \sum_{l=0}^{N} \tilde{f}_l
\]

where

\[
\tilde{f}_l = \begin{cases} 
\frac{\lambda^2}{\lambda^2 + \sigma_l^2} f_l^\infty + \frac{\sigma_l^2}{\lambda^2 + \sigma_l^2} \left( \frac{d_l}{\sigma_l} \right) & \text{for } l = 1, \ldots, r, \\
 f_l^\infty & \text{for } l = r + 1, \ldots, n,
\end{cases}
\]

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and \( f_l^\infty = u_l^T f^\infty, d_l = u_l^T d \) (Fox et al., 2012). Equation (4.15) shows that regularisation solutions are a weighted sum of all vectors in the SVD between the data fit \( \frac{d_l}{\sigma_l} \) and the default solution for the subspace \( \{u_l\}_{l=1}^r \). Outside of that space of the SVD is given by the components of the default solution in \( \{u_l\}_{l=r+1}^N \). The functions \( \frac{\lambda^2}{\lambda^2 + \sigma^2} \) and \( \frac{\sigma^2}{\lambda^2 + \sigma^2} \) both lie in the range \([0, 1] \) for \( \sigma, \lambda \in \mathbb{R}^+ \). Thus, within the span of \( A^{-1} \), the reconstruction gradually shifts between the data fit for high \( \sigma \) and the default solution for low \( \sigma \) vectors.

The weight of the singular vectors in the Tikhonov regularisation and TSVD schemes are plotted in Figure 4.3. The TSVD weights the singular vectors with a step function, cutting off any singular values lower than the cutoff \( \sigma = 0.1 \) in this case. In contrast, Tikhonov regularisation weighs the vectors with the functions in the second plot (the dotted line being the weight of the default solution). The regularisation parameter is \( \lambda = 0.1 \) here; the point where the reconstruction has equal contribution from both terms.

Both regularisation schemes are ways of favouring smoother singular vectors and excluding the lower value ones. Tikhonov regularisation has a smooth transition, whereas the TSVD is abrupt. The abrupt cutoff creates artefacts in the solution image known as Gibbs phenomena (Gottlieb and Shu, 1997), as shown at the end of this chapter.

The method used by Lehtikangas et al. (2016) is the MAP estimate, which is
a slightly more sophisticated form of regularisation, but is essentially the same as Tikhonov regularisation (Fox and Norton, 2016).

4.2 Electromagnetic flow tomography

The SVD was used on the computational model for the EMFT/flowmeter system from Chapter 3. The analysis in this section is done using the no-slip condition, ignoring the boundary integral. This is because the largest singular value vectors, when including the boundary integral, comprise of mostly flow on the boundary. Additionally, the TSVD presented here serves to reinforce the lack of penetrative ability of the EMFT system — which is exacerbated by the boundary integral — while providing an example of regularisation at work.

4.2.1 Singular value decomposition

![Image](image_url)  

Figure 4.4: Normalised singular values for the pipe flowmeter measurements, truncated at $y = 10^{-3}$. Created using 1000 circumferential measurement points and a mesh with 2072 nodes.

The ordered singular values are presented in Figure 4.4, normalised to the highest singular value. If the highest value is taken to be the signal, then there are many vectors that can be said to be “measurable” in the idealised model based on any reasonable signal to noise ratio. Note the change in behaviour of the singular values around the 150–200th value, where the change to an approximate straight line. This is due to the measurements starting to pick up features of the discretisation because there are only approximately 200 nodes around the boundary.
Figure 4.5: Singular vectors of the no-slip EMFT computational model.
The first of the ordered vectors are shown in pairs in Figure 4.5. The flow singular vector in Figure 4.5b has a significant inner product with a constant flow. In contrast, the latter flow singular vectors have a negligible inner product with constant flow. This trend continues for the vast majority of the next 200 or so singular vectors pairs as the number of peaks and troughs seen on the edge of Figures 4.5d and 4.5f increase and they penetrate less into the pipe’s depth. The SVD reinforces the various arguments throughout the previous chapter — that the most distinguishable flow features are those closest the boundary.

The first singular vector also links to the standard flowmeter. The constant flow is measurable in a standard flowmeter system and the two measurements are located at the maxima and minima of the potential in Figure 4.5a.

A conclusion to make about this SVD is that, while Figure 4.4 shows that many pieces of information can be gained by the unrealistic point measurements, almost all of these pieces of information pertain to the outer region of the flow and become increasingly useless.

4.2.2 TSVD flow reconstruction

A TSVD was performed on a data set generated by adding noise with precision $\gamma = 10$ (standard deviation of 0.1) to various phantom flow vectors. Once again, the no-slip condition is used and 1000 point measurements were taken for analysis purposes.

Figure 4.6 is the reconstruction of the flow using data generated by a constant phantom flow, with total flow of $6.283 \text{Tm}^{-1}$ shown in Figure 4.6a. The reconstruction $\tilde{f}_k$ was found for a number of cutoff values $k$. An $L_2$-norm difference was performed between the reconstruction and the phantom flow in order to calculate the error of the reconstruction (Figure 4.6c). This is calculated by: $\text{error}(\tilde{f}_k) = (\tilde{f}_k - f)^T(\tilde{f}_k - f)$. There is a clear drop in the error function until approximately $k = 200$, hence that cutoff was chosen for Figure 4.6b. Beyond the 300 mark, the error increases to the point where the scale of the first 200 are hard to make out. Hence the truncation of the $x$ axis in said plot.

The reconstruction’s singular vectors spectral decomposition in Figure 4.6d was calculated by taking the inner product with the data. Initially there are definite peaks every three singular vectors, as only an odd number of the wiggles (as shown in Figure 4.5f) have a non-zero inner product with the constant. This trend stops around the 25

\footnote{the units are from flow being defined by $f = v_z B_y$}
Figure 4.6: The TSVD for measurements generated by constant phantom flow.

mark where the noise in the data has a non-negligible effect on the reconstruction.

The reconstruction in Figure 4.6b does a very good job in replicating the phantom flow with a bulk flow rate of $6.226\text{Tm}^{-1}$. This is expected as it is a regularisation method and nothing is more regular than a flat image. The fringe effects are called Gibbs phenomena and occur due to the sharp cutoff in Figure 4.3. The image is still a good deterministic solution to the inverse problem despite the Gibbs phenomena because solutions to the mathematical model are distributional solutions. Hence the solution images only have meaning when integrated with a test function, which would blur out the Gibbs phenomena.

An example of a slightly more complicated phantom is shown in Figure 4.7. The TSVD is performed on data generated by a linear stratified phantom flow (from Chapter 6) with the bulk flow of $6.283\text{Tm}^{-1}$, as shown in Figure 4.6a. All of the plots are as
above, with some notable differences in the reconstructed images. Firstly, the error is higher in Figure 4.7c for the majority of \( k \). This is reasonable as the phantom is no longer an axially symmetric flow field. The reconstructed image using \( k = 200 \) has the same linear structure as the phantom, but the Gibbs phenomena is much more extreme. The flow in Figure 4.7b is truncated for the plot at the limits of the phantom in Figure 4.7a. Without this, the linear features become imperceptible.

Once again there are clear structures in the left-most values of the spectrum in Figure 4.7d, which break down as the noise starts to dominate.

The TSVD solution to the inverse problem is once again distributional and is only meaningful when integrated with a test function. However the size of the Gibbs phenomena are such that the solution only converges for test functions significantly larger than the peaks and troughs at \( \theta = 0 \). Increasing the cutoff will decrease the width of
the Gibbs phenomena, however the TSVD becomes dominated by the noise when this is done.

4.3 Summary

What is presented here is a brief example of a deterministic method for solving inverse problems in order to provide a counterpoint for the following chapters. A summary of the pros and cons of deterministic methods is as follows:

- Deterministic methods gain useful information from an ill-posed problem by creating a weighting across all possible states and choosing the “best” one as the solution to the inverse problem.

- The output of a deterministic method is a single, entirely qualitative image with no statement of confidence.

- The choice of solution state is usually found by scanning the space of the regularisation parameter $\lambda$ (or $k$) and choosing the value which looks the best. This process can be very computationally intensive.

- Deterministic methods are less useful when quantitative results are sought. The flow rate is a quantitative value, for example. It is very useful to be able to state how confident one is in the reconstructed values. This is particularly important for EMFT due to the non-penetrative quality of the measurements. Deterministic methods do not provide quantitative estimates and uncertainty.

Another significant problem with deterministic methods pertains to the regularisation parameter $\lambda$. Looking at Tikhonov regularisation (4.4), $\lambda$ is a parameter denoting the trade off between the data misfit and the level of smoothness as dictated by the semi-norm. Thus, it can be viewed as the ratio of two quantities $\frac{\delta}{\gamma}$ where $\gamma$ is a measure of how precise we believe the measurement to be (the precision of the Gaussian distribution that the noise is drawn from), and $\delta$ is the level of smoothness that we wish to see\(^3\). In the Bayesian formulation a probability over these quantities can be calculated given the data and it has been shown that the values for $\lambda$ in regularisation do not lie in the support for the distribution over $\gamma$ and $\delta$ (Fox and Norton, 2016). Moreover, Fox and Norton (2016) showed that no value of $\lambda$ provides a meaningful summary of the data misfit or the semi-norm. This leads to the final point:

\(^3\)called $\delta$ to match the hyperparameter from later chapters.
• A deterministic solution may not lie in the support of likely states from a more robust Bayesian formulation.

Ultimately, deterministic methods make sacrifices in order to gain useful information out of ill-posed problems. They are a means to reduce the potentially infinite space of probable states to pick out a single one. However, as will be seen in the next chapter, finding the solution to an inverse problem does not have to involve solving an ill-posed deterministic problem and can be found as a well-posed probabilistic problem.
Chapter 5

Bayesian Formulation

The Bayesian formulation rephrases the inverse problem described by the forward map

\[ d = Af + n. \]  \hspace{1cm} (5.1)

Where deterministic methods ask:

*Given data, \( d \), and the forward problem (5.1), can we find the true state of the system \( f \)?*  

for which the answer is no, a Bayesian formulation asks

*Given data, \( d \), and the forward problem (5.1), how much more can we know about the state of the system \( f \) than we do currently?*

One can represent the knowledge of a variable \( x \) as a probability distribution \( \pi(x) \) over said variable. This is a central point of Bayesian formulations; probability distributions are representations of our knowledge. This includes knowledge of the state, the measurements, and the system. The fundamental object which we must manipulate changes from vectors into probability distributions in a Bayesian formulation.

This is a more natural way of approaching problems in science — at least philosophically. “The truth” can never be obtained in science as it based on falsification principles: simplistically, one can only disprove things. The question of whether or not a scientific theory is true is not a meaningful question — a concept which can be somewhat unintuitive, as demonstrated by the behaviour some laypeople may demonstrate towards the Theory of Evolution. The more useful question to ask is “how confident we are in a scientific theory?” which is a question suited to the field of probability.

For these reasons, probability is also the natural language of inverse problems.
5.1 Bayesian inference

This section contains an overview of the Bayesian formulation of inverse problems loosely based on the work of Robert (2007), Watzenig and Fox (2009), and von Toussaint (2011).

The goal of any statistical inference is to recover optimal, unbiased knowledge of the true state $f$, given some related noisy data $d$. The probability distribution describing this is called the posterior. This is denoted as $\pi(f|d)$ and read as “the probability of $f$ given $d$”. Bayes rule provides the mechanism by which this can be calculated:

$$
\pi(f|d) = \frac{\pi(d|f)\pi(f)}{\pi(d)}, \tag{5.2}
$$

and was derived by Thomas Bayes, presented posthumously in Richard Price’s *An Essay towards solving a Problem in the Doctrine of Chances* (1763).

On the right hand side of (5.2): $\pi(f)$ is called the prior density and describes the information about $f$ before the inclusion of the data $d$. This will be discussed in Section 5.2.

The likelihood function is $\pi(d|f)$ and contains the information of the measurement process by which the data is gained. It is closely related to the noise distribution of the measurement process. This will be discussed in Section 5.1.1.

The denominator, $\pi(d)$, is a finite normalising constant, important for when the forward map varies. However the EMFT forward map is static and the posterior distribution is therefore also static. Additionally, in sampling problems such as the one in this thesis the posterior does not need to be normalised. Thus, $\pi(d)$ can be ignored in the sampling method presented in this paper.

If one treats probabilities as a representation of knowledge, then Bayes’ theorem is the only way to update knowledge while maintaining consistency, given new information (Robert, 2007).

Using Bayes’ theorem, the posterior can be explicitly constructed up to a normalising constant. The posterior is usually a very high dimensional object (of order $10^2$, up to $10^7$ dimensions). Thus, visualisation and calculation using the posterior is difficult. The “inversion” is no longer about conditioning the inverse mapping to find best state as it is in deterministic methods. The problem is instead one of statistical sampling of a high-dimensional, multivariate probability distribution and “the solution” — that is, estimates of the quantities of interest — is found by calculating expectation values of the posterior distribution over all possible states.
There are two commonly stated issues with the Bayesian formulation when compared to deterministic methods. First is the computational intensiveness of the sampling methods. However, this is not necessarily a problem as recent methods such as the MTC sampler can exceed similar regularisation methods (Fox and Norton, 2016).

The second, perhaps more valid issue is that the Bayesian formulation is unintuitive. Deterministic methods are easier to understand with the standard forward-map-focused education in physics and engineering fields, leading to diagrams like the chart in Figure 4.1. This is not a good reason for why one shouldn’t use a method. However, it is true that more thought and setup goes into a Bayesian inference method than a regularisation method, so the latter is easier to create an uninformed implementation with.

However, even individual samples of the posterior from the Bayesian formulation are informative. Single samples can often provide a better reconstruction than a regularised inversion (Fox, 2008). While many samples allow for quantitative estimates to be made, even a small set of samples can provide a qualitative picture of the nature and scale of uncertainties (Watzenig and Fox, 2009).

On the other hand, the advantages of a Bayesian formulation are numerous. We are sampling the full posterior distribution which allows for the calculation of the expectation values such as the mean, variance, etc. These are quantitative values and thus a Bayesian formulation allows for estimates on values such as energy of a system and the level of confidence in the answer.

The posterior distribution can be explicitly written down with a pen and paper. As stated earlier, the issue is calculation of expectation values and visualisation of this distribution. However, this means that the Bayesian reconstruction approaches the optimal reconstruction merely by adding more samples, provided the modelling is consistent.

In a Bayesian formulation the assumptions are explicitly entered into the method via the distributions \( \pi(f) \) and \( \pi(d|f) \). Additionally, the representation of the spaces, the forward map, and the noise model are all used in the construction of these densities.

Finally, Bayesian methods are much less reliant on subjectivity than deterministic methods. Deterministic methods require a choice of what “looks right”, whereas the posterior of Bayesian methods cover all probable states in the state space, weighted by the probability that they are the “true” one, consistent with the data.

There is always some degree of subjectivity, even if it is just the modelling decisions. Subjectivity additionally enters Bayesian formulations through the choice of represen-
tations as well as the choice of the so-called hyperparameters like \( \delta \) and \( \gamma \). However, in using a hierarchical model\(^1\) the hyperparameters can also be estimated to lessen their subjectivity. This changes the subjective choice from the hyperparameters to the parameters used to describe the hyperparameter distribution. While seemingly just shifting the subjectivity, this does reduce the effect of subjective choice on the output of the method as it is now one step further removed from the quantities of interest. This principle of estimating hyperparameters can be applied recursively, but it is common to only include the hyperparameters (\( \delta \) and \( \gamma \) in this case) into the estimation, and no further.

A debate exists as to whether or not a Bayesian method should be informative. Informative means that one can enter in some preexisting knowledge about the desired state such as the state is smooth. The so called “objective Bayes” methods attempt to create a method as uninformative as possible. Whereas proponents of highly informative methods argue that all knowledge about the true state should be used when solving an inverse problem. Uninformative methods can be very useful for building an understanding of what kind of states the inverse problem tends towards estimating, but for practical application informative methods are often preferred. The uninformative methods used in this work were not ideal (see Section 6.3).

### 5.1.1 Likelihood function

The presence of noise in the measurement process means that the measurements are probabilistic. Looking at the forward map

\[
d = Af + n,
\]  

the noise \( n \) is distributed as the distribution \( \pi_n(n) \). In most cases this is a multivariate Gaussian distribution, but any noise process can be used including highly correlated noise. In EMFT, this noise is iid Gaussian distributed due to the time averaging and the central limit theorem. The source of the noise consists of thermal and shot noise, digitisation error, and external influences. This is the noise in the measurement process only and does not include error in the model.

Given a true state \( f \), the measurement \( d \) is randomly distributed about \( Af \), as the map \( A \) is deterministic. Thus the probability density function of the measurement data

\(^{1}\)explained in Section 5.2.3

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$d$ given said true state is:
\[ \pi(d|f) = \pi_n(d - Af), \quad (5.4) \]
as the change in variables has the determinant of 1. This is a probability density for
$d$, however it is not for the parameter of interest, $f$. Thus, the object $\pi(d|f)$ is often
written as $l(f|d)$ and is called the likelihood function not a “density”.

If $n$ is multivariate Gaussian distributed with variance $\Sigma$, the data is distributed as
\[ \pi(d) \propto \exp \left( -\frac{1}{2} (d - Af)^T \Sigma^{-1} (d - Af) \right). \quad (5.5) \]
The variance used in this thesis is $\Sigma = \gamma^{-1}I$, where $I$ is the identity matrix, and
variance constant, $\gamma$, has units of $V^{-2}$.

### 5.1.2 Summarising the posterior distribution

As stated earlier, the posterior is of very high dimension and thus explicit calculation
of expectation values is unfeasible. Therefore, the posterior distribution is explored
by approximation. In most cases this means sampling. The current state-of-the-art
sampling algorithm is the Monte Carlo Markov chain (MCMC).

A MCMC is a sampling algorithm that generates a Markov chain with a probabilis-
tic transition kernel such that it has the equilibrium distribution of the desired proba-
bility distribution, $\pi(f|d)$. Thus, after the MCMC has run sufficiently long enough to
near the equilibrium density (called the burn-in time), the sequence can be said to be
samples drawn from the posterior, denoted $f_i \sim \pi(f|d)$. MCMC algorithms distinguish
themselves from one another through the choice of the transition kernel.

**Metropolis-Hastings algorithm**

The Metropolis-Hasting algorithm is an algorithm for generating the transition kernel
between steps of the Markov chain. The next step in the chain $f_p$ is drawn from a
proposal density $q(f_p|f_i)$, based on the current step $f_i$. While the choice of proposal
densities is arbitrary, it can greatly affect the efficiency of the sampling algorithm and
can simplify the analytics slightly. The algorithm is as follows:

Let the MCMC currently be at state $f_i$

1. A proposal for the next state is drawn $f_p \sim q(f_p|f_i)$.

2. The Metropolis-Hastings acceptance ratio is calculated as
\[ \alpha(f_i, f_p) = \min \left( 1, \frac{\pi(f_p|d) q(f_p|f_i)}{\pi(f_i|d) q(f_i|f_p)} \right). \quad (5.6) \]
3. Accept the proposal with probability $\alpha(f_i, f_p)$. If accepted, set $f_{i+1} = f_p$, otherwise $f_{i+1} = f_i$.

4. Repeat until desired chain length is achieved.

Choosing a proposal distribution such that $q(f_p|f_i) = q(f_i|f_p)$ is common as it simplifies $\alpha(f_i, f_p)$.

### 5.1.3 Summary statistics

Once a set of samples is generated, the issues of visualisation and quantitative calculation remains. A plot of a single sample can be useful but does not have quantitative value. Quantitative results for probability density functions are generated in the form of summary statistics, specifically expectation values.

The expectation value of a quantity $h(f)$ of a probability density function is given by the integral

$$E[h] = \int h(f)\pi(f)df,$$

where $E[\cdot]$ denotes the expectation value.

Commonly used expectation values are the mean, or first moment:

$$\mu = E[f] = \int f\pi(f)df,$$

and the variance, also called the second central moment:

$$\text{var}(f) = E[f^2] - E[f]^2 = \int f^2\pi(f)df - \mu^2.$$

The related standard deviation is given by

$$\sigma_f = \sqrt{\text{var}(f)}.$$  \hspace{1cm} (5.10)

Most calculated expectation values are various polynomials of $f$ as they closely related to many physical values. For example, the potential energy of a particle is related to the square of position by $U = \frac{1}{2}kx^2$.

### 5.1.4 Monte-Carlo integration

The integrals of the previous section are unfeasible to calculate directly for the posterior distribution and must be approximated. Monte-Carlo integration is used for this approximation.
Given $N$ samples drawn from an arbitrary probability distribution $\pi(f)$, then the Monte-Carlo approximation of an integral in the form of (5.7) is given by

$$\int h(f)\pi(f)df \approx \frac{1}{N} \sum_{i=1}^{N} h(f_i).$$  \hspace{1cm} \text{(5.11)}$$

The error of this approximation is inversely proportional to the number of samples. By the central limit theorem, the variance of the approximation error for correlated samples such as those from an MCMC is given by

$$\text{Var}(\bar{h}_N) = \tau_h \frac{\text{Var}(h)}{N},$$  \hspace{1cm} \text{(5.12)}$$

where $\bar{h}_N$ is the average of $h(f)$ from $N$ samples and $\tau_h$ is the integrated autocorrelation time (IACT). One interpretation of the IACT is that it is the number of correlated samples required to reduce the variance by the same amount one independent sample would. Thus, the IACT is a good measure for how efficient a sampling algorithm is; a low IACT means that fewer steps and therefore calculations are required to obtain the same accuracy as an equivalent large IACT algorithm.

Example IACT times for different MCMC algorithms in the problem of deblurring a grey-scale pixel image range from $\sim 5$ to $\sim 20$ (Fox and Norton, 2016).

## 5.2 Prior models

As one would expect from equation (5.2), the posterior is heavily influenced by the choice of prior. The prior defines both the space of possible states without regard to any data and is a probability distribution over the space. The space of states without any prior assumptions on the system is too expansive to be of any use, as was discussed in the previous chapter. This is also why early investigators into EMFT assumed the shape of the flow field to be axisymmetric. However, if one assumed a different flow, say, a flow field antisymmetric about the $y$-axis, then the results would be very different as demonstrated by Shercliff (1954).

The core components of all image models are the representation of the unknown image and the prior density over the space of said representation. Representation and knowledge are linked and thus a representation should be chosen based on the type of information that one wishes to extract from the system. Prior models contain the information about the representation and can be divided into low-, mid-, and high-level models. These are separated in an increasing level of specificity with low-level models being the most generic such as applying grey-scale values over pixels, or mesh nodes.
For the case of EMFT, the goal is to estimate the total flow rate of a single-phase fluid through a plane. Thus, it is useful to use a low-level representation of the unstructured mesh from Section 3.3.2. For multi-phase fluids such as mixes of oil, water, and air, higher level representations may be used such as the polygonal representations used in ECT (Watzenig and Fox, 2009).

5.2.1 Locally connected Gaussian Markov random field

For the EMFT system a relatively uninformative low-level prior is appropriate for calculating a bulk flow rate. For this purpose a Gaussian Markov random field (GMRF) will be used. This is a multivariate Gaussian prior distribution conditioned using the 2-norm graph Laplacian. Much like in regularisation, where the reconstruction is conditioned on a semi-norm with a derivative matrix, this also favours states with a level of smoothness.

A locally connected linear GMRF is defined by drawing the value of the \(i\)th node as

\[
f_i|\partial_i \sim N \left( \sum_{j \in \partial_i} f_j, \delta^{-1} |\partial_i|^{-1} \right),
\]

where \(\partial_i\) is the set of \(f_j\)s neighbouring the \(i\)th node, \(|\partial_i|\) area or length connecting the nodes. The locally connected part of this refers to the connections between the nodes for the calculation of the Laplacian being only nearest neighbour.

In practice, locally connected linear GMRF priors refine the space of possible states, favouring states with a lower second derivative. The spatial variation of probable states is limited by the parameter \(\delta\), sometimes called a “lumping constant”. The multivariate form of equation 5.13 is:

\[
f|\delta \sim N \left(0, (\delta L)^{-1}\right),
\]

where \(L\) is the discrete Laplacian, with elements defined by

\[
L_{ij} = \int_{\Omega} \nabla \psi_i \cdot \nabla \psi_j d\xi,
\]

and \(\psi\)s are the basis functions from Section 3.3 (Fox and Norton, 2016). The lumping constant \(\delta\) has units \((Tms^{-1})^{-2}\), similar form to the variance constant \(\gamma\). This can be thought of as an indicator for the length scales of spatial variances in possible flow states. Whilst not an actual length scale — in part due to the units not matching that of the state — a state drawn using a larger \(\delta\) will be smoother than on with one with a smaller \(\delta\).
The distribution in equation 5.14 is a non-normalisable and thus forms an improper prior. However, this is not a problem as the posterior is normalisable and well defined.

Modelling the flow as a GMRF is done in some regularisation papers for a MAP estimator (Lehtikangas et al., 2016). However, it is not referred to as a GMRF.

5.2.2 Hyperprior modelling

A further advantage of Bayesian inference is that Bayes rule can be applied recursively. The likelihood function and prior distribution are often dependent on some form of scale parameter, such as the noise variance $\gamma^{-1}$ and the “lumping constant” $\delta$. A very common approach is to choose these as some physically justified value. However, it is also common for there to be unaccounted noise and the model of the system to not be perfect. The modelling itself is usually the largest source of error. Because of this, it is very useful to model these hyperparameters (sometimes called nuisance parameters), $\vartheta = (\delta, \gamma)$, with uncertainty too and to treat the distribution of them as the so-called hyperprior $\pi(\vartheta)$.

Bayes rule including a hyperprior becomes:

$$\pi(f, \vartheta|d) = \pi(f|d, \vartheta)\pi(\vartheta)$$

$$= \frac{\pi(d|f, \vartheta)\pi(f|\vartheta)\pi(\vartheta)}{\pi(d)}.$$ (5.16) (5.17)

Generally this approach brings with it a large increase in computational cost. This cost is more than just adding two more nodes in the mesh as there are many dependencies to $\vartheta$ in the other distributions.

Common methods that do this, while trying to minimise computational costs, include the Gibbs sampler, the one-block algorithm, and the marginal algorithm. Recent work by Fox and Norton (2016) has demonstrated that the marginal algorithm can exceed even regularisation in speed with careful implementation, which is the main advantage of regularisation methods.

5.2.3 Hierarchical stochastic models

The aim of Bayesian inference with hyperpriors is to sample from the distribution $\pi(f, \vartheta|d)$. With the addition of the hyperprior, Bayesian inference can be treated as a hierarchical stochastic model.

Hierarchical stochastic models provide simple strategies for incorporating complicated interactions at different stages of the model’s hierarchy and are useful as soon
as there are predictors at different levels of variation (Gelman, Carlin, Stern, Dunson, Vehtari, and Rubin, 2013; Wikle, Berliner, and Cressie, 1998). The stratification of the model aids the feasibility of computational implementation at high dimension (Wikle et al., 1998).

The quantities in the EMFT model are treated as random variables drawn from the related distributions

\[ \vartheta \sim \pi(\vartheta), \quad (5.18) \]

\[ f|\vartheta \sim N(0, Q^{-1}(\vartheta)), \quad (5.19) \]

\[ d|f, \vartheta \sim N(Af, \Sigma(\vartheta)), \quad (5.20) \]

where \( Q(\vartheta) = \delta L \). This is a hierarchical model because the random variables in (5.19) and (5.20) depend on the previous variables. Naturally, this is difficult to sample from as the distributions of lower hierarchy change based on the higher terms.

There are a few ways in which sampling algorithms have been adapted to handle this. One simple way is to augment the system of equations such that \( f \) and \( \vartheta \) are treated as a combined entity and sampled together. This is inefficient and requires many more samples than other methods. Another is the Gibbs sampler; a sampling algorithm which alternates sampling between drawing from (5.18) and \( f|d, \vartheta \).

There have been improvements to these styles of samplers, such as the One-Block sampler. However the sampling method used in this thesis makes use of the marginal algorithm.

**The marginal algorithm**

The inverse problem of finding the hyperparameters given some data can be solved by eliminating the possible states, \( f \), through marginalisation. Marginalising a joint probability density is defined by

\[ \pi(\vartheta|d) = \int \pi(f, \vartheta|d) df, \quad (5.21) \]

where the variable \( f \) is integrated out.

If the variables \( f \) and \( \vartheta \) are drawn independently, then the full conditional posterior can be written as

\[ \pi(f, \vartheta|d) = \pi(\vartheta|d)\pi(f|\vartheta, d). \quad (5.22) \]

Thus the full conditional can be found by the algorithm:
1. draw from the marginal posterior $\vartheta \overset{iid}{\sim} \pi(\vartheta|d)$.

2. draw from the conditional posterior $f \sim \pi(f|\vartheta,d)$.

The marginal posterior is given by combining (5.17) with (5.22) to get

$$
\pi(\vartheta|d) = \frac{\pi(d|f, \vartheta) \pi(f|\vartheta) \pi(\vartheta)}{\pi(f|\vartheta,d) \pi(d)}.
$$

(5.23)

However, as stated previously, the factor $\pi(d)$ can be ignored for the sampling problem, giving the result

$$
\pi(\vartheta|d) \propto \frac{\pi(d|f, \vartheta) \pi(f|\vartheta) \pi(\vartheta)}{\pi(f|\vartheta,d)}.
$$

(5.24)

The MCMC algorithm that makes use of this for solving inverse problems is called the marginal then conditional (MTC) algorithm.

### 5.3 Marginal then conditional sampling

The MTC sampler was originally proposed and detailed by Fox and Norton (2016). This section details the implementation of the MTC for EMFT applications.

#### 5.3.1 Marginal density

The Gaussian likelihood (5.5) with a diagonal variance matrix and the GMRF prior (5.15) combine in (5.24) to give

$$
\pi(\vartheta|d) \propto \sqrt{\frac{\det(\gamma I) \det(\delta L)}{\det(\gamma A^T A + \delta L)}} \exp\left\{-\frac{1}{2} d^T \left[\gamma - \gamma^2 A(\gamma A^T A + \delta L)^{-1} A^T\right] d\right\} \pi(\vartheta).
$$

(5.25)

Making a change of variables to $(\gamma, \lambda) = (\gamma, \frac{\delta}{\gamma})$, this becomes

$$
\pi(\vartheta|d) \propto (\lambda \gamma)^{n/2} \exp\left\{-\frac{1}{2} [G(\lambda) + \gamma F(\lambda)]\right\} \pi(\vartheta),
$$

(5.26)

where

$$
G(\lambda) = \ln(\det(\Lambda)) \quad \text{and} \quad F(\lambda) = d^T d - d^T A\Lambda^{-1} A^T d,
$$

(5.27, 5.28)

with $\Lambda = A^T A + \lambda L$. 

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5.3.2 Computation of extreme numbers

There is an issue when calculating (5.26). The extreme nature of the numbers involved in said calculation require care, despite the probability distribution, $\pi(\theta|d)$, being only two dimensional.

In a 64-bit floating point number representation the largest number is about $1.8 \times 10^{308}$. Any number larger than that is considered to be $\infty$. Similarly, any number smaller than around $2.2 \times 10^{-308}$ is considered to be 0. Therefore, any calculation involving more extreme numbers will fail.

The first issue is the term $(\lambda\gamma)^{n/2}$. The exponent is proportional to the size of the discretisation which can be over 10000 in some cases. This can be avoided by working in the log-probability.

A more significant problem is the term $G(\lambda) = \ln(\det(\Lambda))$. Taking the eigenvector decomposition the matrix $\Lambda$, the determinant can be written as

$$\det(\Lambda) = \prod_{i=1}^{n} \eta_i,$$

where $\eta_i$ is the $i$th eigenvalue. However, from Chapter 4, the eigenvalues of a generalised map like $\Lambda$ rapidly drop off, rapidly approaching zero. The product of thousands of $\eta_i \approx 10^{-13}$ will be indistinguishable from zero in floating point numbers. This leads to $G(\lambda) \to -\infty$ on a computer.

The problem of calculating the determinants of large matrices can make computations like in (5.25) seem unfeasible, especially because this is a discrete approximation of a continuous system. As discussed in Chapter 3, there is an unbounded difference between the discrete approximation of the differential operator $L$ and the actual laplacian $\nabla^2$. The discretisation, quantified by the number of nodes $n$, taken to the limit $n \to \infty$ leads to $\det(L) \to \infty$. This seemingly destroys the consistency of the discretisation.

Determinants are not well defined in this limit. However, the ratios of determinants are well defined. This can be clearly seen for the ratio of closely related matrices by examining the identity (Gohberg, Goldberg, and Krupnik, 2012)

$$\ln(|I + tF|) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r!} t^r \text{tr}(F^r).$$

(5.30)

The ratio of two similar matrices is close to equalling the identity. This would make $t$ in (5.30) very small and thus, the sum is convergent.

This highlights some of the reasons why a Bayesian framework can be daunting and unpopular: the computational capacity and care required in setting up the method can
be intimidating. While considerations are necessary, the issue is not as bad as it seems because the only thing that needs to be calculated is the acceptance probability $\alpha$. Given a state $\vartheta_i$ and a proposal state $\vartheta_p \approx \vartheta_i$ drawn from the proposal probability density $q(\vartheta_p|\vartheta_i)$, the acceptance probability is

$$\alpha(\vartheta_i, \vartheta_p) = \min \left( 1, \frac{\pi(\vartheta_p|d) q(\vartheta_p|\vartheta_i)}{\pi(\vartheta_i|d) q(\vartheta_i|\vartheta_p)} \right).$$

(5.31)

Looking only at the ratio of the marginal probabilities from (5.26) we get

$$\frac{\pi(\vartheta_p|d)}{\pi(\vartheta_i|d)} = \left( \frac{\lambda_p \gamma_p}{\lambda_i \gamma_i} \right)^{n/2} \exp \left\{ -\frac{1}{2} \left[ G(\lambda_p) - G(\lambda_i) + \gamma_p F(\lambda_p) - \gamma_i F(\lambda_i) \right] \right\} \frac{\pi(\vartheta_p)}{\pi(\vartheta_i)}. \quad (5.32)$$

Because $\vartheta_i \approx \vartheta_p$, the problem involves calculating differences of closely related objects, not the full determinants.

### 5.3.3 Evaluating $G$ and $F$

Both $G(\lambda)$ and $F(\lambda)$ are in fact very simple functions despite their calculation being computationally intense involving large matrices. They are both well-behaved, monotonically increasing functions of a single variable over a reasonable range of $\lambda$ (Fox and Norton, 2016).

A plot of $G(\lambda)$ and $F(\lambda)$ for a large range of $\lambda$ using an example sinusoidal data with added noise $\gamma = 10$ is in Figure (5.1). The previously stated features can clearly be seen when looking at the plots.
Direct calculation of this would be foolhardy with the larger matrices of finer discretisations. Calculating the determinants will lead to hitting computational limits for floating point numbers or yield an insignificant number. Computing the ratio of two insignificant numbers will amplify any computational errors from calculating the determinants. Instead (5.26) should be calculated as a single entity.

In this case \( \vartheta_p \approx \vartheta_i \) and simplifications can be made. One route could be utilising the identity (5.30) found by rearranging (5.26). This is not what is done in this thesis but is included for the sake of completeness.

The path that this work takes is to use Taylor series expansions on the well behaved functions. Fox and Norton (2016) used a 4th order Taylor expansion, but stated that this level is more than was needed. A comparison of various expansions is in Section 5.3.6.

Taking the Taylor expansions of these functions in \( \lambda_p \) about \( \lambda_i \), the ratio (5.26) becomes

\[
\frac{\pi(\vartheta_p | d)}{\pi(\vartheta_i | d)} = \exp \left\{-\frac{1}{2} \sum_{r=1}^{\infty} \left( \frac{G^{(r)}(\lambda_i)}{r!} + \gamma \frac{F^{(r)}(\lambda_i)}{r!} \right) r! \frac{G^{(r)}(\lambda_i)}{r!} \left( \lambda_p - \lambda_i \right)^r + h(\vartheta_p) - h(\vartheta_i) \right\},
\]

(5.33)

where the superscript \( (r) \) refers to the \( r \)th derivative in \( \lambda_p \) and \( h(\vartheta) \) is a shorthand for the remaining terms. The 0th order term cancels out due to the difference, and thus the only important calculations are the derivatives of \( G(\lambda) \) and \( F(\lambda) \).

Using the identities

\[
\frac{\partial}{\partial x} A^{-1} = -A^{-1} \frac{\partial A}{\partial x} A^{-1},
\]

(5.34)

\[
\frac{\partial}{\partial x} |A| = \text{tr} \left( A^{-1} \frac{\partial A}{\partial x} \right), \text{ and}
\]

\[
\frac{\partial}{\partial x} \text{tr}(A) = \text{tr} \left( \frac{\partial A}{\partial x} \right),
\]

(5.35)

(5.36)

the derivatives are given by:

\[
G^{(r)}(\lambda) = (-1)^{r+1} \text{tr}((\Lambda^{-1} L)^r),
\]

(5.37)

and

\[
F^{(r)}(\lambda) = (-1)^{r+1} r! \text{d}^T A(\Lambda^{-1} L)^r \Lambda^{-1} A^T \text{d}.
\]

(5.38)
There is a significant deviation from the work of Fox and Norton (2016) here. Fox and Norton recommend using the Monte Carlo estimate of the trace,

\[
\text{tr}(\Lambda^{-1} L) = E[z^T (\Lambda^{-1} L)^T z],
\]

where each sample \(z_i \sim \text{Uniform}(-1, 1)\) (Meurant, 2009). This is a very accurate and efficient estimator of the trace. It is shown by Meurant that four \(z\) samples are sufficient for the estimate and in my own investigations this was the case. The deviation from the true trace is approximately 0.1% using 4 samples with negligible improvement using any more.

This estimator worked very well for Fox and Norton’s deblurring problem. In the EMFT case, \(G(\lambda_p) - G(\lambda)\) can be of the order \(10^4\) for reasonable discretisations. However, this almost exactly cancels with the \(h(\vartheta)\) terms. The 0.1% error of the gradient can thus result in the estimated gradient of \(\frac{\pi(\vartheta_p | d)}{\pi(\vartheta_i | d)}\) being the wrong sign at times. That is, it can have a higher probability to step away from the bulk of the true probability distribution than towards it.

For this reason the functions (5.37) and (5.38) are calculated through solving the equations without any approximations, as the calculations are more accurate, with little loss of computational time for the level of discretisation used.

The work in this thesis uses a 2nd order Taylor expansion because this has the least error over the range of the chosen proposal distribution. Details about this are found in Section 5.3.6.

5.3.4 Hyperprior

The hyperprior in the EMFT sets the probability that the precision, \(\gamma\), and the smoothness length scale \(\delta\) can be in the reconstruction.

Commonly used hyperpriors are either uninformative priors, designed to minimize any biases introduced by the prior for a desired quantity; or conjugate priors, which lead to posteriors distributions of the same family as the prior, such as exponential distributions, aiding in calculations (Carlin and Louis, 2008; Berger, Strawderman, et al., 1996). Many works, such as Fox and Norton (2016); Agapiou, Bardsley, Papaspiliopoulos, and Stuart (2014); Bardsley (2012), aim for a middle-ground by using a gamma function prior. A gamma function prior is a conditionally conjugate prior and can be considered uninformative (Bardsley, 2012). However, if there is a priori

\(^2\)through the regularisation parameter \(\lambda\)
knowledge on the parameters then an informative prior, such as a Gaussian, can be used (Wikle et al., 1998; Wikle and Hooten, 2006). The MTC sampler does not need the mathematical convenience brought by using a conjugate prior (Fox and Norton, 2016).

The method presented in this thesis was made partially to introduce the Bayesian formulation to the EMFT world, but also to detect groundwater flow. The speed of groundwater can vary drastically from normal river speeds through underground channels, sometimes being measured in centimetres per year. As such, the hyperprior is chosen to be as uninformative of scale as possible.

Additionally, hyperprior is not required to be normalisable because certain improper hyperpriors allow for the marginal posterior to still be normalisable. Thus, I naively used a constant prior as an initial uninformative prior.

**Constant prior**

Using an unbounded constant hyperprior caused problems with the MCMC. Sometimes the MCMC would work and other times the chain would wander off seemingly to infinity. Examination of the log-probability of the marginal density in \( \lambda \) for a chosen value of \( \gamma \) revealed the problem. In Figure 5.2 there is an initial local maximum about where is expected. However as \( \lambda \) increases, the “probability” passes a local minimum and increases to higher “probability” beyond that previous maximum. Thus, if the MCMC’s random walk went beyond the minimum it would increase forever. The MCMC worked when the local maximum was prominent enough and the proposal step-size small enough to not pass the minimum.

Clearly, the constant prior does not lead to a normalisable posterior. This makes sense (with hindsight), as at any point there will always be immensely more of the “probability” to the right than to the left. Thus, the constant is not robust enough to use.

**Jeffreys prior**

It is clear that an unbounded constant prior has more “probability” contained within in every subsequent logarithmic step than the previous one. Thus a change in scale, changes the distribution. What is needed for scale invariance is the log of the prior to be constant, such that any two logarithmic brackets contain the same “probability”.

3quotation marks used here due to the lack of normalisation
This leads to Jeffreys’ prior defined by
\[ \pi(x) = \frac{1}{x}, \]  
(5.40)
This is invariant to any scale change of variables \( y = ax \), as explained in Jeffreys’ original paper (Jeffreys, 1946). The function \( x^{-1} \) is constant in log-space, as seen by examining an infinitesimal part of said function, \( dx/x = du \), under the change of variables \( u = \ln(x) \). This is a clear case where representation matters. The linear change of variables would lead to a very different answer in the constant prior case.

For the MTC, Jeffreys prior is
\[ \pi(\vartheta) = \frac{1}{\delta \gamma}, \]  
(5.41)
\[ = \frac{1}{\lambda \gamma^3}, \]  
(5.42)

taking into account the Jacobian of \( \gamma^{-1} \) from the change of variables. With this, the function \( h(\vartheta) \) in \( \pi(\vartheta|d) \) is
\[ h(\vartheta) = -(n - 6) \ln(\gamma) - (n - 2) \ln(\lambda), \]  
(5.43)

The equivalent plot of the marginal distribution for Jeffreys’ prior can be found in figure 5.3.
5.3.5 Metropolis-Hastings proposal distribution

Common choices of proposal distributions include static Gaussian distributions or top hat distributions. Distributions like these are popular as the proposal distributions have the trait that \( q(\theta_i|\theta_p) = q(\theta_p|\theta_i) \). This simplifies any calculation of the transition probability (5.31). However, static proposal distributions are not useful in this case.

The MTC application of this work is constructed to have scale invariance, thus a top hat distribution of, say unit width, would allow the MCMC random walk to be mobile around the scale of \( \sim 10 \). However relative movement around the scale of \( 10^4 \) would be much slower. Similarly, values around \( 10^{-4} \) would often be skipped over completely. The proposal could even be negative — completely outside the meaningful space for the hyperparameters. Given enough time, the MCMC should accurately sample the distribution, but it is very inefficient. Thus, a proposal step which changes with scale is preferable.

Plotting the distribution through direct calculation of the marginal for a low \( n \) in Figure 5.4, one can see that \( \pi(\theta|d) \) is approximately a log-normal distribution. Log-normal distributions are relatively simple distributions which change with the scale, thus are also appropriate for the proposal distribution.
A log normal distribution is defined by,

$$P(x) = \frac{1}{Sx\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - M)^2}{2S^2}\right),$$  \hspace{1cm} (5.44)

which has the point of highest density at $$x = \exp\left(M - S^2\right)$$, and mean at $$x = \exp\left(M + S^2/2\right)$$. However, this is perfectly Gaussian in log-space.

This proposal was chosen for two reasons. Firstly to reiterate, I am focusing on scale invariance and the log normal distribution is Gaussian in log-space. The log of the hyperparameters are the natural units for the proposal as there is no possibility of the proposal becoming negative and the size of the proposal step adapts to the scale of the parameters. The second reason is that if both $$\lambda_p$$ and $$\gamma_p$$ are drawn independently from log-normal distributions, then there is a simple ratio of forward and backwards probabilities of

$$\frac{q(\vartheta_i|\vartheta_p)}{q(\vartheta_p|\vartheta_i)} = \frac{\lambda_p\gamma_p}{\lambda_i\gamma_i}. \hspace{1cm} (5.45)$$

With this proposal density, the quotient in $$\alpha(\vartheta_i, \vartheta_p)$$ is

$$\frac{\pi(\vartheta_p|d)q(\vartheta_i|\vartheta_p)}{\pi(\vartheta_i|d)q(\vartheta_p|\vartheta_i)} = \exp\left\{-\frac{1}{2} \left[\Delta G + \Delta F - (n - 3) \ln\left(\frac{\gamma_p}{\gamma_i}\right) - n \ln\left(\frac{\lambda_p}{\lambda_i}\right)\right]\right\}, \hspace{1cm} (5.46)$$

where $$\Delta$$ is shorthand for the difference, calculated as described in section 5.3.3.

### 5.3.6 Comparison of approximations

The approximation of $$G$$ and $$F$$ are only necessary for $$\lambda$$, as both are only functions of $$\lambda$$. Thus, this section will be exploring the distribution $$\pi(\lambda|d, \gamma)$$ which can be obtained
by recognising that $\pi(\lambda, \gamma | d) = \pi(\lambda | d, \gamma)\pi(\gamma | d)$. If one assumes perfect knowledge of $\gamma$ then $\pi(\gamma | d) = \pi(\gamma) = \delta(\gamma - \gamma_0)$. Hence, all that is needed is a 1-dimensional MCMC over $\lambda$.

For this section I will set $\gamma_0 = 1000$ as this is a reasonable value for $\gamma$ based on later sections. However, it is important to state here that this work was originally done for a large range of $\gamma$ with little qualitative effect on the comparison and errors. The data used is the same for every method in a comparison and is generated from a shear phantom flow\(^4\) with the additive noise of precision $\gamma_0$.

In approximating the function (5.46), there are a number of different routes which one can take. Each must be judged based on two features: how accurate the approximation is and how computationally intense it is. In the initial comparison of different methods, we can look at just $\Delta G(\lambda)$ using the calculation of (5.37), as the findings can then be easily applied to $\Delta F(\lambda)$. The following comparisons are of first order Taylor series approximation of $\Delta G(\lambda)$.

The methods used to calculate $\Delta G(\lambda)$ are, as they pertain to Figure 5.5:

- Blue dotted line: The “true” method, calculating both $G(\lambda_p)$ and $G(\lambda_i)$ directly through

  \[
  G(\lambda) = \ln(\det(\Lambda(\lambda))) = 2 \sum_{k=1}^{n} \ln(\text{diag}(\sqrt{\Lambda})),
  \]

  where $\sqrt{\Lambda}$ is the Choleski decomposition. This was done purely as a comparison for error calculation. For this to produce a sensible result one must scale the matrix $\Lambda$ by a different constant for each mesh using the identity $\ln(\det(\Lambda)) = \ln(\det(a\Lambda)) - n \ln(a)$. The constants were found by trial and error and thus this is impractical for the final implementation.

- Magenta dot-dashed line: The Monte-Carlo estimate from (5.39).

- Green solid line: The inverse trace method. Solving equation (5.37) by directly calculating the inverse $\Lambda^{-1}$, then multiplying with $L$ to find the trace.

- Red dashed line: The solve trace method. Solving equation (5.37) by calculating $\Lambda^{-1}L$ by solving for $x$ in the equivalent equation

\[
\Lambda x = L.
\]

\(^4\)see Section 6.1.1
The last two methods were both included as I was told that it is better to use successive solves of linear equations instead of calculating the inverses of matrices, so I decided to test that.
(a) Computation time of different calculation methods for different numbers of mesh elements. Linear and log-log plot of same data.

(b) Error of gradient calculations of different methods for different numbers of mesh elements.

Figure 5.5: Scalings of the different methods of calculating first order approximation of $G(\lambda)$. The blue dotted line is the “true” method, the magenta dot-dashed line is the Monte-Carlo estimate, the green solid line is the inverse trance method, and the red dashed line is the solve trace method.
Figure 5.5a shows the computation time of each of these methods for different mesh sizes with 1st order approximations, as quantified by the number of nodes $n$. It is clear upon examination that these methods reach asymptotic behaviour at around the $n \approx 2000$ mark. Table 5.1 shows the gradient of this asymptotic behaviour. The fastest method is the impractical direct calculation, followed by the solve method, then the Monte-Carlo estimator. The slowest time is the inverse trace method, which validates the word-of-mouth wisdom.

<table>
<thead>
<tr>
<th></th>
<th>Solve trace</th>
<th>Inverse trace</th>
<th>MC estimate</th>
<th>“true”</th>
</tr>
</thead>
<tbody>
<tr>
<td>power ($n^p$)</td>
<td>2.62</td>
<td>2.72</td>
<td>2.45</td>
<td>2.29</td>
</tr>
</tbody>
</table>

Following that, Figure 5.5b shows the calculated gradient $\Delta G$ at the reference point $\lambda = 2 \times 10^{-4}$ for the different mesh sizes. The second plot shows the relative difference in gradients using the calculated “true” value as the reference. There is almost no discernible difference between the inverse, solve, and “true” values, as they are the same up to $10^{-13}$. In stark contrast to this, the Monte-Carlo estimator has an error of order $\sim 2 \times 10^{-3}$, backing up the previously stated reasons for why this method is not good for this situation.

Table 5.2: Times to compute derivatives of different methods. The calculation of $F(\lambda)$ is included for comparison.

<table>
<thead>
<tr>
<th></th>
<th>$F(\lambda)$</th>
<th>Solve trace</th>
<th>MC estimate</th>
<th>1st principles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st order</td>
<td>0.0283s</td>
<td>0.0522s</td>
<td>0.104s</td>
<td>0.0795s</td>
</tr>
<tr>
<td>2nd order</td>
<td>0.0375s</td>
<td>0.112s</td>
<td>0.116s</td>
<td>0.0977s</td>
</tr>
<tr>
<td>3rd order</td>
<td>0.0457s</td>
<td>0.180s</td>
<td>0.222s</td>
<td>0.132s</td>
</tr>
<tr>
<td>4th order</td>
<td>0.0571s</td>
<td>0.244s</td>
<td>0.391s</td>
<td>0.132s</td>
</tr>
</tbody>
</table>

Table 5.2 contains the calculation times for higher order Taylor approximations using a mesh with 825 nodes. The $F(\lambda)$ values were calculated using a solve method. An interesting quirk of using a Monte-Carlo estimator is that calculating the second order Taylor approximation is essentially free if first order is calculated. In fact, only the odd order derivatives add significant computation time so it makes sense to use an even order approximation. This was also noted by Fox and Norton (2016).
The 1st principles values of Table 5.2 were calculated by evaluating the function $G(\lambda)$ at nearby points to gain a discrete approximation of the appropriate derivatives. The first order level is essentially calculating the “true” value of $\Delta G$, and the 3rd and 4th order need the same number of points evaluated and as such are equally expensive.

However, there is one important difference between the practical implementation of calculating the “true” value and the 1st principles approximation. That is that an MCMC does not accept every proposal. For each reject step a new calculation of $G(\lambda_p)$ is required for the acceptance probability $\alpha$. However, this is not true for the first principles. Thus, with an ideal acceptance rate of 50% (Roberts, 2012), the “true” method requires on average three expensive Cholesky decomposition calculations, as opposed to the two for the 1st order 1st principles method.

Based on the above figures and discussion, it may seem that the best choice is to use no approximation and calculate $G(\lambda_p) - G(\lambda_i)$ using (5.48) directly. However, this is not a robust approach due to the required matrix scaling, but also in cases with either limited data points or for very large $n$. As discussed just before, the accuracy of calculating determinants decreases as the matrix becomes closer to a singular matrix. That is, as the product of all eigenvalues approaches zero. Adding another node to the mesh will essentially add another positive eigenvalue smaller than all previous ones. Whereas, removing a data point from the map will essentially remove one of the larger eigenvalues. An example of a problem that this can cause is the positive definite matrix $\Lambda$ generated from three data points not being recognised as a positive definite matrix by python’s `numpy.linalg.cholesky()` algorithm on the finer discretisations presented in the above analysis.

Thus, the solve trace method will be used in the final implementation of the MTC.

Figure 5.6 is the true distribution of the marginal density $\pi(\lambda|d, \gamma)$, with the peak lying at $\lambda \approx 6 \times 10^{-4}$. This was calculated on a mesh of 825 nodes with simple sinusoidal data.

The specific mesh was chosen as it is low enough that the true values could be calculated without much difficulty, while remaining accurate when compared to finer meshes. The choice was made qualitatively by running the MTC sampler on the same data set, refining the mesh until there was no perceivable difference in the resultant probability distribution, and then choosing refining a bit more to be safe. Similarly the increased refinement near the domain’s edge was also chosen qualitatively by incrementally lowering the relative characteristic length at the boundary until no perceivable change occurred. This ended up as a tenth the characteristic length of the centre.
Figure 5.6: Example true marginal density $\pi(\lambda|d, \gamma)$ for some sinusoidal data.

What follows are plots of (5.46) for various orders of Taylor expansions of the methods: Monte-Carlo estimator (Magenta dot-dashed), solve (Red dashed), calculating using first principles\(^5\) (blue-dotted line), and the true values in black.

The following sets of figures are of (5.46) calculated with $\lambda_i = 5 \times 10^{-4}$. This is within the bulk of the marginal density — where the MCMC will be for the majority of the time. Figures 5.7 and 5.8 are of the various methods, at various orders of Taylor expansions with $h(\vartheta)$ term calculated in full. Whereas Figures 5.9 and 5.10 are of an equivalent order of Taylor expansion of $h(\vartheta)$.

Figures 5.11 and 5.12 are once again of the Taylor approximation of $h(\vartheta)$, only this time at $\lambda_i = 1 \times 10^{-3}$ to demonstrate how it works further away from the bulk of the distribution.

Two things are immediately evident. Firstly the best approximation is the second order Taylor approximation of $G(\lambda)$, $F(\lambda)$, and $h(\vartheta)$ in Figure 5.9b. Secondly and unsurprisingly, the Monte-Carlo estimator is less accurate than the solve method. This can be quite extreme in some cases. Figure 5.13 shows one example of this where

\(^5\)Once again, this is somewhat redundant as it constituted just solving for the true values of the above terms, and 2nd order and above constituted more evaluations than the actual calculation of (5.46).
the gradient of the approximation is blatantly the wrong sign. This would mean that
the random walk could sometimes be more likely to step away from the bulk of the
marginal distribution than towards it.
Figure 5.7: Approximation methods with full $h(\vartheta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), first principles (blue-dotted line), and the true values (black solid line).
Figure 5.8: Approximation methods with full $h(\vartheta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), and the true values (black solid line).
Figure 5.9: Approximation methods with approximate $h(\theta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), first principles (blue-dotted line), and the true values (black solid line).
Figure 5.10: Approximation methods with approximate $h(\vartheta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), and the true values (black solid line).
Figure 5.11: Approximation methods with approximate $h(\vartheta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), first principles (blue-dotted line), and the true values (black solid line).
Figure 5.12: Approximation methods with approximate $h(\vartheta)$. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), and the true values (black solid line).
Figure 5.13: Example of the estimator having the complete wrong sign. The methods are: Monte-Carlo estimator (Magenta dot-dashed), solve inverse (Red dashed), first principles (blue-dotted line), and the true values (black solid line).
With all of this considered, the approximation method chosen for the MTC sampler used in this thesis is a second order Taylor approximation of \( \ln \left( \frac{\pi(\vartheta_p|d)q(\vartheta_i|\vartheta_p)}{\pi(\vartheta_i|d)q(\vartheta_p|\vartheta_i)} \right) \) in \( \lambda_p \).

The log normal proposal distribution was chosen to have width of \( S_\lambda = 0.4 \), which allows for mobility in the chain, while mostly remaining within the range of good approximation\(^6\).

Examples of the proposal probability density function \( q(\lambda_p|\lambda_i) \) next to the acceptance probability \( \alpha(\lambda_p, \lambda_i) \) are in Figure 5.14 for a single run of the marginal MCMC of \( \lambda \) only (\( \gamma = 1000 \)). The top figures in the subplots are of the acceptance probability with the dotted black line at 1, the dashed blue line is true value of 5.46, and the red solid line is the second order Taylor approximation. Any values of the quotient are capped at 1 in \( \alpha \), but the plotting range is extended to 2 for comparison’s sake.

The log-normal probability density functions looks a little misleading in Figure 5.14 due to them being probability density functions. The “center” of the distribution is where \( \lambda_p = \lambda_i \); where both functions cross the \( \alpha = 1 \) line. The range displayed here is where the vast majority of the random proposals \( \lambda_p \) will fall, given the \( \lambda_i \).

The fringe cases are not very important for two reasons. The log probability is a slowly changing parabola, such as Figure 5.15 which is the log of Figure 5.14a. Thus, any significant changes in the acceptance probability \( \alpha \) will often occur far away from the displayed range. However in cases such as Figure 5.14b, the error at the edge of the range will only serve to widen the estimated marginal posterior \( \pi(\vartheta|d) \), as shown in Figure 5.16.

This widening in the marginal posterior is not a significant issue again for two reasons. Firstly, it is mostly affecting the distribution of the non-physical hyperparameters. The second and most important reason is related to one of the goals of numerical modelling: that the error of the model itself is larger than any numerical error. The modelling error is the difference between the real-world and the idealised model. This will be significant in the vast majority of models, including this one. Thus, the widening of the distribution — an increase in the predicted uncertainties — is not an issue if it is not to a ridiculous degree or does not introduce any bias.

---

\(^6\)This is only for the MCMC over \( \lambda \) and is lowered for the full MCMC over \( \vartheta \) to \( S_\lambda = 0.15 \) and \( S_\gamma = 0.05 \) in order to maintain a good acceptance rate.
Figure 5.14: Comparison of true (blue dashed) and estimated (red solid) $\alpha(\lambda_p, \lambda_i)$ with the proposal probability $q(\lambda_p|\lambda_i)$. 

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Figure 5.15: The log plot of figure 5.14a.

Figure 5.16: Histogram of 5,000 samples generated from a MCMC run, with an overlay of the true marginal distribution — scaled for visualisation purposes.
5.4 Down-sampling

Once the distribution $\pi(\vartheta|d)$ is approximated to a desired level with the random walk it can be down-sampled to create approximately iid. samples. This has the advantage of maintaining about the same variance reducing power as the full chain, while having far fewer samples and therefore calculations. Thus, is preferable for speeding up computation.

The down-sampling rate can be chosen in an informed manner by setting it to the integrated autocorrelation (IACT) from Section 5.1.4:

$$\tau = 1 + \sum_{k=1}^{\infty} \text{corr}(\lambda_0, \lambda_{0:k}),$$

where $\text{corr}(\lambda_0, \lambda_{0:k})$ is the autocorrelation of the sequence in $\lambda$ generated by the random walk. In practice, this is generated by taking a slice of the sequence (after the burn-in time has been removed) and calculating the correlation between that and the rest of the sequence. The autocorrelation for a single run of the MCMC is presented in the Figure 5.17, with an IACT of 9.62.

The IACT represents the number of steps in the correlated random walk required to have the same variance reducing power as a single uncorrelated sample. Thus, if the MCMC down-sampled to every $\lceil \tau \rceil$ steps (10 steps in Figure 5.17), then each sample is essentially independent/uncorrelated.

![Figure 5.17: Example autocorrelation generated from a single MCMC run.](image)

An important thing to note when using the IACT is that correlations are a second order statistic; they are calculated using means and variances. Therefore, the IACT is ideal for Gaussian distributions or cases where the central limit theorem applies. Corollary to this is that it loses meaning as distributions become less Gaussian.
The log-normal distribution is similar to a Gaussian with a more skewed shape. This will affect the autocorrelation differently based on where the reference sub-sequence is mostly located. The probability density function is greater but more condensed for values lower than the mean. So if the sub-sequence is lower than the mean then the calculation will yield a highly correlated sequence. If the sub-sequence is greater than the mean, then it will be calculated as being uncorrelated. These two effects cancel each other out somewhat, but it does mean that the IACT can be variable from run to run. I have calculated IACTs ranging between 4 to 30.

5.5 Full conditional for $f$.

Given a set of samples approximating $\pi(\vartheta|d)$, calculating the full conditional $\pi(f|\vartheta, d)$ is simple. Going back to the hierarchical stochastic model (5.19) combined with Bayes’ rule, the samples for $f$ are treated as random variables drawn like

$$f|\vartheta, d \sim N(\mu_{f|\vartheta,d}, Q_{f|\vartheta,d}^{-1}),$$

where

$$\mu_{f|\vartheta,d} = \gamma(\gamma A^T A + \delta L)^{-1} A^T d, \quad (5.52)$$
$$Q_{f|\vartheta,d} = \gamma A^T A + \delta L. \quad (5.53)$$

This is a Gaussian distribution which makes drawing random samples equivalent to a linear algebra problem. A single independent sample $f|\vartheta, d$ can be gained by solving

$$(\gamma A^T A + \delta L)f = \gamma A^T d + w, \quad (5.54)$$

where $w = w_1 + w_2$ with independent $w_1 \sim N(0, \gamma A^T A)$ and $w_2 \sim N(0, \delta L)$ (Fox and Norton, 2016). Dividing both sides by $\gamma$ yields

$$(A^T A + \lambda L)f = A^T y + w', \quad (5.55)$$

with $w' = w'_1 + w'_2$ now drawn from $w'_1 \sim N(0, A^T A)$ and $w'_2 \sim N(0, \lambda L)$.

The Full conditional posterior $\pi(f, \vartheta|d)$ can thus be generated by using each sample of $\pi(\vartheta|d)$ to draw an independent sample $f|\vartheta, d$ due to the relationship (5.22). Thus, only the one MCMC in $\vartheta$ is required to calculate expectation values over $f$ in $\pi(f, \vartheta|d)$. This fact saves on computational time when compared to Gibbs or one-block samplers.

An interesting note here is that the posterior samples $f|\vartheta, d$ only depend on the ratio of $\lambda = \delta/\gamma$. This forms a further parallel to regularisation methods.
5.6 Marginal then conditional summary

Given a forward map \( A \) and some data \( d \), the MTC algorithm can be summarised as such:

1. Sample from the marginal posterior distribution using a Metropolis-Hastings MCMC with proposal density \( q(\vartheta_p|\vartheta_i) \), and the acceptance probability is given by

\[
\alpha(\vartheta_p, \vartheta_i) = \min\left(1, \frac{\pi(\vartheta_p|d)q(\vartheta_i|\vartheta_p)}{\pi(\vartheta_i|d)q(\vartheta_p|\vartheta_i)}\right),
\]

\[
\pi(\vartheta_p|d)q(\vartheta_i|\vartheta_p) = \exp\left\{-\frac{1}{2} \left[ \Delta G + \Delta F - (n - 3) \ln\left(\frac{\gamma_p}{\gamma_i}\right) - n \ln\left(\frac{\lambda_p}{\lambda_i}\right) \right]\right\},
\]

and \( S \) is diagonal with entries \( S_\lambda = 0.15 \) and \( S_\gamma = 0.05 \). These are chosen to have an acceptance rate of around 50%. If the MCMC is run only over \( \lambda \), then \( S_\lambda = 0.4 \) is ideal.

2. The burn-in of the MCMC is removed from the start of the chain.

3. The IACT, \( \tau \), is calculated for both the chain in \( \lambda \) and in \( \gamma \). The larger of the two is used to down-sample the MCMC output by selecting every \( \lceil \max(\tau_\lambda, \tau_\gamma) \rceil \) sample.

4. The full conditional is then sampled for each \( \lambda \) from the down-sampled chain by solving

\[
(A^T A + \lambda L) f = A^T d + w',
\]

with \( w' = w'_1 + w'_2 \) now drawn from \( w'_1 \sim N(0, A^T A) \) and \( w'_2 \sim N(0, \lambda L) \).

5. Expectations can then be calculated by Monte-Carlo integration:

\[
\int h(f)\pi(f)df \approx \frac{1}{N} \sum_{i=1}^{N} h(f_i).
\]

### 5.6.1 Example of marginal then conditional sampling

A single run of the MCMC is presented here with 10 equidistant measurement points, the data generated from the phantom flow \( f = (x_1 + y_2 + \sqrt{2}) \), multiplied such that...
(a) Phantom flow. The “true” $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean reconstructed flow. Estimated Bulk flow $f_{\text{bulk}} = 10.018 \pm 3.136 \text{Tms}^{-1}$.

(c) Measured potentials

(d) Standard deviation of flow.

Figure 5.18: Simple comparison between the MTC estimates and the phantom flow.

The bulk flow is $10 \text{Tms}^{-1}$, with added iid Gaussian noise of $\gamma = 100$. The phantom mesh has 815 nodes, and the MTC sampler was run on a mesh with 790 nodes.

The MTC using 10000 steps and 100 sub-samples yielded the estimate $\lambda = 0.00165 \pm 0.03352$, where the uncertainty is 2 standard deviations. This is not a useful statistic because the distribution is log-normal. The second order statistics of the log of the samples are more meaningful. As such, for this example $\log_{10}(\lambda) = -3.1667 \pm 0.8270$. This is how the hyperparameters will be displayed in future.

For this run of the MTC sampler the prediction of the bulk flow — the main quantity of interest — was $f_{\text{bulk}} = 10.018 \pm 3.136 \text{Tms}^{-1}$. The total phantom flow is $10 \text{Tms}^{-1}$ so this is very accurate as can be seen by comparing the phantom flow (Figure 5.18a) and the mean reconstruction (Figure 5.18b). Figure 5.18c is of the data gained from the additive noise on the forward mapped phantom flow and Figure 5.18d is of the
standard deviation of the individual nodes in the reconstruction.

This is already an example of all the previously stated advantages that the Bayesian approach has over regularisation methods. The two main advantages are the quantitative calculation of expectation values to get the bulk flow rate and the increase in information which can be gained, such as the confidence in this prediction — shown by the standard deviations.

The figures in 5.19 are examples of other information produced by this MTC run. Figure 5.19a shows the samples of the marginal density without the burn-in. This is a qualitative prediction of what is probable for the regularisation parameter $\lambda$ given the models and data and consequently the smoothness parameter/length scale $\delta$. When the MTC is run in 2D, then it also predicts what the noise level of the data is likely to be, given the model. The exact wording of that last statement will become important in the next chapter.

Figure 5.19b is the calculated autocorrelation. A large IACT of 44.44 was produced for this run and chosen sub-sequence. Figure 5.19c is a comparison of the data and the expected “true” potential calculated as $u_{\text{mean}} = A f_{\text{mean}}$ against the location in $\theta$. There is almost no perceptible difference as $\gamma = 100$ means that the std of the additive noise is 0.01.

Figures 5.19d and 5.19e are further “slices” of the posterior which can be useful. The former shows the samples of the flow $f$ at the node located at $x = (0,0)$. These samples approximately represent, and will tend towards, a Gaussian distribution. Thus the second order statistics of the flow rate as shown in Figure 5.18 are sufficient to fully characterise the conditional distribution for $f$. Finally there is Figure 5.19e, which is of a single sample from the MCMC chain. It was previously stated that even a single sample can provide qualitative results and Figure 5.19e shows that this is true. The approximate scale for the flow, the level of smoothness, and the shape of the flow are three such pieces of information.

Future results will be presented as in Figure 5.18 including a means of displaying the marginal distribution such as in Figure 5.19a.
(a) Samples of marginal distribution.

(b) Autocorrelation of MCMC chain.

(c) Measured potentials (blue dashed line) and $A_{f_{\text{mean}}}$ (red solid line).

(d) Samples of the conditional density of the node at (0,0).

(e) Flow from a single sample.

Figure 5.19: Possible outputs of the MTC.
Chapter 6

Circular Pipe EMFT

6.1 Preliminaries

This chapter examines the behaviour of the MTC sampler as it pertains to circular pipe EMFT. There are a number of things to explore. To examine how real world considerations affect the estimates, the MTC method is applied to different flow profiles (Section 6.2.3) and different number of measurements (Section 6.2.4).

Other useful things to explore do not have real-world analogies. There are a number of modelling decisions — mathematical and statistical — which have influence over the solutions to the inverse problem. Examining how $\gamma$ influences the estimates leads to some interesting insights on the statistical model, covered in Sections 6.2.1 and 6.2.4. Additionally, Chapter 3 discussed the no-slip assumption from fluid dynamics. The inclusion of this assumption has a significant effect on the posterior, explored in Section 6.2.2.

6.1.1 Phantom flows

What follows is a list of the functions describing common, real-world flow fields used as phantoms in the chapter (Lehtikangas et al., 2016). Some of the fields described in the Lehtikangas et al. have been omitted due to their similarities to the other fields\(^1\).

Unless otherwise stated, the following setup for the MTC is used. The pipe was chosen to have unit radius ($R = 1$), and the measurement data was generated by applying the forward map to the various phantom flow fields. The bulk flow rate is set to $10\text{Tms}^{-1}$ for the phantoms. Additive Gaussian iid noise was applied to the

\(^1\)such as the annular flow and parabolic both being axially symmetric with no flow at the boundary
measurement with $\gamma = 10$. This corresponds approximately to a noise level of $\sigma = .1 \approx 2\% \max(|u|)$ for most cases. The phantom flow was generated on an unstructured triangular mesh with 1213 nodes and, in order to avoid performing an inverse crime, the MTC was run on a different unstructured mesh with 815 nodes.

**Constant flow**

The first and most simple flow field is that of a constant. The work in this thesis was initially motivated with the assumption that flow was constant. In the EM flowmeter theory\(^2\) the measurement from axisymmetric flow is equal to that of constant flow. Thus, it is useful to include a constant flow in the analysis.

$$f_{\text{constant}} := f(r, \phi) = f_{\text{max}}.$$

\(^6\)see Chapter 2.

**Parabolic flow**

Parabolic flow is an axisymmetric flow field commonly found in “slow horizontal laminar flow”. The parabolic flow field is given by

$$f_{\text{parabolic}} := f(r, \phi) = f_{\text{max}} \left(1 - \frac{r^2}{R^2}\right).$$

**Stratified flow**

Stratified flow is an asymmetric flow field commonly found in “slow horizontal flow containing different solid materials”. The field is given by

$$f_{\text{stratified}} := f(x, y) = f_{\text{max}} \left(1 - \frac{y}{R}\right).$$

**Quartic flow**

Quartic flow fields are asymmetric and commonly found in “flow after a pipe elbow”. Thus they can be considered similar to the kind of “upstream disturbance” which Williams spoke of (Williams, 1930). This is given by

$$f_{\text{quartic}} := f(x, y) = f_c \left(1 + f_1(x, y) + f_2(x, y) + f_3(x, y)\right),$$

for some constant $f_c$, and

$$f_1(x, y) = \frac{y}{R},$$

$$f_2(x, R) = \frac{y}{R},$$

$$f_3(x, y) = \frac{y}{R},$$
\[
f_2(x, y) = \frac{1}{2} \left( \frac{x + y}{R} \right)^2 - 0.25, \quad (6.6)
\]
\[
f_3(x, y) = \left( \frac{x}{R} \cos \left( \frac{\pi}{9} \right) + \frac{y}{R} \sin \left( \frac{\pi}{9} \right) \right)^3 - 0.5 \left( \frac{x}{R} \cos \left( \frac{\pi}{9} \right) + \frac{y}{R} \sin \left( \frac{\pi}{9} \right) \right), \quad (6.7)
\]
and
\[
f_4(x, y) = \left( \frac{x}{R} \right)^4 - 0.75 \left( \frac{x}{R} \right)^2 + 0.0625. \quad (6.8)
\]

6.2 Results

6.2.1 Effect of hyperparameters $\gamma$ and $\lambda$

Note: all runs of the MTC for this section have the same data — excluding Figure 6.2.

One of the first things noticed when running the MTC sampler is that it overestimates $\gamma$. Figure 6.1 is generated with stratified phantom flow using 16 measurements. The estimate of $\log_{10}(\gamma) = 5.084 \pm 0.056$ in Figure 6.1f is very far from the value of $\gamma = 10$ used to generate the data. This is a recurring feature of this version of the MTC sampler and will be pointed out again when relevant.

Ultimately this is due to it being far more likely for deviations in the data — which we know to be noise — to be from the true flow itself for the given the forward map, the prior information, and the uninformative data. Hence the smoothness/length-scale parameter is predicted to be $\log_{10}(\delta) = -1.357 \pm 0.616$, i.e. very small, allowing for small scale variations in the flow. Sharp changes in the mean flow reconstruction are evident near the electrodes in Figure 6.1b.

This behaviour in the posterior is a consequence of the larger influence of flow near the electrode than flow further away (as discussed in Chapter 3) and the uninformative prior. The electrodes are too far apart for the system to effectively distinguish noise from “truth” in the data. Exemplifying this is Figure 6.2 with 500 measurements\(^3\). The densely placed measurements are much more correlated than the previous figure and thus the noise is more distinguishable from the truth. Figure 6.2c shows that the estimated $\gamma$ is much more reasonable at $\log_{10}(\gamma) = 2.262 \pm 0.552$. This behaviour is a trend as $\gamma$ tends towards the true value as more measurements are added.

There is a clear flaw in the example of Figure 6.2. The measurement spacing is finer than the spacing of the mesh nodes around the boundary. Nearby measurements are highly correlated because of this but measurements that are finer than discrete elements are not a good representation of the EMFT system.

\[^3\text{far too many measurements than practical EMFT system}\]
The measurement kernel (from Chapter 3) has singularities at the measurement points which implies that the trend of over-fitting the data (large $\gamma$) is a feature of the forward map. The combination of the contributions of flow closer to the measurement location being far larger than that of distant flow and the uninformative hyperprior means that the MTC will see the measurements as uncorrelated and will thus over-fit the data.

Figures 6.3 and 6.4 are two examples of the MTC run where $\gamma$ is assumed to be known perfectly. These are generated by choosing the hyperprior to be $\pi(\theta) = \pi(\lambda)\delta(\gamma - \gamma_0)$, where $\gamma_0 = 10$ and 1000 respectively. This situation is not realistic as there will always be uncertainty in measurement noise levels. A more appropriate prior for $\gamma$ could be a Gaussian or similar distribution about the expected value, as unknown factors, such as temperature variations and error in setup will exist. Nonetheless, these figures demonstrate the effect different $\gamma$s have on the reconstruction, which effectively forces a weight on the data misfit side of the reconstruction, then estimates the level of smoothness.

Figure 6.3 is generated with $\gamma$ set to 10: the true value. This does a remarkably good job at reconstructing the phantom flow field and predicting the bulk flow rate, estimating $\log_{10}(\lambda) = -2.090 \pm 0.820$. That would mean that the length scale parameter $\delta \approx 0.1$.

A more extreme version of data over-fitting is in Figure 6.4 where $\gamma$ is chosen as 1000. There is a clear difference between the mean (Figure 6.3b) and phantom as the features of the reconstruction are more local to the measurement locations than the linear 6.3. The marginal density gives $\log_{10}(\lambda) = -4.216 \pm 0.828$ and the length scale parameter $\delta \approx 0.06$, smaller than the previous one, but still within a reasonable range.

Going up another two orders of magnitude in $\gamma$ produces results very similar to Figure 6.1, where $\gamma \approx 10^5$. This estimates $\log_{10}(\lambda) = -6.441 \pm 0.622$ and thus $\log_{10}(\delta) = -1.357 \pm 0.616$, or $\delta \approx 0.044$. A loose conclusion from this is that the parameter $\delta$ does not readily get much smaller than this point.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 9.899 \pm 3.198 \text{Tms}^{-1}$.

(c) Scatter plot of samples from marginal distribution.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.441 \pm 0.622$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.084 \pm 0.056$.

Figure 6.1: Full MTC sampler for stratified flow with 16 measurements. The data is omitted for brevity’s sake as it is the same throughout most of this section.
(a) Simulated data of 500 measurements.

(b) Mean flow, $f_{\text{bulk}} = 9.982 \pm 3.14 \text{Tms}^{-1}$.

(c) Scatter plot of samples from marginal distribution.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -2.985 \pm 1.778$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 2.262 \pm 0.552$.

Figure 6.2: Full MTC sampler for stratified flow with 500 measurements.
(a) Phantom flow, \( f_{\text{bulk}} = 10 \text{Tms}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 9.916 \pm 3.218 \text{Tms}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of marginal distribution, \( \log_{10}(\lambda) = -2.090 \pm 0.820 \).

Figure 6.3: MTC sampler for stratified flow with 16 measurements and \( \gamma = 10 \).
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 9.886 \pm 3.176 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of marginal distribution, 
\[ \log_{10}(\lambda) = -4.216 \pm 0.828. \]

Figure 6.4: MTC sampler for stratified flow with 16 measurements and $\gamma = 1000$. 

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6.2.2 No-slip assumption

Note: all runs of the MTC for this section use the same data.

It was discussed in Chapter 3 that the Green’s function solution to the PDE consisted of the sum of two integrals: the volume integral and a boundary integral. Papers in the field of EMFT use the no-slip condition from fluid dynamics to eliminate the boundary integral. While doing so does simplify the mathematical and numerical models, it also adds a degree of complexity to the inverse problem. If one assumes that the flow has zero velocity at the boundary, then one restricts the space of possible solutions to the inverse problem.

Work such as by Lehtikangas et al. (2016) and Kollár et al. (2014) use the no-slip condition to simplify their mathematical and numerical models. This is contradicted by some of the presented reconstructions, which have a non-zero velocity at the boundary. These chosen states lie outside the space of possible states. This is problematic because the contribution to the measurement from the boundary integral is significant. Using the no-slip condition can significantly affect the solution to the inverse problem and should be consistent with any reconstruction (as explained in Chapter 3).

Figure 6.5 is a comparison between an MTC run with the boundary integral included (slip allowed, on the left) and without it (no-slip condition, on the right) for a stratified phantom flow. This flow field has non-zero flow at the boundary. Qualitatively there is little difference between these two. Both methods predict the bulk flow rate with high accuracy and have similar shaped mean flows. The main difference is the scale of data. The data is created using the forward map based on the respective boundary condition for \( f \). If the condition is wrong in the model, then the reconstruction would also be of the wrong scale. For example, the no-slip condition is not valid in Figure 6.5 but if it is assumed true, then the reconstructed flow field would be almost an order of magnitude larger than it is in reality.

In contrast to this, Figure 6.6 has a parabolic flow profile where the no-slip condition is true. Once again, the left is with a slip possible and the right has no-slip\(^4\). The problem posed in this figure is the opposite of the previous; now the inclusion of the boundary integral will cause problems. The estimated bulk flow rate in the no-slip Figure 6.5d is very accurate to the true rate of 10. However, the prediction in Figure 6.5c is blatantly wrong because the scale of the data is small and the contribution to the measurement from the boundary integral is large when it is included. It is more

\(^4\)That said, I have not limited the space of possible states as should be done because what is presented here is sufficient to make the point.

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likely (in the model) that the measurements are dominated by flow near the boundary. Hence, the prediction is almost an order of magnitude smaller when one wrongly assumes that the flow at the boundary is non-zero.

For the rest of this thesis the boundary integral will be included as it matches all but one of the phantom flow profiles presented.
Figure 6.5: MTC sampler for stratified flow with 16 measurements. Results of with (right) and without (left) the no-slip condition.

(a) Slip data.
(b) No-slip data.
(c) Slip mean flow, $f_{\text{bulk}} = 9.899 \pm 3.198 \text{Tms}^{-1}$.
(d) No-slip mean flow, $f_{\text{bulk}} = 9.931 \pm 3.112 \text{Tms}^{-1}$.
(e) Samples from slip marginal distribution, $\log_{10}(\lambda) = -6.441 \pm 0.622$ and $\log_{10}(\gamma) = 5.084 \pm 0.056$.
(f) Samples from no-slip marginal distribution, $\log_{10}(\lambda) = -6.470 \pm 0.518$ and $\log_{10}(\gamma) = 5.082 \pm 0.056$.
Figure 6.6: MTC sampler for parabolic flow with 16 measurements. Results of with (right) and without (left) the no-slip condition.
6.2.3 Different phantom flows

Figures 6.7, 6.8, 6.9, and 6.10 show single runs of the MTC sampler for constant, parabolic, stratified, and quartic flow respectively. The second order statistics are shown in Table 6.1.

Table 6.1: Second order statistics of the MTC sampler for different flow fields (true \( f_{\text{bulk}} = 10 \text{Tms}^{-1} \)).

<table>
<thead>
<tr>
<th></th>
<th>( f_{\text{bulk}} (\text{Tms}^{-1}) )</th>
<th>( \log_{10} \lambda )</th>
<th>( \log_{10} \gamma )</th>
<th>( \log_{10} \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>9.833 ± 3.144</td>
<td>-6.193 ± 0.540</td>
<td>5.084 ± 0.056</td>
<td>-1.109 ± 0.534</td>
</tr>
<tr>
<td>Parabolic</td>
<td>1.588 ± 3.130</td>
<td>-6.089 ± 0.532</td>
<td>5.082 ± 0.058</td>
<td>-1.007 ± 0.526</td>
</tr>
<tr>
<td>Stratified</td>
<td>9.899 ± 3.198</td>
<td>-6.441 ± 0.622</td>
<td>5.084 ± 0.056</td>
<td>-1.357 ± 0.616</td>
</tr>
<tr>
<td>Quartic</td>
<td>12.651 ± 3.162</td>
<td>-6.685 ± 0.574</td>
<td>5.082 ± 0.058</td>
<td>-1.604 ± 0.564</td>
</tr>
</tbody>
</table>

The estimated bulk flow rates are very accurate for both the stratified and constant cases but completely wrong for the parabolic case. This is to be expected as both the constant and stratified flow fields can be easily interpolated through examining the boundary values. This is contrasted in the parabolic case because the internal shape of the flow fields is not easily interpolated from the boundary values of \( f = 0 \). In between these cases is the quartic flow, which is less accurate than the two smoother phantoms, but does have the true values within the main support of the distribution. The true bulk flow rate is less than \( 2 \sigma \) from the mean in this case. The phantom flow in Figure 6.10a has a somewhat hollow shape, at least on the right hand side. That is, the flow values on the boundary are larger than the nearby interior flow. Hence, the flow is hard to predict because the exact shape of the field is not a simple interpolation.
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 9.833 \pm 3.144\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.193 \pm 0.540$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.084 \pm 0.0354$.

Figure 6.7: Full MTC sampler for constant flow with 16 measurements.
(a) Phantom flow, \( f_{\text{bulk}} = 10 \text{Tms}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 1.588 \pm 3.130 \text{Tms}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of \( \lambda \) from marginal distribution, \( \log_{10}(\lambda) = -6.089 \pm 0.532 \).

(f) Samples of \( \gamma \) from marginal distribution, \( \log_{10}(\gamma) = 5.082 \pm 0.058 \).

Figure 6.8: Full MTC sampler for parabolic flow with 16 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 9.899 \pm 3.198\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.441 \pm 0.622$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.084 \pm 0.056$.

Figure 6.9: Full MTC sampler for stratified flow with 16 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 12.651 \pm 3.162 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.685 \pm 0.374$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.082 \pm 0.058$.

Figure 6.10: MTC sampler for quartic flow with 16 measurements.
6.2.4 Measurement number

For the purposes of examining the effect of differing number of measurement points / electrodes I applied the MTC sampler to the quartic phantom flow. As shown in the previous section, the quartic phantom flow is not easy for an EMFT system to estimate, but it still lies within the bulk of the distribution. Therefore the quartic flow is suitable for this comparison as it is non-trivially estimated.

The electrodes are placed equidistantly around the circumference of the pipe, starting at $\phi = 0$. The second order statistics of these runs are printed in Table 6.2 and corresponds to Figures 6.11 to 6.20.

Table 6.2: Second order statistics of the MTC sampler for different measurements numbers with quartic flow (true $f_{\text{bulk}} = 10\text{Tms}^{-1}$).

<table>
<thead>
<tr>
<th>Measurements</th>
<th>$f_{\text{bulk}}$ (Tms$^{-1}$)</th>
<th>log$_{10}(\lambda)$</th>
<th>log$_{10}(\gamma)$</th>
<th>log$_{10}(\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 electrodes</td>
<td>$13.948 \pm 3.124$</td>
<td>$-8.395 \pm 3.888$</td>
<td>$6.161 \pm 0.126$</td>
<td>$-2.234 \pm 3.886$</td>
</tr>
<tr>
<td>3 electrodes</td>
<td>$14.533 \pm 3.146$</td>
<td>$-6.073 \pm 1.510$</td>
<td>$5.286 \pm 0.056$</td>
<td>$-0.787 \pm 1.510$</td>
</tr>
<tr>
<td>4 electrodes (a)</td>
<td>$13.966 \pm 3.152$</td>
<td>$-6.853 \pm 1.110$</td>
<td>$5.194 \pm 0.058$</td>
<td>$-1.659 \pm 1.106$</td>
</tr>
<tr>
<td>4 electrodes (b)</td>
<td>$12.942 \pm 3.156$</td>
<td>$-7.401 \pm 1.164$</td>
<td>$5.193 \pm 0.058$</td>
<td>$-2.208 \pm 1.160$</td>
</tr>
<tr>
<td>5 electrodes</td>
<td>$13.071 \pm 3.134$</td>
<td>$-7.207 \pm 1.090$</td>
<td>$4.992 \pm 0.054$</td>
<td>$-2.215 \pm 1.088$</td>
</tr>
<tr>
<td>6 electrodes</td>
<td>$13.077 \pm 3.136$</td>
<td>$-8.248 \pm 1.144$</td>
<td>$6.265 \pm 0.140$</td>
<td>$-1.983 \pm 1.136$</td>
</tr>
<tr>
<td>7 electrodes</td>
<td>$12.884 \pm 3.154$</td>
<td>$-7.317 \pm 0.688$</td>
<td>$5.507 \pm 0.066$</td>
<td>$-1.810 \pm 0.682$</td>
</tr>
<tr>
<td>8 electrodes</td>
<td>$12.816 \pm 3.150$</td>
<td>$-6.940 \pm 0.700$</td>
<td>$5.172 \pm 0.060$</td>
<td>$-1.768 \pm 0.700$</td>
</tr>
<tr>
<td>12 electrodes</td>
<td>$12.785 \pm 3.128$</td>
<td>$-7.056 \pm 0.654$</td>
<td>$5.458 \pm 0.070$</td>
<td>$-1.598 \pm 0.646$</td>
</tr>
<tr>
<td>16 electrodes</td>
<td>$12.687 \pm 3.158$</td>
<td>$-6.656 \pm 0.766$</td>
<td>$5.081 \pm 0.058$</td>
<td>$-1.576 \pm 0.556$</td>
</tr>
</tbody>
</table>

The two electrode case in Figure 6.11 is the same as an EM flowmeter. The mean estimate in Figure 6.11b is approximately constant, which matches the theory in Chapter 2. From Chapter 4 we know that the first singular vector has the close correlation with a constant vector. A single voltage measurement from 2 electrodes is effectively gaining the easiest accessible piece of information which is similar to measuring the largest singular vector.

To add to the discussion from Section 6.2.1, it is impossible to determine the noise level from a single voltage measurement in the form of a single number. Hence the relationship between the data misfit and conditioning is completely uncertain, leading to Figure 6.11e. This figure shows that $\lambda$ is estimated to be somewhere between $10^{-15}$
and $10^{-3}$ — a very large range.

In contrast, Figure 6.11f shows that $\gamma$ has a relatively narrow distribution. This fact is more due to the model than any other factors. The two measurement points are very loosely correlated and thus it is more likely that the measurement noise is insignificant in the eyes of the statistical model than there being a significant level of noise.

The 3 electrode case in Figure 6.12 is a refinement on the predicted value of $\lambda$ (Figure 6.12e) from the 2 electrode case. The estimated uncertainty is reduced to the range of $10^{-9}$ to $10^{-4}$. The mean in Figure 6.12b also shows a level of refinement from the constant Figure 6.11b. There is now some level of spatial variation clearly visible. However the level of spatial variation is still small and effectively an interpolation from values near the electrode locations. Additionally the 3 electrode case estimates the largest smoothness parameter of $\delta \approx 0.16$. Once again, the addition of another electrode can be viewed as obtaining an additional piece of information to the constant.

Figure 6.13 is with 4 electrodes and is once again a refinement on the previous cases, gaining another piece of information. However the kind of information/the level of refinement is dependent on the flow field shape. Comparing Figures 6.13a and 6.13c shows that the flow around the measurement points contains none of the extreme values in the phantom. In contrast to this, Figure 6.14 is generated with a different choice of measurement locations, more overlapping the extreme features of the phantom. The reconstructed mean estimate Figure 6.14b is more accurate than Figure 6.13b due to this fact.

The accuracy of the estimates increases incrementally at diminishing levels, continuing up to those shown in Figure 6.20. The decreasing rate of improvement is to be expected from the results in Section 4.2.1. The increasing amount of information is progressively less significant.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 13.948 \pm 3.124 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -8.395 \pm 3.888$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 6.161 \pm 0.126$.

Figure 6.11: MTC sampler for quartic flow with 2 measurements.
(a) Phantom flow, \( f_{\text{bulk}} = 10\text{Tms}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 14.533 \pm 3.146\text{Tms}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of \( \lambda \) from marginal distribution, \( \log_{10}(\lambda) = -6.073 \pm 1.510 \).

(f) Samples of \( \gamma \) from marginal distribution, \( \log_{10}(\gamma) = 5.286 \pm 0.056 \).

Figure 6.12: MTC sampler for quartic flow with 3 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 13.966 \pm 3.152 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.853 \pm 1.110$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.194 \pm 0.058$.

Figure 6.13: MTC sampler for quartic flow with 4 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10\,\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 12.942 \pm 3.156\,\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.401 \pm 1.164$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.193 \pm 0.058$.

Figure 6.14: MTC sampler for quartic flow with 4 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 13.071 \pm 3.134\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.207 \pm 1.090$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 4.992 \pm 0.054$.

Figure 6.15: MTC sampler for quartic flow with 5 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 13.077 \pm 3.136 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -8.248 \pm 1.144$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 6.265 \pm 0.140$.

Figure 6.16: MTC sampler for quartic flow with 6 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tm}^{-1}$. 

(b) Mean flow, $f_{\text{bulk}} = 12.884 \pm 3.154 \text{Tm}^{-1}$. 

(c) Simulated data. 

(d) Flow standard deviation 

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.317 \pm 0.688$. 

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.507 \pm 0.066$. 

Figure 6.17: MTC sampler for quartic flow with 7 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 12.816 \pm 3.150\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.940 \pm 0.700$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.172 \pm 0.060$.

Figure 6.18: MTC sampler for quartic flow with 8 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 12.785 \pm 3.128 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.056 \pm 0.654$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.458 \pm 0.070$.

Figure 6.19: MTC sampler for quartic flow with 12 measurements.
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 12.687 \pm 3.158\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.656 \pm 0.766$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.081 \pm 0.058$.

Figure 6.20: MTC sampler for quartic flow with 16 measurements.
6.2.5 Bulk flow rate

Continuing the comparison of different flow fields, I now examine the effect changing the scale of the quartic flow field has on the estimates. Scaling of the flow transfers directly to scaling of the potential due to the linear nature of the problem. There are two ways that the system can be scaled: with and without also scaling the signal to noise ratio. The noise is characterised by the $\gamma_{\text{true}}$ which is used to add noise to the simulated data.

Table 6.3 contains the second order statistics of the relevant MTC runs, with the first row being from Figure 6.20 in the previous section. Figures 6.20, 6.21, and 6.22 are of quartic phantom flow with bulk flow rates of 10, 1 and 0.1 Tms$^{-1}$ respectively and all share the same noise added to the simulated data ($\gamma_{\text{true}} = 10$). This corresponds with the signal to noise ratios of approximately .8%, 8%, and 80% respectively.

Figures 6.23, 6.24, and 6.25 have quartic phantom flow with bulk flow rates of 100, 1, and 0.1 Tms$^{-1}$ with $\gamma_{\text{true}}$ of 1, 100, and 1000 respectively. These runs maintain a constant signal to noise ratio of about 0.1%. The most significant result here is that the estimates of $\gamma$ and $\delta$ change proportionately with $\gamma_{\text{true}}$.

Overall, the bulk flow is estimates are relatively accurate in these circumstances — estimating the same scale in flow as the corresponding phantoms. This is excluding Figure 6.22 where the noise dominated the data. However, the uncertainty of the estimated flow remained about the same. For the $f_{\text{bulk}} = 10\text{Tms}^{-1}$ case, this may look like a good estimate within 2$\sigma$ of the truth, but for the $f_{\text{bulk}} = 1$ or 0.1Tms$^{-1}$ cases the uncertainty is proportionately huge, and for the $f_{\text{bulk}} = 100\text{Tms}^{-1}$ case the accuracy is very bad.

Table 6.3: Second order statistics of the MTC sampler for different flow field (true $f_{\text{bulk}} = 10\text{Tms}^{-1}$).

<table>
<thead>
<tr>
<th>$f_{\text{bulk}}$</th>
<th>$\gamma_{\text{true}}$</th>
<th>E$[f_{\text{bulk}}]$ (Tms$^{-1}$)</th>
<th>$\log_{10} \lambda$</th>
<th>$\log_{10} \gamma$</th>
<th>$\log_{10} \delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>12.687 ± 3.158</td>
<td>-6.656 ± 0.566</td>
<td>5.081 ± 0.058</td>
<td>-1.576 ± 0.556</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>1.076 ± 3.144</td>
<td>-6.158 ± 0.450</td>
<td>5.081 ± 0.058</td>
<td>-1.076 ± 0.442</td>
</tr>
<tr>
<td>0.1</td>
<td>10</td>
<td>-0.077 ± 3.172</td>
<td>-6.148 ± 0.496</td>
<td>5.082 ± 0.058</td>
<td>-1.066 ± 0.492</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>126.877 ± 3.156</td>
<td>-6.715 ± 0.558</td>
<td>3.096 ± 0.270</td>
<td>-3.619 ± 0.620</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>1.269 ± 3.136</td>
<td>-6.653 ± 0.496</td>
<td>7.059 ± 0.364</td>
<td>0.406 ± 0.656</td>
</tr>
<tr>
<td>0.1</td>
<td>1000</td>
<td>0.154 ± 3.122</td>
<td>-6.708 ± 0.506</td>
<td>8.998 ± 0.948</td>
<td>2.290 ± 1.156</td>
</tr>
</tbody>
</table>
(a) Phantom flow, $f_{\text{bulk}} = 1 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 1.076 \pm 3.144 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.158 \pm 0.450$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.081 \pm 0.058$.

Figure 6.21: MTC sampler for quartic flow with 16 measurements and “true” bulk flow of $1 \text{Tms}^{-1}$, $\gamma_{\text{true}} = 10$. 

159
(a) Phantom flow, $f_{\text{bulk}} = 0.1 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = -0.077 \pm 3.172 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.148 \pm 0.496$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.082 \pm 0.058$.

Figure 6.22: MTC sampler for quartic flow with 16 measurements and “true” bulk flow of $0.1 \text{Tms}^{-1}$, $\gamma_{\text{true}} = 10$. 
(a) Phantom flow, $f_{\text{bulk}} = 100\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 126.877 \pm 3.156\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.715 \pm 0.558$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 3.096 \pm 0.270$.

Figure 6.23: MTC sampler for quartic flow with 16 measurements and “true” bulk flow of 100Tms$^{-1}$, $\gamma_{\text{true}} = 1$. 

161
(a) Phantom flow, $f_{\text{bulk}} = 1 \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 1.269 \pm 3.136 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -6.653 \pm 0.496$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 7.059 \pm 0.364$.

Figure 6.24: MTC sampler for quartic flow with 16 measurements and “true” bulk flow of $1 \text{Tms}^{-1}$, $\gamma_{\text{true}} = 100$
(a) Phantom flow, \( f_{\text{bulk}} = 0.1 \text{Tms}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 0.154 \pm 3.122 \text{Tms}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of \( \lambda \) from marginal distribution, \( \log_{10}(\lambda) = -6.708 \pm 0.506 \).

(f) Samples of \( \gamma \) from marginal distribution, \( \log_{10}(\gamma) = 8.998 \pm 0.948 \).

Figure 6.25: MTC sampler for quartic flow with 16 measurements and “true” bulk flow of 0.1Tms\(^{-1}\), \( \gamma_{\text{true}} = 1000 \).
6.3 Discussion

The Bayesian approach to EMFT systems allow one to describe a probability distribution of likely states, consistent with the data, based on prior assumptions. However, EMFT is a non-penetrative measurement system so it does not provide much information of internal structures. When solving the inverse problem of some given data, the probability of any images with internal structures differing from the flow near the boundary are weighted down significantly. Therefore, the mean of the probability density on image space is an image with a simple interpolation from the well determined values near the boundary — where the measurements are located.

Despite this limitation, the estimate of bulk flow rate, which is of interest in a standard EM flowmeter can be very well determined given the appropriate prior knowledge (e.g. the no-slip condition for negligible flow at the boundary). There is an increasing level of accuracy when more measurements made, at a decreasing rate of improvement.

It is evident that there are structures in the estimated flow reconstructions based on the electrode placement. One might speculate that a random placement of electrodes could eliminate this. However any random placement would also lead to structures based on the electrode locations. It would be possible to improve the results by taking multiple measurements using different random placements, but this is impractical in a real-world situation. The equidistant measurements provide the greatest coverage, hence why they were used in this thesis and in the literature. Although, varying the distance between electrodes could be advantageous for the MTC sampler. The different distances could aid the estimation of $\delta$ and $\gamma$. However, there are greater issues that will get in the way of this in the implementation presented in this work.

It is clear that the prior is not appropriate in this construction of an MTC sampler. The Jeffreys prior was chosen to make the method uninformative to the scale of the system but did not work in this case. This was due in part to a specific feature of the EMFT system: the singularities in the forward map. The estimated measurement precision, $\gamma$, and the smoothness parameter, $\delta$, seem to be determined by two things: the correlation between the measurements as described by the forward map and the similarity in the measured data. However due to the singular nature of the measurement kernel, the measurements are almost completely determined by the local region around a measurement point. Thus, the correlation between measurements will be very low. Much like the 2 electrode measurement case (Figure 6.11), noise is nearly indistinguishable from truth. Combining this lack of correlation with Jeffreys prior
which allows for any scale to be viable, $\gamma$ will be over estimated.

The statement that the system is “uninformative to scale” is not true because the standard deviation of flow is the same for every single case in this chapter ($\sim 1.5$). This can be understood by examining the equation used to calculate samples:

$$ (A^TA + \lambda L)f = A^Td + w', \quad (6.9) $$

where $w = w'_1 + w'_2$ with $w'_1 \sim N(0, A^TA)$ and $w'_2 \sim N(0, \lambda L)$. Equation (6.9) can be rearranged to clearly separate the mean of the distribution and the standard deviation

$$ f = \underbrace{(A^TA + \lambda L)^{-1} A^Td}_{\mu} + \underbrace{(A^TA + \lambda L)^{-1} w'}_{\text{var}}, \quad (6.10) $$

since $w'$ has zero mean and a non-zero standard deviation. The random variables, $w = w'_1 + w'_2$, are computed as

$$ w'_1 = (A^TA + \lambda L)^{-1} A^Tn_1 \quad \text{and} \quad w'_2 = \lambda (A^TA + \lambda L)^{-1} Ln_2, \quad (6.11) $$

where $n_i \sim N(0,1)$.

Sampling this to estimate the standard deviations of $w'$, $w'_1$, and $w'_2$ for a sensible range of $\lambda$ produces Figure 6.26. It is clear upon examination that the standard deviation produced by this is, in practice, independent of $\lambda$. The idea of gaining scale independence from merely using Jeffreys prior is not true. The scale is effectively set by this constant standard deviation of Figure 6.26.
Chapter 7

Mini-Aquifer

7.1 Mini-Aquifer details

The work in this thesis was funded as a part of the SfTI National Science Challenge 10: Inverting Electromagnetics, Tranche 1, the goal of which is to create a system for measuring the flow rate of groundwater using electromagnetic principles. The ideal outcome would be the manufacturing of a portable device that could be used to determine the sources and travel paths of the water in aquifers.

A small-scale, lab-based prototype was built in the University of Canterbury as a test-bed for the design and measurement techniques required for the final system. This was called the mini-aquifer.

7.1.1 Experimental set up

The mini-aquifer tank consists of three sections, as depicted in Figure 7.1. The central tank is where the measurements are made and is to be filled with some obstructing medium such as sand or glass beads. This tank is connected on opposing sides to the flow control tanks; tanks used to control the water flow rate by maintaining a fixed height difference between the two tanks. The water is pumped from one tank to the other to maintain this difference. The boundary between the main tank and the flow control is a plastic barrier with regularly drilled holes, as can be seen in the photographs of Figure 7.2.

Beneath the tank is a large electromagnet, powered by an oscillating current in the tens of Hz\(^1\). The magnetic field strength was proposed to be in the order of 10mT.\(^1\)

\(^1\)The electromagnet was moved from above the tank to below later in the process. This change
Figure 7.1: Mini-aquifer design. Water travels in the \( z \) direction through the measurement plane under an orthogonally vertical magnetic field generated by an external electromagnet. These are respectively depicted in Subfigure (b) as blue dotted-circles and orange arrows. The induced potential is measured with some electrodes, depicted as teal squares in Subfigure (b).

The exact arrangement of measurements was not decided on as of writing this, but the electrodes will be placed inside the main tank itself. In the photos of Figure 7.2 there are three PVC pipes containing a line of electrodes as a possible measurement arrangement. The measurements are to be time averaged, using the oscillations to minimise electrochemical effects at the electrodes. The trial flow rate was 10mm/hr and the expected signals on the order of 10nV — below the noise floor. The measurement and control electronics designs are not relevant to this thesis and are omitted.
(a) Photo of the Mini-aquifer tank.

(b) Closeup photo of the coil used to generate a magnetic field.

Figure 7.2: Photos of the mini-aquifer experimental setup in the University of Canterbury (November 2018).
7.1.2 Experimental errors

Assuming the signal can be filtered perfectly, thereby eliminating the signal induced from the oscillating magnetic field, then the errors in the measurement are:

- **Location error:** The electrode will never be placed exactly where it is assumed to be in the model. This adds an error due to improper modelling of the real-world setup.

- **Measurement disturbance:** The electrodes are physical objects and thus their placement will change the local flow. The flow must move around the electrodes. Additionally the placement of the electrode may compact, or create a channel for the fluid, further changing the flow.

- **Geometric errors:** This error is a combination of errors in the electrode shape and the electrode fluid interface. This could be as simple as the electrode not being a perfect sphere or cylinder due to a dent or manufacturing error.

- **Electrochemical errors:** The voltage at the electrodes can cause electrolysis, changing the state of the fluid. This could cause some buildup of solids at the electrodes or lead to the creation of gasses. Solid buildup and gasses change the electrode fluid interface and the geometry.

- **Background noise:** There will always be some background noise due to electron shot noise, stray EM waves, and thermal noise.

Concerns over geometric and electrochemical errors have been prevalent in electromagnetic flow measurement investigations since its inception. To quote Filloux:

“The most stringent limitation to electric field sampling in the sea is the difficulty in achieving low-noise electrical continuity between measuring circuits and sea water. Even the best matched silver-silverchloride electrodes introduce variable electrochemical signals hard to maintain below a millivolt.” — Filloux (1973)

This concern was also shared by the NSC team which also sought to use silver-silverchloride electrodes. However, there is a problem with contact electrodes which can be clearly seen from the measurement kernel from Chapter 3; there is a singularity at the measurement points. Even for non-point electrodes, the local geometry has a

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2The location and disturbance errors are included as types of geometric errors.
colossal effect on the measurement. This is particularly true for the electrode-fluid interface, which could easily contain some skin oil fingerprints from when it was set up or any other tiny defect.

The issues of geometrical errors are why the more advanced field of ECT, but also inductive flow tomography, uses non-contact measurements (Watzenig and Fox, 2009; Yin et al., 2009).

7.2 Model

The rectangular shape of the mini-aquifer has one distinct difference to an actual groundwater measurement system: the mini-aquifer is confined. Confined aquifers do exist in reality, however not at the depth at which the electrodes can be placed. The inability for current to flow through the plastic tank creates boundary conditions whereas the outdoor case would be only restricted by the surface of the water table. Thus, the Greens functions for the mini-aquifer is the same as (3.50):

\[
g(x|\xi) = -\frac{1}{2\pi} \left( \ln(||x - \xi||) + \sum_{l} \sum_{k} \ln(||x - \xi_{l,k}||) \right),
\]

(7.1)

where \( l, k \in \mathbb{Z} \setminus \{0\} \) and \( \xi_{i,j} = ((-1)^l \xi_1 + IR, (-1)^k \xi_2 + kR) \) and \( R \) is the width of the square. The mini-aquifer cross-section is rectangular whereas this is for a square domain. However, qualitatively the mini-aquifer’s cross-section and a square are approximately the same. As I initially developed this system for a square, this was used for convenience sake.

Everything else in the set-up of this problem is the same as the circular pipe case, excluding the measurement points. Using the placements in Figure 7.2 as inspiration, meshes must be built which are refined towards the measurement points. The meshes used are presented in Figure 7.3.

It is very important to note that there are a few flaws with this model and setup. Most notably, the electrode model of point electrodes is even less accurate when placed inside the flow than on the boundary. If this was the real-world scenario then the electrodes would block the flow and would also thus have a surface charge due to the electrode-fluid boundary. Additionally there may be a no-slip condition at the electrode’s surface. However, this more simple model can nonetheless be used to gain an insight on the system.
(a) Mesh for two measurement columns of electrodes using 1346 nodes.
(b) Mesh for three measurement columns of electrodes using 1912 nodes.

Figure 7.3: Meshes for the mini-aquifer. The electrode locations are in the centre of the dense regions.

7.2.1 Phantom flows

Two different types of phantom flows were used to generate the data. These are:

**Constant flow**

This project was initially motivated with the assumption that flow was constant. The constant flow is defined by

\[
f_{\text{constant}} := f(x, y) = f_{\text{max}}. \tag{7.2}
\]

**Stratified flow**

Stratified flow is more realistic for aquifers as the flow rate decreases with depth. The field is given by

\[
f_{\text{stratified}} := f(x, y) = f_{\text{max}} \left(1 - \frac{y}{R}\right). \tag{7.3}
\]
7.3 Results

7.3.1 Measurable flow rate

Figures 7.4 through 7.7 are of MTC runs using the above setup. The two phantom flows used to generate the data are constant flow fields (figures 7.4 and 7.5) and stratified flow fields (figures 7.6 and 7.7). Two different electrode setups are used, corresponding to the use of two or three electrode columns. These are 6 electrodes (figures 7.4 and 7.6) and 9 electrodes (figures 7.5 and 7.7). Table 7.1 shows the second order statistics of the results.

Table 7.1: Second order statistics of the MTC sampler for different flow field (true \( f_{\text{bulk}} = 10 \text{Tm}^{-1} \)).

<table>
<thead>
<tr>
<th></th>
<th>Electrodes</th>
<th>( f_{\text{bulk}} ) (Tm(^{-1}))</th>
<th>( \log_{10} \lambda )</th>
<th>( \log_{10} \gamma )</th>
<th>( \log_{10} \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>6</td>
<td>9.956 ± 0.996</td>
<td>-6.983 ± 1.116</td>
<td>5.366 ± 0.058</td>
<td>-1.618 ± 1.114</td>
</tr>
<tr>
<td>Constant</td>
<td>9</td>
<td>10.245 ± 0.994</td>
<td>-7.116 ± 0.612</td>
<td>5.904 ± 0.170</td>
<td>-1.212 ± 0.622</td>
</tr>
<tr>
<td>Stratified</td>
<td>6</td>
<td>10.516 ± 0.996</td>
<td>-7.731 ± 1.112</td>
<td>5.247 ± 0.052</td>
<td>-2.484 ± 1.576</td>
</tr>
<tr>
<td>Stratified</td>
<td>9</td>
<td>10.360 ± 0.996</td>
<td>-8.026 ± 0.746</td>
<td>6.055 ± 0.188</td>
<td>-1.971 ± 0.712</td>
</tr>
</tbody>
</table>

The results are very similar to that of the previous chapter, with the estimates of the bulk flow being very accurate. There is a significant point of difference; the accuracy of the measurements do not increase with the addition of the extra electrode column. This is likely to be a side effect from immersing the electrodes in the flow or of the simple flow fields used.

Comparing the electrode locations such as in Figure 7.4c with the corresponding mean reconstructed flow in Figure 7.4b shows an interesting relationship. The flow on either side of the measurement is always very different. This behaviour is clear when considering the flowmeter equation

\[
\nabla^2 u = -\nabla \cdot (v \times B)
\]

The source in this BVP is dipolar due to the divergence. Because of this and the data over-fitting which comes from this model (as discussed in the previous chapter), the most likely flow given the measurements has sharp changes at the electrode locations. This effectively reconstructs a flow profile as the combination of several dipoles at the electrode positions with simple interpolation. The clearest example of this is in Figure 7.7b which almost looks like a 3 \( \times \) 4 pixel image.
(a) Phantom flow, \( f_{\text{bulk}} = 10\text{Tm}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 9.956 \pm 0.996\text{Tm}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of \( \lambda \) from marginal distribution, \( \log_{10}(\lambda) = -6.983 \pm 1.116 \).

(f) Samples of \( \gamma \) from marginal distribution, \( \log_{10}(\gamma) = 5.366 \pm 0.058 \).

Figure 7.4: Full MTC sampler for constant flow with 6 measurements (2 columns of 3 electrodes).
(a) Phantom flow, \( f_{\text{bulk}} = 10 \text{Tms}^{-1} \).

(b) Mean flow, \( f_{\text{bulk}} = 10.245 \pm 0.994 \text{Tms}^{-1} \).

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of \( \lambda \) from marginal distribution, \( \log_{10}(\lambda) = -7.116 \pm 0.612 \).

(f) Samples of \( \gamma \) from marginal distribution, \( \log_{10}(\gamma) = 5.904 \pm 0.170 \).

Figure 7.5: Full MTC sampler for constant flow with 9 measurements (3 columns of 3 electrodes).
(a) Phantom flow, $f_{\text{bulk}} = 10^\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 10.516 \pm 0.996 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.731 \pm 1.112$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 5.247 \pm 0.052$.

Figure 7.6: Full MTC sampler for stratified flow with 6 measurements (2 columns of 3 electrodes).
(a) Phantom flow, $f_{\text{bulk}} = 10\text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 10.360 \pm 0.996\text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -8.026 \pm 0.746$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 6.055 \pm 0.188$.

Figure 7.7: Full MTC sampler for stratified flow with 9 measurements (3 columns of 3 electrodes).
7.3.2 Realistic flow rate

The proposed test values for the velocity and magnetic field are 10mm/hr and 10mT respectively. These values lead to a bulk flow rate of approximately $f_{\text{bulk}} = 2.8 \times 10^{-8}$ Tms$^{-1}$ where the cross sectional area is unity. Use of this value for the constant phantom for an MTC run is presented in Figure 7.8 with the additive noise chosen so that the noise is approximately 1% the signal.

Table 7.2: Second order statistics of the MTC sampler for the realistic aquifer flow rate (true $f_{\text{bulk}} = 10^{-8}$ Tms$^{-1}$).

<table>
<thead>
<tr>
<th>Electrodes</th>
<th>$f_{\text{bulk}}$ (Tms$^{-1}$)</th>
<th>$\log_{10} \lambda$</th>
<th>$\log_{10} \gamma$</th>
<th>$\log_{10} \delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>6</td>
<td>0.006 ± 0.988</td>
<td>-7.026 ± 1.016</td>
<td>26.021 ± 0.160</td>
</tr>
</tbody>
</table>

The results have a large relative uncertainty. As discussed in the previous chapter, the uncertainty is determined by the forward map $A$ and the matrix $L$. Consequently measurements of the expected nanovolt scale are indistinguishable from no measurement at all. The same is true for any phantom flow.
(a) Phantom flow, $f_{\text{bulk}} = 10^{-8} \text{Tms}^{-1}$.

(b) Mean flow, $f_{\text{bulk}} = 0.006 \pm 0.988 \text{Tms}^{-1}$.

(c) Simulated data.

(d) Flow standard deviation

(e) Samples of $\lambda$ from marginal distribution, $\log_{10}(\lambda) = -7.026 \pm 1.016$.

(f) Samples of $\gamma$ from marginal distribution, $\log_{10}(\gamma) = 26.021 \pm 0.160$.

Figure 7.8: Full MTC sampler for realistically scaled constant flow with 6 measurements (2 columns of 3 electrodes).
Chapter 8

Conclusions and Discussion

The goal of this work was twofold: introducing Bayesian inference methodology into EMFT practice and exploring the viability of measuring flows on the scale of groundwater systems. The former was the main focus of this thesis as it is a larger contribution to the scientific community. Additionally, the Bayesian EMFT methodology needed to be developed in order to test the latter.

In this chapter, I provide an overview of the advantages of using Bayesian inference over traditional methods and discuss the specific findings of this body of work. While some of the points made here are not new, such as the advantages of Bayesian methods, they are relatively unknown in the field of EMFT and are an important part of the discussion surrounding the first stated goal.

8.1 Advantages of the Bayesian paradigm

The advantages of Bayesian inference have been presented in numerous papers (von Toussaint, 2011; Robert, 2007; Watzenig and Fox, 2009).

Methods of solving an inverse problem must overcome the ill-posed (or at least ill-conditioned) nature of inverse problems. With the addition of noise, a single data set $d$ can be from an impossibly large set of states $f$. The practice of deterministic inversion is to turn this ill-posed problem into a well-posed problem by restricting the set of possible states or changing the forward map itself such that there is only a single solution state. The biggest problems with this are that the choice of restriction is completely subjective and the output of a method is only qualitatively useful.

The Bayesian formulation is in terms of probability functions — the natural language of noise — and treats probability as one’s knowledge. This approach does not
seek to limit the space of possible possible states in any way. Instead, Bayesian methods create a probability distribution over all of the state-space, weighting the feasibility of each individual state. A Bayesian approach turns the ill-posed inversion problem into a well-posed problem of statistical inference.

At the surface level, Bayesian inference appears to remove the subjectivity from the calculations, however this is not the case. A Bayesian formulation may lessen the influence of subjectivity on the results, but more importantly it is more upfront about the influence of subjectivity. The subjectivity introduced from the assumptions of the system are explicitly entered into the formulation — mostly within the prior. Given prior assumptions and data, the Bayesian approach provides quantitative and optimal estimates.

The most important practical advantage of the Bayesian formulation is the higher level of information contained within the “solution”. Deterministic solutions of an inverse problem are a single qualitative state $\tilde{f}$. In stark contrast to this, the output of a Bayesian formulation is the probability distribution $\pi(f|d)$ or, if hyperparameter sampling is used, $\pi(f, \vartheta|d)$. Thus the amount of information provided is far more extensive than the deterministic case. Calculation of expectation values is possible, as well as the means to characterise the uncertainties in these calculations. This is particularly pertinent in the case where one desires the estimation of physical quantities such as the energy contained within a system.

The method presented in this paper made use of Gaussian properties to simplify the calculations, as done by Fox and Norton (2016). In general, a Bayesian approach can use any probability distribution as it is a sampling problem. Many deterministic methods are minimisation problems. Minimisation, and therefore most deterministic methods do not easily extend to multi-modal distributions. Even more sophisticated methods like the MAP estimator require a single mode, otherwise they can be caught on a lesser local minima. Monte-Carlo Markov chain sampling has the property of being able to traverse the entire space and thus will not remain in local minima.

Bayesian formulations are very versatile and allow for different level representation of the state space such as the polygonal representation of Watzenig and Fox (2009). It is possible to also perform estimation on more abstracted quantities, such as hyperparameters and model error.

Another comparison where regularisation methods are lacking is the so-called “L-curve” by which $\lambda$ is chosen. The entirety of the curve does not lie within the support of the regularising semi-norm and data miss-fit of a more robust Bayesian approach
Thus, regularising does not choose the ideal hyperparameters for the inversion and is not useful for estimating the smoothness or noise of a system.

Assumptions used in a Bayesian method are explicit. An assumption about the prior knowledge of the state, the representations used, and the measurements are encapsulated in the construction of the method (mostly in the prior distribution itself). The usefulness of having explicit assumptions is highlighted when considering the consequence of said assumptions. If there is an unwanted trend in the output of a deterministic method such as regularisation, then there is no way to trace the trend back to the mechanisms of the method. In stark contrast to this, cause and effect of the explicit assumptions in Bayesian methods can be readily traced. An example from within this thesis is the scale-invariant Jeffreys prior assumption contradicting another assumption in the method.

8.1.1 Scale invariance

In Section 5.3.4 Jeffreys prior was introduced with the intention of making the noise and smoothness hyperparameters (\(\gamma\) and \(\delta\)) be uninformative of scale. The ideal outcome of this assumption would make the smoothness and the noise level predictions able to change scales based on the data. A measurement of nanovolts would be treated the same as a measurement in volts.

However, the results from Chapter 6 clearly showed that the overall method is not scale invariant. The uncertainty of the flow field is set to a specific scale (\(\sim 3 \text{ Tms}^{-1}\)). Thus, a measurement of nanovolts are not equivalent to a measurement of volts.

From Equation (6.11), the variance in the reconstructed flow field is dictated entirely by the terms

\[
(A^T A + \lambda L)^{-1} A^T A \quad \text{and} \quad \lambda (A^T A + \lambda L)^{-1} L,
\]

which is constant in both \(\lambda\) and the level of mesh refinement. Thus, the variance is determined by the matrices \(A\) and/or \(L\). This could mean one of two things. Either the locally linear GMRF prior (Section 5.2.1) from which \(L\) comes has some influence on the scale or the radius/size of the pipe essentially sets the level of variance. The latter is more likely because the GMRF prior’s scale of smoothness is dictated by \(\delta\), whereas the radius affects the values of both \(A\) and \(L\).

This supposition is backed up by the uncertainty of the square aquifer reconstructions from Chapter 7 being \(\sim 1 \text{ Tms}^{-1}\). Comparing the area of both cross-sections,
the pipe has \( a_{\text{circle}} = \pi \) and the square has \( a_{\text{square}} = 1 \). These values correlate with the flow rate uncertainties in both the pipe and the aquifer reconstructions.

Thus, trying to achieve scale invariance through the use of an objective prior such as Jefferys prior is fruitless because the geometries of the system itself have an implicit scale. This is most likely due to the form of the measurement kernel. When the domain gets smaller, the distance between the majority of the flow and the electrodes gets smaller. The non-penetrative aspect of the measurement lessens and the reconstruction can thus be more accurate.

With the model in this thesis using point electrodes, the reduction in size to obtain increased accuracy is infinitely scaleable. However with a more realistic model of an electrode, such as the complete electrode model, this would not be the case and the size of the electrodes themselves would provide another implicit scale.

The use of Jefferys prior without realising the implicit scale of the system has another consequence: the overestimation of the precision \( \gamma \). As discussed in Section 6.2.1, the measurement kernels have singularities at the measurement points and, therefore, the measurements are more influenced by nearby flow than distant flow. Due to the singular nature, it is very difficult to correlate the measurements, which gives the sampling method trouble in distinguishing noise from real data. Hence there is the tendency to overestimate \( \gamma \).

In a situation where the measurements are uncorrelated and there is an implicit scale of uncertainty an uninformative prior is not only useless, but even misleading. It would be better to use an informative prior. Given actual experimental data, \( \gamma \) is not difficult to estimate. So if one were to implement this measurement system, any sensibly chosen informative hyperprior would be more appropriate, such as a Gaussian hyperprior about the estimated value of \( \gamma \).

8.1.2 Improvements to the method

Firstly I would like to reiterate that this method of solving an inverse problem using a MTC sampler was made as a base-level example of the possibilities of Bayesian methods. The method here is not a proposal of the best way to do things. Naturally then, there are many possible improvements.

Additionally, it is worth noting that there is a lack of any experimental data to compare with. Thus, the number of improvements which can be definitively talked about are not as expansive as they could be. For example, the no-slip condition;
whether or not this is an appropriate assumption in the context of the pipe — large or small — is unclear. However, there are some modelling choices which clearly can be improved upon.

Continuing on from the previous section, there is room to improve the hyperpriors. If one had the ability to make measurements then a Gaussian estimate of \( \gamma \) would be better, as stated previously. Even without the real-world experimental setup there are a few pieces of information that can be used to estimate the scale. For example, commercial electrodes have a level of precision in their technical specification data sheets. This could be used as an initial estimate and a large distribution about that value would work well. Lacking that, a distribution about the expected signal to noise ratio would also work well. In summary: anything informative would be much better than Jeffreys prior.

Another potential point of concern is the effectively constant uncertainty in the flow fields. In an ideal output of an EMFT sampling algorithm I would expect the uncertainty to increase the further away from the electrode, not remain about the same. I do not know whether this is a false preconception of how the system works or a property of the prior assumptions which I have used. On one hand, the flow near the boundary is closer to the measurements and will be well determined. However, near the centre of the circle the flow is loosely determined by all of the electrodes, not just one. On the other hand, the GMRF prior will smooth out things which are loosely determined by the measurements — much like a form of interpolation. Whether this is too strong an effect, I do not know.

My suspicion is that my preconceptions are wrong. This is not to say that the prior assumptions don’t also contribute to this feature. However, it is hard to predict the outcome of an MCMC sampling algorithm. To put it differently, it is hard to predict the answer to the question “What is the most likely flow field given the data, a forward map, and the assumptions?”. Perhaps with any sensible model of the system and prior assumptions, flow fields such as the quartic flow are just very unlikely.

Regardless of speculation, phantom flows lying outside the support of the samples is problematic in any method of solving an inverse problem and is worth examining.

One further point of improvement would be optimisation. The meshes could be made coarser by using knowledge of the implicit scale. As stated previously, the goal of numerical modelling is to have any numerical errors be insignificant compared to the modelling error. The meshes used in this thesis are more fine than necessary with this goal in mind.
8.2 Electromagnetic flow tomography as an inverse problem

The goal of EMFT systems is very ambitious. On the surface level, it is conceptually very clear. There is a potential generated by a fluid moving through a magnetic field given by:

$$\nabla^2 u = -\nabla \cdot (v \times B).$$  \hspace{1cm} (8.2)

If we measure the potential, then we can infer the velocity field. However, there are many complications to this simple idea.

Firstly, the source having a divergence means that the potential is related to dipoles of \((v \times B)\), as evident by the dipolar expansion in (3.10). Thus, the potential measurement is not determined by the velocity, but more by spatial variation in the velocity. This can be seen by the dipole-like flow estimates about the electrodes in Chapter 7.

A further consequence of the dipolar relation is that the kernel drops off like \(r^{-2}\) in 2D. This means that the measurement system is non-penetrative; the flow nearby influences the measurements far more than distant flow. Consequently, any spatial variation in the internals of the fluid are very difficult to determine and finer structures are essentially invisible.

EMFT measurement systems are very ill-posed. The limited measurements versus the continuous flow space, plus all flow influencing every measurement makes the forward map unreasonably close to a singular matrix. The calculation of determinants will fail if poorly set up due to the numerical error from small numbers being indistinguishable from zero.

With all of that said, EMFT systems can be very powerful if the flow field has simple shapes which can be easily interpolated from the boundary measurements.

8.2.1 Comparison with existing methods

The most extensive recent work in EMFT are by Lehtikangas et al. (2016) and Kollár et al. (2014). These will be the main points of comparison.

Before that, a brief aside into EM flowmeters. The initial findings in this thesis confirm simple EM flowmeters as an effective means of calculating the total flow of a fluid, a fact known since before Shercliff’s work (Shercliff, 1962). The main principle behind EM flowmeters is the two electrode measurement of no-slip axisymmetric flow being the same as that of constant flow with the same bulk flow rate. Since slow,
laminar flow is axisymmetric, the bulk flow rate can be measured given some calibration against known flow rates.

Extending from this axisymmetric result, the two electrode arrangement measurement effectively measures the first singular vector of the system which has a close correlation with the constant flow. Looking at the 2-electrode results from Section 6.2.4, it is almost as close to estimating the true value of the bulk flow as the 16-electrode arrangement. The phantom flow used in that run was a quartic field. Considering that quartic flow is outside of what EM flowmeters are designed for, this is an impressive result.

Much of the EMFT literature assumes the no-slip condition from fluid dynamics. This is a sensible assumption in the continuous mathematical model. However problems can arise when turning this mathematical model into a numerical one. Firstly, the discrete approximation may not be fine enough to effectively represent the boundary layer which arises from this assumption. Secondly and more importantly, with this assumption the numerical model needs to be forced to maintain the zero velocity at the boundary of the discretised domain. Some works such as that by Teshima et al. (1995) do force this boundary condition. However, both Lehtikangas et al. (2016) and Kollár et al. (2014) do not set the velocity to zero at the boundary, despite assuming it to be so.

It ultimately seems that the no-slip condition is useful if the true flow is also zero at the boundary, and vice-versa (see Section 6.2.2). Simply put, it is always better to have a model match the situation being modelled. If the velocity fields used by Lehtikangas et al. (2016) match real-world flow as is stated in the paper, then using the no-slip condition or not would have great effect on the reconstructions. Only two of the presented velocity fields are zero at the boundary. Since both of these groups assume that the no-slip assumption is valid when constructing their forward maps — the forward maps which were used to generate their data — their reconstructions will match their phantom flows.

A direct comparison is not possible for a number of reasons. Firstly is this difference in the forward maps used to generate the data. Section 6.2.2 shows why this is problematic as the results can vary hugely based on the inclusion of the no-slip assumption.

Another point which makes direct comparison difficult is that the results presented in works other than this thesis are qualitative comparisons — whether or not the
flow looks the same. The use of a Bayesian approach in this thesis, which allows for quantitative results, already gives this work an advantage over all deterministic inversion EMFT papers. If one were to directly compare the qualitative looks of the results between these works then the work in this thesis loses out. This is not a fair comparison for a number of reasons. Firstly this work was to introduce Bayesian methods into the EMFT world as a simple working example — not to obtain state-of-the-art results. Secondly is the poor use of the no-slip assumption in other works. Finally, the object that we would be referring to when using the word “looks” in this thesis is the mean of the posterior distribution. It is possible and quite likely that the “true” flow is contained within the bulk of the distribution yet looks different to the the mean. A result such as this is a successful outcome of Bayesian inference.

Ultimately what matters in a comparison of different measurement methods is how useful they are when implemented in the real-world. As neither this method nor the comparison papers have a real-world implementation, this comparison is impossible. In terms of practical implementation, the quantitative output and customisation of the Bayesian approach is far more useful than a deterministic inversion.

One flaw that this work has highlighted in EMFT systems is the use of contact measurement. Using non-physical point electrodes leads to singularities in the measurement kernel. Even if better electrode models were used there will be a high sensitivity to local geometries and the material around the electrodes. This has been noted by many investigators throughout the years and has even been seen in the mini-aquifer group. Investigations into non-contact voltage measurements, such as charge pump arrangements like in the field of electrical capacitance tomography (Watzenig and Fox, 2009), would be of great benefit to the field of EMFT.

A seemingly universal property in all investigations in EMFT, from Williams to now, is that an assumption about the shape of the velocity field is required in order to gain useful results. The GMRF as used here and in the work of Lehtikangas et al. (2016) is a useful, weak assumption. It favours smoothness and is a great advancement on the axisymmetric assumptions from Shercliff’s time.
8.3 Mini-Aquifer project

Conclusions made pertaining to the mini-aquifer project are minimal because the system had not finished development at the time of writing this thesis.

There is one conclusion which can be drawn: the expected measurements on the level of nanovolts will be impossible to reconstruct with this current system. This is due to the implicit scale of the system, spoken of previously. This scale is most likely dictated by the size of the “pipe” and in the case of the mini-aquifer, the standard deviation of the flow is $\sim 0.5 \, \text{Tm/s}$. This means that the expected voltage is indistinguishable from no flow because the level of uncertainty will be overwhelmingly larger in size than any measurement. There are uncountably many different flows which could lead to the same nanovolt data-set. However, only a tiny fraction of them have the scale of flow wanted. The rest will have larger positive and negative flows than reality.

A further reason why the expected measurements on the level of nanovolts will be impossible within this system is the use of contact electrode immersed in the flow. There is an extreme sensitivity in the measurement kernel to features near an electrode. Even without considering electrochemical effects, immersing the electrodes in the medium will change the nearby flow, further altering the measured voltage from that expected of a given flow. This problem has been avoided by either taking measurements outside the flow like done by Longuet-Higgins and Deacon (1949) or through calibration in a standard flowmeter.

8.4 Future work and possibilities

Possible future work which can directly follow on from this thesis are as follows.

- Updating the hyperpriors to more sensible ones.
- Fully exploring the implicit scale of the system.
  - Investigating where it comes from and
  - determining if it is possible to adjust the scale such that the mini-aquifer project can work.
- Introducing a more accurate electrode model into the system.
- Constructing an experimental setup of an EMFT system. This will allow for
– the ability to estimate $\gamma$ and create a better hyperprior,
– validation of approximations for different flow fields,
– the ability to validate methods and objectively be able to talk about usefulness.

• Ultimately, creating a commercially viable EMFT system to obsolete EM flowmeters.
References


Appendix A

Journal and conference publications during research period


