Characterisation of Spherical Splits

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Abstract

We investigate the properties of collections of linear bipartitions of points embedded into $\mathbb{R}^3$, which we call collections of affine splits. Our main concern is characterising the collections generated when the points are embedded into S^2; that is, when the collection of splits is spherical. We find that maximal systems of splits occur for points embedded in general position or general position in S^2 for affine and spherical splits, respectively. Furthermore, we explore the connection of such systems with oriented matroids and show that a maximal collection of spherical splits map to the topes of a uniform, acyclic oriented matroid of rank 4, which is a uniform matroid polytope. Additionally, we introduce the graphs associated with collections of splits and show that maximal collections of spherical splits induce maximal planar graphs and, hence, the simplicial 3-polytopes. Finally, we introduce some methodologies for generating either the hyperplanes corresponding to a split system on an arbitrary embedding of points through a linear programming approach or generating the polytope given an abstract system of splits by utilising the properties of matroid polytopes. Establishing a solid theory for understanding spherical split systems provides a basis for not only combinatorial–geometric investigations, but also the development of bioinformatic tools for investigating non-tree-like evolutionary histories in a three-dimensional manner.

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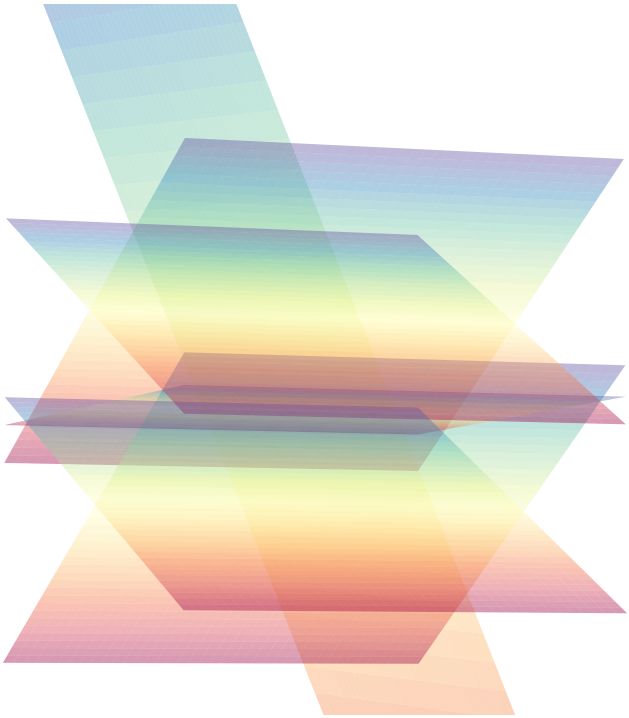


Chapter 1

Introduction

We begin with a brief overview of the relative fields and ideas that will be encountered throughout this work.

A rich theory emerges at the interface of combinatorics and geometry, resulting in the fascinating field of discrete (or combinatorial) geometry. Although there have been myriad outstanding discoveries within the field as a result of the work of many ingenious mathematicians, a wide range of open problems (often very easily stated and seemingly innocuous) still exist. A nice example is the Hadwiger Conjecture: "Can any convex body in n-dimensional space be covered with 2^n (or less) smaller copies of itself?" (Boltjansky and Gohberg, 1985). Matoušek (2002), in his fantastic text on the subject, compared discrete geometry to "an Alpine mountain range", where "convenient paths [...] provide safe trails to a few peaks and lookout points", but "reaching the higher peaks still needs substantial effort".

The field of convex geometry has experienced a relatively recent bloom with the uprise of computational geometry and combinatorial optimisation alongside the progressive development of ever-more-sophisticated computing systems (see, e.g., Preparata and Shamos, 1988; Nemirovski, 2007). Even though the basic premises and ideas present in convex geometry have been investigated since antiquity, the major developments—especially those concerning the combinatorial properties of convex objects—have mainly been conducted within the past couple hundred years, with particularly prolific progression occurring in the 20th century (Grünbaum, 2003).

A natural question, which has arisen in many combinatorial contexts, relates to the concept of separation: We ask in which ways, or how many ways, is a single object

or collection of objects separable, where the object/s of interest may be concrete (e.g., a collection of triangles in the plane) or abstract (e.g., a topological space). Furthermore, higher-level questions of invariance in the notion of separability provide us with a potential means of relating and comparing seemingly disparate concepts. Thus, the study of separability is of great inherent interest from a purely mathematical standpoint, much in the same manner as symmetry.

The concept of separability has strong motivation in terms of classification systems on data sets, which underpins the theories of machine learning and neural networks, both massive areas of research in the modern landscape involving (but not limited to) the development of algorithms to effectively classify data sets (see Bishop, 2006; Rojas, 1996, for good introductions to machine learning and neural networks, respectively). These methods utilise the fundamental idea of (linear or non-linear) separability in order to better understand the structure of a given data set, which is absolutely crucial in the modern age of 'big data', or to make efficient and precise predictions on the classification of new data. This is extremely valuable throughout many scientific disciplines, not to mention societally and commercially (Widrow, Rumelhart, and Lehr, 1994).

In the development of machine learning, a fundamental idea is that of hyperplane arrangements and the subsequent induced separations of a given set of points, which then, in some sense, must be independent. In the 1930s, Whitney and Nakasawa (both of whom have fascinating—and, in the latter case, tragic—stories; see Keith, 2013; Nishimura and Kuroda, 2009, respectively) independently laid the foundations of matroid theory as a generalised axiomatisation of independence. In a very different manner, the theory of oriented matroids was simultaneously developed, predominantly during the 1960s and 1970s, through very different approaches by many different mathematicians. Some of the key players in this period were Rockafellar, who indicated the need for an axiom system in oriented matroid theory (Rockafellar, 1969); Folkman and Lawrence, for whom the fundamental Topological Representation Theorem in the theory was named (Folkman and Lawrence, 1978); Las Vergnas, who set about axiomatising the theory from a graph-theoretic/combinatorial standpoint (Las Vergnas, 1975); and Bland, who approached it from a linear programming duality standpoint (Bland, 1974). Most pertinently to the context of this work is the concept of oriented matroids arising from sign vectors, which generalise the concept of the partitions (i.e., separations) of an n-dimensional space by arrangements of $(n-1)$-dimensional hyperplanes.

The equivalence of the axiom systems of oriented matroids allows us to translate results and contexts originally written in disparate mathematical languages under a rigorous combinatorial backbone. It is still, however, a young field, and there is a feeling that there is much of importance yet to be found.

Graph theory naturally links into all combinatorial fields of mathematics and, hence, has found good use in the fields mentioned above; a prime example of the connection between graph theory and discrete/convex geometry being the Circle Packing Theorem, also known as the Koebe–Andreev–Thurston Theorem (Thurston, 1981, Section 13), which says that, for every connected simple planar graph G, there exists a circle packing in the plane whose intersection graph is isomorphic to G, which has far-reaching consequences (see, e.g., Rodin and Sullivan, 1987). Furthermore, neural networks, which are composed of nodes and (weighted) connections between them (Rojas, 1996), rely on an intrinsic graph structure, and so graph-theoretic results may permit us a deeper understanding of such methods. These examples demonstrate only a tiny sample of the flexibility and connectivity of graph theory throughout diverse mathematical contexts. A particular aspect of graph theory which is relevant to the subject at hand is that of inscribability. An intuitive and visual concept, discerning inscribability amounts to asking which graphs can be realised as polytopes with vertices lying on the unit sphere (Dillencourt and Smith, 1996). However, at present, a satisfying combinatorial characterisation of inscribability is still lacking.

Graph theory has found applications through diverse scientific realms, and one such example in biology is in providing a model for evolution. Trees (i.e., connected acyclic graphs) play an important role in evolutionary modelling, where the assumption that the process of evolution occurs by genetic lines bifurcating over time has been used to develop mathematical methods to infer genetic histories (Morrison, 1996; Farris, 1972). While tree models are, in a large number of cases, sufficient for representing the evolutionary processes which have occurred, they lack the ability to express reticulate phenomena, such as horizontal gene transfer. For this reason, the idea of a phylogenetic network was developed, which may better represent such evolutionary histories coherently (see Huson and Bryant, 2006, for an in-depth survey of the diverse uses of phylogenetic networks in evolutionary studies). The ability to soundly infer historical contexts from data allows researchers to obtain an in-depth understanding of evolutionary mechanisms and to rigorously formulate theories to explain such aberrant phenomena. Hence, providing consistent frameworks for the construction of such

methods is an important facet of mathematical biology.

In the context of phylogenetic networks, the concept of a *circular split system* was introduced by Bandelt and Dress (1992) and developed into a widely used method by Bryant and Moulton (2004). The end-product of their Neighbor-Net method is a collection of circular splits of a set of taxa, which is subsequently used to construct a highly resolved phylogenetic network (called a *splits graph*). Following their lead, we propose the idea of a collection of *spherical* splits as an extension of this idea; it is not entirely a natural extension, as three-dimensional objects tend to have wildly differing properties to those in two dimensions, and so we have attempted to treat the construction and analysis of such collections carefully and thoroughly.

This thesis is organised as follows:

In Chapter 2, we first define the concept of a split as a bipartition of a finite abstract set. This concept naturally extends to an affine split, which is a bipartition induced by an embedding of the set into an ambient space; in our case, we consider embeddings into $\mathbb{R}^3$, where natural linear bipartitions are induced by planes. We establish some fundamental properties of such collections of affine splits, in particular finding a bound on the cardinality of a maximal collection of affine splits, which is tight if and only if the points are embedded into general position. Following this, we look at some fundamental definitions and connections in convex polytope theory and define a collection of polytopal splits to be a collection of affine splits with the points embedded as the vertices of a 3-dimensional polytope, from which it follows that a maximal collection of polytopal splits must be associated with a set of points in general and convex position. Finally, after considering the issue of polytope inscribability, we define spherical splits, which correspond to collections of polytopal splits where the points are constrained to lie on the unit sphere. We detail some difficulties with discerning polytopal collections of splits from spherical ones, and provide a restriction result for spherical splits.

In Chapter 3, we begin by outlining oriented matroid theory from the covector and tope (maximal covector) axiomatisations, introducing both axiom systems and considering the natural equivalence between the two. The properties of, and operations relating to, sign vectors are introduced, and uniformity, acyclicity, and simplicity of oriented matroids are detailed. We follow by defining the Vapnik–Chervonenkis dimension for split collections and showing that a maximal collection of affine splits has VC-dimension 4. By appropriately defining a certain set of sign vectors $\mathcal{T}$ from a maximal collection of affine splits and utilising the fundamental correspondence of Gärtner and Welzl (1994)

between collections satisfying certain properties and sets of topes related to uniform oriented matroids, we establish the connection between maximal collections of affine splits and rank 4 uniform oriented matroids. Following this correspondence, we investigate the lattices associated with oriented matroids. After defining and exemplifying order complexes and shellings, we determine a connection between maximal collections of affine splits and shellable 3-balls. We establish the relationship between maximal sets of polytopal/spherical splits and matroid polytopes, and briefly consider the implications of this. Finally, we discuss the realisability of oriented matroids in the context of collections of affine splits.

In Chapter 4, we investigate graph structures associated with collections of polytopal/spherical splits. We discuss some fundamental graph-theoretic results, including Steinitz's connection between 3-connected planar graphs and 3-polytopes, as well as the non-planarity criteria of Kuratowski and Wagner. Then, we define the graph $G_{\mathscr{S}}$ associated with a collection of splits $\mathscr{S}$ and show that those associated with maximal collections of polytopal/spherical splits are maximal planar graphs. We show that such a graph associated with a collection of spherical splits is inscribable, and attempt to provide a reasonable explanation, in terms of the structure of the splits/embedding, for the case of non-inscribable $G_{\mathscr{S}}$. Finally, we consider some contraction results to determine the subgraphs of $G_{\mathscr{S}}$ induced by the separable subsets of a given split.

In Chapter 5, we detail two methodologies: One for generating a split collection given a set of points, and another for generating the face lattice of a matroid polytope given a set of topes. While these methods are most definitely not optimal, they provide a launchpad for the development of a more comprehensive and efficient system for generating collections of points, hyperplanes, simplices, and/or topes associated with a given input. The first method is a linear programming approach for the determination of topes and hyperplanes given a set of points in $\mathbb{R}^3$, although a natural extension to n dimensions is possible. A derivation and some results of the system are provided. The second method uses a brute-force approach to determine the covector lattice and, using that information, calculate the polytope face lattice of a matroid polytope.

Finally, in Chapter 6, we summarise our findings and discuss the remaining open questions and potential directions for future research.

Chapter 2

Split Systems and Polytopes

We begin with some fundamental definitions.

2.1 Splits

Given a finite set Y, a *split* of Y is a bipartition of the elements of Y into two non-empty parts; that is, a split is comprised of two non-empty sets $A, B \subset Y$, such that $A \cap B = \emptyset$ and $A \cup B = Y$. If A and B comprise a split of Y, we denote the split by $A|B$ (or, equivalently, $B|A$). If $A|B$ is a split and $|A| = i$, then we may call $A|B$ an i-split; it follows that every i-split is an $(n{-}i)$-split, where $n = |Y|$. If $\mathscr{S}$ is a collection of splits, we denote the collection of i-splits by $\mathscr{S}_i$.

Example 1. Let $Y = \{a, b, c, d\}$. Then, if $A = \{a\}$ and $B = \{b, c, d\}$, $A|B$ is a split of Y; in particular, it is a 1-split (or, equivalently, a 3-split). However, if $A = \{a\}$ and $B = \{b, d\}$, then $A|B$ is not a split of Y, as the element c is not included.

Note that, if Y has cardinality n, there are $2^{n-1}{-}1$ possible splits on Y.

2.1.1 Hyperplanes and Affine Splits

In general, a *hyperplane* in an n-dimensional affine space is an $(n{-}1)$-dimensional affine set. For our purposes, as we consider the ambient space to be $\mathbb{R}^3$, when we refer to a hyperplane H, we mean a 2-dimensional plane. That is, $H = \{x \in \mathbb{R}^3 : \langle x, v\rangle = k\}$ for some $v \in \mathbb{R}^3 \setminus \underline{0}$ and $k \in \mathbb{R}$, and where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$. As a hyperplane H is fully determined by v and k, for clarity's sake, we may refer to H as $H(v, k)$.

A set of the form $\{x \in \mathbb{R}^3 : \langle x, v\rangle > k\}$ is called an *open halfspace* and a set of the form $\{x \in \mathbb{R}^3 : \langle x, v\rangle \geq k\}$ is a *closed* halfspace. Given a hyperplane $H = H(v, k)$, we denote the open halfspaces induced by H as $H^+ = \{x \in \mathbb{R}^3 : \langle x, v\rangle > k\}$ and $H^- = \{x \in \mathbb{R}^3 : \langle x, v\rangle < k\}$ for convenience.

Let $\phi : Y \hookrightarrow \mathbb{R}^3$ be an embedding. We call a split $A|B$ of Y an *affine split* (with respect to ϕ) if there exists an open halfspace γ such that $\phi(A) = \phi(Y) \cap \gamma$. We say that the sets A and B are *affinely separable* (or *separable*) if $A|B$ is an affine split with respect to an embedding ϕ.

We denote the collection of all affine splits on Y (with respect to ϕ) by $\mathscr{A}_\phi(Y)$. A collection $\mathscr{S}$ of splits on a finite set Y is *affine* if there exists an embedding $\phi : Y \hookrightarrow \mathbb{R}^3$ such that $\mathscr{S} \subseteq \mathscr{A}_\phi(Y)$. It is important to note that the structure of $\mathscr{A}_\phi(Y)$ is intrinsically dependent on ϕ, and that there will typically be many different non-equivalent affine split collections for any given Y.

Note that, throughout the following, we make use of embeddings of (abstract) sets of points into $\mathbb{R}^3$. This is in order to reconcile with the underlying context in which we wish to utilise spherical splits, where the points represent data labels and the embedding represents the means by which we visualise the data (even though we will not delve into this background territory, herein). Thus, we have kept the embedding, playing the role of intermediary in the bioinformatic process, as we wish to understand which properties of the embedding must be considered in further research.

Example 2. Let $X = \{a, b, c\}$. Then, a collection of splits on X is

$$\mathscr{S} = \big\{\{a\}|\{b, c\}, \{a, b\}|\{c\}\big\}.$$

This collection of splits is depicted, along with an embedding ϕ of X into $\mathbb{R}^2$, in Figure 2.1; note that $\mathscr{S}$ is an affine collection of splits, as the splits are represented by 1-dimensional hyperplanes (the blue lines).

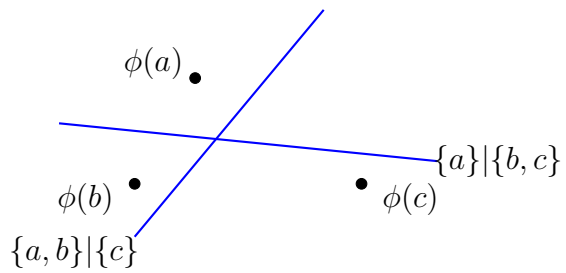


Figure 2.1: Depiction of a collection of affine splits (blue lines) on three vertices.

In investigating collections of combinatorial objects, there is a natural question which arises: When is a collection as large as it can possibly be? Or, more precisely: When is a collection not properly contained in any larger collection? We begin by establishing a property of ϕ which prevents $\mathscr{A}_\phi(X)$ from attaining a maximal number of splits:

Proposition 1. *Let $\phi : X \hookrightarrow \mathbb{R}^3$ be an embedding. If four or more points of $\phi(X)$ are coplanar, then there is an embedding $\psi : X \hookrightarrow \mathbb{R}^3$ such that $\mathscr{A}_\phi(X) \subseteq \mathscr{A}_\psi(X)$ and $\mathscr{A}_\phi(X) \neq \mathscr{A}_\psi(X)$.*

Proof. Let H be a hyperplane such that $|H \cap \phi(X)| > 3$ (as $\phi(X)$ has a 4-coplanar set of points, we can find such an H) and $X' = \{x_i \in X | \phi(x_i) \in H\}$ and, for each split $A|B \in \mathscr{A}_\phi(X)$, let $H_{A|B}$ be a hyperplane separating $\phi(A)$ and $\phi(B)$. Let $\mathcal{X} = \mathbb{R}^3 \setminus \{H_{A|B} : A|B \in \mathscr{A}_\phi(X)\}$. Then, $\phi(X) \subset \mathcal{X}$ and $\mathcal{X}$ is open and so, for each $x_i \in X$, there is an open ball $B(\phi(x_i), \epsilon_i) \subset \mathcal{X}$, where $\epsilon_i > 0$. Let $\epsilon = \min_i \epsilon_i$.

Let v be a unit normal vector to H. For any $u \in X'$, we consider the embedding:

$$\psi(x) = \begin{cases} \phi(x) & x \notin X' \\ \phi(x) + \epsilon v & x = u \\ \phi(x) - \epsilon v & x \in X' \setminus \{u\} \end{cases},$$

with associated affine splits $\mathscr{A}_\psi(X)$. Let $S = A\big|B \in \mathscr{A}_\phi(X)$ and H_S be a plane corresponding to S determined by normal n and constant k. By definition, ϵ is less than the distance from any point in $\phi(X')$ to H_S, which means that the image of any point a in $A \cap X'$ satisfies (without loss of generality) $\psi(a) \in \psi(X) \cap H_S^+$ and, similarly, $\psi(b) \in \psi(X) \cap H_S^-$ for all points b in $B \cap X'$. Thus, $S \in \mathscr{A}_\psi(X)$ and so $\mathscr{A}_\phi(X) \subseteq \mathscr{A}_\psi(X)$. Therefore, every split in $\mathscr{A}_\phi(X)$ is a split in $\mathscr{A}_\psi(X)$.

To see that $\mathscr{A}_\psi(X)$ contains a split that is not in $\mathscr{A}_\phi(X)$, consider the plane H. We know that $H \notin \mathscr{A}_\phi(X)$, as $X' \subset H$. Without loss of generality, let $A = (X \setminus X') \cap H^+$ and $B = (X \setminus X') \cap H^-$. Fix $u \in X'$ and let ψ be defined with respect to u, as above. Then, H will correspond to the split $S_H = \big(A \cup \{u\}\big)\big|\big(B \cup (X' \setminus \{u\})\big)$.

Therefore, $S_H \notin \mathscr{A}_\phi(X)$ but $S_H \in \mathscr{A}_\psi(X)$ and, consequently, $\mathscr{A}_\phi(X)$ is properly contained in $\mathscr{A}_\psi(X)$.

□

We say that a collection of points in $\mathbb{R}^d$ is in *general position* when no $d+1$ points are contained in any (d–1-dimensional) hyperplane. In $\mathbb{R}^3$, this corresponds exactly to disallowing 4-coplanarity: No four points lie on any (2-dimensional) hyperplane. For example, in Figure 2.2, the vertices of the tetrahedron are in general position, but the vertices of the square are not. Note that, for any $n \in \mathbb{N}$ and $d \in \mathbb{N}$, we can always find a set of n points in general position in $\mathbb{R}^d$: For example, a result of Erdős implies that, for sufficiently large N, $(1-\epsilon)N$ points can be placed in general position on the $N \times N$ grid for any $\epsilon > 0$ (Roth, 1951; Froese, Kanj, Nichterlein, and Niedermeier, 2015), which we can then naturally extend to any $N \times N \times \cdots \times N$ grid.

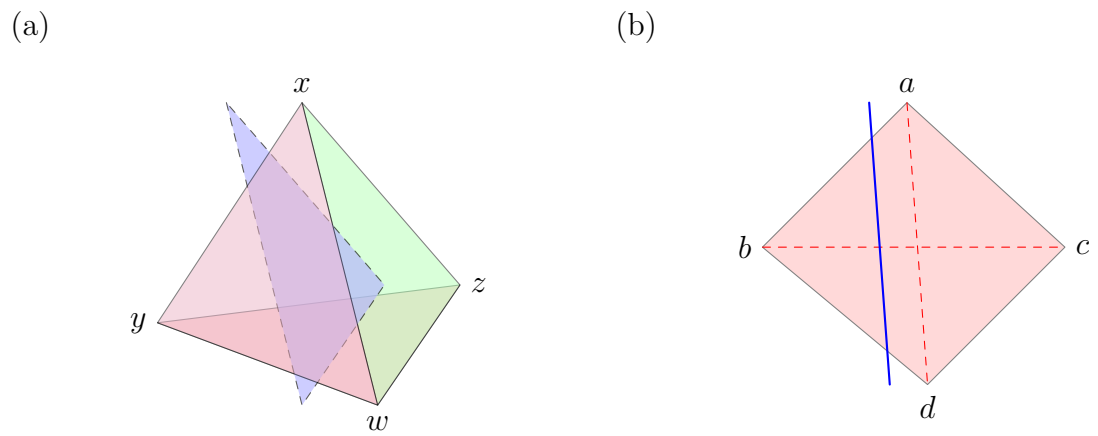


Figure 2.2: (a) Tetrahedron and (b) coplanar square formed by four points in $\mathbb{R}^3$.

In Figure 2.2(a), the blue triangle is a section of a separating hyperplane representing the affine split $\{x, z, w\}|\{y\}$; in Figure 2.2(b), the blue line is a one-dimensional section of the hyperplane representing the split $\{a, c, d\}|\{b\}$, and the red dashed lines correspond to the unrealisable 2-split $\{a, d\}|\{b, c\}$ induced by the 4-coplanarity of the vertices.

Now, we work towards finding an upper bound for the number of affine splits associated with an arbitrary embedding ϕ. For convenience, we define the *integer function* (following the notation of Gärtner and Welzl, 1994) as:

$$\Phi_d(n) := \binom{n}{\leq d} = \sum_{i=0}^{d} \binom{n}{i}.$$

Harding (1967) asked and answered the following question: "What is the number of distinct partitions of a given set of N points in k dimensions that can be thus induced by (k–1)-dimensional hyperplanes?" The following theorem is a direct result of Harding's

theorem (Hwang and Rothblum, 2011, Theorem 1), translated to the context of affine splits (instead of "separable 2-partitions"):

Theorem 2. *Given a finite set of points X and an embedding $\phi : X \hookrightarrow \mathbb{R}^3$ such that the points of $\phi(X)$ are in general position, the number of distinct affine splits in $\mathscr{A}_\phi(X)$ is $\Phi_3(|X|-1)-1$.*

Note that we have the term $\Phi_3\big(|X|-1\big)-1$, and not $\Phi_3\big(|X|-1\big)$ (as in the original), as $X|\emptyset$ is considered a separable 2-partition, but not an affine split.

Proposition 3 follows immediately:

Proposition 3. *Let X be a finite set. If ϕ is an embedding of X into $\mathbb{R}^3$ and $\mathscr{A}_\phi(X)$ is the associated collection of affine splits, then $|\mathscr{A}_\phi(X)| \leq \Phi_3\big(|X|-1\big)-1$.*

Proof. This is a direct consequence of Theorem 2, as the number n of points in general position in X cannot exceed $|X|$. □

The result of Proposition 1 indicates that, with the property that no four points are coplanar, sets of points in general position provide a likely candidate for those on which the collections of affine splits are maximal. We show that any collection of affine splits may be extended to a collection on a set of points in general position.

Proposition 4. *For any collection of affine splits $\mathscr{A}_\phi(X)$, there is an embedding ψ such that $\psi(X)$ is in general position and $\mathscr{A}_\phi(X) \subseteq \mathscr{A}_\psi(X)$.*

Proof. If $\phi(X)$ has no 4-coplanar set of points, then $\phi(X)$ is already in general position. So, suppose that $\phi(X)$ is not in general position and that there exists at least one set of four coplanar points in $\phi(X)$.

By Proposition 1, we can find an embedding $\psi_1 : X \hookrightarrow \mathbb{R}^3$ such that $\mathscr{A}_\phi(X) \subsetneq \mathscr{A}_{\psi_1}(X)$. Now, either $\psi_1(X)$ is in general position, or has at least one set of four coplanar points as well. In the latter case, by Proposition 1, there again exists an embedding $\psi_2 : X \hookrightarrow \mathbb{R}^3$ such that $\mathscr{A}_{\psi_1}(X) \subsetneq \mathscr{A}_{\psi_2}(X)$, and so on. For each ψ_i such that $\psi_i(X)$ is not in general position, we split a 4-coplanar set of points to yield another embedding ψ_{i+1} such that

$$\mathscr{A}_\phi(X) \subsetneq \mathscr{A}_{\psi_1}(X) \subsetneq \cdots \subsetneq \mathscr{A}_{\psi_i}(X) \subsetneq \mathscr{A}_{\psi_{i+1}}(X)$$

is a chain of proper containments.

By Proposition 3, the cardinality of a collection of affine splits on X is bounded by $\Phi_3(|X|-1)-1$, and as $|\mathscr{A}_{\psi_k}|$ is a strictly increasing sequence in k, there must exist some

embedding ω such that $\mathscr{A}_{\psi_k}(X) \subseteq \mathscr{A}_\omega(X)$ and there exists no embedding ψ such that $\mathscr{A}_\omega(X) \subsetneq \mathscr{A}_\psi(X)$. This occurs exactly when $\omega(X)$ has no set of 4-coplanar points and, hence, is in general position.

Therefore, $\mathscr{A}_\phi(X) \subsetneq \mathscr{A}_\omega(X)$ and $\omega(X)$ is in general position.

□

We culminate by showing that the bound on a collection of affine splits is tight and, thus, maximality is achieved for collections of affine splits with respect to embeddings of points into general position:

Proposition 5. *If $\phi(X)$ is in general position, then $|\mathscr{A}_\phi(X)| = \Phi_3(|X|-1)-1$. Furthermore, there exists no embedding $\psi : X \hookrightarrow \mathbb{R}^3$ such that $\mathscr{A}_\phi(X) \subsetneq \mathscr{A}_\psi(X)$.*

Proof. As we have $|X|$ points in general position, Theorem 2 implies that $|\mathscr{A}_\phi(X)| = \Phi_3(|X|-1) - 1$.

Now, suppose that there exists $\psi : X \hookrightarrow \mathbb{R}^3$ such that $\mathscr{A}_\phi(X) \subsetneq \mathscr{A}_\psi(X)$. This means that $|\mathscr{A}_\psi(X)| > \Phi_3(|X|-1)-1$, which, by Theorem 2, implies that there are more than $|X|$ points in X. This is clearly a contradiction, and so $\mathscr{A}_\phi(X)$ is contained in no larger collection of affine splits. □

Proposition 5 motivates the following definition. An affine collection of splits $\mathscr{S}$ on X is *maximal* when $\mathscr{S} = \mathscr{A}_\phi(X)$, where $\phi(X)$ is in general position. For affine splits in $\mathbb{R}^3$, then, maximality occurs when no four points in $\phi(X)$ are coplanar (see Figure 2.2).

The cardinalities of maximal sets of affine splits on n points are shown in Table 4.1, in a later chapter.

Thus, we have a characterisation of maximal collections of affine splits (i.e., those with respect to embeddings in general position) and the cardinality (i.e., $\Phi_3(|X|-1)-1$) of such collections.

2.1.2 Alternate Proof for the Bound

The section above was made much easier (and, perhaps, slightly less transparent) by the use of Theorem 2. We thought it was worthwhile to include our initial derivation

of the bound on a collection of affine splits, as it made use of concepts which are of interest (at least, in our opinion).

To establish the cardinality of a set $\mathscr{S}$ of affine splits, we make use of a specific duality for points and hyperplanes; in particular, the *polar* (Gallier, 2008, Definition 3.3). For a point $x \in \mathbb{R}^3$ and hyperplane $H \subset \mathbb{R}^3$ not containing the origin, define the polar operations $x \mapsto x^\circ$ and $H \mapsto H^\circ$ by

$$x^\circ = \{y \in \mathbb{R}^3 : \langle x, y \rangle = 1\}$$

such that each x maps to a unique hyperplane $x^\circ \in \mathbb{R}^3$, and

$$H^\circ \text{ such that } H = \{x \in \mathbb{R}^3 : \langle H^\circ, x \rangle = 1\},$$

so the polar maps a hyperplane H to a unique point $H^\circ \in \mathbb{R}^3$.

Furthermore, the polar mapping is incidence- and order-preserving (Gallier, 2008); that is,

$$x \in H_+ \Leftrightarrow H^\circ \in x_+^\circ,$$

where H_+ is one of the halfspaces induced by H.

Furthermore, if $x \in S^2$, then x° is the unique hyperplane tangent to S^2 at x.

Note that there are multiple ways that the polar dual of a set has been defined throught the literature (see, e.g., Charney and Davis, 1995). We have chosen this specific definition for the polar as alternative definitions of x° and H° yield other types of sets, which may not be desirable for the matter at hand. For example, if the dual of a point is instead defined by $x^\circ = \{y \in \mathbb{R}^3 : \langle x, y \rangle \leq 1\}$, then x° would not be a hyperplane, but instead the closed halfspace associated with that hyperplane which contains the origin (Gallier, 2008, Definition 3.4).

Proposition 6. *Let X be a set and $\phi : X \hookrightarrow \mathbb{R}^3$ such that $\phi(x_0)$ is the origin for some $x_0 \in X$. There exists a bijective map between a collection of affine splits $\mathscr{A}_\phi(X)$ and the regions cut by the polar arrangement $\mathcal{X}_0 = \{x^\circ : x \in X \setminus \{x_0\}\}$.*

Proof. As $\phi(x_0)$ is the origin, it follows, for every separating hyperplane H of $\phi(X \setminus \{x_0\})$, that $x_0 \in H^-$.

Now, consider two separating hyperplanes H_1 and H_2 such that $H_1^+ \cap \phi(X \setminus x_0) = H_2^- \cap \phi(X \setminus x_0)$. Then, H_1 and H_2 induce the same split in $X \setminus x_0$, but distinct splits in

X. Thus, every split induced by the arrangement of hyperplanes separating $X \setminus \{x_0\}$ is unique.

Now, consider the polar arrangement given by $\mathcal{X}_0$, as defined above. As it is a collection of polar hyperplanes, $\mathcal{X}_0$ cuts $\mathbb{R}^3$ into a collection of disjoint open regions $\mathscr{R}_0$.

Suppose that a split $S \in \mathscr{A}_\phi(X)$ has two representative hyperplanes H_1 and H_2. Then, we have that $H_1^+ \cap \phi(X) = H_2^+ \cap \phi(X)$, and so $\phi(x) \in H_1^+ \Leftrightarrow \phi(x) \in H_2^+$ for all $x \in X$. By the order preservation of the polar, then, we have that

$$H_1^\circ \in \phi(x)^{\circ,+} \Leftrightarrow H_2^\circ \in \phi(x)^{\circ,+},$$

and so H_1° and H_2° lie in the same region R_S. Thus, the split S uniquely corresponds to the region R_S.

To see that there is one region which does not technically correspond to a split, consider the polar region $V = \bigcap_{x \in X \setminus \{x_0\}} \phi(x)^{\circ,-}$. Any point $h \in V$ satisfies $h \in \phi(x)^{\circ,-}$ for all $x \in$X$\setminus\{x_0\}$, and so corresponds to a hyperplane satisfying $\phi(x) \in h^{\circ,-}$ for all $x \in X$ (as $\phi(x_0)$ must be in $h^{\circ,-}$ as well), and so V corresponds to the partition $X|\emptyset$, which is not an affine split.

For any split $S \in \mathscr{A}_\phi(X)$, let $H_S = H(v_S, 1)$ be any hyperplane inducing the split, and let R_S be the region cut by $\mathcal{X}_0$ containing $H_S^\circ = v_S$. Then, the map $\iota : \mathscr{S} \to \mathscr{R}_0$ such that S maps to the region containing v_S is well-defined, as any representative hyperplane of S maps to a unique region R_S, as established above.

Now, let $S, S' \in \mathscr{A}_\mathscr{S}(X)$ such that $S = A|B \neq A'|B' = S'$. Then, without loss of generality, if $A = H_S^+ \cap \phi(X)$ and $A' = H_{S'}^+ \cap \phi(X)$ then there exists at least one element $x \in X \setminus \{x_0\}$ such that $x \in A$ but $x \notin A'$, or vice versa. This implies that H_S° and $H_{S'}^\circ$ are separated by at least one of the x° in $\mathcal{X}_0$ and, so, cannot be in the same region. Hence, ι is injective.

Finally, let v be a point in any region cut by $\mathcal{X}_0$. Then, v° corresponds to the hyperplane $H = \{x \in \mathbb{R}^3 : \langle x, v \rangle = 1\}$, which does not intersect $\phi(X)$ by incidence preservation of the polar and, hence, induces a split S in $\mathscr{A}_\phi(X)$. However, as v is in the same region R_S as any other representative hyperplane of S, the region R_S is in the domain of ι and, so, with the inclusion of $\iota(X|\emptyset) = V$, ι is surjective.

Thus, as a well-defined, injective, and surjective mapping, ι is a bijection between $\mathscr{S}$ and $\mathscr{R}_0$.

□

We demonstrate the bound on the cardinality of an affine split collection by establishing the cardinality of a maximal collection of affine splits:

Proposition 7. *Let X be a set of cardinality n and $\phi(X)$ be an embedding of X into $\mathbb{R}^3$. The number of splits in a maximal collection of affine splits on $\phi(X)$ is $\Phi_3(n-1)-1$.*

Proof. Note that translating the points $\phi(X)$ such that $\phi(x_0)$ is the origin for some point, as in Proposition 6, does not change the structure of the split collection. So, without loss of generality, Proposition 6 holds for an arbitrary embedding into $\mathbb{R}^3$.

Now, suppose that $\phi(X)$ is in general position. By Stanley (2006), the total number of regions—bounded or unbounded—cut by a hyperplane arrangement X is at most $\Phi_3(|X|)$. Therefore, the maximum number of regions cut by the arrangement $\mathcal{X}_0$ is $\Phi_3(|X|-1)$. Furthermore, as the planes are arranged such that every three intersect at a point, as a consequence of the $\phi(X)$ being in general position, the arrangement cuts the maximal amount $\Phi_3(|X|-1)$ of regions.

As, by Proposition 6, the splits are in bijection with the regions cut by $\mathcal{X}_0$, we obtain the desired result by disregarding the one region given by $\iota(X|\emptyset)$:

$$|\mathscr{A}_\phi(X)| = \phi_3(|X|-1)-1.$$

□

As any non-maximal collection of affine splits, by definition, must have less splits than the maximal case, it follows that $\Phi_3(|X|-1)-1$ is the upper bound we desire.

2.2 Polytopes

Before we can define spherical splits, we need to understand a bit about convex polytopes. We may, loosely, think of as polytopes as convex geometric objects with "flat sides". The idea of such objects has existed since antiquity; for example, the Platonic solids are all three-dimensional regular convex polytopes (the simplest of which being the tetrahedron, shown in Figure 2.2(a)). However, the bulk of research into the combinatorial aspects of convex polytopes has been made only in the last century. Grünbaum's *Convex Polytopes* (Grünbaum, 2003) provides a comprehensive survey of convex polytope theory.

The *convex hull* of a set of points X in $\mathbb{R}^d$ is the "smallest" convex set $\text{conv}(X) \subset \mathbb{R}^d$ containing X. More precisely, the convex hull can be defined as the intersection of all convex sets containing X; thus, any other convex set containing X must also contain $\text{conv}(X)$, justifying the use of the term "smallest".

Example 3. If we have the six points in the plane shown in Figure 2.3, then the convex hull is the polygon "wrapped around" the points a, b, c, d, and e. Notice that f lies in the interior of the convex hull.

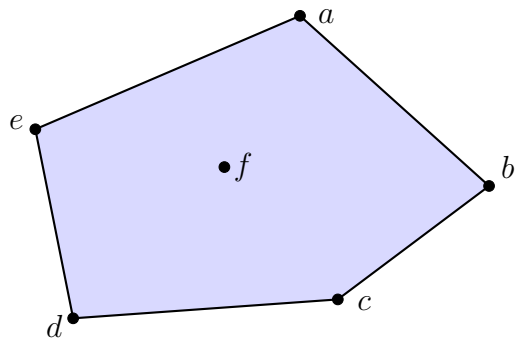


Figure 2.3: Convex hull of six points in the plane.

A *convex polytope* $\mathcal{P}$ can be defined equivalently as the convex hull of a finite number of points or as the finite intersection of a finite number of halfspaces (Matoušek, 2002, Theorem 5.2.2). A polytope $\mathcal{P}$ is called a d-polytope if d is the dimension of the smallest affine subset containing the vertices of $\mathcal{P}$; that is, the dimension of the *affine hull* $\{\sum_{i=1} \lambda_i x_i : x_i \in X, \sum_i \lambda_i = 1\}$, where X is the set of vertices of $\mathcal{P}$. In our case, we will only consider bounded convex 3-polytopes (i.e., those in $\mathbb{R}^3$), which we will simply refer to as polytopes without risk of confusion.

A *face* of a polytope $\mathcal{P}$ is a subset of $\mathcal{P}$ of the form $\mathcal{P} \cap H$, where H is a hyperplane such that $\mathcal{P}$ is fully contained in one of the closed halfspaces induced by H; we call a 0-dimensional face a *vertex*, a 1-dimensional face an *edge*, and a 2-dimensional face a *facet* of $\mathcal{P}$, respectively. The set of all faces $\mathcal{F}(\mathcal{P})$ of a polytope $\mathcal{P}$ has a natural *partial ordering* $\leq$ by inclusion; that is,

$$F \leq G \iff F \subseteq G,$$

for all faces $F, G \in \mathcal{F}(\mathcal{P})$. Note that $\mathcal{P}$ is itself a face, and so $\mathcal{P} \in \mathcal{F}(\mathcal{P})$. Hence, $\mathcal{F}(\mathcal{P})$ is bounded and has well-defined *meet* and *join* operations ($\wedge$ and $\vee$, respectively), given as follows:

$$F \wedge G = F \cap G,$$

and where $F \vee G$ is the smallest face containing both F and G. It follows that $\mathcal{F}(\mathcal{P})$ is a *lattice*, which we call the *face lattice* of $\mathcal{P}$ (see Section 2.2 of Ziegler, 1995, for a more detailed overview of lattices in the context of polytopes).

Example 4. Considering the tetrahedron (as shown in Figure 2.2a), the corresponding face lattice is presented in Figure 2.4. We have that each "tier" of the lattice corresponds to faces of different dimensions; for example, the middle tier contains the edges of $\mathcal{P}$. We also have, for example, $xyz \wedge xyw = xy$ and $yz \vee yw = yzw$.

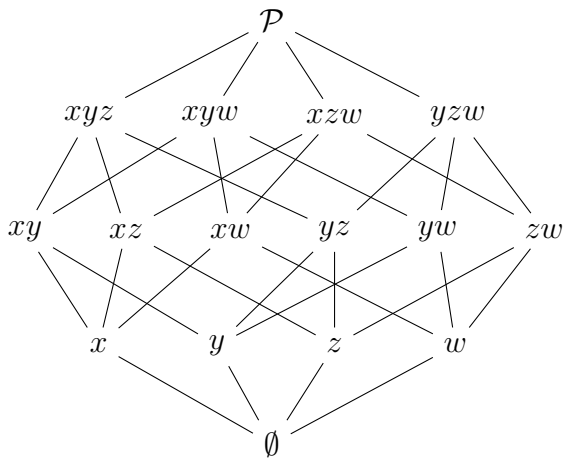


Figure 2.4: The face lattice of the tetrahedron.

2.2.1 Polytopal Splits

For any collection of points $X \in \mathbb{R}^d$, $\operatorname{conv}(X)$ defines an m-polytope (i.e., of dimension $m \leq d$). The set of vertices of the polytope given by $\operatorname{conv}(X)$ are called the *extremal* vertices of X; that is, the extremal vertices are those which do not belong to the interior of any other face of $\operatorname{conv}(X)$ or, equivalently, x is an extremal vertex if and only if

$$\operatorname{conv}\big(X \setminus \{x\}\big) \neq \operatorname{conv}(X).$$

If $\mathscr{A}_\phi(X)$ is a collection of affine splits with respect to an embedding ϕ, we say that $\mathscr{A}_\phi(X)$ is *polytopal* if $\phi(X)$ forms the set of vertices of a 3-polytope. The following theorem provides a characterisation of polytopal collections of splits in terms of the structure of $\mathscr{A}_\phi(X)$:

Theorem 8. *If $\mathscr{S} \subseteq \mathscr{A}_\phi(X)$ is a collection of affine splits such that, for each $x \in X$, $S_x = \{x\}\big|X \setminus \{x\}$ is in $\mathscr{S}$, then $\mathscr{S}$ is a collection of polytopal splits.*

Proof. Let $x \in X$. As S_x is a split for all $y \in X$, $\phi(x)$ is affinely separable from $\phi(X \setminus \{x\})$ and, so, x must not be contained in $\operatorname{conv}\big(\phi(X \setminus \{x\})\big)$. Thus, x must be extremal. As this holds for all $x \in X$, each $\phi(x)$ must be a vertex of $\operatorname{conv}\big(\phi(X)\big)$ and so $\mathscr{S}$ is a collection of polytopal splits.

□

Note that, in general, a collection of affine splits will not be polytopal. For example, the expected number of vertices in the convex hull of n points uniformly distributed in the 3-ball is (asymptotically) on the order of $\sqrt{n}$ (Meilijson, 1990), and on the order of $\log(n)$ when the points are chosen in $\mathbb{R}^3$ using the 3-dimensional normal distribution (Har-Peled, 2011); thus, with increasing n, the likelihood of all n points being extremal will be very low.

Example 5. Figure 2.5 shows two splits (realised as hyperplanes) of a polytope with vertices in general position.

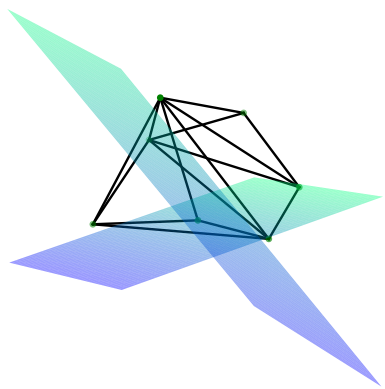


Figure 2.5: Two polytopal splits.

If all points in $\phi(X)$ are vertices of $\operatorname{conv}\big(\phi(X)\big)$ then we say that $\phi(X)$ is *convex independent* (or is in *convex position*, equivalently); that is, for all $x \in X$, we have $\phi(x) \notin \operatorname{conv}\big(\phi(X \setminus \{x\})\big)$. In particular, if $\phi(X)$ is in general position, this implies that, for any five-point subset $X' \subseteq X$, $\operatorname{conv}(X')$ is not a tetrahedron containing one of the points of X'.

We say that $\mathscr{A}_\phi(X)$ is a *maximal* collection of polytopal splits if it is polytopal and

maximal as an affine collection of splits. In particular, this implies that $\phi(X)$ is in general and convex position.

2.2.2 Simplices and Simplicial Complexes

The tetrahedron is an example of a *simplex*: A polytope which is the convex hull of an affinely independent set of vertices. Polytopes may be built by *stacking* (i.e., gluing together at a facet) simplices, provided the resultant set is convex; the triangular bipyramid shown in Figure 2.6 is the result of stacking two tetrahedra. Polytopes constructed in such a way are called *stacked* polytopes.

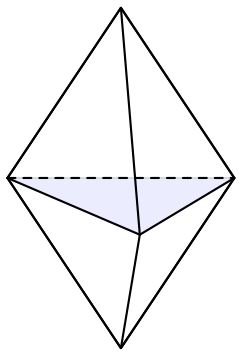


Figure 2.6: The triangular bipyramid, constructed by gluing two tetrahedra together at a facet (shown in blue).

Stacked polytopes composed of tetrahedra are examples of *simplicial complexes*, which are sets of simplices $\mathscr{K}$ satisfying

K1. Every face of a simplex in $\mathscr{K}$ is also in $\mathscr{K}$, and

K2. The non-empty intersection of any two simplices $T_1, T_2 \in \mathscr{K}$ is a face of both T_1 and T_2.

Example 6. For instance, considering the triangular bipyramid in Figure 2.6, we can form a simplicial complex $\mathscr{K}_{bipy}$ if we consider the collection including the two tetrahedra and all of the external faces, edges, and vertices of the polytope, **as well as** their mutual internal facet (in order to guarantee conditions K1 and K2); as such,

- $\mathscr{K}_{bipy}$ is a simplicial 3-complex, as the largest dimension of any simplex is three;
- $\mathscr{K}_{bipy}$ is a *pure* simplicial complex, as all of the maximal simplices (tetrahedra) have the same dimension; and

– The *boundary* of $\mathscr{K}_{bipy}$—that is, the subcomplex consisting of all lower-dimensional (i.e., $d < 3$) simplices only contained in one of the maximal simplices—is the triangular bipyramid.

We will revisit simplicial complexes in Section 3.4, in the context of certain lattices induced by maximal collections of splits.

2.2.3 Polytope Inscribability

Two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are *combinatorially equivalent* if $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{Q})$ are isomorphic as lattices.

A polytope is *inscribable* if it is combinatorially equivalent to a polytope with all vertices lying on the sphere S^2. An example of a non-inscribable polytope (in fact, the smallest non-inscribable polytope with respect to number of faces) is shown in Figure 2.7 (Grünbaum, 1963, see Figure 4.6, also). For a comprehensive modern overview of polytope scribability (that is, inscribability and the dually related circumscribability), the reader is referred to Chen and Padrol (2017). We will also consider inscribability, in terms of the graphs related to polytopes, in section 4.3.

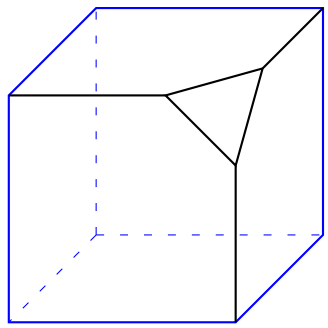


Figure 2.7: The face-minimal uninscribable polytope (i.e., the non-inscribable polytope with the least number of faces).

2.3 Spherical Splits

A collection of splits $\mathscr{S}$ is *spherical* if $\mathscr{S} \subset \mathscr{A}_\phi(X)$ for some embedding $\phi : X \hookrightarrow \mathbb{R}^3$ such that $\phi(X) \subset S^2$ (i.e., the $\phi(x)$ all lie on the unit sphere). Note that, as they are particular cases of affine splits, the above results (in particular, Propositions 3 and 5) hold for spherical splits.

A classification of the polytopal collections of splits $\mathscr{S} = \mathscr{A}_\phi(X)$ for which we may inscribe the points while retaining the structure of $\mathscr{S}$ would be useful. We call a polytopal collection of splits $\mathscr{A}_\phi(X)$ *deformable* to a spherical collection of splits if the points $\phi(X)$ can be moved onto a sphere without changing any split $S \in \mathscr{S}$. Then, a polytopal collection of splits is spherical if it is deformable to a spherical collection of splits. We conjecture the existence of a non-deformable polytopal sets of splits, but lack a characterisation of such a case at present.

Conjecture 1. *There exists a collection of maximal polytopal splits which is not deformable to a collection of spherical splits.*

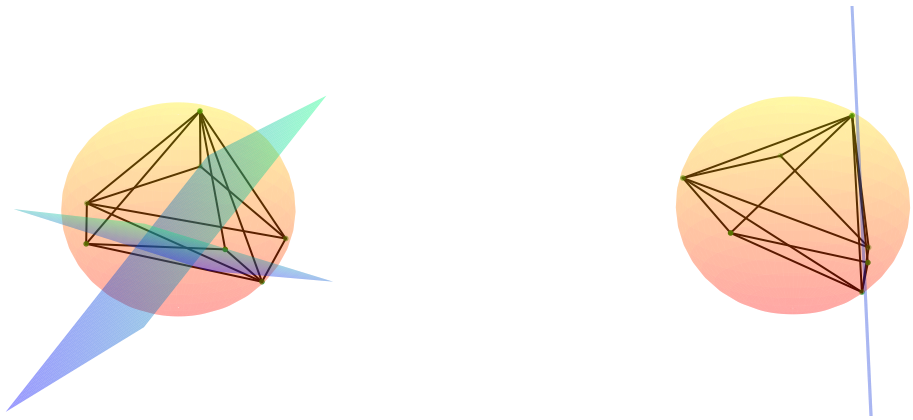
We believe that such an example may be constructed by demonstrating that there exists a collection of $\Phi_3(n-1)$ convex simple curves in the plane partitioning a set of n points in general position which cannot be deformed into true circles without at least one of the points necessarily crossing one of the curves. If true, this would imply that there is an intrinsic geometric/combinatorial quality which more strongly categorises spherical splits than the definition given.

If $\phi(X)$ is in general position, then it corresponds to a maximal collection of affine splits by definition. Thus, the maximal collections of spherical splits are precisely the collections of maximal affine splits where $\phi(X)$ is in general position on S^2 (and, hence, in general and convex position).

It is worth noting that a general collection of affine splits will not be spherical (there will be non-extremal vertices), and that there exist affine split collections such that all points of $\phi(X)$ are extremal, but $\mathscr{A}_\phi(X)$ is not spherical—this happens exactly when $\operatorname{conv}\big(\phi(X)\big)$ is the boundary of a non-inscribable polytope. However, if Conjecture 1 is not true, then all maximal polytopal collections of splits are deformable to maximal collections of spherical splits, which would provide a very strong relationship between the polytopal and spherical splits, especially considering Proposition 4.

Example 7. The polytope of Example 5 is shown inscribed in the sphere in Figure 2.8, with the same edge set. Two spherical splits are shown in Figure 2.8(a). In normalising the vertex co-ordinates, the four vertices forming a "kite" on the right side have moved such that the edge down the center, which originally corresponded to a polytopal 2-split, is not a spherical 2-split (see Figure 2.8(b)). This does not necessarily mean that the system is not deformable, however, it exemplifies that the process of embedding the vertices affects the structure of the collection of splits. If the top vertex were moved

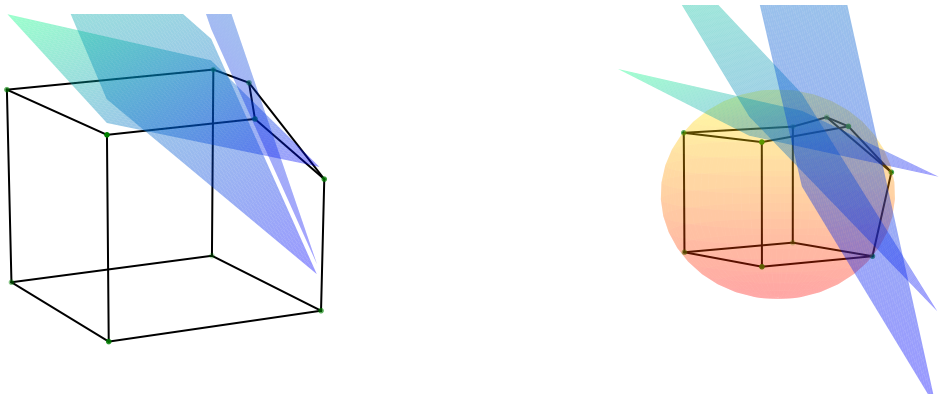
along the sphere "clockwise" (in the plane), eventually the 2-split associated to the edge would become realisable.



(a) (b)

Figure 2.8: (a) Two spherical splits on an inscribed polytope; and (b) side view of a split showing a change in a 2-split due to the embedding.

A few polytopal splits of the face-minimal non-inscribable polytope and the same for the polytope after normalising its vertex co-ordinates are shown in Figure 2.9. Note that, after normalisation, it can be observed that the edges no longer correspond to a convex polytope. Furthermore, before and after normalisation, there are 108 and 119 total affine splits on the vertices (as determined by the linear program detailed in Chapter 5), respectively. This demonstrates that, as we would expect, the collection of polytopal splits on this polytope is not deformable to the full collection of spherical collection of splits on the associated inscribed polytope. However, neither are maximal, as a consequence of the cubic (lower left) corner.



(a) (b)

Figure 2.9: (a) Polytopal splits on the face-minimal non-inscribable polytope; and (b) spherical splits on the normalised vertices of the polytope.

For a collection $\mathscr{S}$ of splits, we define the *restriction* of the collection to $Y \subseteq X$ to be

$$\mathscr{S}|_Y = \{A \cap Y \big| B \cap Y : A|B \in \mathscr{S} \text{ and } A \cap Y \neq \emptyset \neq B \cap Y\}.$$

Note that we have $|\mathscr{S}|_Y| < |\mathscr{S}|$ for any $Y \subsetneq X$.

Proposition 9. *Let X be a set of points and $\phi : X \hookrightarrow S^2$ be an embedding in general position, such that $\mathscr{S} = \mathscr{A}_\phi(X)$ is a maximal collection of spherical splits on X. If $A \subseteq X$, then the restriction of $\mathscr{S}$ to A is a maximal collection of spherical splits.*

Proof. Let $A \subseteq X$. As $\phi(X)$ is in general position, every subset of $\phi(X)$ must also be. Thus, the restriction of $\phi(X)$ to $\phi(A)$ is in general position on the sphere, and induces a maximal collection of affine splits by Proposition 5. Thus, $\mathscr{A}_\phi(A)$, as a maximal collection of affine splits on a set of points on the sphere, is a maximal collection of spherical splits on $\phi(A)$ by definition. Furthermore, as no point was moved, each split in $\mathscr{A}_\phi(A)$ must be the restriction of at least one split in $\mathscr{A}_\phi(X)$, and so $\mathscr{A}_\phi(A) = \mathscr{A}_\phi(X)|_A$.

□

By the same logic, a corollary applying to affine splits (which we will make use of in the next chapter) immediately follows:

Corollary 10. *Let X be a set of points and $\phi : X \hookrightarrow \mathbb{R}^3$ be an embedding in general position, such that $\mathscr{S} = \mathscr{A}_\phi(X)$ is a maximal collection of affine splits on X. If $A \subseteq X$, then the restriction of $\mathscr{S}$ to A is also a maximal collection of affine splits.*

Chapter 3

Oriented Matroids and Topes

An oriented matroid may be defined in a number of equivalent ways. Here, we use the covector and tope axiomatisations, as topes naturally correspond to open cells in arrangements of hyperplanes (or pseudospheres). In this section, the definitions follow those of Björner, Vergnas, Sturmfels, White, and Ziegler (1999) and Richter-Gebert and Ziegler (1997).

3.1 Sign Vectors and Covectors

A *sign vector* is simply a vector with entries in $\{+, -, 0\}$.

The concept of a sign vector generalises the idea of a partition of a set X. If $|X| = n$ and $[n] = (1, 2, \cdots, n)$ is an ordering of the elements x_i of X, then a sign vector C is a vector of length n such that the ith entry C_{x_i} corresponds to x_i, where $+$ entries correspond to points on the positive side of the partition, $-$ to those on the negative side, and 0 to those lying on the separation (which we consider to be a hyperplane, in our context).

The entry of C associated with an element $x \in X$ is denoted by C_x. The *zero set* of a sign vector C is the set $z(C) = \{x \in X : C_x = 0\} \subseteq X$. The *zero vector* $\underline{0}$ is simply the sign vector with all zero entries. Similarly, we define $C^+ = \{x \in X : C_x = +\}$ and $C^- = \{x \in X : C_x = -\}$.

For sign vectors $C, D \in \{+, -, 0\}^X$, the *composition* $C \circ D$ is defined (co-ordinate wise)

by:

$$(C \circ D)_x = \begin{cases} C_x & \text{if } C_x \neq 0, \\ D_x & \text{otherwise.} \end{cases}$$

Furthermore, we define:

$$-C_x = \begin{cases} - & \text{if } C_x = +, \\ + & \text{if } C_x = -, \\ 0 & \text{if } C_x = 0. \end{cases}$$

Thus, if $C \in \mathcal{L}$, $-C$ is the covector satisfying $(-C)_x = -C_x$ for all $x \in X$. Furthermore, for a collection of covectors $\mathcal{C} \subseteq \mathcal{L}$, $-\mathcal{C}$ is the collection $\{-C : C \in \mathcal{C}\}$.

Finally, for $C, D \in \{+, -, 0\}^X$, the *separation* $s(C, D)$ is defined as

$$s(C, D) = \{x \in X : C_x = -D_x \neq 0\}.$$

A set of sign vectors $\mathcal{L} \subseteq \{+, -, 0\}^X$ is the set of *covectors* of an oriented matroid if and only if it satisfies the following covector axioms (Björner *et al.*, 1999, Definition 4.1.1):

C1. $\underline{0} \in \mathcal{L}$;

C2. $C \in \mathcal{L}$ implies $-C \in \mathcal{L}$ (equivalently, $\mathcal{L} = -\mathcal{L}$);

C3. $C, D \in \mathcal{L}$ implies $C \circ D \in \mathcal{L}$; and

C4. if $C, D \in \mathcal{L}$ and $x \in s(C, D)$, then there exists $E \in \mathcal{L}$ such that $E_x = 0$ and $E_y = (C \circ D)_y = (D \circ C)_y$ for all $y \notin s(C, D)$.

An oriented matroid $\mathcal{M}$ is fully determined by a set of covectors satisfying the above axioms, and so we may denote $\mathcal{M}$ by $(X, \mathcal{L})$. We call X the *ground set* of $\mathcal{M}$. An oriented matroid $\mathcal{M}$ is *loop-free* if, for all $x \in X$, there exists $C \in \mathcal{L}$ such that $C_x \neq 0$. Furthermore, $\mathcal{M}$ is called *acyclic* if there is a $C \in \mathcal{L}$ such that $C_x = +$ for all $x \in X$.

The *cocircuits* of $\mathcal{L}$ are the covectors with minimal non-zero entries (or minimal *support*), and the *rank* of an oriented matroid is $|z(C)| + 1$ for any cocircuit C; that is, one more than the number of zeros in any cocircuit.

By the Topological Representation Theorem of Folkman and Lawrence (Björner *et al.*, 1999, Theorem 1.4.1), every rank $d + 1$ oriented matroid $\mathcal{M}$ can be represented as

an arrangement of oriented pseudospheres (of dimension d–1) in S^d (or, equivalently, $\mathcal{M}$ describes a cell decomposition of the d-sphere). Hence, we see a clear and desirable relationship between polytopal/spherical collections of splits and rank 4 oriented matroids.

3.2 Topes

The Folkman–Lawrence topological representation implies that every rank $d+1$ oriented matroid induces a cell decomposition of S^d, which naturally extends to a cell decomposition of $\mathbb{R}^d$; that is, by considering the extension of the (d–1)-dimensional (pseudo)spheres to (d–1)-dimensional (pseudo-)hyperplanes. In such a way, every rank $d+1$ oriented matroid corresponds to a hyperplane (or pseudo-hyperplane) arrangement in $\mathbb{R}^d$. The maximal covectors of an oriented matroid (which, as we will see shortly, also fully determine the oriented matroid) correspond precisely to the cells cut by the associated arrangement.

Uniformity of an oriented matroid, in a general sense, indicates that the hyperplanes of the corresponding arrangement are in general position, where an arrangement $\mathcal{H}$ in $\mathbb{R}^d$ is in general position if (Stanley, 2006):

$$\{H_1, \cdots, H_p\} \subset \mathcal{H},\ p \leq d \Rightarrow \dim(H_1 \cap \cdots \cap H_p) = d - p,$$
$$\{H_1, \cdots, H_p\} \subset \mathcal{H},\ p > d \Rightarrow H_1 \cap \cdots \cap H_p = \emptyset.$$

More precisely, an oriented matroid of rank d on X is *uniform* if all of its cocircuits have exactly d–1 zero entries.

Let X be an (implicitly ordered) set and $\mathcal{T} \subseteq \{+,-,0\}^X$. Then, $\mathcal{T}$ is the set of *maximal covectors* (or *topes*) of a uniform oriented matroid $\mathcal{M}$ with ground set X if and only if it satisfies the following three axioms (from Björner *et al.*, 1999, attributed as "Lawrence's axioms", Lawrence, 1983):

T1. $\mathcal{T} \neq \emptyset$ and $\mathcal{T} \neq \underline{0}$;

T2. $\mathcal{T} = -\mathcal{T}$; and

T3. if $T \in \{+,-,0\}^X$, $T \neq \underline{0}$, satisfies $T^+ \subseteq S^+$ and $T^- \subseteq S^-$ for some $S \in \mathcal{T}$, then either there is a tope $S \in \mathcal{T}$ such that $T \circ S \in \mathcal{T}$ and $T \circ (-S) \notin \mathcal{T}$, or $T \circ U \in \mathcal{T}$ for all $U \in \{+,-\}^X$.

Then, $\mathcal{T}$ uniquely determines the covectors of a uniform oriented matroid $\mathcal{M}$ (and, hence, the oriented matroid $\mathcal{M}$) through the construction (see Section 3.8 of Björner *et al.*, 1999):

$$\mathcal{L} = \{C \in \{+, -, 0\}^X : C \circ \mathcal{T} \subseteq \mathcal{T}\}. \tag{†}$$

Considering that topes must satisfy the covector axiom C3, it follows that all topes $T \in \mathcal{T}$ have the same support and, thus, the same zero set $z(T) := E_0$. We call E_0 the set of *loops* of $\mathcal{M}$. Two elements $x, y \in X \setminus E_0$ are called *parallel* if the following condition holds:

$$C_x = 0 \Leftrightarrow C_y = 0 \text{ for all } C \in \mathcal{L}.$$

If an oriented matroid does not have loops or distinct parallel elements, then it is called *simple*.

Lemma 11. *If an oriented matroid $\mathcal{M}$ is acyclic, then it is simple.*

Proof. By covector axiom C2, the topes of $\mathcal{M}$ have identical support $X \setminus E_0$; therefore, in an acyclic oriented matroid, as

$$z((+, +, \cdots, +)) = \emptyset,$$

all topes have full support and $E_0 = \emptyset$, and so $\mathcal{M}$ is loop-free and has no parallel elements. Thus, an acyclic orented matroid is simple.

□

We see that topes and covectors are, in a sense, interchangeable: Given a collection $\mathcal{L}$ of covectors, the collection of topes $\mathcal{T}$ may be identified simply as the collection of maximal covectors (i.e., those with maximal support); on the other hand, given a collection of topes $\mathcal{T}$, we may obtain the corresponding collection of covectors $\mathcal{L}$ using the equation above (†).

3.3 Vapnik–Chervonenkis Dimension and Spherical Splits

The Vapnik–Chervonenkis (VC)-*dimension*, as introduced by Chervonenkis and Vapnik (1971), of a pair $(X, \mathscr{R})$, where $\mathscr{R} \subseteq 2^X$, is the maximum cardinality of a set $Y \subseteq X$ such that $\mathscr{R}|_Y = 2^Y$; in which case, we can say that Y is *shattered* by $\mathscr{R}$. We denote the VC-dimension of the pair $(X, \mathscr{R})$ by $\dim_{VC}$.

Example 8. For a simple example, let $X = \{a, b, c\}$, $Y_1 = \{a, b\}$, $Y_2 = \{a, c\}$, and $\mathscr{R} = \{\{a, b\}, \{b, c\}, \{a\}, \{c\}\}$. Then, it follows that

$$\mathscr{R}|_{Y_1} = \{\{a, b\}, \{b\}, \{a\}, \emptyset\} \text{ and } \mathscr{R}|_{Y_2} = \{\{a\}, \{c\}\},$$

and, so, Y_1 is shattered by $\mathscr{R}$ (as $\mathscr{R}|_{Y_1} = 2^{Y_1}$), but Y_2 is not.

Furthermore, it can be seen that $\mathscr{R}$ will not be able to shatter X and, thus, the VC-dimension of $(X, \mathscr{R})$ is 2.

The VC-dimension, in its original context, measures the capacity of a statistical classifier; that is, the ability the classifier has to capture the complexity of the space it is imposed upon. In the combinatorial context (which aligns more with ours), the VC-dimension measures the ability of a family of sets to separate finite sets of points (Adams and Nobel, 2012). A VC-dimension of 4, then, implies that the family $\mathscr{R}$ has the capacity to separate sets of four points of X. An important consequence of the VC-dimension in machine learning is that a space of sets F in $\mathbb{R}^d$ is finitely learnable if and only if there is a finite bound on the cardinality of a subset of $\mathbb{R}^d$ which may be shattered by F (Natarajan, 1989, Theorem 7).

In order to define the VC-dimension of a collection of splits, we must ensure that we have a pair $(X, \mathscr{R}_{\mathscr{S}})$ which matches the required structure. To that end, we define the VC-dimension of a collection of splits $\mathscr{S}$ to be the VC-dimension of the range space $(X, \mathscr{R}_{\mathscr{S}})$, where $\mathscr{R}_{\mathscr{S}} = \{A : A|B \in \mathscr{S}\} \cup \{X, \emptyset\}$. In this way, $\mathscr{R}_{\mathscr{S}} \subset 2^X$ preserves the information of $\mathscr{S}$, and we can calculate the VC-dimension associated with a collection of splits.

Proposition 12. *Let $\mathscr{S}$ be an affine collection of splits on X. The pair $(X, \mathscr{R}_{\mathscr{S}})$ has* VC-*dimension less than or equal to* 4*, with equality when $\mathscr{S}$ is maximal.*

Proof. We will first show that if $Y \subset X$ is shattered by $\mathscr{R}_{\mathscr{S}}$, then $|Y| < 5$.

First, suppose that $\mathscr{S}$ is a maximal collection of affine splits. Let $Y \subseteq X$, such that $|Y| = 5$. Consider the restriction $\mathscr{S}|_Y$: As $\mathscr{S}$ is a maximal collection of affine splits on X, $\mathscr{S}|_Y$ corresponds exactly to a maximal collection of affine splits on Y by Corollary 10 and, so, we have that $\left|\mathscr{S}|_Y\right| = \Phi_3(4) - 1 = 14$ by Proposition 5. As each split corresponds to two subsets A and B in $\mathscr{R}_{\mathscr{S}}$, this means that the maximal number of subsets cut by $\mathscr{S}|_Y$ (in addition to X and $\emptyset$) is

$$2\left|\mathscr{S}|_Y\right| + 2 \leq 30 < 32 = \left|2^Y\right|.$$

As the two sets have different cardinalities, they cannot be equal and, so, Y is not shattered by $\mathscr{R}_{\mathscr{S}}$ and the VC-dimension of $(X, \mathscr{R}_{\mathscr{S}})$ must be less than or equal to 4.

Let $Z \subseteq X$, such that $|Z| = 4$ and suppose $\mathscr{S} = \mathscr{A}_\phi(X)$ such that $\phi(X)$ is in general position. As $\mathscr{S}$ is maximal, no four points in $\phi(X)$—and, hence, $\phi(Z)$—are coplanar, by definition. Thus, if A is any non-empty proper subset of Z, then $\phi(A)$ cannot separate $\phi(Z \backslash A)$, and there exists a separating hyperplane H such that $H^+ \cap \phi(Z) \big| H^- \cap \phi(Z)$ is an affine split of Z corresponding to $A \big| Z \setminus A$.

As this holds for all subsets of Z, it is shattered by $\mathscr{R}_{\mathscr{S}}$ and, thus, the VC-dimension of $(X, \mathscr{R}_{\mathscr{S}})$ is 4.

□

We are now at a point to establish an important connection between maximal collections of affine splits and oriented matroids, due to a fundamental correspondence given by Gärtner and Welzl in their substantial development of the Vapnik–Chervonenkis dimension in the context of oriented matroids (Gärtner and Welzl, 1994). In this way, every maximal collection of splits $\mathscr{S}$ corresponds to an oriented matroid $\mathcal{M}_{\mathscr{S}}$.

Given a collection of splits $\mathscr{S}$, we define

$$\mathcal{T} = \{T : T_a = + \text{ for } a \in A, T_b = - \text{ for } b \in B, \text{ for all } A|B \in \mathscr{S}\} \cup \{T_+, -T_+\},$$

where $T_+ = (+, +, \cdots, +)$ has length $|X|$. Note that, as $A|B = B|A$, we have, for all $T \in \mathcal{T}$, that $-T \in \mathcal{T}$ as well.

Theorem 13. *Let $\mathscr{S} = \mathscr{A}_\phi(X)$ be a maximal collection of affine splits on X and let $\mathcal{T}$ be defined as above. Then, $\mathcal{T}$ corresponds to the set of topes of a unique, acyclic, and simple rank 4 uniform oriented matroid $\mathcal{M}$ with ground set X.*

Proof. By Proposition 5, a maximal collection of affine splits $\mathscr{S}$ on X has cardinality $|\mathscr{S}| = \Phi_3(|X|-1)-1$. So, we have

$$|\mathcal{T}| = 2|\mathscr{S}| + 2 = 2\big(\Phi_3(|X|-1)-1\big) + 2 = 2\big(\Phi_3(|X|-1)\big).$$

We also have that $\mathcal{T} = -\mathcal{T}$, and $\dim_{VC}|\mathcal{T}| = \dim_{VC}|\mathscr{S}| = 4$, by Proposition 12.

Then, by (Gärtner and Welzl, 1994, Theorem 50), $\mathcal{T}$ is naturally isomorphic to the set of topes of a uniform oriented matroid $\mathcal{M}$ of rank 4 with X as ground set.

By definition, $(+,+,\cdots,+) \in \mathcal{T}$ and, so, $\mathcal{M}$ is acyclic. Therefore, by Lemma 11, $\mathcal{M}$ is simple.

Finally, suppose there exists another acyclic and loop-free rank 4 oriented matroid $\mathcal{M}'$ on X, such that the topes $\mathcal{T}'$ of $\mathcal{M}'$ satisfy $\mathcal{T} \subseteq \mathcal{T}'$. Then, we have $|\mathcal{T}| \leq |\mathcal{T}'|$. In analogy to Proposition 5, however, the number of topes is bounded by $2\Phi_3(|X|-1)$ and, so, $|\mathcal{T}| = |\mathcal{T}'|$.

Hence, as $\mathcal{T} \subseteq \mathcal{T}'$, we have $\mathcal{T} = \mathcal{T}'$ (up to reorientation), and it follows that $\mathcal{M}' = \mathcal{M}$. So, we may unambiguously refer to $\mathcal{M}$ as **the** uniform oriented matroid $\mathcal{M}_{\mathscr{S}}$ with ground set X corresponding to $\mathscr{S}$.

□

3.4 The Big and Affine Face Lattices

For each partially ordered set (or poset) P with a unique global minimum $\underline{0}$ (i.e., there exists no $C \in P$ such that $C < \underline{0}$), we define the *poset rank* of an element $C \in P$ by the length of the interval $[\underline{0}, C] = \{X \in P : \underline{0} \leq X \leq C\}$. The topes of an oriented matroid $\mathcal{M}$ are the maximal rank elements of a lattice associated to $\mathcal{M}$, which we will discuss in this section.

This lattice emerges from adjoining the set of covectors $\mathcal{L}$ with a global maximum $\hat{1}$ and endowing the resultant set with the (co-ordinate wise) partial order $\leq$ defined by $0 < +$, $0 < -$, and where $+$ and $-$ are incomparable; it is called the *big face lattice* $\mathcal{F}_{big}(\mathcal{L})$.

The *join* of two covectors $C, D \in \mathcal{L}$ in $\mathcal{F}_{big}(\mathcal{L})$ is defined by

$$C \vee D = \begin{cases} C \circ D = D \circ C & \text{if } s(C,D) = \emptyset, \\ \hat{1} & \text{otherwise.} \end{cases}$$

For any tope $T \in \mathcal{T}$, no covector $C \in \mathcal{L}$ satisfies $T \leq C \leq \hat{1}$ (in which case, we say that $\hat{1}$ *covers* T), and so the topes are, as mentioned above, the maximal rank elements, or *coatoms*, of $\mathcal{F}_{big}(\mathcal{L})$. Furthermore, the *atoms* of $\mathcal{F}_{big}(\mathcal{L})$ are the elements which cover $\underline{0}$; these are precisely the cocircuits (i.e., the minimal rank elements) in $\mathcal{L}$.

Example 9. Consider the following set of covectors on a two-point set (it is not difficult

to verify that it satisfies the covector axioms):

$$\mathcal{L} = \{(+,+),(+,-),(-,+),(-,-),(+,0),(0,+),(-,0),(0,-),(0,0)\}.$$

Then, we have $\underline{0} < (0,+) < (+,+)$ and, so, the poset rank of $(+,+)$ is 1. However, $(+,+)$ and $(-,0)$ are incomparable. The join of $(+,0)$ and $(0,+)$ is $(+,+)$ and $(+,0) \vee (0,-) = (+,-)$, but $(+,0) \vee (-,0) = \hat{1}$.

Furthermore, $(+,+)$ is a coatom (and, hence, a tope) and $(+,0)$ is an atom (and, hence, a cocircuit).

If we have an oriented matroid $(X, \mathcal{L})$ and $y \in X$ is an element which is not a loop, we call the triple $(X, \mathcal{L}, y)$ an *affine oriented matroid*. For such an affine oriented matroid $(X, \mathcal{L}, y)$, we define:

$$\mathcal{L}_y^+ = \{C \in \mathcal{L} : y \in C^+\}.$$

By adjoining $\mathcal{L}_y^+$ with $\hat{1}$ and the induced partial order from $\mathcal{F}_{big}(\mathcal{L})$, we obtain the *affine face lattice* $\hat{\mathcal{L}}_y^+$.

In the context of polytopal and spherical splits, consider the affine oriented matroid $(X, \mathcal{L}, p)$, where $p \in X$ and the split $p|X \setminus \{p\}$ corresponds to the positive tope $(+,+,\cdots,+)$ (which we may assume without loss of generality, by reorientation). Then, $\hat{\mathcal{L}}_p^+$ is equivalent to the lattice structure of $\mathscr{S}$ (in that each split $S \in \mathscr{S}$ is equivalent to a coatom of $\hat{\mathcal{L}}_p^+$), as the splits $S \in \mathscr{S}$, by choice of orientation, satisfy $p \in H^+$ for any $H \in \mathcal{H}_S$.

Note that, as all points of $\phi(X)$ are extremal, any $p \in X$ will give the same result with an appropriate reorientation of the planes in $\mathcal{H}_S$ and, so, without loss of generality, we may simply denote the affine face lattice corresponding to $\mathscr{S}$ as $\mathcal{F}_{\mathscr{S}} = \hat{\mathcal{L}}^+$ (i.e., $\hat{\mathcal{L}}^+$ is equivalent—up to reorientation—to $\hat{\mathcal{L}}_p^+$, for any $p \in X$).

Given a partially ordered set $(P, \leq)$, we define the *order complex* $\Delta_{ord}(P)$ as the simplicial complex with the elements of P as vertices and the (finite) *chains* $x_1 < x_2 < \cdots < x_i$, $x_i \in P$, as the simplices (that is, the collection of simplices is generated by the totally ordered chains in P). Note that this situation is a bit more abstract than the simplicial complex $\mathscr{K}_{bipy}$ presented in Example 6, as each element of P is a vertex and any totally ordered subset of P is a simplex in the order complex. For a good collection of lecture notes on poset topology—the first of which introduces the order complex—the reader is referred to Wachs (2006).

Example 10. $\mathscr{K}_{bipy}$ is the order complex of the poset shown in Figure 3.1, with $x > y$ if x is above y:

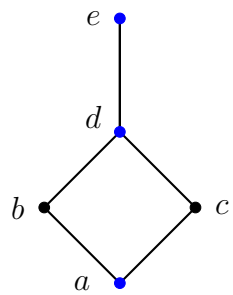


Figure 3.1: The poset P such that $\Delta_{ord}(P) = \mathscr{K}_{bipy}$.

Note that there are two maximal chains ($a < b < d < e$ and $a < c < d < e$) in the poset shown in Figure 3.1, each with four totally ordered subsets of length 3, six of length 2, and four of length 1. In addition, the two maximal simplices intersect in one chain of length three (i.e., $a < d < e$; the blue vertices in the figure), which corresponds to the face shown in blue in Figure 2.6. Furthermore, note that reversing the order will not change the order complex of a poset: If $x < y < z$ is a chain in a poset, then $z < y < x$ in the poset given by reversing the order (which we call the *order dual*).

We call a simplicial d-complex $\mathscr{K}$ *shellable* if it admits an ordering $s_1, s_2, \cdots, s_n$ (which we call a *shelling*) of the collection of its maximal simplices s, such that

$$s_k \cap \left(\bigcup_{i=1}^{k-1} s_i \right)$$

is a non-empty pure simplicial (d–1)-complex $\mathcal{K}$ for all $2 \leq k \leq n$; that is, considering the addition of s_k to the subcomplex $\mathscr{K}_i$ given by the (union of the) collection s_i, $i < k$, s_k is joined to $\mathscr{K}_i$ along a pure $(d-1)$-subcomplex of its boundary. This definition of a shelling is particular for simplicial complexes (Ziegler, 1995); for more general (e.g., non-pure) complexes, see the definition in, for example, Ziegler (1998).

Example 11. The simplicial complex $\mathscr{K}_{bipy}$ of Example 6 given by the triangular bipyramid clearly has a shelling: T_1 (a three-dimensional simplex) is glued to T_2 at a face (a 2-dimensional simplex), and so the ordering (T_1, T_2) is a shelling of $\mathscr{K}_{bipy}$.

Finally, Björner *et al.*, 1999, Theorem 4.5.7(i) gives us that, as $\mathcal{M} = (X, \mathcal{L})$ is a rank 4 uniform oriented matroid defined by a maximal collection of polytopal/spherical splits, the order complex of the associated affine face lattice $\Delta_{ord}(\hat{\mathcal{L}}^+)$ is a shellable 3-ball;

that is, the union of all simplices $s \in \Delta_{ord}(\hat{\mathcal{L}}^+)$ is homeomorphic to the unit ball in $\mathbb{R}^3$.

Example 12. To demonstrate the level of complexity of such a shelling, consider the order complex associated to the affine lattice $\hat{\mathcal{L}}^+_{\text{Tet}}$ given by the topes corresponding to a maximal collection of polytopal splits on the tetrahedron. There are eight maximal covectors in $\hat{\mathcal{L}}^+_{\text{Tet}}$, each of which corresponds to a simplicial region in space, and so covers at most four covectors (i.e., facets of the simplices) which, in turn, cover three covectors (i.e., the edges bounding the facets) which, finally, cover two cocircuits (i.e., the corresponding vertices) each. Thus, we have an upper bound,

$$8 \times 4 \times 3 \times 2 = 192,$$

on the number of simplices in $\Delta_{ord}(\hat{\mathcal{L}}^+_{\text{Tet}})$. Furthermore, the union of all such simplices is homeomorphic to the 3-ball, by the above result.

For fixed $p \in X$, $\mathcal{F}_{\mathscr{S}}$ is isomorphic to $\hat{\mathcal{L}}^+_p$ through the mapping $h_p : \mathcal{F}_{\mathscr{S}} \to \hat{\mathcal{L}}^+$ defined by

$$h_p(S) = T_S,$$

where T_S is the tope of $\mathcal{M}_S$ corresponding to S such that $T_p = +$, and h_p is extended to the non-maximal elements of the lattices by using the natural meet operations on $\mathscr{S}$ and $\mathcal{T}$, such that

$$h_p(S_1 \wedge S_2) = h_p(S_1) \wedge h_p(S_2),$$

for all $S_1, S_2 \in \mathscr{S}$. In brief, the essence here is that splits correspond to dual regions, which we know are in one-to-one correspondence with the topes. The geometric situation of the regions corresponding to splits induces an adjacency property, which we use to determine the lattice $\mathcal{F}_{\mathscr{S}}$. In the same way, topes have a natural adjacency property through the separation, and these properties coincide: S_1 is adjacent to S_2 in $\mathscr{S}$ if and only if they differ by a single (polar) point; T_1 is adjacent to T_2 in $\mathcal{T}$ if and only if their separation is a single element. In this way, the same lattices are induced by $\mathscr{S}$ and $\mathcal{T}$, and the splits correspond, in a one-to-one and onto fashion, to the affine subset of the topes with a single fixed element p. Thus, the induced (geometric and covector) lattices are equivalent. The mapping above indicates this (however imprecisely).

This gives another characterisation of $\mathscr{S}$, as the set of coatoms of a lattice combinatorially equivalent to a shelling of the 3-ball. Whether **every** such shelling of the 3-ball gives us a maximal collection of polytopal splits is a harder question.

Generalising the upper bound in Example 12, we have the following result:

Proposition 14. *If $\mathscr{S}$ is a maximal collection of polytopal splits, the number of simplices in $\Delta_{ord}(\mathcal{F}_{\mathscr{S}})$ is bounded above by $24\Phi_3(|X|-1)$.*

Proof. Using the same reasoning as in Example 12, as each region cut by an arrangement corresponding to $\mathscr{S}$ must be simplicial, we have a maximum of 4 (faces) $\times$ 3 (edges) $\times$ 2 (vertices) $= 24$ chains descending from each maximal element in $\mathcal{F}_{\mathscr{S}}$, of which there are $\Phi_3(|X|-1)$.

□

3.5 Matroid Polytopes

Let $\mathcal{M}$ be an oriented matroid with ground set X and $A \subseteq X$, and define the set $\mathscr{C}|_A = \{C|_A : z(C) \subseteq A\}$; that is, $\mathscr{C}|_A$ is the set of all covectors $C \in \mathcal{L}$ such that the zero set of C is contained in A. We call $\mathcal{M}[A] := (A, \mathscr{C}|_A)$ the *restriction* of $\mathcal{M}$ to A. Furthermore, we define the *contraction* $\mathcal{M}/A$ of $\mathcal{M}$ by A as

$$\mathcal{M}/A = \{C|_{X\setminus A} : C \in \mathcal{L} \text{ and } A \subseteq z(C)\}.$$

A set $F \subseteq X$ is a *face* of $\mathcal{M}$ if there exists a covector $C \in \mathcal{L}$ such that $z(C) = F$ and $C^+ = X \setminus F$. A face is an *extreme point* if the rank of $\mathcal{M}[F]$ is 1.

If $\{x\}$ is an extreme point for all $x \in X$, then $\mathcal{M}$ is called a *matroid polytope.*

Proposition 15. *The uniform oriented matroid $\mathcal{M}$ associated with a maximal collection of polytopal splits is a matroid polytope.*

Proof. Let $x \in X$. As each split $S_x = x\big|X \setminus \{x\}$ is a polytopal split, there exists a tope $T_x = \{-, -, \cdots, +, \cdots, -\} \in \mathcal{T}$ such that $T_x^+ = x$. As $-T_x \in \mathcal{T}$ also, we have $-T_x^+ = X \setminus \{x\}$.

As both $-T_x$ and T_+ are in $\mathcal{T}$, and as $\mathcal{M}$ is acyclic, by covector axiom C4 there exists a covector $C_x \in \mathcal{M}$ such that $C_x^0 = x$, and so $\{x\}$ is a face of $\mathcal{M}$. Now, to see that $\mathcal{M}[\{x\}]$ has rank 1, observe that $\mathscr{C}|_x := \{C|_x : z(C) \subseteq x\}$, and so there must only be one element, C_x, in $\mathscr{C}|_x$. Thus, $\mathcal{M}[\{x\}]$ must have rank 1.

As x was chosen arbitrarily, $\mathcal{M}[\{x\}]$ has rank 1 for all $x \in X$, and so $\mathcal{M}$ is a matroid polytope.

Furthermore, as $\mathcal{M}$ is uniform, it is a uniform matroid polytope.

$\square$

The following result (Björner *et al.*, 1999, Proposition 9.1.2(b)) gives us a concrete means to calculate the face lattice of a given matroid polytope, which will prove to be useful later.

Proposition 16. *Let $\mathcal{M}$ be a uniform matroid polytope on ground set X. A subset $F \subsetneq X$ is a face of $\mathcal{M}$ if and only if $\mathcal{M}/F$ is acyclic.*

As a matroid polytope of rank 4, the set of faces of a matroid polytope $\mathcal{M}$, which we will call the *polytope face lattice* $\hat{\mathcal{L}}_P$, is the face lattice of a linear-piecewise cell decomposition of S^2; furthermore, as $\mathcal{M}$ is uniform, the polytope face lattice $\hat{\mathcal{L}}_P$ is a simplicial (or triangulated) sphere (Björner *et al.*, 1999, Section 9.1).

We define an *m-weak configuration of points and pseudocircles* (or *m-weak PPC configuration*) on a finite point set X to be the pair $\mathscr{P} = (P, L)$, where P is a point set $(p_x)_{x \in X} \in S^2$ and L is a set of simple closed curves in S^2, satisfying the following conditions:

PPC1 For each $l \in L$, there exist at least three points in P lying on l;

PPC2 For any three points $p_{x_1}, p_{x_2}, p_{x_3}$ in P, there exists a unique curve in l_{x_1,x_2,x_3} that contains all of them; and

PPC3 Each pair of distinct curves in L that shares at least m points in P intersects (transversally) at most twice.

The main theorem in Miyata (2018, Theorem 4.1), loosely implies that, for every rank 4 uniform oriented matroid $\mathcal{M}$, there exists a 2-weak PPC configuration realizing $\mathcal{M}$, in which every curve passes through exactly three points (in which case, we say the PPC is in *general position*). The corollary below follows, as a result:

Corollary 17. *The uniform matroid polytope $\mathcal{M}$ associated with a maximal collection of polytopal splits is representable as a 2-weak PPC configuration.*

Example 13. An example of a 2-weak PPC on four points in the plane is shown in Figure 3.2. Note that each pair of curves intersects at two points (and, thus, intersects at most twice transversally).

Finally, although we have not delved very deeply, there is a well-deserved concluding

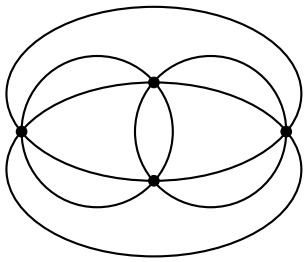


Figure 3.2: A 2-weak configuration of points and pseudocircles (PPC) on four points.

remark (Richter-Gebert and Ziegler, 1997): Matroid polytopes provide an invaluable lens through which we may investigate the theory of convex polytopes. In particular, they provide precise combinatorial representations of convex polytopes, which the PL-sphere model lacks. Although there are some shortcomings in the dual theory of matroid polytope face lattices (e.g., the order dual of the face lattice of a matroid polytope is not, in general, the face lattice of a matroid polytope), applying the dual theory of oriented matroids to matroid polytopes has allowed for the consideration of non-polytopal spheres in the investigation of the realisability properties of polytopes, providing what Richter-Gebert and Ziegler called "perhaps the most powerful single tool ever developed for polytope theory."

3.6 Realisability

Although the realisability of an oriented matroid $\mathcal{M}$ can be considered in many different ways, a good definition (Richter-Gebert and Ziegler, 1997) is that an oriented matroid $\mathcal{M}$ of rank $d+1$ is *realisable* if there exists a vector configuration X such that $\mathcal{M} = (X, \mathcal{L}_X)$, where $\mathcal{L}_X$ is the collection $\{C_X(y) : y \in \mathbb{R}^d\}$ generated by the function

$$C_X(y) = (\text{sign}(y^T x_1), \cdots, \text{sign}(y^T x_n)).$$

A realisable rank d oriented matroid $\mathcal{M}$ can be identified by a *chirotope* $\chi : X^d \to \{-, 0, +\}$ (0 is not in the image of χ if $\mathcal{M}$ is uniform), such that the realisation $\phi : X \to \mathbb{R}^{d-1}$ satisfies

$$\chi(x_1, x_2, x_3, \cdots, x_d) = \text{signdet}\big(\overline{\psi}(x_1), \overline{\psi}(x_2), \overline{\psi}(x_3), \cdots, \overline{\psi}(x_n)\big),$$

where $\overline{\psi}(x) = \binom{\psi(x)}{1}$; that is, the vector obtained by appending a 1 to $\psi(x)$ (Björner *et al.*, 1999).

Furthermore, even if we do not fully know the structure of χ, with partial information of χ—the orientations of the simplices, which may be generated from a simplicial complex—we may obtain the $\psi(x_i)$ using non-linear optimisation methods (see Firsching, 2015). With the additional constraint that each $\psi(x_i)$ is a unit vector, we will obtain an inscribed set of points $\psi(X) \subset S^{d-2}$. This provides us with a potential method for developing point–split collections starting from more abstract data.

In this thesis, given a collection of affine splits $\mathscr{S}$, we obtained an oriented matroid $\mathcal{M}_{\mathscr{S}}$ through an *a priori* existing geometric structure: An arrangement of real hyperplanes. Thus, we know that $\mathcal{M}$ is realisable when it is obtained from a collection of affine splits:

Proposition 18. *The oriented matroid $\mathcal{M}_{\mathscr{S}}$ induced by a maximal affine collection of splits $\mathscr{S}$ is realisable.*

Proof. We know that $\mathscr{S}$ is in bijective correspondence with a representative collection of $\Phi_3(|X|-1)$ hyperplanes $\mathcal{H}_{\mathscr{S}}$, and so may be represented as a hyperplane arrangement in $\mathbb{R}^3$.

As every hyperplane arrangement $\mathcal{H}$ gives rise to an oriented matroid (for example, using the construction in Section 6.1.3 of Richter-Gebert and Ziegler, 1997, where, using the fact that each hyperplane $H \in \mathcal{H}$ naturally divides $\mathbb{R}^3$ into positive and negative halfspaces, the intersection of $\mathcal{H}$ and S^2 induces a cell decomposition on S^2, in which each cell corresponds to a sign vector in $\{0, +, -\}^{|\mathcal{H}|}$, where the ith entry indicates the position of the cell with respect to the circle cut by the ith hyperplane H_i; the collection of all such sign vectors forms the set of covectors $\mathcal{L}$ of the associated oriented matroid), it follows that each arrangement of hyperplanes is in bijective correspondence with a realisable oriented matroid (up to reorientation).

Thus, $\mathcal{M}_{\mathscr{S}}$ is realisable.

□

Summarising the results of this chapter, the following theorem follows from and strengthens Theorem 13 and Proposition 18, along with the discussions of Gärtner and Welzl (1994) and Björner *et al.* (1999):

Theorem 19. *Let X be a set and $\phi : X \hookrightarrow \mathbb{R}^3$ be an embedding. Then, $\mathscr{A}_\phi(X)$ is a maximal collection of affine splits if and only if $\mathcal{M}_{\mathscr{S}}$ (as defined above) is a unique,*

acyclic, realisable, rank 4 *uniform oriented matroid with ground set* X.

Realisability of an oriented matroid is an inherently difficult property to test for—determining the realisability of an oriented matroid has been determined to be polynomially equivalent to the existential theory of the reals (see Fukuda, Miyata, and Moriyama, 2012) and, thus, NP-hard—and our result relies upon the knowledge that realisable oriented matroids are in correspondence with real hyperplane arrangements. This is by no means trivial, and this topic could be easily expanded on in order to acquire further technical results.

Chapter 4

Graphs Induced by Collections of Splits

As we have found a natural relationship between spherical collections of splits and polytopes, a combinatorial aspect worth investigation is gained through Steinitz's seminal correspondence between polytopes and planar graphs (see Theorem 21, below). To this end, we define a graph structure related to polytopal split collections and explore the related properties. We assume some, but not much, prior knowledge of graph theory; for a thorough introductory text, the reader is referred to Bollobás (2002).

4.1 Graphs and Split Systems

Let $\mathscr{S}$ be a collection of splits on X. We define the graph $G_{\mathscr{S}}$ associated with $\mathscr{S}$ to be the graph with vertex set X and edge set $E_{\mathscr{S}} = \{\{a,b\} : \{a,b\}|X \setminus \{a,b\} \in \mathscr{S}\}$; that is, $G_{\mathscr{S}} = (X, E_{\mathscr{S}})$.

Lemma 20. *Let $\mathscr{S}$ be a maximal collection of polytopal splits on X with respect to an embedding ϕ. Then, $\{a,b\}\big|X \setminus \{a,b\} \in \mathscr{S}$ if and only if conv$\big(\{\phi(a), \phi(b)\}\big)$ is an edge of conv$\big(\phi(X)\big)$.*

Proof. Suppose that $S = \{a,b\}\big|X \setminus \{a,b\} \in \mathscr{S}$. Note that $\phi(X)$ is a set of extremal points (as the $\phi(X)$ is in general and convex position) and, so, $\text{conv}\big(\phi(X)\big)$ is a polytope with set of vertices being all points of $\phi(X)$. Furthermore, as $\{a,b\}$ and $X \setminus \{a,b\}$ are separated by a hyperplane in the embedding, $\text{conv}\big(\phi(\{a,b\})\big) \bigcap \text{conv}\big(\phi(X \setminus \{a,b\})\big) = \emptyset$.

Let ab denote $\operatorname{conv}(\{\phi(a), \phi(b)\})$, and ℓ be the line through $\phi(a)$ and $\phi(b)$. First, we ensure that ℓ does not intersect $\operatorname{conv}\big(\phi(X \setminus \{a,b\})\big)$. Suppose there exists $c \in \ell \cap \operatorname{conv}\big(\phi(X \setminus \{a,b\})\big)$. As $c \in \operatorname{conv}\big(\phi(X \setminus \{a,b\})\big)$, we have that either $\phi(a) \in \operatorname{conv}\big(\phi(X \setminus \{b\})\big)$ or $\phi(b) \in \operatorname{conv}\big(\phi(X \setminus \{a\})\big)$, which contradicts that each $x \in \phi(X)$ is extremal. Thus, no such c exists, and so ℓ and $\operatorname{conv}\big(\phi(X \setminus \{a,b\})\big)$ are disjoint.

Now, define P_ℓ to be a plane with normal ℓ, and let π be the projection onto P_ℓ. Then, as both $\phi(a)$ and $\phi(b)$ lie on ℓ, $\pi(\phi(a)) = \pi(\phi(b))$ is a point $p \in P_\ell$. Furthermore, $\pi\big(\operatorname{conv}(\phi(X \setminus \{a,b\}))\big)$ is a convex polygon $C \subset P_\ell$.

As $\pi^{-1}(p) = \ell$, and as $\ell \cap \operatorname{conv}\big(\phi(X \setminus \{a,b\}\big) = \emptyset$, we conclude that p and C are also disjoint. Thus, by the hyperplane separation theorem (see, e.g., Rockafellar, 1970, Theorem 11.1), there exists a line $L \subset P_\ell$ such that C is properly contained in an open half-plane cut by L, and (without loss of generality) which passes through p, from which it follows that L does not intersect C. Affinely extending L by ℓ yields a hyperplane H_{ab} which contains $\phi(a)$, $\phi(b)$, and ab, but does not intersect any other point of $\operatorname{conv}\big(\phi(X)\big)$.

As $\operatorname{conv}\big(\phi(X)\big)$ is supported by H_{ab} at $\phi(\{a,b\})$, ab is, by definition, an edge of $\operatorname{conv}\big(\phi(X)\big)$.

For the converse, suppose that ab is an edge of $\operatorname{conv}\big(\phi(X)\big)$. Then, there exists a hyperplane passing through $\phi(a)$ and $\phi(b)$ which does not intersect the interior of $\operatorname{conv}\big(\phi(X)\big)$. Hence, $\phi(a)$ and $\phi(b)$ can be separated from $\phi\big(X \setminus \{a,b\}\big)$ and so $\{a,b\} \big| X \setminus \{a,b\} \in \mathscr{A}_\phi(X)$.

□

We call a graph G *planar* if it can be drawn in the plane such that no two edges of G intersect. We can characterise planarity of a graph in terms of the forbidden minors K_5 (the complete graph on five vertices) and $K_{3,3}$ (the "utility graph") shown in Figure 4.1; that is, a planar graph contains no subgraph which is a subdivision of either minor.

The seminal results that a planar graph has forbidden subdivisions and minors were, respectively, established by Kuratowski (1930) and Wagner (1937).

A planar graph G is a *maximal* planar graph if no edge can be added to G without violating the planarity of G; equivalently, each face of G is bounded by three edges (and, thus, is triangular). As such, we call a maximal planar graph *triangulated*.

(a) (b)

Figure 4.1: The forbidden minors for planarity (a) K_5 and (b) $K_{3,3}$.

A graph $G = (V, E)$ is *connected* if there exists a path between every pair of vertices $u, v \in V$; that is, if there is a sequence of edges $u, a_1, a_1a_2, \cdots a_{n-1}a_n, a_n, v$ in $E \times E$. Furthermore, we call the graph *k-vertex-connected* (or, simply, *k-connected*) if it has more than k vertices and remains connected when any set of less than k vertices are removed.

Example 14. Both K_5 and $K_{3,3}$ are connected, where K_5 is 4-connected and $K_{3,3}$ is 3-connected. The graph given by the edges and vertices (which we call the 1-*skeleton*) of the tetrahedron, as shown in Figure 4.2, is also 3-connected (the tetrahedron was portrayed as a polytope in Figure 2.2a).

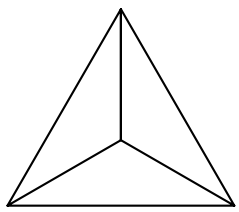


Figure 4.2: The 1-skeleton of the tetrahedron.

Steinitz's characterisation of the boundary complexes (i.e., 1-skeletons) of 3-polytopes as the 3-connected planar graphs (Steinitz, 1922), in slightly more modern terms, is as follows:

Theorem 21 (Steinitz's Theorem)**.** *A graph $\mathscr{G}$ is realisable as a 3-polytope if and only if $\mathscr{G}$ is planar and 3-connected.*

The result was published, in full, in conjunction with Rademascher (Steinitz and Rademascher, 1934) in 1934, after which the result seemingly fell into obscurity until 1962, when the result was used by Grünbaum and Motzkin (1962) to demonstrate that there

exist polyhedral graphs in which there does not exist a simple path through all the vertices, in response to a problem posed by Balinski the prior year (Balinski, 1961). This is hardly exceptional in mathematics, but, for what Klee had dubbed the "second landmark" (after Euler's theorem) in the theory of convex polytopes (Klee, 1966), it is not unremarkable that such a seminal result should lie fallow for so long. For more detail on the connection between planar graphs and 3-polytopes, see Grünbaum, 2003, Chapter 13.

Proposition 22. *If $\mathscr{S} = \mathscr{A}_\phi(X)$ is a maximal polytopal collection of splits, then $G_\mathscr{S}$ is maximal planar and 3-connected.*

Proof. If $\mathscr{S}$ is a collection of polytopal splits, this implies that $\phi(X)$ is the set of vertices of $\mathcal{P} = \operatorname{conv}\big(\phi(X)\big)$, and, as each $S \in \mathscr{S}_2$ is an edge of $\mathcal{P}$ by Lemma 20, the graph $G_\mathscr{S}$ must be the 1-skeleton of $\mathcal{P}$.

By Steinitz's Theorem, as $G_\mathscr{S}$ is the 1-skeleton of a polytope in $\mathbb{R}^3$, it corresponds to a unique (up to combinatorial equivalence) 3-connected planar graph.

Now, suppose that there exists a face F of $\mathcal{P}$ such that F is bounded by more than three edges. However, this implies that four points lie on a plane—namely, the hyperplane supporting $\mathcal{P}$ at F—and, thus, $\phi(X)$ is not in general position. This contradicts the maximality of $\mathscr{S}$, and F must be triangular.

Therefore, $G_\mathscr{S}$ is triangulated and, hence, is maximal planar.

□

The following result follows immediately from Proposition 22, as subgraphs of planar graphs are, again, planar (considering forbidden minors):

Corollary 23. *If $\mathscr{S} \subsetneq \mathscr{A}_\phi(X)$ with respect to an embedding ϕ such that $\phi(X)$ is in general and convex position, then $G_\mathscr{S}$ is planar.*

Note that, depending on the structure of $\mathscr{S}_2$, $G_\mathscr{S}$ may not even be connected!

As each maximal planar graph on n vertices has $3n - 6$ edges (see Diestel, 2005, Corollary 4.2.10), we must have that $|\mathscr{S}_2| = 3|X| - 6$ for a maximal collection of polytopal or spherical splits. Accordingly, the numbers of splits $|\mathscr{S}^{max}|$ and 2-splits $|\mathscr{S}_2^{max}|$ of a maximal planar graph are given in Table 4.1.

Table 4.1: Numbers of maximal collections of splits and edges of the corresponding graphs for maximal polytopal split collections on X.

$\lvert X\rvert$	$\lvert\mathscr{S}^{max}\rvert$	$\lvert\mathscr{S}_2^{max}\rvert$
2	1	0
3	3	3
4	7	6
5	14	9
6	25	12
7	41	15
8	63	18
9	92	21
10	129	24
11	177	27
12	231	30
13	298	33
14	377	36
15	469	39
16	575	42
17	696	45
18	833	48
19	987	51
20	1,159	54
⋮	⋮	⋮
40	9,918	114
41	10,699	117
⋮	⋮	⋮
85	98,854	249
86	102,425	252
⋮	⋮	⋮
182	988,441	540
183	1,004,913	543
⋮	⋮	⋮

4.2 Examples of Graphs and Polytopes Induced by Collections of Splits

In this section, we provide some basic examples of maximal collections of splits with their associated graphs and polytopes. For each of $|X| = 4$ and 5, there is only one maximal planar graph, and so the following two examples are comprehensive for those cases.

Example 15 (Four points)**.** Let $X = \{a, b, c, d\}$. Then, we have that the collection of all splits on X is given by

$$\mathscr{S} = \{abc|d, abd|c, acd|b, bcd|a, ab|cd, ac|bd, ad|bc\}$$

and so

$$\mathscr{S}_2 = \{ab|cd, ac|bd, bc|ad\}.$$

Again, note that

$$|\mathscr{S}| = 7 = \Phi_3(|X|-1)-1.$$

We have $V(G_{\mathscr{S}}) = X$ and, as each 2-split gives us two edges, we have

$$E(G_{\mathscr{S}}) = \{ab, ac, bc, cd, bd, ad\}.$$

We can see, from Figure 4.3a, that $G_{\mathscr{S}}$ is maximal planar. When $G_{\mathscr{S}}$ is inscribed in the sphere, the corresponding polytope is the tetrahedron, as shown in Figure 4.3b.

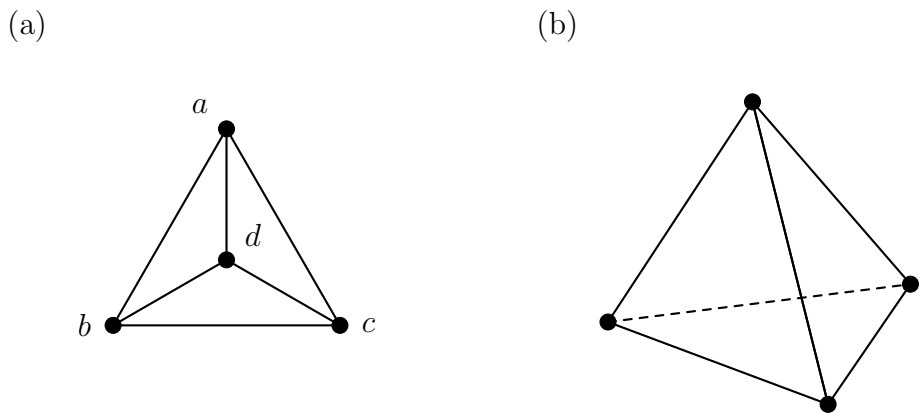


Figure 4.3: (a) The 2-split graph $G_{\mathscr{S}}$ for the maximal collection of splits on four vertices; and (b) the corresponding polytope, the tetrahedron.

Example 16 (Five points)**.** Let $X = \{a, b, c, d, e\}$. The unique (up to combinatorial isomorphism) maximal collection of polytopal splits on X is given by

$$\mathscr{S}_1 = \{a|bcde, b|acde, c|abde, d|abce, e|abcd\}, \text{ and}$$

$$\mathscr{S}_2 = \{ab|cde, ac|bde, ae|bcd, bc|ade, bd|ace, be|acd, cd|abe, ce|abd, de|abc\}.$$

We have that $\mathscr{S}_3 = -\mathscr{S}_2$ and $\mathscr{S}_4 = -\mathscr{S}_1$ and so, by identifying these, we get:

$$|\mathscr{S}| = |\mathscr{S}_1| + |\mathscr{S}_2| = 14 = \Phi_3(|X|-1)-1.$$

Note that $ad|bce$ is not included as a split; otherwise, we would have $|\mathscr{S}| > \Phi_3(|X|-1)-1$ and $|\mathscr{S}|$ would exceed the cardinality of a maximal collection of affine splits.

So, we have $E(G_{\mathscr{S}}) = \{ab, ac, ae, bc, bd, be, cd, ce, de\}$. We can see, from Figure 4.4a, that $G_{\mathscr{S}}$ is maximal planar. With the addition of the edge ad, we would have $G_{\mathscr{S}} = K_5$, the complete graph on five vertices (see Figure 4.5, below), which is non-planar. When $G_{\mathscr{S}}$ is inscribed in the sphere, the corresponding polytope is the triangular bipyramid, as shown in Figure 4.4b.

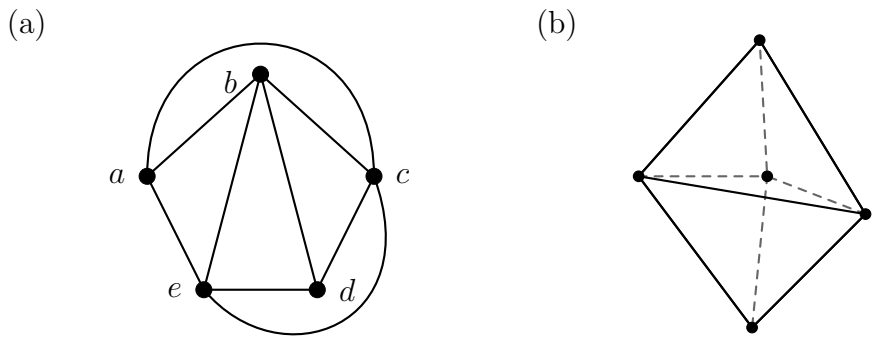


Figure 4.4: (a) The 2-split graph $G_{\mathscr{S}}$ for the maximal collection of splits on five vertices; and (b) the corresponding polytope, the triangular bipyramid.

4.3 Graph Inscribability

A graph is said to be *inscribable* if it is combinatorially equivalent to the 1-skeleton of an inscribable polytope. Combinatoric conditions for inscribability have been a long-standing topic of interest in graph theory and, although no firm necessary and sufficient

condition has been established as yet, many criteria for inscribability have been discerned (for a relatively comprehensive overview, the reader is referred to Dillencourt and Smith, 1996). The vertex-minimal non-inscribable graph is K_5, which is shown, in $\mathbb{R}^3$, in Figure 4.5.

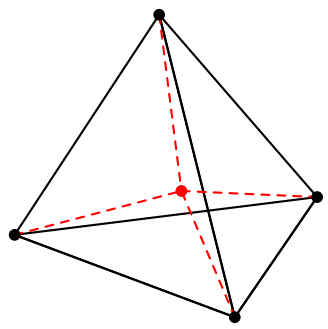


Figure 4.5: The vertex-minimal non-inscribable graph K_5, shown here in $\mathbb{R}^3$.

A forbidden minor for inscribability (Grünbaum, 1963) is shown (in blue) in Figure 4.6 (this was used to discern the face-minimal non-inscribable polytope in Figure 2.7).

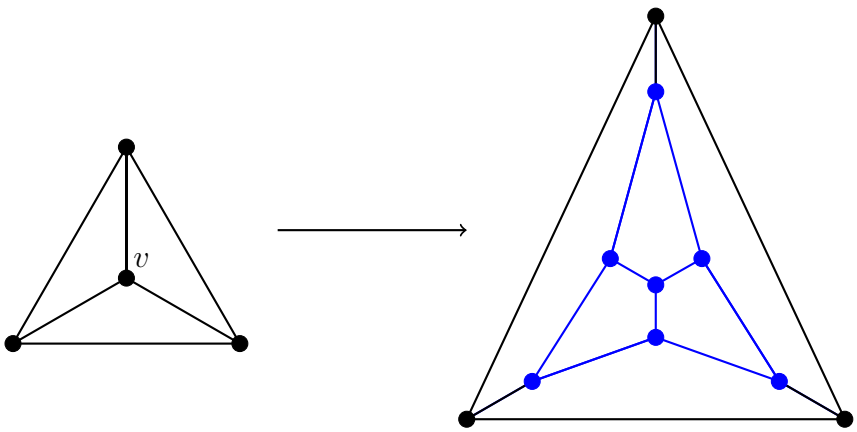


Figure 4.6: Replacing the vertex v in the tetrahedron with a forbidden subgraph to form the 1-skeleton of the face-minimal uninscribable polytope.

Proposition 24. *If $\mathscr{S}$ is a maximal collection of spherical splits, then $G_{\mathscr{S}}$ is inscribable.*

Proof. As $\mathscr{S}$ is maximal, $G_{\mathscr{S}}$ is maximal planar and, hence, a triangulation.

First, if $G_{\mathscr{S}}$ is 4-connected, then it is inscribable by Theorem 3.3 of Dillencourt and Smith (1996).

As $\phi(X)$ is in convex position, the (stereographic) projection of $\phi(X)$ into the plane yields no 4-cocircular set of points (i.e., no four points lie on any circle in the plane). Hence, $G_{\mathscr{S}}$ is *Delaunay realisable*; that is, realisable as a triangulation (or, more generally, a tesselation) where all vertices on the boundary of a common interior face are cocircular, and no points lie on the interior of any such circumcircle. Note that $G_{\mathscr{S}}$ may not be a Delaunay triangulation but, instead, equivalent to one through a sequence of "flips" (see, e.g., Cheung, 2009); hence, it is realisable as a Delaunay triangulation (for an informative overview of the Delaunay triangulation and its dual, the Voronoi diagram, the reader is referred to Aurenhammer, 1991).

By Lemma 2.2 of Dillencourt and Smith (1996), a planar graph G is Delaunay realisable if and only if the graph G' obtained by stellating (that is, adding a vertex and connecting it to all vertices on the boundary of the face) the unbounded face f of G is inscribable. Consider the graph $H_{\mathscr{S}}$ obtained by deleting v: $H_{\mathscr{S}}$ is still triangulated, as deleting a vertex of degree n in a triangulation yields an n-polygonal face, and so deleting v leaves a triangle T. Thus, $H_{\mathscr{S}}$ is also Delaunay realisable. By appropriate (stereographic) rotation, we can consider T to be the unbounded face, and the graph obtained by stellation of T is simply $G_{\mathscr{S}}$. Thus, as $G_{\mathscr{S}}$ was obtained by stellating the unbounded face T of the Delaunay realisable graph $H_{\mathscr{S}}$, $G_{\mathscr{S}}$ is inscribable.

Therefore, as $\mathscr{S}$ is a maximal collection of spherical splits, $G_{\mathscr{S}}$ is an inscribable graph.

□

An immediate result follows for non-maximal collections $\mathscr{S}$:

Corollary 25. *If $\mathscr{S}$ is non-maximal, but $G_{\mathscr{S}}$ is 4-connected, then it is inscribable.*

For example, the split collection on the octahedron is non-maximal (as it has multiple sets of 4-coplanar vertices), but the octahedron is 4-connected and inscribable, as shown in Figure 4.7.

We are faced with an interesting question: With an embedded set of points $\phi(X)$ and consequent collection of splits $\mathscr{S}$, when is it the case that $G_{\mathscr{S}}$ is not inscribable? By considering the range of conditions given by Dillencourt and Smith (1996), we can at least discern that this implies that the graph $G_{\mathscr{S}}$ must be non-Delaunay realisable (along with any graph obtained by deleting a vertex of $G_{\mathscr{S}}$), non-Hamiltonian (as well as any graph obtained by removing any pair of adjacent vertices), and at most 3-connected, among other conditions. These conditions provide us with a loose idea of

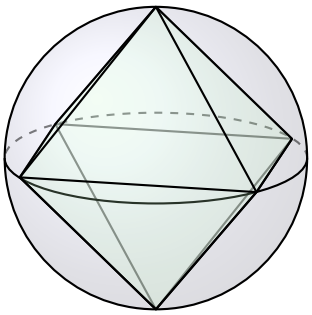


Figure 4.7: The octahedron inscribed in the sphere.

what is happening with the points; for example, the non-Delaunay property suggests that $\phi(X)$ has a high level of coplanarity, as the deletion of any point will still induce a non-Delaunay realisable graph. However, a more in-depth review of both the graph-theoretic and geometric conditions is necessary to have any hope of characterising the exact conditions in which an embedding induces a non-inscribable graph.

4.4 Contractions, Deletions, and Restrictions

The set of all vertices adjacent to a vertex a in a graph is called the *neighbourhood* $N(a)$ of a; that is, the neighbourhood is the set $N(a) = \{v \in V(G) \setminus \{a\} : \text{there exists } e \in E(G) \text{ such that } e = \{v, a\}\}$. Similarly, the neighbourhood of an edge $\{a, b\}$ is defined as the union of the neighbourhoods of the vertices a and b.

For an edge $e = \{a, b\} \in E(G)$, we define the *edge contraction* of G with respect to e to be the graph $G\rfloor_e$ obtained by removing the edge e, identifying the vertices a and b as a single vertex a', and connecting the neighbourhood of $\{a, b\}$ to a'.

Furthermore, for a subgraph $H \subseteq G$, we define the *contraction* of G with respect to H to be the graph $G\rfloor_H$ obtained by contracting all edges in H. It is clear to see that an edge contraction is just a contraction of the subgraph $\{a, b\}$; i.e., $G\rfloor_e = G\rfloor_{\{a,b\}}$. A subgraph contraction is shown in Figure 4.8.

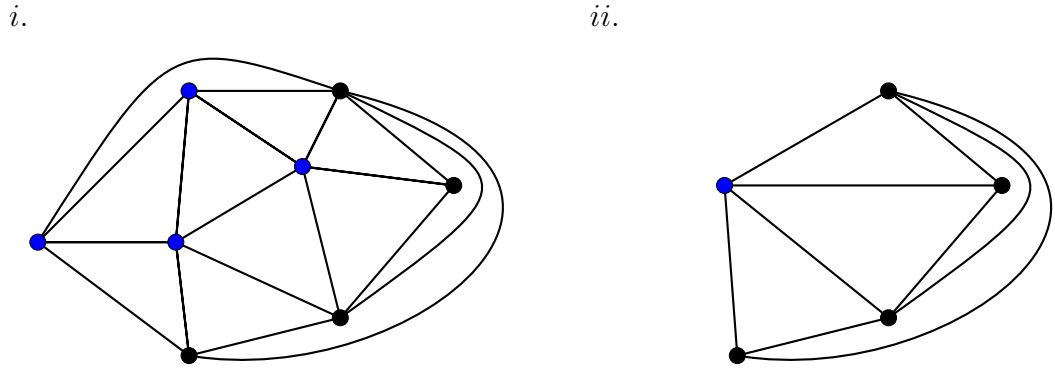


Figure 4.8: (*i.*) A maximal planar graph on eight vertices and (*ii.*) the result of contracting each edge adjacent to two blue vertices.

We call a cycle F in a planar, 3-connected graph G a *face-enclosing cycle* if it forms a face of a polytope with 1-skeleton G. Every face-enclosing cycle, thus, is a simple cycle in G induced on the vertices of F.

Lemma 26. *Let F be a face-enclosing cycle in a planar, 3-connected graph G. Then, the graph $G \setminus F$ obtained by deleting F from G (i.e., the subgraph of G induced on $V(G) \setminus V(F)$) is connected.*

Proof. By a main result of Tutte (1963) (or, in a slightly more modern light, Theorem 1 of Bruhn, 2004), a face-enclosing cycle, as a simple element of the cycle space of the finite 3-connected graph G, is a *peripheral cycle*. However, peripheral cycles are exactly those which are non-separating—that is to say, the subgraph $G \setminus F$ is connected—which is exactly the property we require.

□

Proposition 27. *If $\mathscr{S}$ is a maximal collection of polytopal or spherical splits and $A|B \in \mathscr{S}$, then $G_{\mathscr{S}}[A]$ and $G_{\mathscr{S}}[B]$—that is, the subgraphs induced in $G_{\mathscr{S}}$ by A and B, respectively—are connected.*

Proof. Let H be a plane corresponding to $A|B$, and denote by V_H the vertices given by $H \cap E(G_{\mathscr{S}})$, by E_H the edges given by intersecting H with faces of $G_{\mathscr{S}}$, and let F be the face bounded by E_H (as E_H will be a cycle, given that no face of $G_{\mathscr{S}}$ is interior to $\mathrm{conv}(\phi(X))$). Finally, let G_F be the graph $G_F = (V_H, E_H)$.

Now, consider the graph $G_{A,F}$ given by $G_{\mathscr{S}}[V(A) \cup V_F]$ (i.e., the graph induced on A and the vertices of G_F), in which the subdivided edges link the vertices of A and H. It follows that $G_{A,F}$ is planar and 3-connected and that the deletion of F in $G_{A,F}$ is simply $G_{\mathscr{S}}[A]$.

Then, as F is a face-enclosing cycle of $G_{A,F}$ and as $G_{A,F} \setminus G_F = G_{\mathscr{S}}[A]$, Lemma 26 implies that $G_{\mathscr{S}}[A]$ is connected. We may just as well have chosen B instead of A and, so, the result holds for $G_{\mathscr{S}}[B]$, as well.

Note that this result applies to polytopal and spherical splits alike, as the proof relies primarily only on the convexity of the sets considered. □

We arrive at a relatively nice characterisation of the pairs of subgraphs of $G_{\mathscr{S}}$ which are related to spherical splits:

Proposition 28. *Let $\mathscr{S}$ be a maximal collection of spherical splits with associated graph $G_{\mathscr{S}}$ and $S = A|B \in \mathscr{S}$. Then, the graphs $G_{\mathscr{S}}\rfloor_A$ and $G_{\mathscr{S}}\rfloor_B$ are planar, connected, and inscribable.*

Proof. Both of the graphs $G_{\mathscr{S}}\rfloor_A$ and $G_{\mathscr{S}}\rfloor_B$ are contracted from $G_{\mathscr{S}}$ and, so, must be planar: No new edges are generated and, thus, no new edge crossings can occur. As $G_{\mathscr{S}}$ is connected and the graph $G_{\mathscr{S}}[A]$ is connected (by Proposition 27), $G_{\mathscr{S}}\rfloor_B$ must also be connected, given that there exists at least one edge between A and B (otherwise, $G_{\mathscr{S}}$ would not be connected). The same reasoning follows for $G_{\mathscr{S}}\rfloor_A$.

Finally, let ϕ be an embedding of $V(G_{\mathscr{S}})$ into S^2. Then, $G_{\mathscr{S}}[A]$ is Delaunay realisable as a subset of a Delaunay realisable graph and, so, the graph obtained by stellating the unbounded face is inscribable.

To see that the vertices disconnected from B must form a face, suppose that F is the face formed by deleting B. Then, F is a simple cycle on a collection of vertices $V(F) = \{a_i\} \subseteq A$. Suppose that there exists a vertex a_k in F such that a_k is not adjacent to any vertex of B in G. Without loss of generality, we can suppose that both neighbours a_{k-1} and a_{k+1} of a_k in F are adjacent to the vertices b_1 and b_2, respectively, in B.

Now, if $b_1 = b_2 = b$, then $a_{k-1}a_{k+1}b$ must be a triangle and so a_1a_2 is an edge, contradicting that F is a simple cycle. So, suppose that $b_1 \neq b_2$. Then, as a_k is not adjacent to any vertex in B, neither $a_ka_{k-1}b_1$ or $a_ka_{k+1}b_2$ are triangles, and so there

exists a face in $G_{\mathscr{S}}$ bounded by at least five vertices (see Figure 4.9). As this definitely cannot be the case, it follows that each vertex in F must be adjacent to a vertex of B.

Thus, the graph $G_{\mathscr{S}}\downharpoonleft_B$ is equivalent to the graph given by stellating F in $G_{\mathscr{S}}[A]$, and is hence inscribable. The symmetric argument gives the result for $G_{\mathscr{S}}\downharpoonleft_A$.

□

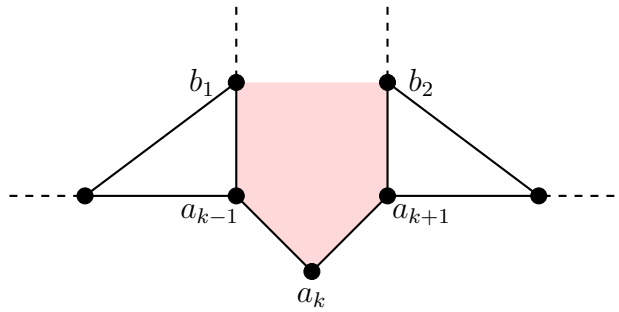


Figure 4.9: Illustration of the argument in the proof of Proposition 28.

Finally, given a graph G, when can we say there must exist a collection of polytopal/spherical splits with corresponding graph G? Considering the following example, we see that the problem lies in the embedding of the graph, not just the structure of the graph.

Example 17. Consider the graph and the two embeddings on the sphere (brought to the front side of the sphere) shown in Figure 4.10: For the same graph (Figure 4.10(a)), we get a different collection of splits, depending on where the middle vertices are moved. In Figures 4.10(b) and (c), the same exemplary hyperplane is shown in red. We see that the induced split by the same hyperplane is different, indicating that the change of embedding has altered the structure of the associated split.

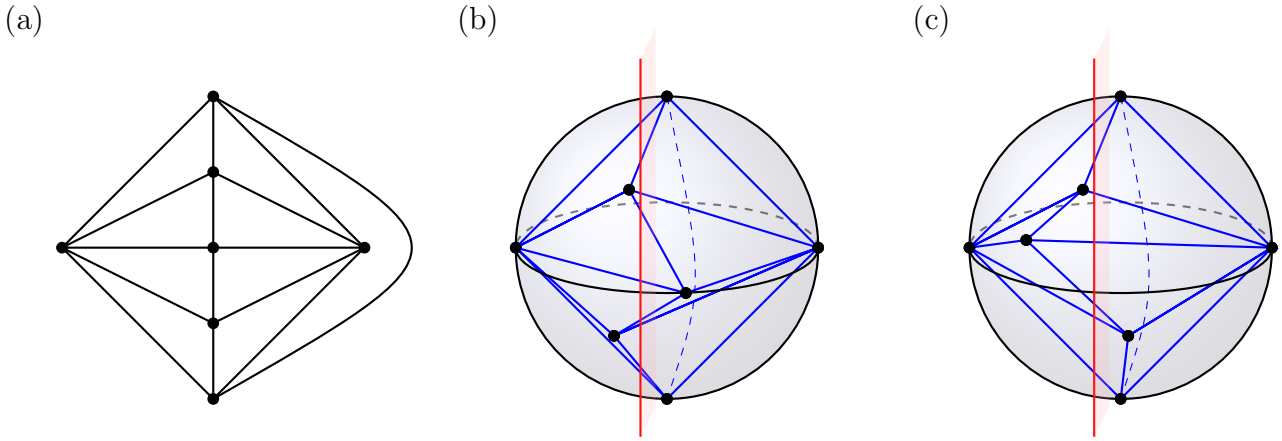


Figure 4.10: A graph and two different embeddings on the sphere showing different induced splits.

We close this chapter with our second open question. We have discovered some necessary conditions for the graph induced by a spherical split system to be inscribable, but are still lacking sufficient conditions.

Question 1. *What are the sufficient conditions for $G_{\mathscr{S}}$ to correspond to a maximal collection of polytopal or spherical splits?*

It seems intuitive that the split structure must be related to an interplay between the cycle space and embedding of a graph but, for graphs on $n > 5$ vertices, this relationship is not trivial to deduce. However, it seems likely that we can generate maximal collections, perhaps by exploiting Delaunay-type properties.

Overall, it remains unknown whether we can fully characterise collections of spherical splits from the structure of a graph alone. Those splits which are not 1-, 2-, or 3-splits are dependent not only on the structure of G, but how we embed it. Hence, there may, unfortunately, be no good graph-theoretic characterisation of collections of spherical splits. However, as we have seen, we can at least draw some conclusions about the relationships between the structures $\mathscr{S}$ and $G_{\mathscr{S}}$.

Chapter 5

Generation Methods

Finally, we combine our findings on affine, polytopal, and spherical splits, in order to provide some methods for the calculation and/or visualisation of such collections, depending on the data given.

5.1 Convex Optimisation

Our first method takes a collection of points in $\mathbb{R}^3$ as input and generates the corresponding collection of separating hyperplanes and topes, using a linear programming approach. This method was primarily used to generate images for illustrative purposes, but provides a potential starting point for the development of more sophisticated split generation software.

5.1.1 Derivation

Assume we have, as input, n points $X = \{x_i\}$ in $\mathbb{R}^3$ (i.e., we have an implicit embedding of the x_i). We detail a linear program to determine whether any bipartition $A|B$ of X is a split of the points.

If A and B are separable, then there exists a hyperplane $H = H(v, k)$ such that

$$\langle v, a\rangle > k \text{ for all } a \in A \text{ and } \langle v, b\rangle < k \text{ for all } b \in B.$$

As the sets are finitely separable, we can reformulate these equations as

$$\langle v, a\rangle \geq k + \epsilon \text{ and } \langle v, b\rangle \leq k - \epsilon,$$

for some sufficiently small $\epsilon \in \mathbb{R}$. Multiplying both sides of each equation by $\frac{1}{\epsilon}$, we obtain

$$\langle \frac{v}{\epsilon}, a \rangle \geq \frac{k}{\epsilon} + 1 \text{ and } \langle \frac{v}{\epsilon}, b \rangle \leq \frac{k}{\epsilon} - 1,$$

where $H(\frac{v}{\epsilon}, \frac{k}{\epsilon})$ is simply another equivalent way of writing H, and so is equivalent to $H(v, k)$. Implementing this equivalence, multiplying the first inequality by -1, and shifting k to the other side, we get

$$-\langle v, a \rangle + k \leq -1 \text{ and } \langle v, b \rangle - k \leq -1,$$

which is then condensed into the inequality of the linear program below.

Note that, as our objective is simply to determine the existence of a plane H, we set the objective function to $\underline{0} \in \mathbb{R}^4$ for the sake of simplicity.

Let $A = \{a_1, \cdots, a_{n_1}\} \subset \mathbb{R}^3$ and $B = \{b_1, \cdots, b_{n_2}\} \subset \mathbb{R}^3$ form a bipartition of X (i.e., $n_1 + n_2 = n$ and $A \cap B = \emptyset$) and let $\tilde{h} = (h_1, h_2, h_3, k)^T \in \mathbb{R}^4$ represent a potential separating hyperplane $H(v, k)$ of A and B. Setting

$$A = \begin{pmatrix} -a_1 & 1 \\ \cdots & \cdots \\ -a_{n_1} & 1 \\ b_1 & -1 \\ \cdots & \cdots \\ b_{n_2} & -1 \end{pmatrix} \in \mathbb{R}^{n \times 4} \text{ and } b = \begin{pmatrix} -1 \\ -1 \\ \cdots \\ -1 \\ \cdots \\ -1 \end{pmatrix} \in \mathbb{R}^n,$$

where the a_i and b_i are row vectors, and maximising the inequality

$$A\tilde{h} \leq b,$$

feasibility of the linear program is equivalent to the separability of A and B.

Thus, a brute-force check (however inefficient) of all possible bipartitions of X will yield all the affinely separable bipartitions—that is, the affine splits—along with the separating hyperplanes.

To implement this, we define the point matrix $P \in \mathbb{R}^{n \times 4}$ by (assuming the x_i are row

vectors):

$$P = \begin{pmatrix} x_1 & -1 \\ x_2 & -1 \\ \vdots & \vdots \\ x_n & -1 \end{pmatrix},$$

and, noting that it is not necessary in A for the a_i to be clustered nor to be above the b_i, we may take a selection of the x_i and multiply their corresponding rows by -1 to create a bipartition of X. To do this, let $T \in \{-1, 1\}^n$ (i.e., T is a vector of length n with entries in $\{-1, 1\}$) and multiply the ith row of P by T_i to obtain P_T. In this way, the linear program becomes testing the feasibility of

$$P_T \tilde{h} \leq b, \text{ for all } T \in \{-1, 1\}^n,$$

with the output being a maximum of $\Phi_3(n-1)$ hyperplanes H and their corresponding $\{-1, 1\}$-vectors T. Note that each T corresponds exactly to a tope! In this way, we can obtain the tope set $\mathcal{T}$, which, in turn, determines the oriented matroid $\mathcal{M}$ corresponding to the collection of splits on X.

5.1.2 Implementation and Results

The linear program was implemented using Python 3.7.2, where the linear program was carried out with the `linprog` function of the scipy.optimize package with objective function $(0, 0, 0, 0)$, bounds $(-\infty, \infty)$, and tolerance 10^{-7}. The full tope collection (i.e., $\{-1, 1\}^N$) was generated using a modified Gray code method (see, e.g., Doran, 2007). All output hyperplanes were normalised (except for those with normal $\underline{0}$) to avoid plotting issues.

In the worst case, we will have to perform 2^n iterations to determine all hyperplanes; however, we do have a stopping criteria—twice the maximal cardinality of $\mathscr{S}$—and, so, we may have better luck than that: Even for $n = 15$, we have that $2\Phi_3(n-1)$ is approximately 3% of 2^n. This suggests that better heuristics for the topes T may dramatically increase the speed of the program. Furthermore, if the topes can be determined *a priori* (perhaps by an appropriate distance-based clustering method), the calculation will be much quicker.

With some preliminary testing, the program has proven to generate all topes consistently: The last element tested was $(-1, -1, \cdots, -1)$, which was always in $\mathcal{T}$, and

$|\mathcal{T}| = 2\phi_3(N-1)$, which means that all potential topes were traversed and only those corresponding to hyperplane separations were chosen. This means that the collection of hyperplanes is incidentally doubled but, using the symmetry of $\mathcal{T}$ (due to the way that the tope collection was generated), in plotting we only require the first half of the collection of hyperplanes. Unfortunately, this also means that all 2^n potential topes must be tested, with subsequent computational costs. If we only need the hyperplanes, we need only test the first half of the tope collection, halving the execution time.

The average time taken for the program to compute $\mathcal{T}$ and the hyperplanes for n points is approximately on the order of 2^{n-7} seconds. Table 5.1 shows the time elapsed for the program to generate all topes and hyperplanes for some (relatively low) values of n. All times are the averages of three replicates. The program was carried out in Windows 10 64-bit on an AMD A9-9425 Dual-Core (3.1 GHz base) processor.

n	4	5	6	7	8	9	10	11	12	13
Time (s)	0.082	0.169	0.367	0.887	1.832	3.735	8.468	17.69	34.988	77.78

Table 5.1: Time taken (in seconds) for the linear program to calculate all hyperplanes and splits on n randomly generated points on S^2.

An arrangement of hyperplanes generated by the linear program for the vertices of the regular tetrahedron are shown in Figure 5.1.

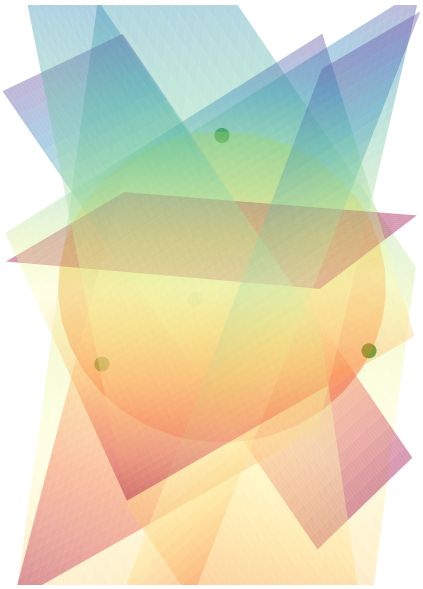


Figure 5.1: Splits on the vertices (green points) of a regular tetrahedron inscribed in the sphere, represented by a generated hyperplane arrangement.

The corresponding topes are $(1,1,1,1)$, $(1,1,1,-1)$, $(1,1,-1,1)$, $(1,1,-1,-1)$, $(1,-1,1,1)$, $(1,-1,1,-1)$, $(1,-1,-1,1)$, $(1,-1,-1,-1)$, $(-1,1,1,1)$, $(-1,1,1,-1)$, $(-1,1,-1,1)$, $(-1,1,-1,-1)$, $(-1,-1,1,1)$, $(-1,-1,1,-1)$, $(-1,-1,-1,1)$, and $(-1,-1,-1,-1)$; the respective corresponding hyperplanes (given in the form $H = (h_1, h_2, h_3, k)$) are $(0,0,0,1)$, $(-0.707,-1.225,-0.5,0.5)$, $(-0.707,1.225,-0.5,0.5)$, $(-1.414,0,-1,0)$, $(1.414,0,-0.5,0.5)$, $(0.707,-1.225,-1,0)$, $(0.707,1.225,-1,0)$, and $(0,0,-1.5,-0.5)$.

Note that the vacuous region—corresponding to the tope $(1,1,1,1)$— was not plotted in Figure 5.1, as the corresponding hyperplane was $(0,0,0,1)$.

A hyperplane arrangement generated for six normally distributed points—thus, corresponding to an affine collection of splits—is shown in Figure 5.2.

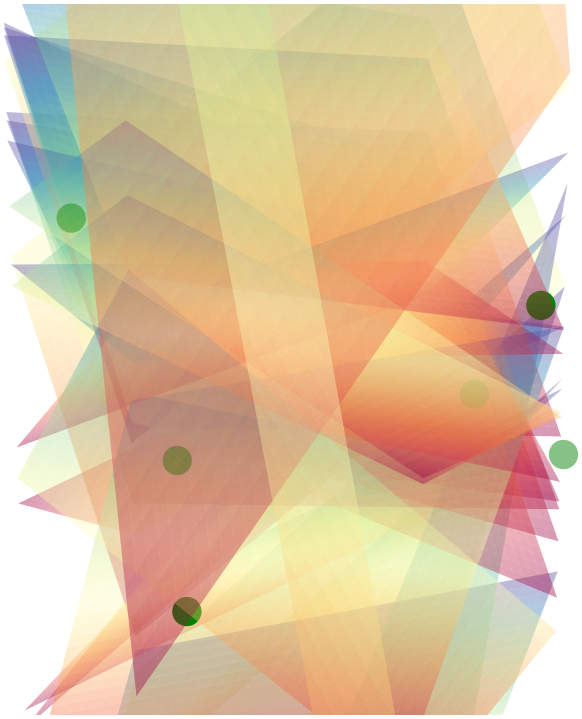
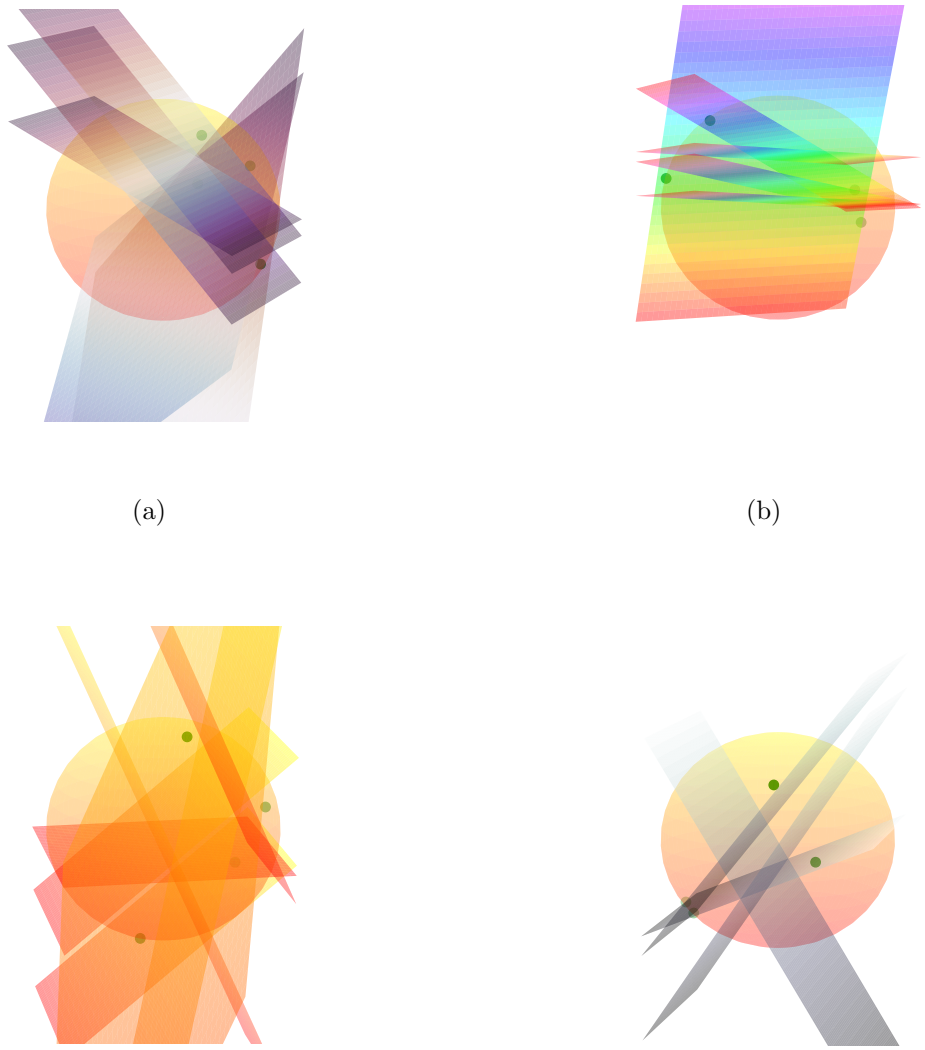


Figure 5.2: An affine split collection represented as a hyperplane arrangement.

Note that, even with only six points, the corresponding system of hyperplanes becomes very cluttered (as there are a total of 26 hyperplanes to plot!). This suggests that an alternate means of visualisation is necessary.

Some hyperplane arrangements corresponding to randomly generated sets of four points on the sphere given by the linear program are shown in Figure 5.3.



(a) (b) (c) (d)

Figure 5.3: Spherical split collections represented as hyperplane arrangements.

5.2 Face Lattice Generation

Our second method generates the polytope face lattice from a collection of topes, using a result for matroid polytopes described in Section 3.5, where we showed that a maximal collection of polytopal splits uniquely determines a rank 4 uniform matroid polytope $\mathcal{M}$, the face lattice of which is equivalent to a simplicial sphere. We also have a concrete means for obtaining the face lattice of $\mathcal{M}$, by means of finding which

subsets $A \subsetneq X$ lead to acyclic contractions.

5.2.1 Derivation

Given a collection of splits $\mathscr{S} = \{A_i|B_i\}$ on a finite set X, we may immediately associate a collection of topes $\mathcal{T} \subset \{+,-,0\}^X$ to $\mathscr{S}$ (with some ordering on X). From this collection of topes, we then, can generate the collection of covectors $\mathcal{L}$ by the brute-force method at the end of Section 3.2. This is extremely unwieldy: Even though we only need to generate covectors with zero sets of cardinalities 1, 2, and 3, we still need to test all

$$2\Phi_3(|X|-1)\binom{|X|}{1}+2\Phi_3(|X|-1)\binom{|X|}{2}+2\Phi_3(|X|-1)\binom{|X|}{3} = 2\Phi_3(|X|-1)\big(\Phi_3(|X|)-1\big)$$

possible covectors against each of the $2\Phi_3(|X|-1)$ elements of $\mathcal{T}$.

However, we can generate all *subtopes* (the elements $S \in \mathcal{L}$ such that $|z(S)| = 1$), by making use of the following lemma (Björner *et al.*, 1999, Lemma 4.2.2.(c)):

Lemma 29. *Let $T \in \mathcal{T}$. There exists a tope $T_1 \in \mathcal{T}$ such that $s(T, T_1) = \{x\}$ if and only if there is a subtope $S \in \mathcal{L}$ such that $S < T$ and $z(S) = \{x\}$.*

Thus, we only need to compare each pair of topes, leading to on the order of N^2 operations. Then, we can repeat the same procedure on the subtopes (treating them as a tope set, in their own right) to yield the sub-subtopes (having zero set of cardinality 2) and, then, repeat once more on the sub-subtopes to obtain the cocircuits (with zero set of cardinality 3). In this way, we can generate $\mathcal{L}$ much more effectively.

After obtaining $\mathcal{L}$, we can test whether $\mathcal{M}/A$ is acyclic for all 3- and 2-point subsets A of X (as $\mathcal{M}$ has rank 4, $|z(C)| \leq 3$ for all $C \in \mathcal{L}$) in order to find the 2- and 1-faces (i.e., facets and edges) of the lattice, respectively (we already know that each $x \in X$ is a face, as $\mathcal{M}$ is a matroid polytope). We can do this efficiently with $\{-1, 0, 1\}$-vectors by finding the $\{0, 1\}$-vectors and taking the indices of their zero sets, as this gives exactly those sets in X who restrict to acyclic oriented matroids.

From this information, we can construct the unique simplicial sphere (i.e., triangulation) given by $\mathcal{M}$; in fact, if we only need the 1-skeleton, then we only need to consider the 2-point subsets of X.

This may provide a more efficient way to compute $G_{\mathscr{S}}$, if we are only given information of the splits. Furthermore, given a set of points distributed in convex and general

position, we can obtain the face lattice through the topes generated by linear program detailed above (see section 5.1).

5.2.2 Implementation and Results

The face lattice program was implemented in Python 3.7.2, using only the numpy package.

As a preliminary result, the output for the covector lattice calculated from the topes of the tetrahedron (as listed in Section 5.1) by the covector generation portion of the face lattice program are as follows:

Topes (16): (1, 1, 1, 1), (1, 1, 1, -1), (1, 1, -1, 1), (1, 1, -1, -1), (1, -1, 1, 1), (1, -1, 1, -1), (1, -1, -1, 1), (1, -1, -1, -1), (-1, 1, 1, 1), (-1, 1, 1, -1), (-1, 1, -1, 1), (-1, 1, -1, -1), (-1, -1, 1, 1), (-1, -1, 1, -1), (-1, -1, -1, 1), and (-1, -1, -1, -1);

Subtopes (32): (-1, 0, 1, 1), (0, -1, -1, 1), (0, -1, -1, -1), (1, 1, -1, 0), (-1, 0, 1, -1), (1, 0, 1, 1), (-1, -1, -1, 0), (-1, 0, -1, -1), (0, 1, 1, 1), (0, -1, 1, -1), (-1, -1, 0, 1), (1, -1, 1, 0), (-1, 1, 0, -1), (-1, 1, -1, 0), (1, 1, 0, 1), (-1, 1, 1, 0), (1, 0, -1, -1), (1, -1, 0, -1), (1, -1, -1, 0), (0, 1, -1, -1), (0, 1, -1, 1), (-1, -1, 1, 0), (1, -1, 0, 1), (1, 0, -1, 1), (1, 1, 0, -1), (1, 1, 1, 0), (-1, 1, 0, 1), (-1, -1, 0, -1), (0, -1, 1, 1), (0, 1, 1, -1), (-1, 0, -1, 1), and (1, 0, 1, -1);

Sub-subtopes (24): (0, 0, 1, -1), (0, 0, 1, 1), (-1, 0, -1, 0), (0, -1, 0, 1), (0, 1, -1, 0), (0, 1, 0, -1), (1, 0, -1, 0), (0, -1, -1, 0), (1, -1, 0, 0), (1, 0, 0, -1), (-1, 0, 0, 1), (0, 0, -1, -1), (-1, 1, 0, 0), (0, -1, 1, 0), (1, 0, 1, 0), (0, 1, 1, 0), (-1, 0, 1, 0), (-1, -1, 0, 0), (0, 0, -1, 1), (-1, 0, 0, -1), (1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 0, 1), and (0, -1, 0, -1);

Cocircuits (8): (-1, 0, 0, 0), (1, 0, 0, 0), (0, 0, 0, 1), (0, 0, -1, 0), (0, 0, 1, 0), (0, -1, 0, 0), (0, 1, 0, 0), and (0, 0, 0, -1).

The corresponding face lattice is (where the points are denoted $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$):

Facets (4): (1,2,3), (0,1,2), (0,1,3), and (0,2,3);

Edges (6): (0,1), (1,3), (0,3), (2,3), (1,2), and (0,2); and

Vertices (4): (1), (0), (2), and (3).

The lattice is depicted in Figure 5.4. Compared with Figure 2.4, if we adjoin top and

bottom elements to the lattice and relabel 0, 1, 2, and 3 as x, y, z, and w, we see that the lattices are equivalent.

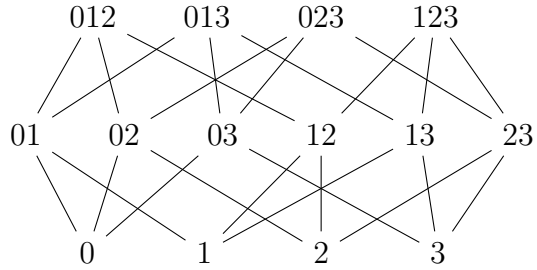


Figure 5.4: The face lattice generated from the topes of the tetrahedron.

Chapter 6

Conclusions

Finally, we summarise the findings of all we have done in this work and suggest directions for future research.

6.1 Conclusions

The concept and properties of affine, polytopal, and spherical splits have been introduced, along with the connections of collections of polytopal/spherical splits in the context of (maximal planar) graphs and (rank 4 uniform) oriented matroids.

In Chapter 2, splits, as bipartitions of elements, formed the starting point of our investigation. By embedding the (abstract) elements into the (concrete) ambient space $\mathbb{R}^3$, we were able to use the additional structure of linear subspaces (i.e., 2-dimensional hyperplanes) to define the idea of an affine split, from which we could discern fundamental combinatorial properties. The geometric properties of the embedding we used turned out to be the deciding factor in the combinatorics of the collection of affine splits, with a higher degree of coplanarity in the structure inducing a higher degree of redundancy in the collection. Thus, the minimal level of coplanarity, achieved when the points were embedded into general position, provided the maximal collection of affine splits.

Following on from this, in Section 2.2, we looked into the connection between affine splits and convex polytopes, showing that the polytopal collections of splits—those where the points are embedded as the vertices of a polytope—are exactly those for which all trivial splits (i.e., of the form $\{x\}|X \setminus \{x\}$) occur. Consequently, the maximal

polytopal collections of splits correspond exactly to those embeddings of points into convex and general position. In imposing a convexity condition on the embedded points, we created a stronger characterisation of the associated collection of splits.

In Section 2.3, spherical split collections were defined as those which correspond to embeddings of points into the sphere S^2. In light of this, we understand that all maximal collections of spherical splits must be those corresponding to embeddings of points into convex and general position in S^2. Intuitively, this is not much of a leap from the polytopal case, but we found polytope inscribability to be a very subtle topic. Whether or not there is a gap between the maximal polytopal and maximal spherical collections of splits remains a main open question of this work.

Then, in Chapter 3, we introduced oriented matroids through the covector and tope axiom sets and reviewed the basic definitions and results pertaining to the theory. Following this, we detailed the Vapnik–Chervonenkis dimension, which provided the context for bridging between maximal systems of affine splits and rank 4 uniform oriented matroids, through Theorem 13. We showed that these oriented matroids are acyclic, simple, and realisable, as well as corresponding to matroid polytopes if the split collection is polytopal or spherical. Furthermore, by investigating the big and affine face lattices, we found that the order complex of the lattice induced by a maximal collection of spherical splits is homeomorphic to a shelling of the 3-ball.

Next, in Chapter 4, we investigated the relationship between polytopal/spherical collections of splits and the graphs induced by the 1-skeletons of the associated polytopes, or, equivalently, the collection of 2-splits. We found that maximal collections of spherical splits induce inscribable 3-connected maximal planar graphs. Additionally, we deduced some contraction and restriction results for the subgraphs induced by the sets partitioned by a split. In forging graph-theoretic connections with spherical splits, we have gained additional combinatorial tools with which we may further investigate such collections. Considering the nuances of obtaining a split collection from a graph, we must consider what lies between combinatorics and geometry: The way in which we embed a graph has an effect on the geometry of the system which is created. Therefore, it is crucial to look more in-depth at the way in which we embed graphs to gain a better understanding of how to (or whether we can, with any consistency) generate collections of affine, polytopal, and/or spherical splits from a given graph.

Finally, in Chapter 5, a couple computational methods were outlined. The first was a linear program for generating the topes and hyperplanes of small point sets, which was

used to generate some of the illustrations in the text. The second method generated the polytope face lattice given the topes of a matroid polytope, and can work together with the aforementioned program to provide more combinatorial information about the structure of a point set.

6.2 Future Work

The theory we have built around spherical splits here is hardly complete; in fact, we have only scratched the surface. Much is yet to be uncovered, especially in terms of the relations to oriented matroid theory and the further reaches such research may provide. Additionally, there is still much yet to be understood from the aspect of inscribability, for which a deeper exploration of both graph-theoretic and geometric contexts is required.

As for future research directions, we reiterate the major open questions:

(Conjecture 1). Does every maximal collection of polytopal splits correspond to a maximal collection of spherical splits? We believe there may be an answer to the contrary (which is the content of the conjecture), and that the counterexample may be constructed as a forbidden deformation of closed convex curves in the plane to true circles, given a static point configuration. This may be similar to other questions asked, and may or may not be easily answerable.

If the conjecture is false, then this implies that all point sets in convex and general position induce a maximal collection of spherical splits, giving us a stronger combinatorial–geometric criterion for such collections. However, if it is true, then we are provided with evidence that spherical splits, in some sense, are influenced by the Euclidean properties of the ambient space more strongly than polytopal splits. This intuitively makes sense, as the defining characteristic of spherical splits is "more than" that of polytopal splits.

(Question 1). What are the sufficient conditions for a graph $G_{\mathscr{S}}$ to be induced by a collection of polytopal or spherical splits?

An answer to this question may lead towards a better understanding of the properties of polytopal/spherical splits, and may have some bearing on the theory of graph inscribability. However, it seems that this question may need more than just a little thought.

We hope that this work motivates new discoveries, be it in an abstract sense or a down-to-earth one.

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